Phys 442

ELECTRICITY & MAGNETISM 3

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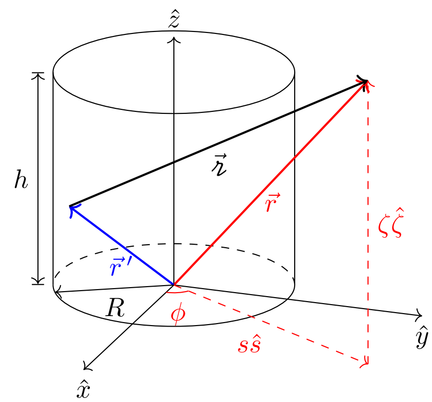
1 Coordinates and Symmetry

A clever choice of coordinates systems typically makes solving a problem considerably easier. Mathematically, this is due to *Noether's Theorem*. A typical three dimensional Lagrangian will have three dependent generalized coordinates $L = L(x, y, z) = L(s, \theta, \zeta) = \cdots$. However, if one can identify generalized coordinates q that make the Lagrangian invariant $\frac{\partial L}{\partial q} = 0$, then the *Euler-Lagrange* equations are considerably similar,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \implies \frac{\partial L}{\partial \dot{q}} = \text{const.} \implies L \propto \dot{q}$$

As such, the number of equations that remain to solved has been reduced.

2 First Assignment?



A1.1: Use cylindrical coordinates with ζ along the axis of the cable,

$$V(\zeta) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{C}} \frac{\mathrm{d}\rho}{\mathscr{r}}$$

Where $\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$, $\vec{r}\,'$ is the source point and $\vec{r}$ is the field point. The entire cylinder is the set of all source points $\vec{r}\,'$ that are contained inside $|\vec{r}\,'| \leq R$.

$$\vec{r} = \zeta\hat{\zeta}$$
$$\vec{r}\,' = s'\hat{s}' + \zeta'\hat{\zeta}$$

$$V(\zeta) = \frac{\rho}{4\pi\epsilon_0}\int_{\mathcal{C}} \frac{\mathrm{d}V}{|\vec{r} - \vec{r}\,'|}$$

Where $\mathrm{d}V = s\mathrm{d}s\mathrm{d}\theta\mathrm{d}\zeta$. One can then find the electric *field* by doing $\vec{E} = -\vec{\nabla}V = E_\zeta\hat{\zeta} = -\frac{\partial V}{\partial \zeta}\hat{\zeta}$

A1.2:

Between the two conductors, there will be a radial electric field $\vec{E} = E(s)\hat{s}$ and parallel magnetic field $\vec{B} = B(s)\hat{\zeta}$. Outside the two conductors, there will be no electric or magnetic field.

$$E^{\parallel}_{\text{vac}} = 0$$
$$E^{\perp}_{\text{vac}} = \frac{\sigma}{\epsilon_0}$$
$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$$

For part g), use Laplace's equation $\nabla^2 V = 0$. In cylindrical coordinates, Laplace's equation is,

$$\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) = 0$$

Cylindrical coordinates gives us the following symmetries $\frac{\partial V}{\partial \phi} = \frac{\partial V}{\partial \zeta} = 0$. Solving this system gives the potential in terms of s: $V(s) = \cdots$. Then the electric field can then be obtained via $\vec{E} = -\vec{\nabla} V$.
A1.3: Using cylindrical coordinates once again, the electric field is going to be radial outwards to the uniform charge density. For the uniform density cylinder, construct a Gaussian surface cylindrically around the cylinder. For the current density cylinder, the current density is the current per cross sectional area. Construct an Amperian loop,

$$\oint_{\mathcal{A}} \mathrm{d}\vec{\ell} \cdot \vec{B} = \mu I_{\text{enc}}$$

Part e), finding the vector potential,

$$A(\vec{r}) = \frac{\mu_0}{4\pi} \int_{\mathcal{C}} \mathrm{d}\tau' \, \frac{\vec{J}(\vec{r})}{\mathscr{r}}$$

Evidently, $\hat{s}$ and $\hat{s}'$ are in *different* directions. Solving such an equation yields,

$$A(s) = \frac{\mu_0}{4\pi} \int_0^{2\pi} \mathrm{d}\phi' \int_0^a s' \mathrm{d}s' \int_{-\infty}^{\infty} \mathrm{d}\zeta' \, \frac{J(s)}{\left| s\hat{s} - s'\hat{s}' - \zeta'\hat{\zeta} \right|}$$

Recognize the structure of the potential integral,

$$V(s) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{C}} \mathrm{d}\tau' \, \frac{\rho(r')}{\mathscr{r}} = \frac{\rho_0}{4\pi\epsilon_0} \int_{\mathcal{C}} \frac{\mathrm{d}\tau'}{\mathscr{r}}$$

Comparing to the vector potential, we have an equivalent integral (up to a constant).

$$A(s) = \frac{\mu_0 J_0}{4\pi} \int_{\mathcal{C}} \frac{\mathrm{d}\tau'}{\mathscr{r}}$$

For question f), use the definition of $\vec{B}$ in terms of $\vec{A}$,

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

Further, recall that if $\vec{E} = -\vec{\nabla} V$, then by Stoke's theorem for some loop $\mathcal{L}$,

$$V = -\int_{\mathcal{L}} \mathrm{d}\vec{\ell} \cdot \vec{E}$$

3 Conservation Laws

Beginning with one of Maxwell's equations,

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Taking the divergence of the above equation,

$$\cancelto{0}{\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{B} \right)} = \mu_0 \vec{\nabla} \cdot \vec{J} + \mu_0 \epsilon_0 \frac{\partial}{\partial t} \left(\vec{\nabla} \cdot \vec{E} \right)$$

Luckily, the divergence of a curl is always 0. Dividing by relevant constants we obtain the following conservation law,

$$0 = \vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} \tag{3.1}$$

This is a conservation of charge. It is a **local** conservation law because it holds for all points in space $\vec{r}$. Intuitively, is claims that the rate of charge of charge at a point is equal to the amount of current following in or out of the take point.

A2.1: Again using cylindrical coordinates $\vec{r} = s\hat{s} + \zeta\hat{\zeta}$. Let the current flow in such a way that the magnetic field points along the ζ-axis. Let $\mathcal{L}$ be an Amperian loop with one side at distance $|\vec{r}| \to \infty$,

$$\int_{\mathcal{L}} \mathrm{d}\vec{\ell} \cdot \vec{B} = \mu_0 I_{\mathrm{enc}}$$

The same equation can be reused to calculate the vector potential for a Gaussian surface $\mathcal{S}$,

$$\int_{\mathcal{L}} \mathrm{d}\vec{\ell} \cdot \vec{A} = \int_{\mathcal{S}} \mathrm{d}\vec{a} \cdot \vec{B} = \Phi$$

Where Φ is the magnetic flux through $\mathcal{S}$. Furthermore, the energy required to set up a magnetic field is,

$$W = \frac{1}{2\mu_0} \int_{\mathcal{C}} \mathrm{d}\tau B^2 = \frac{1}{2} \int_{\mathcal{C}} \mathrm{d}\tau \vec{J} \cdot \vec{A} = \frac{1}{2} L I^2$$

Where L is the self-inductance of the solenoid.

4 Poynting's Theorem

First we begin with two of Maxwell's equations,

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{4.1}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{4.2}$$

Computing the inner product between eq. (4.1) and $\vec{B}$, and the inner product between eq. (4.2) and $\vec{E}$ and taking a difference,

$$\vec{B} \cdot \left(\vec{\nabla} \times \vec{E}\right) - \vec{E} \cdot \left(\vec{\nabla} \times \vec{B}\right) = -\frac{\partial}{\partial t}\left(\frac{\epsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2\right) = \mu_0 \vec{E} \cdot \vec{J}$$

Letting $\frac{\epsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2$ be the **electromagnetic energy density** u, we have the following identity,

$$\vec{\nabla} \cdot \left(\vec{E} \times \vec{B}\right) = -\mu_0 \frac{\partial u}{\partial t} - \mu_0 \vec{E} \cdot \vec{J} \tag{4.3}$$

Physically eq. (4.3) corresponds to a conservation of energy. We refer to the term $\frac{1}{\mu_0}\left(\vec{E} \times \vec{B}\right)$ as the Poynting vector $\vec{S}$ as it determines the direction of electromagnetic radiation. The Poynting vector $\vec{S}$ represents the power density.

$$\vec{\nabla} \cdot \vec{S} + \frac{\partial u}{\partial t} + \vec{E} \cdot \vec{J} = 0 \tag{4.4}$$

Much like eq. (3.1), eq. (4.4) is a local conservation of *energy*. The only algebraic difference is the term $\vec{E} \cdot \vec{J}$. If there is a flowing charge $\vec{J}$ through an electric field $\vec{E}$, then there is work done on the charge. By Gauss's theorem, the energy leaving through a surface $\mathcal{S}$ per unit time is,

$$\int_{\mathcal{V}} \vec{\nabla} \cdot \vec{S} \mathrm{d}\tau = \oint_{\mathcal{S}} \mathrm{d}\vec{a} \cdot \vec{S}$$

and the E-M energy in the volume $\mathcal{V}$ is given by,

$$\int_{\mathcal{V}} \mathrm{d}\tau u$$

Where again, u is the electromagnetic energy density. If we integrate over eq. (4.4),

$$\int_{\mathcal{V}} \mathrm{d}\tau \left(\vec{\nabla} \cdot \vec{S} + \frac{\partial u}{\partial t} + \vec{E} \cdot \vec{J}\right) = \oint_{\mathcal{S}} \mathrm{d}\vec{a} \cdot \vec{S} + \int_{\mathcal{V}} \mathrm{d}\tau \frac{\partial u}{\partial t} + \int_{\mathcal{V}} \mathrm{d}\tau \vec{E} \cdot \vec{J}$$

Each term in eq. (4.4) has it's purpose illuminated. The final term $\int_{\mathcal{V}} \mathrm{d}\tau \vec{E} \cdot \vec{J}$ corresponds to the work done on moving charges $\vec{J}$ in the volume $\mathcal{V}$. In it important to note that there are no terms that corresponding to "magnetic work".

Consider the work done to move a charge q a displacement $\mathrm{d}\vec{\ell}$ by E-M forces,

$$\begin{aligned}
\mathrm{d}W &= \mathrm{d}\vec{\ell} \cdot \vec{F} \\
&= \mathrm{d}\vec{\ell} \cdot q\left(\vec{E} + \vec{v} \times \vec{B}\right) \\
&= \vec{v}\mathrm{d}t \cdot q\left(\vec{E} + \vec{v} \times \vec{B}\right) \\
&= q\mathrm{d}t\left(\vec{v} \cdot \vec{E}\right) + q\mathrm{d}t \underbrace{\left(\vec{v} \cdot \left\{\vec{v} \times \vec{B}\right\}\right)}_{=0} \\
&= q\mathrm{d}t\left(\vec{v} \cdot \vec{E}\right)
\end{aligned}$$

So for a continuous charge distribution we have that $\mathrm{d}q = \rho \mathrm{d}\tau$ and $\rho\vec{v} = \vec{J}$. Which means that the rate of work done on the charge ρ in the volume $\mathcal{V}$ (i.e. creating the current density $\vec{J}$) is,

$$\dot{W} = \int_{\mathcal{V}} \mathrm{d}\tau \vec{E} \cdot \vec{J}$$

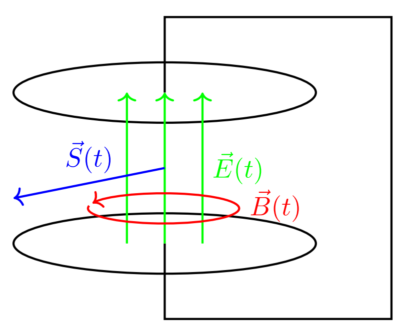
We can interpret this as the work done per unit time rearranging the charge in $\mathcal{V}$. One again eq. (4.4) is given by

$$\vec{\nabla} \cdot \vec{S} + \frac{\partial u}{\partial t} + \vec{E} \cdot \vec{J} = 0$$

With the following interpretations,

- $\vec{\nabla} \cdot \vec{S}$: amount of radiation energy leaving the point $\vec{r}$
- $\frac{\partial u}{\partial t}$: increase in E-M energy at the point $\vec{r}$
- $\vec{E} \cdot \vec{J}$: the amount of work done on charges at the point $\vec{r}$

As an illustrative example, consider a parallel plate capacitor with an electric field $\vec{E}$ between them.



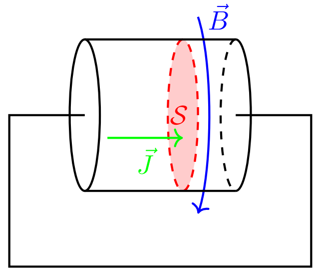
We ave that the magnetic field points in the $\hat{\phi}$ direction, $\vec{B} = V\hat{\phi}$. The electric field $\vec{E} = E\hat{\zeta}$, and Poynting vector are $\vec{S} = S\hat{s}$. We have that the radiation through the surface $\mathcal{S}$,

$$\int_{\mathcal{S}} \mathrm{d}\vec{a} \cdot \vec{S} = -(2\pi a h)S$$

Therefore $\frac{\partial U}{\partial t} = -(2\pi a h)S$ corresponding to the amount of energy flowing out of the capacitor and therefore,

$$U = \int_0^\infty \mathrm{d}t(-2\pi a h S) = \frac{1}{2}CV^2$$

Ex 8.1:



Inside the conductor the electric field moves parallel to its axis $\vec{E} = \frac{V_0}{\ell}\hat{\zeta}$. The magnetic field is then given by,

$$\vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial E}{\partial t} + \mu_0 \vec{J}$$

Integration over the surface $\mathcal{S}$,

$$\int_{\mathcal{S}} \mathrm{d}\vec{a} \cdot \left(\vec{\nabla} \times \vec{B}\right) = \mu_0 \int_{\mathcal{S}} \mathrm{d}\vec{a} \cdot \vec{J} = \mu_0 I_{\text{enc}}$$

Therefore computing this integral $\int \mathrm{d}\vec{\ell} \cdot \vec{B}$ yields,

$$\vec{B} = \frac{\mu_0 I}{2\pi a}\hat{\phi}$$

Moreover, the Poynting vector is given by,

$$\begin{aligned}\vec{S} &= \frac{1}{\mu_0}\vec{E} \times \vec{B} \\ &= \frac{1}{\mu_0}\frac{V_0}{\ell}\hat{\zeta} \times \frac{\mu_0 I}{2\pi a}\hat{\phi} \\ &= -\frac{V_0 I}{2\pi a \ell}\hat{s}\end{aligned}$$

Therefore the radiation flux,

$$\int_{\mathcal{S}} \mathrm{d}\vec{a} \cdot \vec{S} = -\frac{V_0 I}{2\pi a \ell}\int_{\mathcal{S}} \mathrm{d}a = -V_0 I$$

Which is exactly the amount of Joule heating for a current I though a wire with voltage V_0 across it. Using $V = IR$,

$$\int_{\mathcal{S}} \mathrm{d}\vec{a} \cdot \vec{S} = -I^2 R$$

Ex 8.2 (Griffiths Problem 8.13): A long thin solenoid of radius a has a time dependent current $I_s(t)$ flowing around it. Encircling the solenoid is a ring of radius b with current $I_r(t)$ $(b \gg a)$ passing through it. The ring has resistance R. There is an induced electro-motive-force in the ring due to the solenoid,

$$\mathcal{E} = -\dot{\Phi}_S = -\frac{\partial}{\partial t}\left(\pi a^2 B_S\right)$$

Where $B_s = \mu_0 n I_s$. The EMF $\mathcal{E}$ must also equal $\mathcal{E} = I_r R$. Therefore,

$$I_r = -\frac{1}{R}\left(\mu_0 \pi a^2 n\right)\dot{I}_S$$

In order to calculate the electric and magnetic fields just outside solenoid, recognize that $\vec{B}_s = B_s(t)\hat{z}$ point along the axis of the solenoid. Similarly recognize that $\vec{E} = E\hat{\phi}$. Therefore the Poynting vector is given by,

$$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B}_s = \frac{1}{\mu_0}E B_s \hat{s} = ?$$

We first need to calculate $\vec{E}$ and $\vec{B}_s$. The magnetic field is known to be $\vec{B} = \mu_0 n I_s \hat{z}$ on axis and $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$,

$$\int \mathrm{d}\vec{a}\cdot\vec{\nabla}\times\vec{E} = -\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}\vec{a}\cdot\vec{B}$$

$$\int \mathrm{d}\vec{\ell}\cdot\vec{E} = -\dot{\Phi} = 2\pi a E$$

Which gives,

$$\vec{E} = \frac{\dot{\Phi}}{2\pi a}\hat{\phi}$$

The magnetic field off axis and outside the solenoid due to the ring is given by,

$$\mathrm{d}\vec{B}_r(s) = \frac{\mu_0 I}{4\pi}\frac{\mathrm{d}\vec{\ell}\times\vec{\mathscr{r}}}{\mathscr{r}^2}$$

Where $\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$ and we take $\vec{r}\,' = b\hat{s}'$ and $\vec{r} = z\hat{z}$.

$$\vec{\mathscr{r}} = z\hat{z} - b\hat{s}'$$

We will take the infinitesimal loop to be $\mathrm{d}\vec{\ell} = b\mathrm{d}\phi'\hat{\phi}'$.

$$\begin{aligned}\mathrm{d}\vec{\ell}\times\vec{\mathscr{r}} &= \left(b\hat{\phi}'\mathrm{d}\hat{\phi}'\right)\times(z\hat{z} - b\hat{s}') \\ &= az\mathrm{d}\phi'\hat{s}' + b^2\mathrm{d}\phi'\hat{z}\end{aligned}$$

We integrating around the loop $\mathcal{L}$, all of the contributions in the $\hat{s}'$ directions will cancel out.

$$\int_{\mathcal{L}} \mathrm{d}\vec{\ell}\times\vec{\mathscr{r}} = \cdots$$

Thus,

$$\begin{aligned}\vec{B}_r &= \frac{\mu_0 I_r}{4\pi}\int\frac{b^2\mathrm{d}\phi'\hat{z}}{(z^2+b^2)^{3/2}} \\ &= \frac{\mu_0 I_r b^2 2\pi\hat{z}}{4\pi(z^2+b^2)^{3/2}} \\ &= \frac{\mu_0 b^2}{2(z^2+b^2)^{3/2}} I_r\hat{z}\end{aligned}$$

Therefore the Poynting vector points radial outward,

$$\begin{aligned}\vec{S} &= \frac{1}{\mu_0}\vec{E}_r\times\vec{B}_r \\ &= \frac{1}{\mu_0}\left(\frac{\pi a^2\mu_0 n\dot{I}_s}{2\pi a}\hat{\phi}\right)\times\left(\frac{\mu_0 b^2}{2(z^2+b^2)^{3/2}}I_r\hat{z}\right) \\ &= \frac{\mu_0}{4}an\dot{I}_s\frac{b^2}{(z^2+b^2)^{3/2}}I_r\hat{s}\end{aligned}$$

Now that the Poynting vector is known, one can calculate the power radiated from the system.

$$\begin{aligned}P &= \int \mathrm{d}\vec{a}\cdot\vec{S} \\ &= \frac{\mu_0}{4}an\dot{I}_s I_r b^2\int \mathrm{d}za\mathrm{d}\phi\hat{s}\cdot\frac{1}{(z^2+b^2)^{3/2}}\hat{s}\end{aligned}$$

$$
\begin{aligned}
&= \frac{\mu_0}{4} an\dot{I}_s I_r b^2 (2\pi a) \int \mathrm{d}z \frac{1}{\left(z^2+b^2\right)^{3/2}} \\
&= \frac{\mu_0}{4} an\dot{I}_s I_r b^2 (2\pi a) \frac{2}{b^2} \quad \text{Integral Table} \\
&= \mu_0 \pi a^2 n \dot{I}_s I_r
\end{aligned}
$$

But we know that $\mu_0 \pi a^2 n \dot{I}_s = -I_r R$. Therefore $P = -I_r^2 R$ as expected.