
Phys 442

ELECTRICITY & MAGNETISM 3

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Latest versions of all my course notes are available at **www.tcfraser.com/coursenotes**.

1 Coordinates and Symmetry

A clever choice of coordinates systems typically makes solving a problem considerably easier. Mathematically, this is due to *Noether's Theorem*. A typical three dimensional Lagrangian will have three dependent generalized coordinates $L = L(x, y, z) = L(s, \theta, \zeta) = \dots$. However, if one can identify generalized coordinates q that make the Lagrangian invariant $\frac{\partial L}{\partial q} = 0$, then the *Euler-Lagrange* equations are considerably similar,

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0 \implies \frac{\partial L}{\partial \dot{q}} = \text{const.} \implies L \propto \dot{q}$$

As such, the number of equations that remain to be solved has been reduced.

2 First Assignment?

A1.1: Use cylindrical coordinates with ζ along the axis of the cable,

$$V(\zeta) = \frac{1}{4\pi\epsilon_0} \int_C \frac{d\rho}{r}$$

Where $\vec{r} = \vec{r} - \vec{r}'$, \vec{r}' is the source point and \vec{r} is the field point. The entire cylinder is the set of all source points \vec{r}' that are contained inside $|\vec{r}'| \leq R$.

$$\begin{aligned}\vec{r} &= \zeta \hat{\zeta} \\ \vec{r}' &= s' \hat{s}' + \zeta' \hat{\zeta}\end{aligned}$$

$$V(\zeta) = \frac{\rho}{4\pi\epsilon_0} \int_C \frac{dV}{|\vec{r} - \vec{r}'|}$$

Where $dV = s ds d\theta d\zeta$. One can then find the electric field by doing $\vec{E} = -\vec{\nabla}V = E_{\zeta} \hat{\zeta} = -\frac{\partial V}{\partial \zeta} \hat{\zeta}$

A1.2:

Between the two conductors, there will be a radial electric field $\vec{E} = E(s)\hat{s}$ and parallel magnetic field $\vec{B} = B(s)\hat{\zeta}$. Outside the two conductors, there will be no electric or magnetic field.

$$\begin{aligned}E_{\text{vac}}^{\parallel} &= 0 \\ E_{\text{vac}}^{\perp} &= \frac{\sigma}{\epsilon_0} \\ \vec{\nabla} \cdot \vec{E} &= \frac{\rho}{\epsilon_0}\end{aligned}$$

For part g), use Laplace's equation $\nabla^2 V = 0$. In cylindrical coordinates, Laplace's equation is,

$$\nabla^2 V = \frac{1}{s} \frac{\partial}{\partial s} \left(s \frac{\partial V}{\partial s} \right) = 0$$

Cylindrical coordinates gives us the following symmetries $\frac{\partial V}{\partial \phi} = \frac{\partial V}{\partial \zeta} = 0$. Solving this system gives the potential in terms of s : $V(s) = \dots$. Then the electric field can then be obtained via $\vec{E} = -\vec{\nabla}V$.

A1.3: Using cylindrical coordinates once again, the electric field is going to be radial outwards to the uniform charge density. For the uniform density cylinder, construct a Gaussian surface cylindrically around the cylinder. For the current density cylinder, the current density is the current per cross sectional area. Construct an Amperian loop,

$$\oint_{\mathcal{A}} d\vec{l} \cdot \vec{B} = \mu I_{\text{enc}}$$

Part e), finding the vector potential,

$$A(\vec{r}) = \frac{\mu_0}{4\pi} \int_C d\tau' \frac{\vec{J}(\vec{r}')}{r}$$

Evidently, \hat{s} and \hat{s}' are in *different* directions. Solving such an equation yields,

$$A(s) = \frac{\mu_0}{4\pi} \int_0^{2\pi} d\phi' \int_0^a s' ds' \int_{-\infty}^{\infty} d\zeta' \frac{J(s)}{|s\hat{s} - s'\hat{s}' - \zeta'\hat{\zeta}|}$$

Recognize the structure of the potential integral,

$$V(s) = \frac{1}{4\pi\epsilon_0} \int_C d\tau' \frac{\rho(r')}{r} = \frac{\rho_0}{4\pi\epsilon_0} \int_C \frac{d\tau'}{r}$$

Comparing to the vector potential, we have an equivalent integral (up to a constant).

$$A(s) = \frac{\mu_0 J_0}{4\pi} \int_C \frac{d\tau'}{r}$$

For question f), use the definition of \vec{B} in terms of \vec{A} ,

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

Further, recall that if $\vec{E} = -\vec{\nabla}V$, then by Stoke's theorem for some loop \mathcal{L} ,

$$V = - \int_{\mathcal{L}} d\vec{l} \cdot \vec{E}$$

Todo (TC Fraser): Figure out O'Donovan

3 Conservation Laws

Beginning with one of Maxwell's equations,

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Taking the divergence of the above equation,

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) \overset{0}{=} \mu_0 \vec{\nabla} \cdot \vec{J} + \mu_0 \epsilon_0 \frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{E})$$

Luckily, the divergence of a curl is always 0. Dividing by relevant constants we obtain the following conservation law,

$$0 = \vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} \quad (3.1)$$

This is a conservation of charge. It is a **local** conservation law because it holds for all points in space \vec{r} . Intuitively, it claims that the rate of change of charge at a point is equal to the amount of current following in or out of the take point.

A2.1: Again using cylindrical coordinates $\vec{r} = s\hat{s} + \zeta\hat{\zeta}$. Let the current flow in such a way that the magnetic field points along the ζ -axis. Let \mathcal{L} be an Amperian loop with one side at distance $|\vec{r}| \rightarrow \infty$,

$$\int_{\mathcal{L}} d\vec{l} \cdot \vec{B} = \mu_0 I_{\text{enc}}$$

The same equation can be reused to calculate the vector potential for a Gaussian surface \mathcal{S} ,

$$\int_{\mathcal{L}} d\vec{l} \cdot \vec{A} = \int_{\mathcal{S}} d\vec{a} \cdot \vec{B} = \Phi$$

Where Φ is the magnetic flux through \mathcal{S} . Furthermore, the energy required to set up a magnetic field is,

$$W = \frac{1}{2\mu_0} \int_C d\tau B^2 = \frac{1}{2} \int_C d\tau \vec{J} \cdot \vec{A} = \frac{1}{2} LI^2$$

Where L is the self-inductance of the solenoid.

4 Poynting's Theorem

First we begin with two of Maxwell's equations,

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (4.1)$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \quad (4.2)$$

Computing the inner product between eq. (4.1) and \vec{B} , and the inner product between eq. (4.2) and \vec{E} and taking a difference,

$$\vec{B} \cdot (\vec{\nabla} \times \vec{E}) - \vec{E} \cdot (\vec{\nabla} \times \vec{B}) = -\frac{\partial}{\partial t} \left(\frac{\epsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2 \right) = \mu_0 \vec{E} \cdot \vec{J}$$

Letting $\frac{\epsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2$ be the **electromagnetic energy density** U , we have the following identity,

$$\vec{\nabla} \cdot (\vec{E} \times \vec{B}) = -\mu_0 \frac{\partial U}{\partial t} - \mu_0 \vec{E} \cdot \vec{J} \quad (4.3)$$

Physically eq. (4.3) corresponds to a conservation of energy. We refer to the term $\frac{1}{\mu_0} (\vec{E} \times \vec{B})$ as the Poynting vector \vec{S} as it determines the direction of electromagnetic radiation,

$$\vec{\nabla} \cdot \vec{S} + \frac{\partial U}{\partial t} + \vec{E} \cdot \vec{J} = 0 \quad (4.4)$$

Much like eq. (3.1), eq. (4.4) is a local conservation of *energy*. The only algebraic difference is the term $\vec{E} \cdot \vec{J}$. If there is a flowing charge \vec{J} through an electric field \vec{E} , then there is work done on the charge.