Phys 476

GENERAL RELATIVITY

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Version: 1.0

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Latest versions of all my course notes are available at **www.tcfraser.com**.

1 Introduction

1.1 History

The first lecture was a summary of astrophsyical history from around ∼200BC to today. I elected not to take notes as it was pretty standard stuff and a lot of slides. Sorry.

2 Tensor Formalism

At the core of General Relativity is the mathematics of differential geometry. Differential geometry requires the idea of tensors, a generalization of vectors and matricies and forms that can handle messy geometries and metrics.

Let V be a vector space of finite dimension. Any V is isomorphic to $\mathbb{R}^{n+1}$ through the coefficients of a chosen basis. Let the basis of V be given by,

$$\{e_i\}_{i=0,\ldots,n}$$

Then any vector $v \in V$ is expressible by,

$$v = \sum_{i=0}^{n} v^i e_i$$

Where v^i are the i-th coefficients of the vector v with respect to the basis $\{e_i\}$.

2.1 Einstein Summation Rule

For convience let's provide a new, shorter notation for the vector v.

$$v^i e_i = v^0 e_0 + \ldots + v^n e_n = \sum_{i=0}^{n} v^i e_i$$

Effectively, we have just **dropped the summation sign**. The einstein summation rule is as follows:

If there are two identical indicies, 1 "up" and 1 "down", it means that a summation is secretly present, it's just be removed for convience. Note that the i in this case is *dummy index*.

$$v^i e_i = v^\alpha e_\alpha = v^j e_j$$

Here v^i are the components of vector $v \in V$ and are real numbers. $v^i \in \mathbb{R}, \forall i \in \{0, \ldots, n\}$.

Note v^i is called the vector v when i is the set $\{0, \ldots, n\}$, but can also be called the i-th component of v when i has a fixed value $i \in \{0, \ldots, n\}$.

2.2 Examples of Basis for V

The values of e_i or the i's themselves can take on many possible values.

- cartesian coordinates t, x, y, z
- spherical coordinates t, r, ϕ, θ
- etc.

Each of the above examples is the space $V = \mathbb{R}^4$ (with some bounds for spherical coordinates).

2.3 Dual Vector Space

The dual vector space of V denoted V^* is also isomorphic to $\mathbb{R}^{n+1}$ and is built from the space of linear forms on V.

$$V^* = \{w : V \to \mathbb{R} \mid w(\alpha v_1 + \beta v_2) = \alpha w(v_1) + \beta w(v_2)\}$$

where $v_1, v_2 \in V$ and $\alpha, \beta \in \mathbb{R}$.
In Quantum Mechanics, the vectors are the bras and the elements of the dual space (called the covectors) are the kets.

We note,

$$\left\{f^i\right\}_{i=0,\ldots,n}$$

is the basis for V^* is defined by the kronecker symbol δ,

$$f^j(e_j) = \delta^j{}_i$$

$$\delta^j{}_i = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

An element in V^* is $w = w_i f^i$. w_i are the components of the covector w. Note that for a **finite dimensional vector space**,

$$V^{**} = V$$

2.4 Bilinear Maps

Introduce a bilinear map $B(v, w)$ where $B : V \times V \to \mathbb{R}$ where,

$$B(\alpha v_1 + \beta v_2, w) = \alpha B(v_1, w) + \beta B(v_2, w)$$

and the same for the other parameter w.

Examples include the inner product (otherwise known as the scale or dot product).
Bilinear forms are bilinear maps such that the following conditions are true:

- symmetric: $B(v, w) = B(w, v)$
- non-degenerated: $B(v, w) = 0 \quad \forall v \implies w = 0$

Playing with indicies,

$$\begin{aligned} B(v, w) &= B(v^\alpha e_\alpha, w^\beta e_\beta) \\ &= v^\alpha B(e_\alpha, w^\beta e_\beta) \quad \text{By linearity} \\ &= v^\alpha w^\beta B(e_\alpha, e_\beta) \quad \text{By linearity} \end{aligned}$$

A bilinear map used in this way provides a way to eliminate the headache of complicated cross sums. Define new notation,

$$B(e_\alpha, e_\beta) \equiv g_{\alpha\beta}$$

Where $g_{\alpha\beta}$ is a real number $\mathbb{R}$ because α and β are summed over.

$$B(v, w) = v^\alpha w^\beta g_{\alpha\beta} = v^\alpha g_{\alpha\beta} w^\beta = w^\beta g_{\alpha\beta} v^\alpha$$

All of the above terms are commutative because in the end, it represents a sum over all α, β.

$$B(v, w) = \underbrace{v^0 w^0 g_{00} + \ldots + v^2 w^3 g_{2,3} + \ldots + v^n w^n g_{nn}}_{(n+1)^2 \text{terms}}$$

2.5 Distance and Norms

To define a distance in a vector space, we can use norms. In this case, $g_{\alpha\beta}$ would be called the metric. The Euclidean metric (with respect to a cartesian basis) for example would be,

$$g_{\alpha\beta} = \begin{cases} 1 & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases}$$

We can also choose to enforce that the basis be orthonormal,

$$B(e_i, e_j) = \begin{cases} \pm 1 & i = j \\ 0 & i \neq j \end{cases}$$

Note that the potential for a negative norm means the notion of positive definiteness is no longer gauranteed.

2.6 Signatures of Metrics

We call the signature of the metric the number of $+1$'s and -1's appearing in g_{ij} when dealing with the orthonormal basis. Signature is denoted as:

$$(p, q) = \left(\underbrace{p}_{\text{postive}}, \underbrace{q}_{\text{negative}} \right)$$

For example,

- Euclidean metric: $(n + 1, 0)$
- Minkowski metric: $(n, 1)$

Note the order of the signature is chosen to be (p, q) and not (q, p) by convention.

2.7 Covectors from Vectors

Note that v^i was called the vector and w_i was called the covector. This notation seems to indicate that conversion between V and V^* is notationally equivalent to raising and lowering the indicies.

We call the following opperation "Lowering the index using the metric".

$$\underbrace{v^\alpha}_{\text{components of vector}} \mapsto g_{\alpha\beta} v^\beta = \underbrace{v_\alpha}_{\text{components of covector}}$$

In use,

$$B(v, w) = v^\alpha g_{\alpha\beta} w^\beta = \underbrace{v_\beta}_{\text{bra}} \underbrace{w^\beta}_{\text{ket}}$$

2.8 Linear Map on V to V

$$M : V \to V$$

Where M is a matrix. An the map is equivalent to $v \to Mv \in V$. Some definition,

$$(Mv)^{\alpha} = \underbrace{M^{\alpha}{}_{\beta}}_{\text{Matrix(components)}} v^{\beta}$$

Note that $M^{\alpha}{}_{\beta} \in \mathbb{R}$ for α and β fixed. Example: The identity matrix is denoted $\delta^{\alpha}{}_{\beta} = \mathbb{I}$.

2.9 Scalar Product on Dual Space

Introduce a scalar product for the covectors w.

$$w, t \in V^*$$

$$w \cdot t = w_{\alpha} h^{\alpha\beta} t_{\beta}$$

Where $h^{\alpha\beta}$ is symmetric and non-degenerate.
So how is the scalar product between the dual and normal space related? Specifically how are $g_{\alpha\beta}$ and $h^{\alpha\beta}$ connected? Well,

$$\begin{aligned} v^{\alpha} g_{\alpha\beta} w^{\beta} &= v^{\alpha} w_{\alpha} \\ &= v_{\gamma} h^{\gamma\alpha} w_{\alpha} \\ &= v^{\nu} g_{\nu\gamma} h^{\gamma\alpha} w_{\alpha} \\ &= v^{\nu} g_{\nu\gamma} h^{\gamma\alpha} g_{\alpha\mu} w^{\mu} \end{aligned}$$

Since this is true for any v and w we require that,

$$h^{\gamma\alpha} g_{\alpha\mu} = \delta^{\gamma}{}_{\mu}$$

This means we say that the metric h is the inverse of the metric g. Convention on V^*: we denote the metric $g^{\alpha\beta}$ (the indicies are "up").

2.10 Invariance of Scalar Product

Let us say we have a matrix $M : v \to \tilde{v} = Mv, w \to \tilde{w} = Mw$ and that M preserves the scalar product.

$$\tilde{v} \cdot \tilde{w} = v \cdot w \qquad \forall v, w$$

Examine,

$$M^{\gamma}{}_{\alpha} v^{\alpha} g_{\alpha\beta} M^{\beta}{}_{\rho} w^{\rho} = v^{\alpha} g_{\alpha\beta} w^{\beta}$$

Use communitivity and dummyness of indicies to obtain,

$$v^{\alpha} M^{\gamma}{}_{\alpha} g_{\alpha\rho} M^{\rho}{}_{\beta} w^{\beta} = v^{\alpha} g_{\alpha\beta} w^{\beta}$$

Drop outer covectors v and w to get,

$$M^{\gamma}{}_{\alpha} g_{\alpha\rho} M^{\rho}{}_{\beta} = g_{\alpha\beta} \tag{2.1}$$

Note that this expression is consistent with the Einstein summation convention.

An example of an M on euclidean space could be a rotation matrix, or the identity.

When M satisfies 2.1, it is said to be orthogonal. It $\det(M) = 1$ then we say that M is *special*.

2.11 Trace of M

What is the trace of M?

$$\text{Tr}(M) = M^\alpha{}_\alpha = M^0{}_0 + \ldots + M^n{}_n$$

This is just a notationally convention. It is the sum of the diagonal terms of M.

2.12 Tensor Product

A tensor product makes a linear map a multi-linear map.

Theorem:
Let E and F be 2 vector spaces (with finite dimensionality.)
$\exists$ a unique (!) set (up to isomorphism) $E \otimes F$ such that if f is a bilinear map $f : E \times F \to \mathbb{R}$ then $\exists$ a linear map $f^* : E \otimes F \to \mathbb{R}$ such that $f = f^* \circ \phi$ with

$$\begin{tikzcd}
E \times F \arrow[d, "\phi"'] \arrow[dr, "f"] & \\
E \otimes F \arrow[r, "f^*"'] & \mathbb{R}
\end{tikzcd}$$

Then we have,

$$\text{Lin}(E \otimes F, \mathbb{R}) \cong \text{Bin}(E \times F, \mathbb{R})$$
$$\text{Lin}(f^*, \mathbb{R}) \cong \text{Bin}(f, \mathbb{R})$$

where '$\cong$' is used to denote *isomorphic*.

Properties:
Basis for $E \otimes F$ is $e_\alpha \otimes g_\alpha$ where e_α is the basis for E and g_α is the basis for F. For $a \in \mathbb{R}$ and $t, v \in E$, $u, w \in F$,

- $\dim(E \otimes F) = \dim(E)\dim(F)$
- $a(v \otimes w) = (av) \otimes w = v \otimes (aw)$
- $(v + t) \otimes w = v \otimes w + t \otimes w$
- $v \otimes (w + u) = v \otimes w + v \otimes u$
- $a \otimes w = aw$
- $\mathbb{R} \otimes F = F$

Note that $V^* \otimes V^* \cong \text{Bin}(V \times V, \mathbb{R})$. To motivate this, let $f^\alpha \otimes f^\beta$ be the basis for $V^* \otimes V^*$, and then a general element in $V^* \otimes V^*$ is,

$$t = t_{\alpha\beta} f^\alpha \otimes f^\beta$$

Note that $t_{\alpha\beta}$ is just a set of numbers. Then the tensor product is expanded as follows,

$$\begin{aligned}
t(v \otimes w) &= t(v^\alpha e_\alpha \otimes w^\beta e_\beta) \\
&= t_{\gamma\delta}(f^\gamma \otimes f^\delta)(v^\alpha e_\alpha \otimes w^\beta e_\beta) \\
&= t_{\gamma\delta} v^\alpha w^\beta (f^\gamma \otimes f^\delta)(e_\alpha \otimes e_\beta) \quad \text{By linearity}
\end{aligned}$$

$$
\begin{aligned}
&= t_{\gamma\delta} v^\alpha w^\beta f^\gamma(e_\alpha) f^\delta(e_\beta) \quad \text{By foiling and definition of } f \\
&= t_{\gamma\delta} v^\alpha w^\beta \delta^\gamma \alpha \delta^\delta{}_\beta \\
&= t_{\gamma\delta} v^\gamma w^\beta \delta^\delta{}_\beta \quad \text{By sifting property of } \delta \\
&= t_{\gamma\delta} v^\gamma w^\delta \quad \text{By sifting property of } \delta \text{ again}
\end{aligned}
$$

Since $t(v \otimes w)$ is the tensor product $V^* \otimes V^*$ and $t_{\gamma\delta}$ is the components of the bilinear form, one can see the connection $V^* \otimes V^* \cong \text{Bin}(V \times V, \mathbb{R})$.

Tensors allow one to write bilinear maps as linear maps. What about multi-linear maps?

Tensors:
A tensor of rank (k, l) is a multilinear map

$$\underbrace{V^* \times \cdots \times V^*}_{k} \times \underbrace{V \times \cdots \times V}_{l} \to \mathbb{R}$$

which transforms *well* under the change of basis of V and V^*.

Tensor	Rank
vectors	$(1, 0)$
covectors	$(0, 1)$
scalar	$(0, 0)$
metric	$(0, 2)$
inverse metric	$(2, 0)$
matrix	$(1, 1)$

The set of tensors of rank (k, l) is a vector space of dimension n^{k+l} (if V has dimension n). Checking with the examples above motivates this fact.

Using the basis $e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_k} \otimes f^{\beta_1} \otimes \cdots \otimes f^{\beta_k}$

$$T = T^{\alpha_1 \alpha_2 \cdots \alpha_k}{}_{\beta_1 \beta_2 \cdots \beta_l} e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_k} \otimes f^{\beta_1} \otimes \cdots \otimes f^{\beta_k}$$

For fixed α_i and β_i this is a real number in $\mathbb{R}$. These are the *components of the tensor*.

By abuse of notation we will call $T^{\alpha_1 \alpha_2 \cdots \alpha_k}{}_{\beta_1 \beta_2 \cdots \beta_l}$ the tensor.

We are talking about these transformations as change of basis of V and V^*. Examples:

- rotations (boost)
- change of coordinates from cartesian to spherical, cylindrical, etc.

We can have a linear change of basis $\tilde{x}^\mu = A^\mu{}_\nu x^\nu$.

Example:

Cartesian	Polar
$e_1 = \vec{i}$	$\tilde{e}_1 = e_r$
$e_2 = \vec{j}$	$\tilde{e}_2 = e_\theta$

Example:

$$\tilde{e}_\alpha = \underbrace{\frac{\partial x^\nu}{\partial \tilde{x}^\alpha}}_{\text{Jacobian}} e_\nu = A^\nu{}_\alpha e_\nu$$

Note: *Up in the denominator means down on the original coordinates (LHS).*
For example,

$$\begin{array}{l|l} x^1 = x & \tilde{x}^1 = r \\ x^2 = y & \tilde{x}^2 = \theta \end{array}$$

$$\tilde{e}_1 = e_r = \frac{\partial x^1}{\partial \tilde{x}^1} e_1 + \frac{\partial x^2}{\partial \tilde{x}^1} e_2 = \cos\theta e_1 + \sin\theta e_1$$
$$\tilde{e}_2 = e_\theta = \frac{\partial x^1}{\partial \tilde{x}^2} e_1 + \frac{\partial x^2}{\partial \tilde{x}^2} e_2 = -r\sin\theta e_1 + r\cos\theta e_1$$

Vectors in multiple basis:

$$v = v^\nu e_\nu = \tilde{v}^\nu \tilde{e}_\nu$$

With conversion of basis given by,

$$\tilde{e}_\alpha = A^\nu{}_a l e_\nu$$

Thus substituting in,

$$v^\nu e_\nu = \tilde{v}^\alpha A^\nu{}_\alpha e_\nu$$

$$v^\nu = \tilde{v}^\alpha A^\nu{}_\alpha$$

But with A as a Jacobian,

$$v^\nu = \frac{\partial x^\nu}{\partial \tilde{x}^\alpha} \tilde{v}^\alpha$$
$$\tilde{v}^\alpha = \frac{\partial \tilde{x}^\alpha}{\partial x^\nu} v^\nu$$

But what about the dual space?
By definition,

$$\tilde{f}^\beta\left(\tilde{e}_\nu\right) = \delta^\beta_\mu = \tilde{f}^\beta\left(A^\alpha{}_\nu e_\alpha\right) = A^\alpha{}_\nu \tilde{f}^\beta\left(e_\alpha\right)$$

Let $\tilde{f}^\beta\left(e_\alpha\right)$ be expressed as $\tilde{f}^\beta = B^\beta{}_\gamma f^\gamma$

$$\begin{aligned} \tilde{f}^\beta\left(\tilde{e}_\nu\right) &= A^\alpha{}_\nu B^\beta{}_\gamma f^\gamma\left(e_\alpha\right) \\ &= A^\alpha{}_\nu B^\beta{}_\gamma \delta^\gamma{}_\alpha \\ &= B^\beta{}_\gamma A^\gamma{}_\nu \\ &= \delta^\beta{}_n u \end{aligned}$$

Thus B is the inverse of A.

What is performing *well*? A tensor is performming well if it's compponents transform as

$$T^{\nu_1\nu_2\cdots\nu_k}{}_{\alpha_1\alpha_2\cdots\alpha_l} \to \frac{\partial \tilde{x}_1^\nu}{\partial x_1^\beta} \cdots \frac{\partial \tilde{x}_k^\nu}{\partial x_k^\beta} \frac{\partial x_1^\gamma}{\partial \tilde{x}_1^\alpha} \cdots \frac{\partial x_k^\gamma}{\partial \tilde{x}_k^\alpha} T^{\beta_1\beta_2\cdots\beta_k}{}_{\gamma_1\gamma_2\cdots\gamma_l} = \tilde{T}^{\nu_1\nu_2\cdots\nu_k}{}_{\alpha_1\alpha_2\cdots\alpha_l}$$

If you find something like $T^\alpha{}_\beta$ is it a tensor? **No! You must check if it transforms well.**

$$\frac{\partial}{\partial x^\nu} v^\alpha \quad \text{This is not a tensor.}$$

The derivative here prevents it from being well-formed. In the future we will define a derivative that allows a tensor to transform well.

2.13 Operations on Tensors

- Add (with matching rank): $T^{\alpha_1\alpha_2}{}_{\beta_1\beta_2} + C^{\alpha_1\alpha_2}{}_{\beta_1\beta_2}$.
- Contraction (partial trace): $\mathcal{T}(k,k) \to \mathcal{T}(k-1,k-1)$.
  - $T^{\alpha_1\cdots\alpha_i\cdots\alpha_k}{}_{\beta_1\cdots\beta_j\cdots\beta_l} \to T^{\alpha_1\cdots\alpha_i\cdots\alpha_k}{}_{\beta_1\cdots\alpha_j\cdots\beta_l}$
- "Outer" Product (Gluing together tensors)
  - $\mathcal{T}(k,l) \times \mathcal{T}(k',l') \to \mathcal{T}(k+k', l+l')$
  - $(T_1, T_2) \to T_1 T_2$
  - $T_1 T_2 \to T_1^{\nu_1\cdots\nu_k}{}_{\alpha_1\cdots\alpha_l} T_2^{\beta_1\cdots\beta_k}{}_{\gamma_1\cdots\gamma_l}$
  - **Example:** $(v^\alpha . w_\beta) \to v^\alpha \otimes w_\beta = v^\alpha w_\beta$. (In QM this is $|\phi\rangle\langle\varphi|$)

The metric $g_{\alpha\beta}$ can change the rank of a tensor. Recall a metric is rank $(0,2)$ is symmetric and is non-degenerate.

Example:
Changing from rank $(1,0)$ to rank $(0,1)$:

$$v^\alpha \to v_a = g_{\alpha\beta} v^\beta$$

Changing from rank $(2,2)$ to rank $(4,0)$:

$$c^{\alpha\beta}{}_{\gamma\delta} \to c_{\alpha\beta\gamma\delta} = g_{\alpha\rho} g_{\beta\eta} c^{\rho\eta}{}_{\gamma\delta}$$

Changing from rank $(2,2)$ to a different rank $(2,2)$:

$$c^{\alpha\beta}{}_{\gamma\delta} \to c^{\alpha}{}_{\beta}{}^{\gamma}{}_{\delta} = g_{\beta\rho} g^{\gamma\eta} c^{\alpha\rho}{}_{\eta\delta}$$

2.14 Facts About Tensors

Order Matters:

$$\begin{aligned} c^{\alpha}{}_{\beta}: &\quad V^* \times V \to \mathbb{R} \\ c_{\alpha}{}^{\beta}: &\quad V \times V^* \to \mathbb{R} \\ c^{\beta}_{\alpha}: &\quad \text{Nothing. Don't do this.} \end{aligned}$$

Equality between tensors:
As tensors, indicies must match:

Position of indicies is matching: $c^{\alpha}{}_{\gamma}{}^{\delta} = T^{\alpha}{}_{\gamma}{}^{\delta}$

Position of indicies is **not** matching: $c^{\alpha}{}_{\gamma}{}^{\delta} \neq T^{\alpha}{}_{\gamma\delta}$

But for fixed α, γ, δ, one can abuse the notation a bit:

$$c^{\alpha}{}_{\gamma}{}^{\delta} = T^{\alpha}{}_{\gamma\delta} \quad \text{Try to avoid this.}$$

2.15 Outer Product and Contraction

Example:
Outer Product: $M^{\alpha}{}_{\beta}M^{\gamma}{}_{\delta} = C^{\alpha}{}_{\beta}{}^{\gamma}{}_{\delta}$
Contraction: $M^{\alpha}{}_{\beta}M^{\beta}{}_{\delta} = C^{\alpha}{}_{\beta}{}^{\beta}{}_{\delta} = C^{\alpha}{}_{\gamma}$

Example:
Outer product and contraction: $C^{\alpha\beta}{}_{\gamma\delta}T^{\gamma\delta}{}_{\rho} = A^{\alpha\beta}{}_{\rho}$
This doesn't make sense: $C^{\alpha\beta}{}_{\gamma\gamma}T^{\gamma\delta}{}_{\rho} =??$
Note, when there is a "+" sign we can be "loose" with the indicies. Here the dual indicies **do not** indicate a summation. This acts as an abuse of notation, but is sometimes difficult to avoid.

$$C^{\alpha\gamma}T_{\gamma}{}^{\delta} + F_{\gamma}{}^{\delta}A^{\alpha\gamma}$$

2.16 Interpretation of Tensors

By looking at the indicies, how can we interpret the physical meaning of the tensor object?

Tensor	Interpretation
v^{ν}	vector
v_{ν}	covector
$M^{\alpha}{}_{\beta}$	matrix (α rows, β columns)
$M^{\alpha}{}_{\alpha}$	contracted matrix (trace)
$M^{\alpha\gamma}{}_{\delta}$	matrix whose elements are vectors themselves ($\cdot^{\gamma}{}_{\delta}$ is the matrix)
$M^{\alpha\gamma}{}_{\delta}$	vector with matrix components (M^{α} is the vector)
$R^{\alpha\beta}{}_{\gamma\delta}$	matrix of matricies *

*For example, if $\dim V = 4$, $R^{\alpha\beta}{}_{\gamma\delta}$ has $4^4 = 256$ components. Note however, there can be many symmetries that reduce the number of unique components.

2.17 Symmetry of Tensor

We can always build a symmetric and antisymmetric part of a tensor $T^{\alpha\beta}$. Let's look at the case of 2 indicies α, β:

Symmetric Part:

$$T_{(\alpha\beta)} = \frac{1}{2}\left(T_{\alpha\beta} + T_{\beta\alpha}\right)$$
$$T_{(\alpha\beta)} = T_{(\beta\alpha)}$$

Antisymmetric Part:

$$T_{[\alpha\beta]} = \frac{1}{2}\left(T_{\alpha\beta} - T_{\beta\alpha}\right)$$
$$T_{[\alpha\beta]} = -T_{[\beta\alpha]}$$

Note that for all tensors $T^{\alpha\beta} = T^{(\alpha\beta)} + T^{[\alpha\beta]}$. This acts as the decomposition into odd and even symmetries of the tensor.

For more indicies:

$$T^{(\alpha\beta)}{}_{[\gamma\delta]} = \frac{1}{4}\left(T^{\alpha\beta}{}_{\gamma\delta} + T^{\beta\alpha}{}_{\gamma\delta} - T^{\alpha\beta}{}_{\delta\gamma} - T^{\beta\alpha}{}_{\delta\gamma}\right)$$

What does $T^{(\alpha\beta\gamma)}$ mean? For that we will need a permutation group.

3 Physics Review

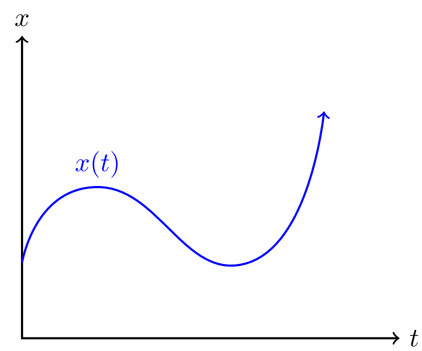
Moving away from tensors for a moment...

3.1 Newtonian Physics

According to Galileo and Newton, we got the interpretation that both space and time if flat $\left(\mathbb{R}^3\right)$ and is absolute. More specifically, all clocks have the same time on them if they are started at the same time.

Built on cartesian coordinate system: $(\vec{x}, t)$. With this we say that an object is at position $\vec{x}$ at time t. They are *outcomes of measurements*. In General Relativity, the notion of coordinates can be quite different.

Consider a particle (2d spacetime):



Typically, x is drawn as the ordinate (y-axis) and t as the abscissa (x-axis).

Spacetime diagram:
In a spacetime diagram, t is drawn as the ordinate.