An Alphabet of Leakage Measures

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***Abstract*—We introduce a family of information leakage measures called *maximal* α, β*-leakage*, parameterized by real numbers α and β. The measure is formalized via an operational definition involving an adversary guessing an unknown function of the data given the released data. We obtain a simple, computable expression for the measure and show that it satisfies several basic properties such as monotonicity in β for a fixed α, non-negativity, data processing inequalities, and additivity over independent releases. Finally, we highlight the relevance of this family by showing that it bridges several known leakage measures, including maximal α-leakage ($\beta = 1$), maximal leakage ($\alpha = \infty, \beta = 1$), local differential privacy ($\alpha = \infty, \beta = \infty$), and local Rényi differential privacy ($\alpha = \beta$).**

I. Introduction

How much information does an observation released to an adversary reveal/leak about correlated sensitive data? This fundamental question arises in many secrecy and privacy problems whenever data about users is stored in a database (e.g., social networks and cloud-based services) and a certain level of information leakage is unavoidable in exchange for certain services. The release of an observation to an adversary could be either an inadvertent result of a design via a side channel or intentional, e.g., in the context of querying databases. Various measures of information leakage have been proposed over past few years, see e.g., [1]–[9].

For any leakage measure, one of the key challenges is to associate an operational interpretation in terms of its definition. Only a few leakage measures possess such an operational meaning. For example, the works in [3], [5], [6], which pertain to the release of observation due to a side channel, measure privacy in terms of an adversary's gain in *guessing* the sensitive data after observing the released data. Issa *et al.* [5] introduce maximal leakage (MaxL), which quantifies the maximal logarithmic gain in the probability of correctly guessing any arbitrary function of the original data from the released data. Liao *et al.* [6] later generalized maximal leakage to a family of leakages, maximal α-leakage (Max-αL), that allows tuning the measure to specific applications. In particular, similar to MaxL, Max-αL quantifies the maximal logarithmic gain in a monotonically increasing power function (dependent on α) applied to the probability of correctly guessing.

Among leakage measures motivated by worst-case adversaries, differential privacy (DP) [7] has emerged as the gold standard. A differentially private algorithm guarantees that its outputs restrict the adversary from distinguishing between neighboring data entries. When privacy guarantees have to be provided in a distributed setting, local DP (LDP) [8], [9] provides such strong guarantees for every pair of (users) data entries. In the context of composing DP outputs sequentially, Rényi differential privacy (RDP) [4] has emerged as a meaningful variant to compute overall DP guarantees. Specifically, RDP relaxes DP based on the Rényi divergence.

No single measure of privacy/information leakages suits all the scenarios in practice. In this paper, we undertake the study of unifying various measures of information leakage so that the leakage measure can be tailored to different settings depending on the context. Motivated by [5], [6], we introduce a leakage measure, *maximal* α, β*-leakage*, which is parameterized by two real numbers α and β. We obtain a simple computable expression for it and show that this family of measures encompasses a host of existing leakage measures: in particular, Max-αL ($\beta = 1$), MaxL ($\alpha = \infty, \beta = 1$), LDP ($\alpha = \beta = \infty$), and local Rényi differential privacy (LRDP) ($\alpha = \beta$) — a notion of RDP defined analogous to LDP (see Figure 1). An important consequence of our result is an operational interpretation of LDP and LRDP. We note that this subsumes an operational meaning of LDP given by Issa *et al.* [5] via defining maximal realizable leakage, a notion of leakage measure concerned with worst-case analysis similar to LDP. However, maximal α, β-leakage is defined in terms of average-case analysis (in the spirit of MaxL and Max-αL) and it still recovers LDP which concerns with worst-case scenario by exploiting the interplay between the parameters α and β. We also show that this general leakage measure satisfies all the axiomatic properties of a measure of information leakage, including non-negativity, equality to zero if and only if the original data and the released data are independent of each other, and data-processing inequalities. This also infers that LDP satisfies both the data processing (i.e., post-processing and linkage) inequalities unlike DP which does not satisfy the linkage inequality [10].

II. Preliminaries

We begin by reviewing the definitions of some existing information leakage measures, in particular, maximal α-leakage (which subsumes maximal leakage) and local (Rényi) differential privacy.

Definition 1 (Maximal α-leakage [6])**.** *Let P_{XY} be a joint distribution, where X and Y represent the original data and the released data, respectively. The maximal α-leakage from X to Y, for $\alpha \in (1, \infty)$, is defined as*

$$\mathcal{L}_{\alpha}^{\max}(X \to Y)$$

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$$= \sup_{U-X-Y} \log \frac{\max_{P_{\hat{U}|Y}} \sum_{u,y} P_{UY}(u,y) P_{\hat{U}|Y}(u|y)^{\frac{\alpha-1}{\alpha}}}{\max_{P_{\hat{U}}} \sum_u P_U(u) P_{\hat{U}}(u)^{\frac{\alpha-1}{\alpha}}}, \quad (1)$$

where U represents any randomized function of X that takes values in a arbitrary finite alphabet and $\hat{U}$ is an estimator of U with the same support as U.

Maximal α-leakage is a generalization of another measure of information leakage, the maximal leakage [5]. In particular, the latter recovers the former when $\alpha = \infty$. Liao *et al.* [6] showed that

$$\mathcal{L}_\alpha^{\max}(X \to Y) = \sup_{P_{\tilde{X}}} I_\alpha^{\mathrm{S}}(\tilde{X}; Y), \quad (2)$$

where the supremum is over all the probability distributions $P_{\tilde{X}}$ on the support of P_X and $I_\alpha^{\mathrm{S}}(\cdot;\cdot)$ is the Sibson mutual information of order α [11].

Definition 2 (Local differential privacy [8], [9])**.** *Given a conditional distribution $P_{Y|X}$, the local differential privacy (LDP) is defined as*

$$\mathcal{L}^{\mathrm{LDP}}(X \to Y) = \max_{\substack{y \in \mathcal{Y},\\ x,x' \in \mathcal{X}}} \log \frac{P_{Y|X}(y|x)}{P_{Y|X}(y|x')}. \quad (3)$$

We may define local Rényi differential privacy as a generalization of local differential privacy based on the Rényi divergence [12].

Definition 3 (Local Rényi differential privacy)**.** *Given a conditional distribution $P_{Y|X}$, the local differential privacy (LRDP) is defined as*

$$\mathcal{L}^{\mathrm{LRDP}}(X \to Y) = \max_{\substack{y \in \mathcal{Y},\\ x,x' \in \mathcal{X}}} \frac{1}{\alpha - 1} \log \sum_y P_{Y|X}(y|x')^{1-\alpha} P_{Y|X}(y|x)^{\alpha}. \quad (4)$$

It can be verified using L'Hôpital's rule that LRDP simplifies to LDP as $\alpha \to \infty$.

III. Maximal α, β-Leakage

Motivated by the definitions of maximal leakage and maximal α-leakage, we introduce maximal α, β-leakage as follows.

Definition 4 (Maximal α, β-leakage)**.** *Given a conditional distribution $P_{Y|X}$, the maximal α, β-leakage from X to Y for $\alpha \in (1, \infty)$ and $\beta \in [1, \infty)$ is defined as*

$$\mathcal{L}_{\alpha,\beta}(X \to Y) := \sup_{P_X} \sup_{U \to X \to Y} \frac{\alpha}{\alpha - 1} \log \frac{\max_{P_{\hat{U}|Y}} \left[\sum_y P_Y(y) \left(\sum_u P_{U|Y}(u|y) P_{\hat{U}|Y}(u|y)^{\frac{\alpha-1}{\alpha}} \right)^{\beta} \right]^{1/\beta}}{\max_{P_{\hat{U}}} \sum_u P_U(u) P_{\hat{U}}(u)^{\frac{\alpha-1}{\alpha}}}. \quad (5)$$

where $\hat{U}$ represents an estimator taking values from the same arbitrary finite alphabet as U. It is defined by continuous extension for $\alpha = \infty$ or $\beta = \infty$

We remark that the definition of maximal α, β-leakage in (5) nearly recovers the definition of maximal α-leakage from (1) (and thus maximal leakage also) when $\beta = 1$. While the definition of Max-αL does not include the supremum over P_X, as shown in (2), Max-αL has an implicit supremum over P_X, so including this supremum does not change the value. We have included the supremum in the definition of maximal α, β-leakage in order to recover some of the worst-case measures such as LDP and LRDP as special cases. One can view the introduction of β into the summation in the numerator in (5) as allowing a continuous transition from a simple average over y (at $\beta = 1$) to a maximum over y (at $\beta = \infty$). This maximum over y is present in the definition of maximal *realizable* leakage [5, Definition 8], which corresponds to $\alpha = \beta = \infty$, and has been shown to be equal to LDP. This allows us to view maximal leakage and maximal realizable leakage as two corner points of the inner optimization problem in (5) for $\alpha = \infty$ and $\beta = 1$ and $\beta = \infty$, respectively (see also Fig. 1).

The following theorem simplifies the expression of α, β-leakage in (5).

Theorem 1. *Maximal α, β-leakage defined in* (5) *simplifies to*

$$\mathcal{L}_{\alpha,\beta}(X \to Y) = \max_{x'} \sup_{P_{\tilde{X}}} \frac{\alpha}{(\alpha-1)\beta} \log \sum_y P_{Y|X}(y|x')^{1-\beta} \left(\sum_x P_{\tilde{X}}(x) P_{Y|X}(y|x)^{\alpha} \right)^{\beta/\alpha}, \quad (6)$$

where $P_{\tilde{X}}$ is a probability distribution on the support of P_X.

A detailed proof for Theorem 1 is given in Section V-A. For $\beta \le \alpha$, the quantity inside the log in (6) is concave in $P_{\tilde{X}}$; thus the supremum over $P_{\tilde{X}}$ can be efficiently solved using convex optimization techniques. As we will show in Section IV, for $\beta \ge \alpha$, the supremum over $P_{\tilde{X}}$ can be replaced by a maximum over $x \in \mathcal{X}$. Thus, in either case the quantity in (6) can be efficiently computed for finite alphabets.

Like other leakage measures, maximal α, β-leakage satisfies several basic properties such as non-negativity, data processing inequalities and additivity, as shown in the following theorem.

Theorem 2. *For $\alpha \in (1, \infty)$ and $\beta \in [1, \infty)$, maximal α, β-leakage*

1) *is monotonically non-decreasing in β for a fixed α;*
2) *satisfies data processing inequalities, i.e., for the Markov chain $X - Y - Z$:*

$$\mathcal{L}_{\alpha,\beta}(X \to Z) \le \mathcal{L}_{\alpha,\beta}(X \to Y) \quad (7a)$$

$$\mathcal{L}_{\alpha,\beta}(X \to Z) \le \mathcal{L}_{\alpha,\beta}(Y \to Z). \quad (7b)$$

3) *is non-negative, i.e.,*

$$\mathcal{L}_{\alpha,\beta}(X \to Y) \ge 0 \quad (8)$$

with equality if and only if X and Y are independent.

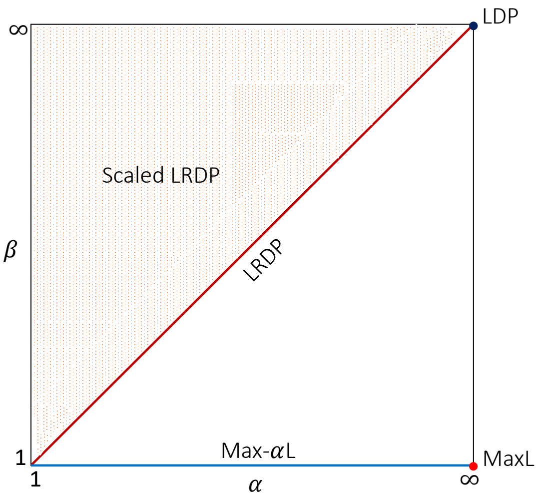

Fig. 1. Relationship between maximal α, β-leakage and other leakage measures. In particular, the family of measures includes as special cases maximal α-leakage (Max-αL), maximal leakage (MaxL), local differential privacy (LDP), and local Rényi differential privacy (LRDP).

4) *satisfies additivity: i.e., if* (X_1, Y_1) *and* (X_2, Y_2) *are independent, then*

$$\mathcal{L}_{\alpha,\beta}(X_1, X_2 \to Y_1, Y_2) = \sum_{i\in\{1,2\}} \mathcal{L}_{\alpha,\beta}(X_i \to Y_i). \tag{9}$$

A detailed proof for Theorem 2 is given in Section V-B.

Remark 1. Although maximal α, β-leakage is monotonic in only one of its orders, if we consider a reparameterization in which $\tau \in [0, 1]$ and $\beta = \frac{\alpha}{1-\tau(1-\alpha)}$, the new leakage measure is non-increasing in τ for a fixed α, and non-decreasing in α for a fixed τ. A detailed proof is given in Section V-C.

IV. Relationship between Maximal α, β-Leakage, and Other Measures

As mentioned earlier, maximal α, β-leakage recovers maximal α-leakage (and thus maximal leakage) when $\beta = 1$. The choices of α and β help to recover other leakage measures such as a scaled LRDP for $\alpha \le \beta$, LRDP for $\alpha = \beta$, and LDP for $\alpha = \infty$ and $\beta = \infty$, as shown in Fig. 1. We also show a simplification to the leakage measure when $\alpha = \infty$ and β is arbitrary. We now present these in detail.

When $\alpha \le \beta$, we have

$$\begin{aligned}
&\mathcal{L}_{\alpha,\beta}(X \to Y) \\
&= \max_{x'} \sup_{P_{\tilde{X}}} \frac{\alpha}{(\alpha-1)\beta} \log \sum_y P_{Y|X}(y|x')^{1-\beta} \\
&\quad \left(\sum_x P_{\tilde{X}}(x) P_{Y|X}(y|x)^{\alpha} \right)^{\beta/\alpha} &(10)\\
&= \max_{x'} \max_{x} \frac{\alpha}{(\alpha-1)\beta} \log \sum_y P_{Y|X}(y|x')^{1-\beta} P_{Y|X}(y|x)^{\beta} &(11)
\end{aligned}$$

where (11) follows because the argument of the logarithm in (10) is convex in $P_{\tilde{X}}$ and so the supremum is attained at an extreme point. This quantity is a scaled LRDP of order β. Furthermore, if $\alpha = \beta$, the expression in (11) reduces to

$$\begin{aligned}
&\mathcal{L}_{\beta,\beta}(X \to Y) \\
&= \max_{x'} \max_{x} \frac{1}{\beta-1} \log \sum_y P_{Y|X}(y|x')^{1-\beta} P_{Y|X}(y|x)^{\beta}, &(12)
\end{aligned}$$

which is exactly LRDP of order β. If $\alpha = \infty$ and $\beta = \infty$, then

$$\begin{aligned}
\mathcal{L}_{\infty,\infty}(X \to Y) &= \max_{x'} \log \max_y \frac{\max_x P_{Y|X}(y|x)}{P_{Y|X}(y|x')} &(13)\\
&= \max_{x,x',y} \log \frac{P_{Y|X}(y|x)}{P_{Y|X}(y|x')} &(14)
\end{aligned}$$

which is LDP. So $\mathcal{L}_{\infty,\beta}(X \to Y)$ passes from maximal leakage at $\beta = 1$ to LDP at $\beta = \infty$.

If $\alpha = \infty$ but β is arbitrary, then we have

$$\begin{aligned}
&\mathcal{L}_{\infty,\beta}(X \to Y) \\
&= \max_{x'} \frac{1}{\beta} \log \sum_y P_{Y|X}(y|x')^{1-\beta} \max_x P_{Y|X}(y|x)^{\beta}. &(15)
\end{aligned}$$

This quantity differs from LRDP of order β only in that the max over x is inside the summation over y rather than outside.

Finally, we propose the following extension for maximal α, β-leakage so as to include the data of multiple users:

$$\mathcal{L}_{\alpha,\beta}(X^n \to Y) := \sup_{P_{X^n}} \max_i \sup_{U \to X^n \to Y} \frac{\alpha}{(\alpha-1)\beta} \log$$
$$\frac{\displaystyle\sup_{P_{\hat{U}|X_{-i},Y}} \sum_{x_{-i},y} P_{X_{-i},Y}(x_{-i},y) \left[\sum_u \frac{P_{U|X_{-i},Y}(u|x_{-i},y)}{P_{\hat{U}|X_{-i},Y}(u|x_{-i},y)^{-\frac{\alpha-1}{\alpha}}} \right]^{\beta}}{\displaystyle\sup_{P_{\hat{U}|X_{-i}}} \sum_{x_{-i}} P_{X_{-i}}(x_{-i}) \left[\sum_u \frac{P_{U|X_{-i}}(u|x_{-i})}{P_{\hat{U}|X_{-i}}(u|x_{-i})^{-\frac{\alpha-1}{\alpha}}} \right]^{\beta}}. \tag{16}$$

where X^n and X_i represent a dataset with n entries and the i^{th} entry of X^n, respectively. It is clear that this measure encompasses Max-αL ($\beta = 1$), MaxL ($\alpha = \infty, \beta = 1$), LDP ($\alpha = \infty, \beta = \infty$), and LRDP ($\alpha = \beta$) for $n = 1$, all of which concern with data of a single user. We conjecture that, for $n > 1$, it recovers DP [7] ($\alpha = \infty, \beta = \infty$) and RDP [4] ($\alpha = \beta$) which concern with data of multiple users and protect privacy between neighbouring databases.

V. Proofs

A. Proof of Theorem 1

For $\alpha \in (1, \infty)$ and $\beta \in [1, \infty)$, we first bound $\mathcal{L}_{\alpha,\beta}(X \to Y)$ from above and then, give an achievable scheme.

Upper Bound: Consider the optimization in the denominator of (5):

$$\max_{P_{\hat{U}}} \sum_u P_U(u) P_{\hat{U}}(u)^{\frac{\alpha-1}{\alpha}}. \tag{17}$$

This is solved by

$$P_U(u)P_{\hat{U}}(u)^{-1/\alpha} = \nu \tag{18}$$

for some constant ν. So we have

$$P_{\hat{U}}(u) = \frac{P_U(u)^\alpha}{\sum_{u'} P_U(u')^\alpha}. \tag{19}$$

Thus the denominator becomes

$$\sum_u P_U(u)\left(\frac{P_U(u)^\alpha}{\sum_{u'} P_U(u')^\alpha}\right)^{\frac{\alpha-1}{\alpha}} = \left(\sum_u P_U(u)^\alpha\right)^{\frac{1}{\alpha}}. \tag{20}$$

Similarly the numerator becomes

$$\left[\sum_y P_Y(y)\left(\sum_u P_{U|Y}(u|y)^\alpha\right)^{\beta/\alpha}\right]^{1/\beta}. \tag{21}$$

Thus, the logarithmic term in (5) reduces to

$$\log \frac{\left[\sum_y P_Y(y)\left(\sum_u P_{U|Y}(u|y)^\alpha\right)^{\beta/\alpha}\right]^{1/\beta}}{\left(\sum_u P_U(u)^\alpha\right)^{1/\alpha}} \tag{22}$$

$$= \log \frac{\left[\sum_y P_Y(y)^{1-\beta}\left(\sum_u P_{U,Y}(u,y)^\alpha\right)^{\beta/\alpha}\right]^{1/\beta}}{\left(\sum_u P_U(u)^\alpha\right)^{1/\alpha}} \tag{23}$$

$$= \frac{1}{\beta}\log \sum_y P_Y(y)^{1-\beta}\left[\frac{\sum_u P_U(u)^\alpha P_{Y|U}(y|u)^\alpha}{\sum_u P_U(u)^\alpha}\right]^{\frac{\beta}{\alpha}}. \tag{24}$$

Using Jensen's inequality and the Markov chain $U - X - Y$, we have

$$P_{Y|U}(y|u)^\alpha = \left(\sum_x P_{X|U}(x|u)P_{Y|X}(y|x)\right)^\alpha \tag{25}$$

$$\leq \sum_x P_{X|U}(x|u)P_{Y|X}(y|x)^\alpha. \tag{26}$$

So maximal α, β-leakage may be bounded from above by

$$\mathcal{L}_{\alpha,\beta}(X \to Y) \leq \sup_{P_X} \sup_{U \to X \to Y} \frac{\alpha}{(\alpha-1)\beta} \log \sum_y P_Y(y)^{1-\beta} \left[\frac{\sum_{u,x} P_U(u)^\alpha P_{X|U}(x|u)P_{Y|X}(y|x)^\alpha}{\sum_u P_U(u)^\alpha}\right]^{\frac{\beta}{\alpha}} \tag{27}$$

$$\leq \sup_{P_X} \sup_{P_{\tilde{X}}} \frac{\alpha}{(\alpha-1)\beta} \log \sum_y P_Y(y)^{1-\beta} \left[\sum_x P_{\tilde{X}}(x) P_{Y|X}(y|x)^\alpha\right]^{\frac{\beta}{\alpha}} \tag{28}$$

where

$$P_{\tilde{X}}(x) = \frac{\sum_u P_U(u)^\alpha P_{X|U}(x|u)}{\sum_u P_U(u)^\alpha}. \tag{29}$$

Lower Bound: The proof is based on the expression in (24) as well as "shattering" method. Consider a random variable U such that $U \to X \to Y$ form a Markov chain and $H(X|U) = 0$. For each x, let $\mathcal{U}_x$ be a finite set such that $U = u \in \mathcal{U}_x$ if and only if $X = x$ and $\mathcal{U} = \bigcup_{x\in\mathcal{X}} \mathcal{U}_x$. Moreover, given $X = x$ let U be uniformly distributed on $\mathcal{U}_x$. That is,

$$P_{U|X}(u|x) = \begin{cases} \frac{1}{|\mathcal{U}_x|} & \text{for all } u \in \mathcal{U}_x \\ 0 & \text{otherwise,} \end{cases} \tag{30}$$

and so

$$P_{Y|U}(y|u) = \begin{cases} P_{Y|X}(y|x) & \text{for all } u \in \mathcal{U}_x \\ 0 & \text{otherwise.} \end{cases} \tag{31}$$

Therefore, we have

$$\frac{\sum_u P_U(u)^\alpha P_{Y|U}(y|u)^\alpha}{\sum_u P_U(u)^\alpha} \tag{32}$$

$$= \frac{\sum_{x\in\mathcal{X}}\sum_{u\in\mathcal{U}_x}\left(\frac{P_X(x)P_{U|X}(u|x)}{P_{X|U}(x|u)}\right)^\alpha P_{Y|U}(y|u)^\alpha}{\sum_{x\in\mathcal{X}}\sum_{u\in\mathcal{U}_x}\left(\frac{P_X(x)P_{U|X}(u|x)}{P_{X|U}(x|u)}\right)^\alpha} \tag{33}$$

$$= \frac{\sum_x |\mathcal{U}_x|^{1-\alpha} P_X(x)^\alpha P_{Y|X}(y|x)^\alpha}{\sum_x |\mathcal{U}|^{1-\alpha} P_X(x)^\alpha}. \tag{34}$$

So we may bound maximal α, β-leakage from below by

$$\mathcal{L}_{\alpha,\beta}(X \to Y) \geq \sup_{P_X} \sup_{\mathcal{U}_x} \frac{\alpha}{(\alpha-1)\beta} \log \sum_y P_Y(y)^{1-\beta} \left(\frac{\sum_x |\mathcal{U}_x|^{1-\alpha} P_X(x)^\alpha P_{Y|X}(y|x)^\alpha}{\sum_x |\mathcal{U}|^{1-\alpha} P_X(x)^\alpha}\right)^{\frac{\beta}{\alpha}} \tag{35}$$

$$= \sup_{P_X, P_{\tilde{X}}} \frac{\alpha}{(\alpha-1)\beta} \log \sum_y P_Y(y)^{1-\beta} \Big(\sum_x P_{\tilde{X}}(x) P_{Y|X}(y|x)^\alpha\Big)^{\frac{\beta}{\alpha}} \tag{36}$$

where here

$$P_{\tilde{X}}(x) = \frac{|\mathcal{U}_x|^{1-\alpha} P_X(x)^\alpha}{\sum_x |\mathcal{U}|^{1-\alpha} P_X(x)^\alpha}, \tag{37}$$

and we have used the fact that any distribution $P_{\tilde{X}}(x)$ can be reached with appropriate choice of $|\mathcal{U}_x|$, assuming $P_X(x) > 0$ for all x; this condition can be assumed because any P_X is arbitrarily close to a distribution with full support. Thus, combining (28) and (36), we have

$$\mathcal{L}_{\alpha,\beta}(X \to Y) = \sup_{P_X} \sup_{P_{\tilde{X}}} \frac{\alpha}{(\alpha-1)\beta} \log \sum_y P_Y(y)^{1-\beta} \left(\sum_x P_{\tilde{X}}(x) P_{Y|X}(y|x)^\alpha\right)^{\frac{\beta}{\alpha}}. \tag{38}$$

Since the choice of P_X only impacts P_Y, and the supremum of a convex function is attained at an extreme point, we may simplify (38) as follows.

$$\mathcal{L}_{\alpha,\beta}(X \to Y) = \max_{x'} \sup_{P_{\tilde{X}}} \frac{\alpha}{(\alpha-1)\beta} \log \sum_y P_{Y|X}(y|x')^{1-\beta} \left(\sum_x P_{\tilde{X}}(x) P_{Y|X}(y|x)^\alpha \right)^{\beta/\alpha}. \tag{39}$$

B. Proof of Theorem 2

Monotonicity in β: For $\alpha \in (1,\infty)$, $\beta_1, \beta_2 \in [1,\infty)$ and $\beta_2 > \beta_1$, consider the argument of the logarithm in (6):

$$\sum_y P_{Y|X}(y|x')^{1-\beta_1} \left[\sum_x P_{\tilde{X}}(x) P_{Y|X}(y|x)^\alpha \right]^{\frac{\beta_1}{\alpha}} \tag{40}$$

$$= \sum_y P_{Y|X}(y|x') \Bigg[P_{Y|X}(y|x')^{-\alpha} \sum_x P_{\tilde{X}}(x) P_{Y|X}(y|x)^\alpha \Bigg]^{\frac{\beta_2\beta_1}{\alpha\beta_2}} \tag{41}$$

$$\leq \Bigg[\sum_y P_{Y|X}(y|x') \Big(P_{Y|X}(y|x')^{-\alpha} \sum_x P_{\tilde{X}}(x) P_{Y|X}(y|x)^\alpha \Big)^{\frac{\beta_2}{\alpha}} \Bigg]^{\frac{\beta_1}{\beta_2}} \tag{42}$$

$$= \left[\sum_y P_{Y|X}(y|x')^{1-\beta_2} \left(\sum_x P_{\tilde{X}}(x) P_{Y|X}(y|x)^\alpha \right)^{\frac{\beta_2}{\alpha}} \right]^{\frac{\beta_1}{\beta_2}} \tag{43}$$

where the inequality results from applying Jensen's inequality to the concave function $f:\ x \to x^p\ (x \geq 0,\ p < 1)$. For $\alpha \in (1,\infty)$ and $\beta \in [1,\infty)$, the function $f: t \to \frac{\alpha}{(\alpha-1)\beta} \log t$ is increasing in $t > 0$. Therefore, we have

$$\frac{\alpha}{(\alpha-1)\beta_1} \log \sum_y P_{Y|X}(y|x')^{1-\beta_1} \Bigg[\sum_x P_{\tilde{X}}(x) P_{Y|X}(y|x)^\alpha \Bigg]^{\frac{\beta_1}{\alpha}}$$
$$\leq \frac{\alpha}{(\alpha-1)\beta_2} \log \sum_y P_{Y|X}(y|x')^{1-\beta_2} \Bigg[\sum_x P_{\tilde{X}}(x) P_{Y|X}(y|x)^\alpha \Bigg]^{\frac{\beta_2}{\alpha}}. \tag{44}$$

Taking maximum over x' and supremum over $P_{\tilde{X}}$ complete the proof.

Data processing inequalities: Let random variables X, Y, Z form a Markov chain, i.e., $X - Y - Z$. Based on the expression of maximal α,β-leakage in (38) we first prove that

$$\mathcal{L}_{\alpha,\beta}(X \to Z) \leq \mathcal{L}_{\alpha,\beta}(X \to Y). \tag{45}$$

For any $y \in \mathcal{Y}$, let

$$g(y) = \left(\sum_x P_{\tilde{X}}(x) P_{Y|X}(y|x)^\alpha \right)^{\frac{1}{\alpha}} \tag{46}$$

and

$$c_z(y) = \frac{P_Y(y)\ P_{Z|Y}(z|y)}{P_Z(z)} \tag{47}$$

such that $\sum_y c_z(y) = 1$. Applying Jensen's inequality to the convex function $f:\ x \to x^p\ (x \geq 0,\ p \geq 1)$, we have

$$\sum_y c_z(y) \left(\frac{g(y)}{P_Y(y)} \right)^\beta \geq \left(\sum_y c_z(y) \frac{g(y)}{P_Y(y)} \right)^\beta, \tag{48}$$

and using the definition of $c_z(y)$ in (47) we get

$$\sum_y P_Y(y)\ P_{Z|Y}(z|y) \left(\frac{g(y)}{P_Y(y)} \right)^\beta \geq P_Z(z) \left(\sum_y \frac{P_{Z|Y}(z|y)}{P_Z(z)} g(y) \right)^\beta. \tag{49}$$

We now take a summation over z. Thus,

$$\sum_z \sum_y P_Y(y)\ P_{Z|Y}(z|y) \left(\frac{g(y)}{P_Y(y)} \right)^\beta \geq \sum_z P_Z(z) \left(\sum_y \frac{P_{Z|Y}(z|y)}{P_Z(z)} g(y) \right)^\beta. \tag{50}$$

Therefore,

$$\sum_y P_Y(y)^{1-\beta} g(y)^\beta \geq \sum_z P_Z(z)^{1-\beta} \left(\sum_y P_{Z|Y}(z|y) g(y) \right)^\beta. \tag{51}$$

Thus, using the definition of $g(y)$ in (46) we have

$$\sum_y P_Y(y)^{1-\beta} \left(\sum_x P_{\tilde{X}}(x) P_{Y|X}(y|x)^\alpha \right)^{\frac{\beta}{\alpha}}$$
$$\geq \sum_z P_Z(z)^{1-\beta} \Bigg(\sum_y P_{Z|Y}(z|y) \Big(\sum_x P_{\tilde{X}}(x) P_{Y|X}(y|x)^\alpha \Big)^{\frac{1}{\alpha}} \Bigg)^\beta \tag{52}$$

$$= \sum_z P_Z(z)^{1-\beta} \Bigg(\sum_y \Bigg(\sum_x \Big(P_{\tilde{X}}(x)^{\frac{1}{\alpha}} P_{Z|Y}(z|y) P_{Y|X}(y|x) \Big)^\alpha \Bigg)^{\frac{1}{\alpha}} \Bigg)^\beta \tag{53}$$

$$\geq \sum_z P_Z(z)^{1-\beta} \Bigg(\Bigg(\sum_x \Big(\sum_y P_{\tilde{X}}(x)^{\frac{1}{\alpha}} P_{Z|Y}(z|y)$$

$$P_{Y|X}(y|x)\Big)^{\alpha}\Big)^{\frac{1}{\alpha}}\Bigg)^{\beta} \quad (54)$$

$$=\sum_z P_Z(z)^{1-\beta}\Bigg(\sum_x P_{\tilde{X}}(x)\Big(\sum_y P_{Z|Y}(z|y) P_{Y|X}(y|x)\Big)^{\alpha}\Bigg)^{\frac{\beta}{\alpha}} \quad (55)$$

$$=\sum_z P_Z(z)^{1-\beta}\left(\sum_x P_{\tilde{X}}(x)P_{Z|X}(z|x)^{\alpha}\right)^{\frac{\beta}{\alpha}} \quad (56)$$

where

- (54) follows because p-norm satisfies the triangle inequality for $p \in (1,\infty)$,
- (56) follows because the Markov chain $X-Y-Z$ holds.

For $\alpha \in (1,\infty)$ and $\beta \in [1,\infty)$, the function $f : t \to \frac{\alpha}{(\alpha-1)\beta}\log t$ is increasing in $t>0$. Therefore, we have

$$\frac{\alpha}{(\alpha-1)\beta}\log\sum_y P_Y(y)^{1-\beta}\left(\sum_x P_{\tilde{X}}(x)P_{Y|X}(y|x)^{\alpha}\right)^{\frac{\beta}{\alpha}} \quad (57)$$

$$\geq \frac{\alpha}{(\alpha-1)\beta}\log\sum_z P_Z(z)^{1-\beta}\left(\sum_x P_{\tilde{X}}(x)P_{Z|X}(z|x)^{\alpha}\right)^{\frac{\beta}{\alpha}}. \quad (58)$$

Taking suprema over P_X and $P_{\tilde{X}}$ completes the proof. We now prove that

$$\mathcal{L}_{\alpha,\beta}(X\to Z) \leq \mathcal{L}_{\alpha,\beta}(Y\to Z) \quad (59)$$

based on the definition of maximal α,β-leakage in (5). Let

$$f(P_{UZ}) = \frac{\alpha}{\alpha-1}\log \frac{\max_{P_{\hat{U}|Z}}\left[\sum_z P_Z(z)\left(\sum_u P_{U|Z}(u|z)P_{\hat{U}|Z}(u|z)^{\frac{\alpha-1}{\alpha}}\right)^{\beta}\right]^{1/\beta}}{\max_{P_{\hat{U}}}\sum_u P_U(u)P_{\hat{U}}(u)^{\frac{\alpha-1}{\alpha}}}. \quad (60)$$

For the Markov chain $X-Y-Z$, we have

$$\begin{aligned}\mathcal{L}_{\alpha,\beta}(X\to Z) &= \sup_{P_X}\sup_{U\to X\to Z} f(P_{UZ}) && (61)\\ &= \sup_{P_X}\sup_{U\to X\to Y\to Z} f(P_{UZ}) && (62)\\ &\leq \sup_{P_X}\sup_{U\to Y\to Z} f(P_{UZ}) && (63)\\ &\leq \sup_{P_Y}\sup_{U\to Y\to Z} f(P_{UZ}) && (64)\\ &= \mathcal{L}_{\alpha,\beta}(Y\to Z)\end{aligned}$$

where (62) follows because P_{UZ} are the same under the Markov chains $U-X-Z$ and $U-X-Y-Z$, and (64) follows from the fact that a subset of all distributions P_Y is reachable from the distribution P_X.

Non-negativity: Consider the logarithmic term in (6):

$$\begin{aligned}&\log\sum_y P_{Y|X}(y|x')^{1-\beta}\left(\sum_x P_{\tilde{X}}(x)P_{Y|X}(y|x)^{\alpha}\right)^{\frac{\beta}{\alpha}} && (65)\\ &\geq \log\sum_y P_{Y|X}(y|x')^{1-\beta}\Big(\sum_x P_{\tilde{X}}(x)P_{Y|X}(y|x)\Big)^{\beta} && (66)\\ &= \log\sum_y P_{Y|X}(y|x')\left(\frac{\sum_x P_{\tilde{X}}(x)P_{Y|X}(y|x)}{P_{Y|X}(y|x')}\right)^{\beta} && (67)\\ &\geq \log\left(\sum_y P_{Y|X}(y|x')\ \frac{\sum_x P_{\tilde{X}}(x)P_{Y|X}(y|x)}{P_{Y|X}(y|x')}\right)^{\beta} && (68)\\ &= \log\left(\sum_{x,y} P_{\tilde{X}}(x)P_{Y|X}(y|x)\right)^{\beta} = \log 1 = 0 && (69)\end{aligned}$$

where both inequalities follow from applying Jensen's inequality to the convex function $f:\ x\to x^p\ (x\geq 0,\ p\geq 1)$ and the fact that logarithmic functions are increasing. Equality holds in the first inequality if and only if for any $y\in\mathcal{Y}$, $P_{Y|X}(y|x)$ are the same for all $x\in\mathcal{X}$. Thus, we have

$$P_{Y|X}(y|x) = P_Y(y) \quad x\in\mathcal{X}, y\in\mathcal{Y} \quad (70)$$

which means X and Y are independent. This condition is also sufficient for equality in the second inequality.

Additivity: We have $P_{X_1Y_1X_2Y_2} = P_{X_1Y_1}\cdot P_{X_2Y_2}$. To prove the additivity in (9), using Theorem 1 it suffices to show that

$$\begin{aligned}&\sup_{P_{\tilde{X}_1,\tilde{X}_2}}\sum_{y_1,y_2} P_{Y_1Y_2|X_1X_2}(y_1,y_2|x_1',x_2')^{1-\beta}\\ &\left(\sum_{x_1,x_2} P_{\tilde{X}_1,\tilde{X}_2}(x_1,x_2)P_{Y_1Y_2|X_1X_2}(y_1,y_2|x_1,x_2)^{\alpha}\right)^{\beta/\alpha}\\ &= \sup_{\substack{P_{\tilde{X}_i}\\ i\in 1,2}}\prod_{i=1}^{2}\Big(\sum_{y_i} P_{Y|X}(y_i|x_i')^{1-\beta}\Big(\sum_{x_i} P_{\tilde{X}_i}(x_i)\\ &\qquad P_{Y_i|X_i}(y_i|x_i)^{\alpha}\Big)^{\beta/\alpha}\Big), && (71)\end{aligned}$$

for every x_1', x_2'. We simplify LHS in (71) as

$$\begin{aligned}&\sup_{P_{\tilde{X}_1,\tilde{X}_2}}\sum_{y_1,y_2} P_{Y_1Y_2|X_1X_2}(y_1,y_2|x_1',x_2')^{1-\beta}\\ &\Big(\sum_{x_1,x_2} P_{\tilde{X}_1,\tilde{X}_2}(x_1,x_2)P_{Y_1Y_2|X_1X_2}(y_1,y_2|x_1,x_2)^{\alpha}\Big)^{\beta/\alpha}\\ &= \sup_{P_{\tilde{X}_1,\tilde{X}_2}}\sum_{y_1,y_2} P_{Y_1|X_1}(y_1|x_1')^{1-\beta}P_{Y_2|X_2}(y_2|x_2')^{1-\beta}\\ &\Big(\sum_{x_1,x_2} P_{\tilde{X}_1,\tilde{X}_2}(x_1,x_2)P_{Y_1|X_1}(y_1|x_1)^{\alpha}P_{Y_2|X_2}(y_2|x_2)^{\alpha}\Big)^{\beta/\alpha}. && (72)\end{aligned}$$

Let $k(y_1) = \sum_{x_1} P_{\tilde{X}_1}(x_1)P_{Y_1|X_1}(y_1|x_1)^{\alpha}$, for all y_1, so that we can define a set of probability distributions over $\mathcal{X}_1$ as

$$P_{\hat{X}_1}(x_1|y_1) = \frac{P_{\tilde{X}_1}(x_1)P_{Y_1|X_1}(y_1|x_1)^{\alpha}}{k(y_1)}. \quad (73)$$

Thus, (72) is equal to

$$\sup_{P_{\tilde{X}_1,\tilde{X}_2}} \sum_{y_1,y_2} P_{Y_1|X_1}(y_1|x_1')^{1-\beta} P_{Y_2|X_2}(y_2|x_2')^{1-\beta} \Big(\sum_{x_1,x_2} k(y_1) P_{\hat{X}_1|Y_1}(x_1|y_1) P_{\tilde{X}_2|\tilde{X}_1}(x_2|x_1) P_{Y_2|X_2}(y_2|x_2)^{\alpha}\Big)^{\beta/\alpha} \tag{74}$$

$$= \sup_{P_{\tilde{X}_1}, P_{\tilde{X}_2|X_1}} \sum_{y_1} P_{Y_1|X_1}(y_1|x_1)^{1-\beta} \big(\sum_{x_1} P_{\tilde{X}_1}(x_1) P_{Y_1|X_1}(y_1|x_1)^{\alpha}\big)^{\frac{\beta}{\alpha}} \max_{\tilde{y}_1} \sum_{y_2} P_{Y_2|X_2}(y_2|x_2')^{1-\beta} \Big(\sum_{x_1,x_2} P_{\hat{X}_1|Y_1}(x_1|\tilde{y}_1) P_{\tilde{X}_2|\tilde{X}_1}(x_2|x_1) P_{Y_2|X_2}(y_2|x_2)^{\alpha}\Big)^{\beta/\alpha} \tag{75}$$

$$= \sup_{P_{\tilde{X}_1}, P_{\tilde{X}_2|X_1}} \sum_{y_1} P_{Y_1|X_1}(y_1|x_1)^{1-\beta} \big(\sum_{x_1} P_{\tilde{X}_1}(x_1) P_{Y_1|X_1}(y_1|x_1)^{\alpha}\big)^{\frac{\beta}{\alpha}} \sum_{y_2} P_{Y_2|X_2}(y_2|x_2')^{1-\beta} \Big(\sum_{x_1,x_2} P_{\hat{X}_1|Y_1}(x_1|y_1^*) P_{\tilde{X}_2|\tilde{X}_1}(x_2|x_1) P_{Y_2|X_2}(y_2|x_2)^{\alpha}\Big)^{\beta/\alpha} \tag{76}$$

We now define

$$P_{\hat{X}_2}(x_2) = \sum_{x_1} P_{\hat{X}_1|Y_1}(x_1|y_1^*) P_{X_2|X_1}(x_2|x_1),$$

which is a probability distribution over $\mathcal{X}_2$. Then, (76) is equal to

$$\sup_{\substack{P_{\tilde{X}_1},\\ P_{\hat{X}_2}}} \sum_{y_1} P_{Y_1|X_1}(y_1|x_1')^{1-\beta} \big(\sum_{x_1} P_{\tilde{X}_1}(x_1) P_{Y_1|X_1}(y_1|x_1)^{\alpha}\big)^{\frac{\beta}{\alpha}} \sum_{y_2} P_{Y_2|X_2}(y_2|x_2')^{1-\beta} \big(\sum_{x_2} P_{\hat{X}_2}(x_2) P_{Y_2|X_2}(y_2|x_2)^{\alpha}\big)^{\frac{\beta}{\alpha}} \tag{77}$$

$$= \sup_{\substack{P_{\tilde{X}_i}\\ i\in 1,2}} \prod_{i=1}^{2} \Big(\sum_{y_i} P_{Y|X}(y_i|x_i')^{1-\beta} \big(\sum_{x_i} P_{\tilde{X}_i}(x_i) P_{Y_i|X_i}(y_i|x_i)^{\alpha}\big)^{\beta/\alpha}\Big). \tag{78}$$

This proves (71), and hence the additivity.

C. Proof for Remark 1

Let $\tau \in [0,1]$ and $\beta = \frac{\alpha}{1-\tau(1-\alpha)}$. We may re-write the expression of maximal α,β-leakage in (6) in terms of α and τ, as follows.

$$\mathcal{L}_{\alpha,\tau}(X \to Y) = \max_{x'} \sup_{P_{\tilde{X}}} \frac{1-\tau(1-\alpha)}{\alpha-1} \log \sum_{y} P_{Y|X}(y|x')^{\frac{(1-\tau)(1-\alpha)}{1-\tau(1-\alpha)}} \left(\sum_{x} P_{\tilde{X}}(x) P_{Y|X}(y|x)^{\alpha}\right)^{\frac{1}{1-\tau(1-\alpha)}}. \tag{79}$$

We claim that this leakage measure is non-increasing in τ for a fixed α, and non-decreasing in α for a fixed τ. Since β is decreasing in τ, the first claim, that the measure is non-increasing in τ, is equivalent to it being non-decreasing in β, which we have already proved in Section V-B. To prove the second claim, we first prove the following lemma, which provides a still other representation of the leakage measure.

Lemma 1. *The measure defined in* (79) *can be represented by*

$$\mathcal{L}_{\alpha,\tau}(X \to Y) = \max_{x'} \sup_{P_{\tilde{X}}} \inf_{Q_Y} \frac{1}{\alpha-1} \log \sum_{x,y} P_{\tilde{X}}(x) P_{Y|X}(y|x)^{\alpha} \left(Q_Y(y)^{\tau} P_{Y|X}(y|x')^{1-\tau}\right)^{1-\alpha}. \tag{80}$$

Remark 2. Some of the relationships to other measures become clear from this lemma. Namely, if $\tau = 1$, then we see the definition of Sibson mutual information as

$$\inf_{Q_Y} D_{\alpha}(P_{XY} \| P_X \times Q_Y). \tag{81}$$

If $\tau = 0$, then we see the definition of LRDP as

$$\max_{x,x'} D_{\alpha}(P_{Y|X=x} \| P_{Y|X=x'}). \tag{82}$$

Proof of Lemma 1: Consider any $\gamma \in (-\inf, 0] \cup [1, \inf)$, and any constants $C(y)$ for $y \in \mathcal{Y}$. Furthermore, consider the optimization problem

$$\inf_{Q_Y} \sum_{y} C(y) Q_Y(y)^{\gamma}. \tag{83}$$

γ is in the range where (83) is convex in Q_Y, so it is solved by setting the derivative of $Q_Y(y)$ to a constant:

$$\nu = \frac{\partial}{\partial Q_Y(y)} \sum_{y} C(y) Q_Y(y)^{\gamma} = C(y)\gamma Q_Y(y)^{\gamma-1}. \tag{84}$$

We can see that the optimal choice is therefore

$$Q_Y(y) = \frac{C(y)^{1/(1-\gamma)}}{\sum_{y'} C(y')^{1/(1-\gamma)}}. \tag{85}$$

Thus (83) becomes

$$\frac{\sum_y C(y) C(y)^{\gamma/(1-\gamma)}}{\left(\sum_{y'} C(y')^{1/(1-\gamma)}\right)^{\gamma}} = \left(\sum_{y} C(y)^{1/(1-\gamma)}\right)^{1-\gamma}. \tag{86}$$

In our case, we have $\gamma = \tau(1-\alpha) < 0$, and

$$C(y) = \sum_{x} P_{\tilde{X}}(x) P_{Y|X}(y|x)^{\alpha} P_{Y|X}(y|x')^{(1-\tau)(1-\alpha)}. \tag{87}$$

Applying the result in (86) to our case, we find that (80) is equal to

$$\max_{x'} \sup_{P_{\tilde{X}}} \frac{1}{\alpha-1} \log \left[\sum_y \left(\sum_x P_{\tilde{X}}(x) P_{Y|X}(y|x)^{\alpha} \right.\right.$$
$$\left.\left. P_{Y|X}(y|x')^{(1-\tau)(1-\alpha)} \right)^{\frac{1}{1-\tau(1-\alpha)}} \right]^{1-\tau(1-\alpha)} \tag{88}$$
$$= \max_{x'} \sup_{P_{\tilde{X}}} \frac{1-\tau(1-\alpha)}{\alpha-1} \log \sum_y P_{Y|X}(y|x')^{\frac{(1-\tau)(1-\alpha)}{1-\tau(1-\alpha)}}$$
$$\left(\sum_x P_{\tilde{X}}(x) P_{Y|X}(y|x)^{\alpha} \right)^{\frac{1}{1-\tau(1-\alpha)}} \tag{89}$$

which is precisely (79). Given Lemma 1, we prove that $\mathcal{L}_{\alpha,\tau}(X \to Y)$ is non-decreasing in α for a fixed τ as follows. We may write the objective function in (80) as

$$\frac{1}{\alpha-1} \log \sum_{x,y} P_{\tilde{X}}(x) P_{Y|X}(y|x)$$
$$\left(\frac{P_{Y|X}(y|x)}{Q_Y(y)^{\tau} P_{Y|X}(y|x')^{1-\tau}} \right)^{\alpha-1}. \tag{90}$$

This expression is non-decreasing in α due to the fact that, for any distribution P_Z, and any constants $C(z)$,

$$\frac{1}{\alpha-1} \log \sum_z P_Z(z) C(z)^{\alpha-1} \tag{91}$$

is non-decreasing in α for $\alpha > 1$.

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