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CE20BTECH11034

1)

a) $f(x) = x^T A x + b$ → convex function $(\nabla^2 f(x) > 0)$ PSD

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad b = [1]$$

↓ Symmetric

$$\nabla f = 2Ax = 0.$$

$$\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 2x - y + z = 0$$

$$-x + 2y = 0 \quad \Rightarrow x = z$$

$$-x + z = 0 \quad \Rightarrow x = 2y.$$

$$2x - \frac{x}{2} + x = 0 \Rightarrow \underline{x = 0} \quad \underline{y = 0} \ \& \ \underline{z = 0}$$

x^* → minimiser $\quad x^* = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$f(x^*) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + 1$$

$$f(x^*) = 0 + 1 = \underline{1} \rightarrow \text{global } \underline{\text{minimum}}$$

b) $f(x) = \|Ax-b\|^2 \quad \mathbb{R}^2 \to \mathbb{R}$ → Convex function $[\nabla^2 f \geq 0]$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

$$\nabla f(x) = 2A^T(Ax-b) = 0$$

$$\Rightarrow 2A^TAx - 2A^Tb = 0$$

$$A^TA = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 13 \\ 13 & 21 \end{bmatrix}$$

$$A^Tb = \begin{matrix} 1 \\ 2 \end{matrix} \begin{bmatrix} 20 \\ 30 \end{bmatrix}$$

$$\nabla f(x) = 0 \Rightarrow \begin{bmatrix} 14 & 13 \\ 13 & 21 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 15 \end{bmatrix}$$

$$14x + 13y = 10$$

$$13x + 21y = 15$$

$$\Rightarrow x^* = \begin{bmatrix} 0.12 \\ 0.64 \end{bmatrix} \to \text{global minima}$$

$$f(x^*) = \left\| \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 0.12 \\ 0.64 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \right\|^2 = 0.199 \to \text{global minimum}$$

c) $f(x) = \|Ax-b\|^2 : \mathbb{R}^3 \to \mathbb{R}$ → convex function $[\nabla^2 f \geq 0]$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 3 & 1 & 9 \\ 4 & 1 & 0 \\ 2 & 1 & 4 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \\ 9 \end{bmatrix}$$

$$\nabla f(x) = 2A^T(Ax-b) = 0$$

$$\Rightarrow A^TAx = A^Tb$$

$$A^TA = \begin{bmatrix} 34 & 19 & 40 \\ 19 & 23 & 23 \\ 40 & 23 & 102 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} 28 \\ 24 \\ 52 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 34 & 19 & 40 \\ 19 & 23 & 23 \\ 40 & 23 & 102 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 28 \\ 24 \\ 52 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} 34x + 19y + 40z &= 28 \\ 19x + 23y + 23z &= 24 \\ 40x + 23y + 102z &= 52 \end{aligned}\right\} \Rightarrow x^* = \begin{bmatrix} 0.05745 \\ 0.6568 \\ 0.3391 \end{bmatrix}$$

global minima

$$f(x^*) = \|Ax^*-b\|^2 = 56.99 \rightarrow \text{global minimum}$$