Algebraic Circuit Complexity

A journey

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Outline

Some History...

The "Holy Grail" of Complexity Theory is the $P \overset{?}{\neq} NP$ problem. This problem has been unexpectedly hard to solve.

Leslie Valiant hypothesised $VP \overset{?}{\neq} VNP$ problem as a stepping stone towards the $P \overset{?}{\neq} NP$ problem.
Recent works in Complexity Theory have been trying to show the separation between various classes that have sprung out of Valiant's initial classes.

Why do we need Circuits?

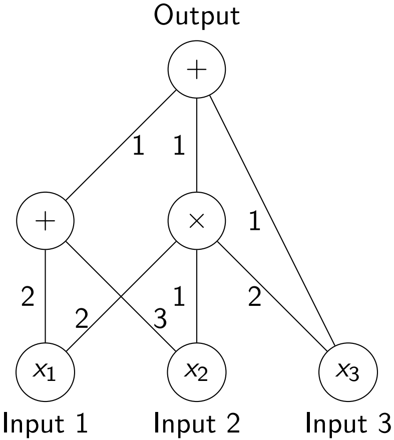
"What is the best way to compute a given polynomial $f(x_1, \cdots, x_n)$ from basic operations such as $+$ and $\times$?" This is the main motivating problem in the field of arithmetic circuit complexity. The notion of complexity of a polynomial is measured via the size of the smallest arithmetic circuit computing it. Arithmetic circuits provide a robust model of computation for polynomials. [Sap15]

Algebraic Circuit

Circuit

An **Algebraic Circuit** is formally a **Directed Acyclic Graph** with a unique *sink vertex* called the *root* such that each internal vertex is labelled as $+$ or $\times$.

Example



In this example, the final output will be $2x_1 + 3x_2 + 2x_1x_2 + x_3$

Complexity Classes

There are classes corresponding the **P** and **NP** called **Valiant's P (VP)** and **Valiant's NP (VNP)**.

- **VP**: Consists of polynomials whose size and degree are both polynomially bounded.
- **VNP**: Set of all polynomials $f \in \mathbb{F}[x_1, x_2, \cdots, x_n]$ such that there exists a polynomial in VP, $g \in \mathbb{F}[x_1, \cdots, x_n, y_1, \cdots, y_m]$, with $m = poly(n)$ and,

$$f(\mathbf{x}) = \sum_{\mathbf{a} \in \{0,1\}^m} g(\mathbf{x}, \mathbf{a})$$

It is easy to see $VP \subseteq VNP$.

The Permanent

It is known that the determinant $\in$ VP.
Permanent is similar to the determinant and is hard to compute (at least till now) and is thought to be in VNP (excluding VP).

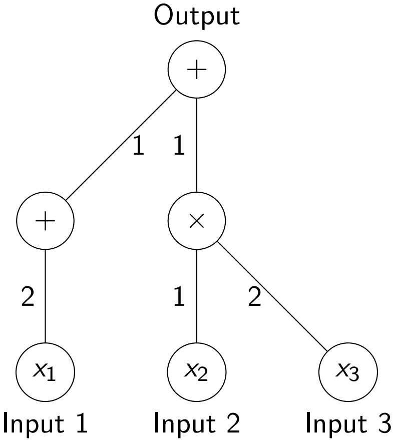
Formula

Formula

A formula is the same as a circuit with the constraint that any node present in the formula can have **at most one** outgoing edge.

These model the way we perform calculations on a piece of paper rather than a computer program.

Example



In this example, the final output will be $2x_1 + 2x_2x_3$

Complexity Classes

VF

The set of polynomials, $f \in \mathbb{F}[x_1, x_2, \cdots, x_n]$ that can be represented by a formula of size $poly(n)$ is called **VF**.

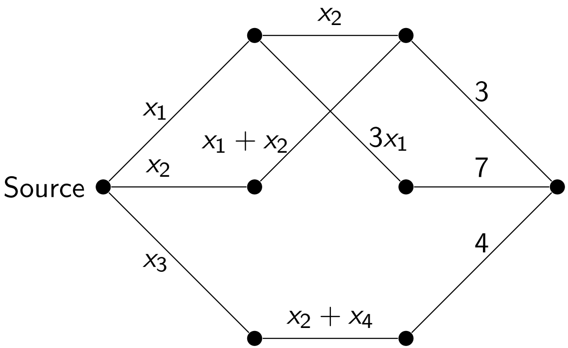
It is easy to see $VF \subseteq VP$.

Algebraic Branching Programs

ABP

An ABP is a *layered directed acyclic graph* with edges labelled with linear polynomials. There is a *source vertex (s)* and a *sink vertex (t)*.

Example



The final polynomial is calculated as

$$(x_3 \cdot (x_2 + x_4) \cdot 4) + (x_2 \cdot (x_1 + x_2) \cdot 3) + (x_1 \cdot x_2 \cdot 3) + (x_1 \cdot 3x_1 \cdot 7)$$

Complexity Classes

VBP

The class **VBP** is defined as the set of polynomials having size $poly(n)$.

It is known that $VF \subseteq VBP \subseteq VP \subseteq VNP$ and these separations are hypothesised to be strict.

Lower bounds on Sum of ABP's

Since finding general lower bounds has been proven to be hard, we will deal with finding lower bounds for *sum of special forms of ABP's* and relate them to general ABP's.
We will analyze **smABP's** and **ROABP's**

Set-Multilinear ABP

Consider a partition of a set of variables $\{X_1, X_2 \cdots X_d\}$.

Set-Multilinear

A *set-multilinear* polynomial is one in which is *homogeneous*, each variable has *individual degree at-most* 1*(multilinear)* and each monomial has *exactly one variable* from each of the d sets.

read-once Oblivious ABP

An ABP is said to be *oblivious* if, for each layer, all the edge labels are *univariate* polynomials in a single variable.
An ABP is called a **read-once oblivious ABP (ROABP)** if each variable appears in at most one layer. [BDS23b]

Hardness Bootstrapping

There is a theorem that shows how lower bounds on $\sum smABP$ can lead to the separation of classes. It states,

Theorem

Theorem 1: *Let n, d be integers such that $d = O(log\ n/log\ log\ n)$. Let $P_{n,d}$ be a set-multilinear polynomial in VNP of degree d. If $P_{n,d}$ cannot be computed by a $\sum smABP$ of width $poly(n)$, then $VBP \neq VNP$.*

[BDS23a]

Outline

We will perform a sequence of structural transformations to the ABP.

- First *homogenize* the ABP, ie. we modify the ABP such that each vertex of the ABP computes a homogeneous polynomial.
- In addition, we will ensure that the ABP has d layers and all the edge labels are *linear forms*.
- Then we set-multilinearize the ABP. This step is efficient for low-degrees only as we obtain a sum of $d^{O(d)}$ smABP's.

This means that superpolynomial bounds for $\sum$smABP imply the same for ABPs, albeit in the low degree regime.

Simulating ABP using $\sum$smABP

Lemma 1

If a set-multilinear polynomial, with d sets each of size $\leq n$ can be calculated by a ABP of *size* s, it can be calculated by a sum of smABP's of total width $d^{O(d)}s$.[BDS23b]

This can be directly used to prove **Theorem 1**.

The proof

Firstly, using the assumption that it cannot be computed note that the width of the $\sum$smABP is, $n^{\omega(1)}$, and $d^{O(d)}s \geq n^{\omega(1)}$. Using the fact $d = O(\log n/\log\log n)$ we get $d^{O(d)} = poly(n)$, hence $s = n^{\omega(1)}$, and we have our desired seperation.

Proof of Lemma 1

Lemma 2: *A degree d polynomial f that can be computed by an ABP of size s can also be computed by an homogeneous ABP of width s and length d.* [BDS23b]
Lemma 3: *If a homogenous ABP of width w and length d for a set-multilinear polynomial is needed, then the polynomial can be calculated by a $\sum$smABP of width $d!w$.* [BDS23b] These can be combined to prove the **Lemma 1**.

Homogenization

To homogenize an ABP we will

- Divide each vertex, v into d vertices such that each vertex $v^{(i)}$ computes the homogeneous part with degree i.
- Now, we will arrange all the vertex that compute the same degree in the same layer. This gives *width s* and length d, hence **Lemma 2**.
- This works because a vertex $v^{(i)}$ can only have edges going to $u^{(i)}$ or $u^{(i+1)}$ (each edge is labelled by a linear form)

ABP to $\sum$smABP

We begin by writing the ABP for a sm-polynomial in its IMM form,

$$P_{n,d} = \prod_{i}^{d} M_i,$$

Here each M is a matrix with $w \times w$ entries that are linear forms. Write each $M_i = \sum_{j=1}^{d} M_{ij}$, such that each M_{ij} has terms from the set X_j.

$$P_{n,d} = \prod_{i}^{d} \sum_{j=1}^{d} M_{ij}$$

[BDS23b]

ABP to $\sum$smABP

We can ignore the non-set-multilinear terms, i.e. ignore the products having terms like $M_{ij}, M_{i'j}$. This gives the form

$$P_{n,d} = \sum_{\pi \in S_d} \prod_{i=1}^{d} M_{i\pi(i)},$$

Hence we have a sum of d! smABP's each of width w.
[BDS23b]

Approximate Circuits...

There are a few polynomials that can be computed approximately in VP, if we just provide a little degree of freedom.
Instead of feeding polynomials in the inputs of the circuit we feed polynomials in ϵ and then calculate the polynomial $g \in \mathbb{F}[\epsilon][x_1, \cdots, x_n]$.
We say $f \in \mathbb{F}[x_1, \cdots, x_n]$ is approximated by g to an order of approximation M if

$$g(\boldsymbol{x}, \epsilon) = \epsilon^M f(\boldsymbol{x}) + \epsilon^{M+1} Q(\boldsymbol{x}, \epsilon)$$

Approximate Circuits...

If the size of g over the field $\mathbb{F}[\epsilon]$ is in VP, then f is in $\overline{\mathsf{VP}}$.
If we restrict the class such that the polynomials in ϵ used must have polynomial-sized circuits themselves, we get a new class called $\overline{\mathsf{VP}}_\epsilon$. This is also called *presentable VP*.

Similar classes can be defined in the cases of VNP.

Debordering VNP

It can be easily seen $VP \subseteq \overline{VP}_\epsilon \subseteq \overline{VP}$, and $VNP \subseteq \overline{VNP}_\epsilon \subseteq \overline{VNP}$. There is a theorem that relates these

Theorem 2: *Over any finite field $VNP = \overline{VNP}_\epsilon$.*

This gives new containments $VP \subseteq \overline{VP}_\epsilon \subseteq VNP$.

[BDS23a]

To prove this we will show $VNP \subseteq \overline{VNP}_\epsilon$ and $\overline{VNP}_\epsilon \subseteq VNP$. The first containment is trivial; for showing the second one, we need some techniques like *Interpolation* and *Valiant's Criterion*.

Interpolation

Consider a polynomial $P(x_1, \cdots, x_n, y)$ and $deg_y P = d$ then the polynomial can be written as

$$P(x_1, \cdots, x_n, y) = P_0(x_1, \cdots, x_n) + P_1(x_1, \cdots, x_n)y + \cdots + P_d(x_1, \cdots, x_n)y^d$$

Consider d distinct constants $\alpha_0, \cdots, \alpha_n$ and let $P(\alpha_i) = P(x_1, \cdots, x_n, \alpha_i)$. Each of the P_i can be calculated using the following matrix multiplication.

$$\begin{bmatrix} P_0 \\ P_1 \\ \vdots \\ P_d \end{bmatrix} = \begin{bmatrix} 1 & \alpha_0 & \cdots & \alpha_0^d \\ 1 & \alpha_1 & \cdots & \alpha_1^d \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_d & \cdots & \alpha_d^d \end{bmatrix}^{-1} \begin{bmatrix} P(\alpha_0) \\ P(\alpha_1) \\ \vdots \\ P(\alpha_d) \end{bmatrix}$$

[Sap15]

#P/poly

In complexity theory, P/poly refers to the set of problems that can be solved using polynomial size circuits.
#P refers to the problem of finding the number of satisfying assignments for a problem.
Hence it is easy to understand that #P/poly refers to the class of problems that are related to finding the number of satisfying assignments using small size circuits.

Valian't Criterion

Let $f = \sum_e c_e \boldsymbol{x}^{\boldsymbol{e}}$ *be a polynomial in* n *variables of degree* $poly(n)$ *over a field* $\mathbb{F}$*. Suppose that there exists a string function* $\phi : \{0,1\}^* \to \{0,1\}^*$ *in* $\#P/poly$ *such that* $\phi(\langle \boldsymbol{e} \rangle) = \langle c_{\boldsymbol{e}} \rangle$*. Then, the polynomial* f *is in VNP over the field* $\mathbb{F}$*.* [BDS23a]

Continuation of debordering...

Now that *interpolation* and *Valiant's Criterion* are covered, we can complete our proof. We have access to $f \in \overline{\mathsf{VNP}}_\epsilon$ using the approximation

$$g(\boldsymbol{x}, \epsilon) = \epsilon^M f(\boldsymbol{x}) + \epsilon^{M+1} Q(\boldsymbol{x}, \epsilon)$$

This is of the hypercube form,

$$g(\boldsymbol{x}, \epsilon) = \sum_{\boldsymbol{a} \in \{0,1\}^m} h(\boldsymbol{x}, \boldsymbol{a}, \epsilon)$$

...

- We will extract the coefficients of $\epsilon^M \boldsymbol{x}^{\boldsymbol{e}}$ in g using *interpolation* and taking the interpolation points to be the *roots of unity*. Consequently $c_{\boldsymbol{e}}$ can be obtained as a hypercube sum of an *exponential degree* circuit of *polynomial size*
- Using finite field arithmetic and the closure of the Boolean class #P under exponential sums, we can go to the *boolean world*
- Thus, we can show that the algebraic circuit above can be replaced by a (multi-output) *Boolean circuit* of polynomial size and the hypercube sum computing the coefficient function is demonstrated in $\#P/poly$
- Now using Valiant's criterion we can claim $\overline{VNP}_\epsilon \subseteq VNP \implies \overline{VNP}_\epsilon = VNP$

[BDS23a]

References

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