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# Phys 442

## ELECTRICITY & MAGNETISM 3

University of Waterloo

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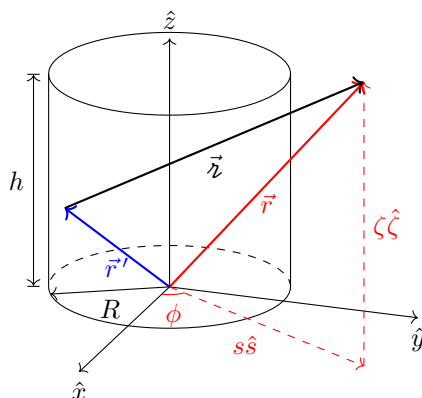
## 1 Coordinates and Symmetry

A clever choice of coordinates systems typically makes solving a problem considerably easier. Mathematically, this is due to *Noether's Theorem*. A typical three dimensional Lagrangian will have three dependent generalized coordinates  $L = L(x, y, z) = L(s, \theta, \zeta) = \dots$ . However, if one can identify generalized coordinates  $q$  that make the Lagrangian invariant  $\frac{\partial L}{\partial q} = 0$ , then the *Euler-Lagrange* equations are considerably similar,

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0 \implies \frac{\partial L}{\partial \dot{q}} = \text{const.} \implies L \propto \dot{q}$$

As such, the number of equations that remain to be solved has been reduced.

## 2 First Assignment?



**A1.1:** Use cylindrical coordinates with  $\zeta$  along the axis of the cable,

$$V(\zeta) = \frac{1}{4\pi\epsilon_0} \int_C \frac{d\rho}{z}$$

Where  $\vec{z} = \vec{r} - \vec{r}'$ ,  $\vec{r}'$  is the source point and  $\vec{r}$  is the field point. The entire cylinder is the set of all source points  $\vec{r}'$  that are contained inside  $|\vec{r}'| \leq R$ .

$$\begin{aligned} \vec{r} &= \zeta \hat{\zeta} \\ \vec{r}' &= s' \hat{s}' + \zeta' \hat{\zeta} \end{aligned}$$

$$V(\zeta) = \frac{\rho}{4\pi\epsilon_0} \int_C \frac{dV}{|\vec{r} - \vec{r}'|}$$

Where  $dV = s ds d\theta d\zeta$ . One can then find the electric field by doing  $\vec{E} = -\vec{\nabla}V = E_{\zeta} \hat{\zeta} = -\frac{\partial V}{\partial \zeta} \hat{\zeta}$

**A1.2:**

Between the two conductors, there will be a radial electric field  $\vec{E} = E(s)\hat{s}$  and parallel magnetic field  $\vec{B} = B(s)\hat{\zeta}$ . Outside the two conductors, there will be no electric or magnetic field.

$$\begin{aligned} E_{\text{vac}}^{\parallel} &= 0 \\ E_{\text{vac}}^{\perp} &= \frac{\sigma}{\epsilon_0} \\ \vec{\nabla} \cdot \vec{E} &= \frac{\rho}{\epsilon_0} \end{aligned}$$

For part g), use Laplace's equation  $\nabla^2 V = 0$ . In cylindrical coordinates, Laplace's equation is,

$$\nabla^2 V = \frac{1}{s} \frac{\partial}{\partial s} \left( s \frac{\partial V}{\partial s} \right) = 0$$

Cylindrical coordinates gives us the following symmetries  $\frac{\partial V}{\partial \phi} = \frac{\partial V}{\partial \zeta} = 0$ . Solving this system gives the potential in terms of  $s$ :  $V(s) = \dots$ . Then the electric field can then be obtained via  $\vec{E} = -\vec{\nabla}V$ .

**A1.3:** Using cylindrical coordinates once again, the electric field is going to be radial outwards to the uniform charge density. For the uniform density cylinder, construct a Gaussian surface cylindrically around the cylinder. For the current density cylinder, the current density is the current per cross sectional area. Construct an Amperian loop,

$$\oint_{\mathcal{A}} d\vec{\ell} \cdot \vec{B} = \mu I_{\text{enc}}$$

Part e), finding the vector potential,

$$A(\vec{r}) = \frac{\mu_0}{4\pi} \int_{\mathcal{C}} d\tau' \frac{\vec{J}(\vec{r}')}{r}$$

Evidently,  $\hat{s}$  and  $\hat{s}'$  are in *different* directions. Solving such an equation yields,

$$A(s) = \frac{\mu_0}{4\pi} \int_0^{2\pi} d\phi' \int_0^a s' ds' \int_{-\infty}^{\infty} d\zeta' \frac{J(s)}{|s\hat{s} - s'\hat{s}' - \zeta'\hat{\zeta}|}$$

Recognize the structure of the potential integral,

$$V(s) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{C}} d\tau' \frac{\rho(r')}{r} = \frac{\rho_0}{4\pi\epsilon_0} \int_{\mathcal{C}} \frac{d\tau'}{r}$$

Comparing to the vector potential, we have an equivalent integral (up to a constant).

$$A(s) = \frac{\mu_0 J_0}{4\pi} \int_{\mathcal{C}} \frac{d\tau'}{r}$$

For question f), use the definition of  $\vec{B}$  in terms of  $\vec{A}$ ,

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

Further, recall that if  $\vec{E} = -\vec{\nabla}V$ , then by Stoke's theorem for some loop  $\mathcal{L}$ ,

$$V = - \int_{\mathcal{L}} d\vec{\ell} \cdot \vec{E}$$

### 3 Conservation Laws

Beginning with one of Maxwell's equations,

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Taking the divergence of the above equation,

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) \overset{0}{=} \mu_0 \vec{\nabla} \cdot \vec{J} + \mu_0 \epsilon_0 \frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{E})$$

Luckily, the divergence of a curl is always 0. Dividing by relevant constants we obtain the following conservation law,

$$0 = \vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} \quad (3.1)$$

This is a conservation of charge. It is a **local** conservation law because it holds for all points in space  $\vec{r}$ . Intuitively, it claims that the rate of change of charge at a point is equal to the amount of current following in or out of the take point.

**A2.1:** Again using cylindrical coordinates  $\vec{r} = s\hat{s} + \zeta\hat{\zeta}$ . Let the current flow in such a way that the magnetic field points along the  $\zeta$ -axis. Let  $\mathcal{L}$  be an Amperian loop with one side at distance  $|\vec{r}| \rightarrow \infty$ ,

$$\int_{\mathcal{L}} d\vec{\ell} \cdot \vec{B} = \mu_0 I_{\text{enc}}$$

The same equation can be reused to calculate the vector potential for a Gaussian surface  $\mathcal{S}$ ,

$$\int_{\mathcal{L}} d\vec{\ell} \cdot \vec{A} = \int_{\mathcal{S}} d\vec{a} \cdot \vec{B} = \Phi$$

Where  $\Phi$  is the magnetic flux through  $\mathcal{S}$ . Furthermore, the energy required to set up a magnetic field is,

$$W = \frac{1}{2\mu_0} \int_{\mathcal{C}} d\tau B^2 = \frac{1}{2} \int_{\mathcal{C}} d\tau \vec{J} \cdot \vec{A} = \frac{1}{2} LI^2$$

Where  $L$  is the self-inductance of the solenoid.

## 4 Poynting's Theorem

First we begin with two of Maxwell's equations,

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (4.1)$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \quad (4.2)$$

Computing the inner product between eq. (4.1) and  $\vec{B}$ , and the inner product between eq. (4.2) and  $\vec{E}$  and taking a difference,

$$\vec{B} \cdot (\vec{\nabla} \times \vec{E}) - \vec{E} \cdot (\vec{\nabla} \times \vec{B}) = -\frac{\partial}{\partial t} \left( \frac{\epsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2 \right) = \mu_0 \vec{E} \cdot \vec{J}$$

Letting  $\frac{\epsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2$  be the **electromagnetic energy density**  $u$ , we have the following identity,

$$\vec{\nabla} \cdot (\vec{E} \times \vec{B}) = -\mu_0 \frac{\partial u}{\partial t} - \mu_0 \vec{E} \cdot \vec{J} \quad (4.3)$$

Physically eq. (4.3) corresponds to a conservation of energy. We refer to the term  $\frac{1}{\mu_0} (\vec{E} \times \vec{B})$  as the Poynting vector  $\vec{S}$  as it determines the direction of electromagnetic radiation. The Poynting vector  $\vec{S}$  represents the power density.

$$\vec{\nabla} \cdot \vec{S} + \frac{\partial u}{\partial t} + \vec{E} \cdot \vec{J} = 0 \quad (4.4)$$

Much like eq. (3.1), eq. (4.4) is a local conservation of *energy*. The only algebraic difference is the term  $\vec{E} \cdot \vec{J}$ . If there is a flowing charge  $\vec{J}$  through an electric field  $\vec{E}$ , then there is work done on the charge. By Gauss's theorem, the energy leaving through a surface  $\mathcal{S}$  per unit time is,

$$\int_{\mathcal{V}} \vec{\nabla} \cdot \vec{S} d\tau = \oint_{\mathcal{S}} d\vec{a} \cdot \vec{S}$$

and the E-M energy in the volume  $\mathcal{V}$  is given by,

$$\int_{\mathcal{V}} d\tau u$$

Where again,  $u$  is the electromagnetic energy density. If we integrate over eq. (4.4),

$$\int_{\mathcal{V}} d\tau \left( \vec{\nabla} \cdot \vec{S} + \frac{\partial u}{\partial t} + \vec{E} \cdot \vec{J} \right) = \oint_{\mathcal{S}} d\vec{a} \cdot \vec{S} + \int_{\mathcal{V}} d\tau \frac{\partial u}{\partial t} + \int_{\mathcal{V}} d\tau \vec{E} \cdot \vec{J}$$

Each term in eq. (4.4) has its purpose illuminated. The final term  $\int_V d\tau \vec{E} \cdot \vec{J}$  corresponds to the work done on moving charges  $\vec{J}$  in the volume  $V$ . It is important to note that there are no terms that correspond to “magnetic work”.

Consider the work done to move a charge  $q$  a displacement  $d\vec{\ell}$  by E-M forces,

$$\begin{aligned} dW &= d\vec{\ell} \cdot \vec{F} \\ &= d\vec{\ell} \cdot q(\vec{E} + \vec{v} \times \vec{B}) \\ &= \vec{v} dt \cdot q(\vec{E} + \vec{v} \times \vec{B}) \\ &= q dt (\vec{v} \cdot \vec{E}) + q dt \underbrace{(\vec{v} \cdot \{\vec{v} \times \vec{B}\})}_{=0} \\ &= q dt (\vec{v} \cdot \vec{E}) \end{aligned}$$

So for a continuous charge distribution we have that  $dq = \rho d\tau$  and  $\rho \vec{v} = \vec{J}$ . Which means that the rate of work done on the charge  $\rho$  in the volume  $V$  (i.e. creating the current density  $\vec{J}$ ) is,

$$\dot{W} = \int_V d\tau \vec{E} \cdot \vec{J}$$

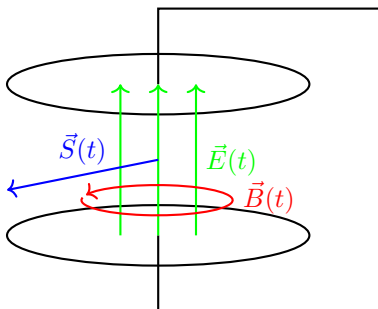
We can interpret this as the work done per unit time rearranging the charge in  $V$ . One again eq. (4.4) is given by

$$\vec{\nabla} \cdot \vec{S} + \frac{\partial u}{\partial t} + \vec{E} \cdot \vec{J} = 0$$

With the following interpretations,

- $\vec{\nabla} \cdot \vec{S}$ : amount of radiation energy leaving the point  $\vec{r}$
- $\frac{\partial u}{\partial t}$ : increase in E-M energy at the point  $\vec{r}$
- $\vec{E} \cdot \vec{J}$ : the amount of work done on charges at the point  $\vec{r}$

As an illustrative example, consider a parallel plate capacitor with an electric field  $\vec{E}$  between them.



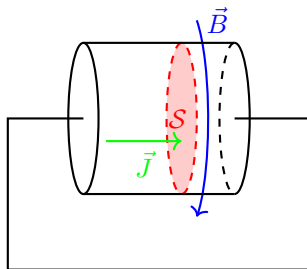
We have that the magnetic field points in the  $\hat{\phi}$  direction,  $\vec{B} = V\hat{\phi}$ . The electric field  $\vec{E} = E\hat{\zeta}$ , and Poynting vector are  $\vec{S} = S\hat{s}$ . We have that the radiation through the surface  $S$ ,

$$\int_S d\vec{a} \cdot \vec{S} = -(2\pi ah)S$$

Therefore  $\frac{\partial U}{\partial t} = -(2\pi ah)S$  corresponding to the amount of energy flowing out of the capacitor and therefore,

$$U = \int_0^\infty dt (-2\pi ahS) = \frac{1}{2} CV^2$$

**Ex 8.1:**



Inside the conductor the electric field moves parallel to its axis  $\vec{E} = \frac{V_0}{\ell} \hat{\zeta}$ . The magnetic field is then given by,

$$\vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial E}{\partial t} + \mu_0 \vec{J}$$

Integration over the surface  $\mathcal{S}$ ,

$$\int_{\mathcal{S}} d\vec{a} \cdot (\vec{\nabla} \times \vec{B}) = \mu_0 \int_{\mathcal{S}} d\vec{a} \cdot \vec{J} = \mu_0 I_{\text{enc}}$$

Therefore computing this integral  $\int d\vec{\ell} \cdot \vec{B}$  yields,

$$\vec{B} = \frac{\mu_0 I}{2\pi a} \hat{\phi}$$

Moreover, the Poynting vector is given by,

$$\begin{aligned} \vec{S} &= \frac{1}{\mu_0} \vec{E} \times \vec{B} \\ &= \frac{1}{\mu_0} \frac{V_0}{\ell} \hat{\zeta} \times \frac{\mu_0 I}{2\pi a} \hat{\phi} \\ &= -\frac{V_0 I}{2\pi a \ell} \hat{s} \end{aligned}$$

Therefore the radiation flux,

$$\int_{\mathcal{S}} d\vec{a} \cdot \vec{S} = -\frac{V_0 I}{2\pi a \ell} \int_{\mathcal{S}} da = -V_0 I$$

Which is exactly the amount of Joule heating for a current  $I$  though a wire with voltage  $V_0$  across it. Using  $V = IR$ ,

$$\int_{\mathcal{S}} d\vec{a} \cdot \vec{S} = -I^2 R$$