
Phys 442

ELECTRICITY & MAGNETISM 3

University of Waterloo

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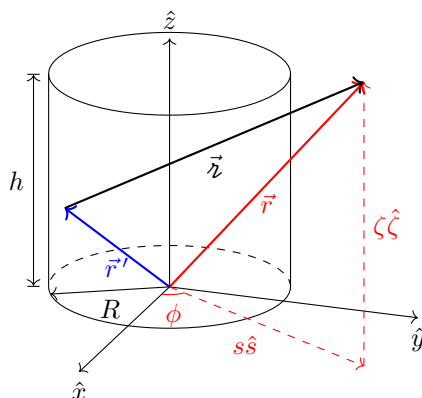
1 Coordinates and Symmetry

A clever choice of coordinates systems typically makes solving a problem considerably easier. Mathematically, this is due to *Noether's Theorem*. A typical three dimensional Lagrangian will have three dependent generalized coordinates $L = L(x, y, z) = L(s, \theta, \zeta) = \dots$. However, if one can identify generalized coordinates q that make the Lagrangian invariant $\frac{\partial L}{\partial q} = 0$, then the *Euler-Lagrange* equations are considerably similar,

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0 \implies \frac{\partial L}{\partial \dot{q}} = \text{const.} \implies L \propto \dot{q}$$

As such, the number of equations that remain to be solved has been reduced.

2 First Assignment?



A1.1: Use cylindrical coordinates with ζ along the axis of the cable,

$$V(\zeta) = \frac{1}{4\pi\epsilon_0} \int_C \frac{d\rho}{z}$$

Where $\vec{z} = \vec{r} - \vec{r}'$, \vec{r}' is the source point and \vec{r} is the field point. The entire cylinder is the set of all source points \vec{r}' that are contained inside $|\vec{r}'| \leq R$.

$$\begin{aligned} \vec{r} &= \zeta \hat{\zeta} \\ \vec{r}' &= s' \hat{s}' + \zeta' \hat{\zeta} \end{aligned}$$

$$V(\zeta) = \frac{\rho}{4\pi\epsilon_0} \int_C \frac{dV}{|\vec{r} - \vec{r}'|}$$

Where $dV = s ds d\theta d\zeta$. One can then find the electric field by doing $\vec{E} = -\vec{\nabla}V = E_{\zeta} \hat{\zeta} = -\frac{\partial V}{\partial \zeta} \hat{\zeta}$

A1.2:

Between the two conductors, there will be a radial electric field $\vec{E} = E(s)\hat{s}$ and parallel magnetic field $\vec{B} = B(s)\hat{\zeta}$. Outside the two conductors, there will be no electric or magnetic field.

$$\begin{aligned} E_{\text{vac}}^{\parallel} &= 0 \\ E_{\text{vac}}^{\perp} &= \frac{\sigma}{\epsilon_0} \\ \vec{\nabla} \cdot \vec{E} &= \frac{\rho}{\epsilon_0} \end{aligned}$$

For part g), use Laplace's equation $\nabla^2 V = 0$. In cylindrical coordinates, Laplace's equation is,

$$\nabla^2 V = \frac{1}{s} \frac{\partial}{\partial s} \left(s \frac{\partial V}{\partial s} \right) = 0$$

Cylindrical coordinates gives us the following symmetries $\frac{\partial V}{\partial \phi} = \frac{\partial V}{\partial \zeta} = 0$. Solving this system gives the potential in terms of s : $V(s) = \dots$. Then the electric field can then be obtained via $\vec{E} = -\vec{\nabla}V$.

A1.3: Using cylindrical coordinates once again, the electric field is going to be radial outwards to the uniform charge density. For the uniform density cylinder, construct a Gaussian surface cylindrically around the cylinder. For the current density cylinder, the current density is the current per cross sectional area. Construct an Amperian loop,

$$\oint_{\mathcal{A}} d\vec{\ell} \cdot \vec{B} = \mu I_{\text{enc}}$$

Part e), finding the vector potential,

$$A(\vec{r}) = \frac{\mu_0}{4\pi} \int_{\mathcal{C}} d\tau' \frac{\vec{J}(\vec{r}')}{r}$$

Evidently, \hat{s} and \hat{s}' are in *different* directions. Solving such an equation yields,

$$A(s) = \frac{\mu_0}{4\pi} \int_0^{2\pi} d\phi' \int_0^a s' ds' \int_{-\infty}^{\infty} d\zeta' \frac{J(s)}{|s\hat{s} - s'\hat{s}' - \zeta'\hat{\zeta}|}$$

Recognize the structure of the potential integral,

$$V(s) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{C}} d\tau' \frac{\rho(r')}{r} = \frac{\rho_0}{4\pi\epsilon_0} \int_{\mathcal{C}} \frac{d\tau'}{r}$$

Comparing to the vector potential, we have an equivalent integral (up to a constant).

$$A(s) = \frac{\mu_0 J_0}{4\pi} \int_{\mathcal{C}} \frac{d\tau'}{r}$$

For question f), use the definition of \vec{B} in terms of \vec{A} ,

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

Further, recall that if $\vec{E} = -\vec{\nabla}V$, then by Stoke's theorem for some loop \mathcal{L} ,

$$V = - \int_{\mathcal{L}} d\vec{\ell} \cdot \vec{E}$$

3 Conservation Laws

Beginning with one of Maxwell's equations,

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Taking the divergence of the above equation,

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) \overset{0}{=} \mu_0 \vec{\nabla} \cdot \vec{J} + \mu_0 \epsilon_0 \frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{E})$$

Luckily, the divergence of a curl is always 0. Dividing by relevant constants we obtain the following conservation law,

$$0 = \vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} \quad (3.1)$$

This is a conservation of charge. It is a **local** conservation law because it holds for all points in space \vec{r} . Intuitively, it claims that the rate of change of charge at a point is equal to the amount of current following in or out of the take point.

A2.1: Again using cylindrical coordinates $\vec{r} = s\hat{s} + \zeta\hat{\zeta}$. Let the current flow in such a way that the magnetic field points along the ζ -axis. Let \mathcal{L} be an Amperian loop with one side at distance $|\vec{r}| \rightarrow \infty$,

$$\int_{\mathcal{L}} d\vec{\ell} \cdot \vec{B} = \mu_0 I_{\text{enc}}$$

The same equation can be reused to calculate the vector potential for a Gaussian surface \mathcal{S} ,

$$\int_{\mathcal{L}} d\vec{\ell} \cdot \vec{A} = \int_{\mathcal{S}} d\vec{a} \cdot \vec{B} = \Phi$$

Where Φ is the magnetic flux through \mathcal{S} . Furthermore, the energy required to set up a magnetic field is,

$$W = \frac{1}{2\mu_0} \int_{\mathcal{C}} d\tau B^2 = \frac{1}{2} \int_{\mathcal{C}} d\tau \vec{J} \cdot \vec{A} = \frac{1}{2} LI^2$$

Where L is the self-inductance of the solenoid.

4 Poynting's Theorem

First we begin with two of Maxwell's equations,

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (4.1)$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \quad (4.2)$$

Computing the inner product between eq. (4.1) and \vec{B} , and the inner product between eq. (4.2) and \vec{E} and taking a difference,

$$\vec{B} \cdot (\vec{\nabla} \times \vec{E}) - \vec{E} \cdot (\vec{\nabla} \times \vec{B}) = -\frac{\partial}{\partial t} \left(\frac{\epsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2 \right) = \mu_0 \vec{E} \cdot \vec{J}$$

Letting $\frac{\epsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2$ be the **electromagnetic energy density** u , we have the following identity,

$$\vec{\nabla} \cdot (\vec{E} \times \vec{B}) = -\mu_0 \frac{\partial u}{\partial t} - \mu_0 \vec{E} \cdot \vec{J} \quad (4.3)$$

Physically eq. (4.3) corresponds to a conservation of energy. We refer to the term $\frac{1}{\mu_0} (\vec{E} \times \vec{B})$ as the Poynting vector \vec{S} as it determines the direction of electromagnetic radiation. The Poynting vector \vec{S} represents the power density.

$$\vec{\nabla} \cdot \vec{S} + \frac{\partial u}{\partial t} + \vec{E} \cdot \vec{J} = 0 \quad (4.4)$$

Much like eq. (3.1), eq. (4.4) is a local conservation of *energy*. The only algebraic difference is the term $\vec{E} \cdot \vec{J}$. If there is a flowing charge \vec{J} through an electric field \vec{E} , then there is work done on the charge. By Gauss's theorem, the energy leaving through a surface \mathcal{S} per unit time is,

$$\int_{\mathcal{V}} \vec{\nabla} \cdot \vec{S} d\tau = \oint_{\mathcal{S}} d\vec{a} \cdot \vec{S}$$

and the E-M energy in the volume \mathcal{V} is given by,

$$\int_{\mathcal{V}} d\tau u$$

Where again, u is the electromagnetic energy density. If we integrate over eq. (4.4),

$$\int_{\mathcal{V}} d\tau \left(\vec{\nabla} \cdot \vec{S} + \frac{\partial u}{\partial t} + \vec{E} \cdot \vec{J} \right) = \oint_{\mathcal{S}} d\vec{a} \cdot \vec{S} + \int_{\mathcal{V}} d\tau \frac{\partial u}{\partial t} + \int_{\mathcal{V}} d\tau \vec{E} \cdot \vec{J}$$

Each term in eq. (4.4) has its purpose illuminated. The final term $\int_V d\tau \vec{E} \cdot \vec{J}$ corresponds to the work done on moving charges \vec{J} in the volume V . It is important to note that there are no terms that correspond to “magnetic work”.

Consider the work done to move a charge q a displacement $d\vec{\ell}$ by E-M forces,

$$\begin{aligned} dW &= d\vec{\ell} \cdot \vec{F} \\ &= d\vec{\ell} \cdot q(\vec{E} + \vec{v} \times \vec{B}) \\ &= \vec{v} dt \cdot q(\vec{E} + \vec{v} \times \vec{B}) \\ &= q dt (\vec{v} \cdot \vec{E}) + q dt \underbrace{(\vec{v} \cdot \{\vec{v} \times \vec{B}\})}_{=0} \\ &= q dt (\vec{v} \cdot \vec{E}) \end{aligned}$$

So for a continuous charge distribution we have that $dq = \rho d\tau$ and $\rho \vec{v} = \vec{J}$. Which means that the rate of work done on the charge ρ in the volume V (i.e. creating the current density \vec{J}) is,

$$\dot{W} = \int_V d\tau \vec{E} \cdot \vec{J}$$

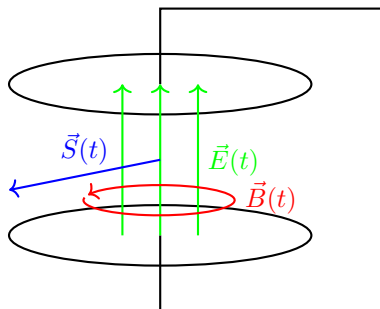
We can interpret this as the work done per unit time rearranging the charge in V . One again eq. (4.4) is given by

$$\vec{\nabla} \cdot \vec{S} + \frac{\partial u}{\partial t} + \vec{E} \cdot \vec{J} = 0$$

With the following interpretations,

- $\vec{\nabla} \cdot \vec{S}$: amount of radiation energy leaving the point \vec{r}
- $\frac{\partial u}{\partial t}$: increase in E-M energy at the point \vec{r}
- $\vec{E} \cdot \vec{J}$: the amount of work done on charges at the point \vec{r}

As an illustrative example, consider a parallel plate capacitor with an electric field \vec{E} between them.



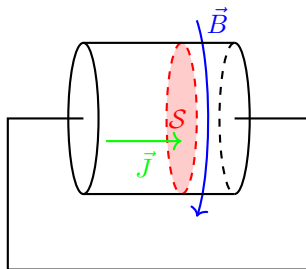
We have that the magnetic field points in the $\hat{\phi}$ direction, $\vec{B} = B\hat{\phi}$. The electric field $\vec{E} = E\hat{\zeta}$, and Poynting vector are $\vec{S} = S\hat{s}$. We have that the radiation through the surface S ,

$$\int_S d\vec{a} \cdot \vec{S} = -(2\pi ah)S$$

Therefore $\frac{\partial U}{\partial t} = -(2\pi ah)S$ corresponding to the amount of energy flowing out of the capacitor and therefore,

$$U = \int_0^\infty dt (-2\pi ahS) = \frac{1}{2} CV^2$$

Ex 8.1:



Inside the conductor the electric field moves parallel to its axis $\vec{E} = \frac{V_0}{\ell} \hat{\zeta}$. The magnetic field is then given by,

$$\vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{J}$$

Integration over the surface \mathcal{S} ,

$$\int_{\mathcal{S}} d\vec{a} \cdot (\vec{\nabla} \times \vec{B}) = \mu_0 \int_{\mathcal{S}} d\vec{a} \cdot \vec{J} = \mu_0 I_{\text{enc}}$$

Therefore computing this integral $\int d\vec{\ell} \cdot \vec{B}$ yields,

$$\vec{B} = \frac{\mu_0 I}{2\pi a} \hat{\phi}$$

Moreover, the Poynting vector is given by,

$$\begin{aligned} \vec{S} &= \frac{1}{\mu_0} \vec{E} \times \vec{B} \\ &= \frac{1}{\mu_0} \frac{V_0}{\ell} \hat{\zeta} \times \frac{\mu_0 I}{2\pi a} \hat{\phi} \\ &= -\frac{V_0 I}{2\pi a \ell} \hat{s} \end{aligned}$$

Therefore the radiation flux,

$$\int_{\mathcal{S}} d\vec{a} \cdot \vec{S} = -\frac{V_0 I}{2\pi a \ell} \int_{\mathcal{S}} da = -V_0 I$$

Which is exactly the amount of Joule heating for a current I though a wire with voltage V_0 across it. Using $V = IR$,

$$\int_{\mathcal{S}} d\vec{a} \cdot \vec{S} = -I^2 R$$

Ex 8.2 (Griffiths Problem 8.13): A long thin solenoid of radius a has a time dependent current $I_s(t)$ flowing around it. Encircling the solenoid is a ring of radius b with current $I_r(t)$ ($b \gg a$) passing through it. The ring has resistance R . There is an induced electro-motive-force in the ring due to the solenoid,

$$\mathcal{E} = -\dot{\Phi}_S = -\frac{\partial}{\partial t} (\pi a^2 B_s)$$

Where $B_s = \mu_0 n I_s$. The EMF \mathcal{E} must also equal $\mathcal{E} = I_r R$. Therefore,

$$I_r = -\frac{1}{R} (\mu_0 \pi a^2 n) \dot{I}_s$$

In order to calculate the electric and magnetic fields just outside solenoid, recognize that $\vec{B}_s = B_s(t) \hat{z}$ point along the axis of the solenoid. Similarly recognize that $\vec{E} = E \hat{\phi}$. Therefore the Poynting vector is given by,

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}_s = \frac{1}{\mu_0} E B_s \hat{s} = ?$$

We first need to calculate \vec{E} and \vec{B}_s . The magnetic field is known to be $\vec{B} = \mu_0 n I_s \hat{z}$ on axis and $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$,

$$\int d\vec{a} \cdot \vec{\nabla} \times \vec{E} = -\frac{d}{dt} \int d\vec{a} \cdot \vec{B}$$

$$\int d\vec{\ell} \cdot \vec{E} = -\dot{\Phi} = 2\pi a E$$

Which gives,

$$\vec{E} = \frac{\dot{\Phi}}{2\pi a} \hat{\phi}$$

The magnetic field off axis and outside the solenoid due to the ring is given by,

$$d\vec{B}_r(s) = \frac{\mu_0 I}{4\pi} \frac{d\vec{\ell} \times \vec{r}}{r^2}$$

Where $\vec{r} = \vec{r} - \vec{r}'$ and we take $\vec{r}' = b\hat{s}'$ and $\vec{r} = z\hat{z}$.

$$\vec{r} = z\hat{z} - b\hat{s}'$$

We will take the infinitesimal loop to be $d\vec{\ell} = b d\phi' \hat{\phi}'$.

$$d\vec{\ell} \times \vec{r} = (b\hat{\phi}' d\phi') \times (z\hat{z} - b\hat{s}') \\ = az d\phi' \hat{s}' + b^2 d\phi' \hat{z}$$

We integrate around the loop \mathcal{L} , all of the contributions in the \hat{s}' directions will cancel out.

$$\int_{\mathcal{L}} d\vec{\ell} \times \vec{r} = \dots$$

Thus,

$$\vec{B}_r = \frac{\mu_0 I_r}{4\pi} \int \frac{b^2 d\phi' \hat{z}}{(z^2 + b^2)^{3/2}} \\ = \frac{\mu_0 I_r b^2 2\pi \hat{z}}{4\pi (z^2 + b^2)^{3/2}} \\ = \frac{\mu_0 b^2}{2(z^2 + b^2)^{3/2}} I_r \hat{z}$$

Therefore the Poynting vector points radial outward,

$$\vec{S} = \frac{1}{\mu_0} \vec{E}_r \times \vec{B}_r \\ = \frac{1}{\mu_0} \left(\frac{\pi a^2 \mu_0 n \dot{I}_s}{2\pi a} \hat{\phi} \right) \times \left(\frac{\mu_0 b^2}{2(z^2 + b^2)^{3/2}} I_r \hat{z} \right) \\ = \frac{\mu_0}{4} a n \dot{I}_s \frac{b^2}{(z^2 + b^2)^{3/2}} I_r \hat{s}$$

Now that the Poynting vector is known, one can calculate the power radiated from the system.

$$P = \int d\vec{a} \cdot \vec{S} \\ = \frac{\mu_0}{4} a n \dot{I}_s I_r b^2 \int dz a d\phi \hat{s} \cdot \frac{1}{(z^2 + b^2)^{3/2}} \hat{s}$$

$$\begin{aligned}
&= \frac{\mu_0}{4} a n \dot{I}_s I_r b^2 (2\pi a) \int dz \frac{1}{(z^2 + b^2)^{3/2}} \\
&= \frac{\mu_0}{4} a n \dot{I}_s I_r b^2 (2\pi a) \frac{2}{b^2} \quad \text{Integral Table} \\
&= \mu_0 \pi a^2 n \dot{I}_s I_r
\end{aligned}$$

But we know that $\mu_0 \pi a^2 n \dot{I}_s = -I_r R$. Therefore $P = -I_r^2 R$ as expected.

A2.2:

a,b) Answers in Griffiths.

c) Consider parallel metal strips with height h and width w where $h \ll w$. A current flows down one plate and up the other. The system will act as a capacitor. The magnetic field outside will be zero and non-negative inside.

d) Griffiths 8.1

A2.3:

Positive and negative charge build up on the surfaces between the capacitor. Of course, there will be a time varying current $I(t)$, electric field $\vec{E}(t)$ and magnetic field $\vec{B}(t)$.

5 Stress Energy Tensor

Last week we looked at conservation laws and we found,

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \quad (\text{charge})$$

and,

$$\vec{\nabla} \cdot \vec{S} + \frac{\partial u}{\partial t} + \vec{J} \cdot \vec{E} = 0 \quad (\text{energy})$$

This week we will continue with momentum and angular momentum and then we will examine the Maxwell stress tensor; the field equivalent for force in Newton's second law. But first we will look at momentum.

5.1 Momentum

Consider two charges $+q_1$ and $-q_2$ with velocities \vec{v}_1 and \vec{v}_2 . The electric field at point 2 due to charge 1 will be denoted \vec{E}_1 . Analogously for \vec{B}_1 . The net force acting on charge q_2 is then,

$$\begin{aligned}
\vec{F}_{2;E} &= q_2 \vec{E}_1 & \vec{F}_{2;B} &= q_2 \vec{v}_2 \times \vec{B}_1 \\
\vec{F}_{1;E} &= q_1 \vec{E}_2 & \vec{F}_{1;B} &= q_1 \vec{v}_1 \times \vec{B}_2
\end{aligned}$$

One will notice that $\vec{F}_{1;B}$ and $\vec{F}_{2;B}$ are not equal and opposite forces like $\vec{F}_{1;E}$ and $\vec{F}_{2;E}$ are. What does this say about Newton's third law?

$$\sum \dot{\vec{p}}_i = \sum \vec{F}_{\text{net}}$$

We forgot about the fact that the electric and magnetic fields carry not only energy (via \vec{S}) but momentum as well. Recall that for photons,

$$\begin{aligned}
E &= hf = \hbar\omega \\
p &= \frac{h}{\lambda} = \hbar k
\end{aligned}$$

Therefore we have that,

$$E = pc$$

Therefore knowing the energy density of the field gives you then momentum density of the field. The momentum density will be denoted \vec{g} .

$$\vec{g} = \frac{1}{c^2} \vec{S} = \mu_0 \epsilon_0 \vec{S} = \frac{1}{4\pi c} \vec{E} \times \vec{B}$$

The force per unit volume $\vec{f} = \Delta \vec{F} / \Delta \tau$ acting on a particle is given by the Lorentz force.

$$\vec{f} = \rho \vec{E} + \vec{J} \times \vec{B}$$

Which when expanded out is,

$$\vec{f} = \epsilon_0 (\vec{\nabla} \cdot \vec{E}) \vec{E} + \left(\frac{1}{\mu_0} \vec{\nabla} \times \vec{B} - \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \times \vec{B}$$

Todo (TC Fraser): Inject hand out Which after some algebra yields,

$$\vec{f} = \epsilon_0 \left((\vec{\nabla} \cdot \vec{E}) \vec{E} + (\vec{E} \cdot \vec{\nabla}) \vec{E} \right) + \frac{1}{\mu_0} \left((\vec{\nabla} \cdot \vec{B}) \vec{B} + (\vec{B} \cdot \vec{\nabla}) \vec{B} \right) - \vec{\nabla} \left(\frac{\epsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2 \right) - \mu_0 \epsilon_0 \frac{\partial \vec{S}}{\partial t} \quad (5.1)$$

We now introduce **Maxwell's stress energy tensor** T with components,

$$T_{ij} = \epsilon_0 \left(E_i E_j - \frac{1}{2} \delta_{ij} E^2 \right) + \frac{1}{\mu_0} \left(B_i B_j - \frac{1}{2} \delta_{ij} B^2 \right) \quad (5.2)$$

Where $\vec{E} = \sum_i E_i \hat{e}_i$ and $\vec{B} = \sum_i B_i \hat{e}_i$. We now have that eq. (5.1) gives,

$$\vec{f} = \vec{\nabla} \cdot T - \epsilon_0 \mu_0 \frac{\partial \vec{S}}{\partial t}$$

As defined $\vec{F} = \int_{\mathcal{V}} d\tau \vec{f}$ is the net mechanical force acting on the matter in a volume \mathcal{V} . Therefore,

$$\begin{aligned} \frac{d}{dt} \vec{p}_{\text{mech}} &= \int_{\mathcal{V}} d\tau \vec{f} \\ &= \int_{\mathcal{V}} d\tau \left(\vec{\nabla} \cdot T - \epsilon_0 \mu_0 \frac{\partial \vec{S}}{\partial t} \right) \\ &= \oint_S d\vec{a} \cdot T - \frac{d}{dt} \int_{\mathcal{V}} d\tau \epsilon_0 \mu_0 \vec{S} \end{aligned}$$

We usually define the second term here to be the momentum contained in the electromagnetic field,

$$\vec{p}_{\text{em}} = \int_{\mathcal{V}} d\tau \epsilon_0 \mu_0 \vec{S}$$

Therefore the conservation of momentum is,

$$\frac{d}{dt} (\vec{p}_{\text{mech}} + \vec{p}_{\text{em}}) = \oint_S d\vec{a} \cdot T \quad (5.3)$$

To draw intuition from continuum mechanics, the **Cauchy stress tensor** is a representation of the total forces acting on a chunk \mathcal{V} of a material due to the neighboring pieces $\mathcal{N}(\mathcal{V})$. Each neighboring chunk $n(\mathcal{V})$ can exert parallel or shear forces on \mathcal{V} . This defines a matrix on force components on each face of \mathcal{V} . Let $\vec{f} = \sigma \cdot d\vec{a}$ where σ is a rank 2 (3d) tensor. We call σ the Cauchy stress tensor such that,

$$\vec{f} = \sigma \cdot d\vec{a}$$

The divergence of the Maxwell stress tensor is,

$$\frac{\partial}{\partial x_i} T_{ij} = \epsilon_0 \left(\frac{\partial E_i}{\partial x_i} E_j + E_i \frac{\partial E_j}{\partial x_i} - \frac{1}{2} \delta_{ij} \frac{\partial E^2}{\partial x_i} \right) + \frac{1}{\mu_0} \left(\frac{\partial B_i}{\partial x_i} B_j + B_i \frac{\partial B_j}{\partial x_i} - \frac{1}{2} \delta_{ij} \frac{\partial B^2}{\partial x_i} \right)$$

Which in vector notation is much simpler **Todo (TC Fraser): Verify this expression**,

$$\vec{\nabla} \cdot T = \epsilon_0 \left((\vec{\nabla} \cdot \vec{E}) \vec{E} + (\vec{E} \cdot \vec{\nabla}) \vec{E} \right) - \frac{1}{2} \epsilon_0 \vec{\nabla} \cdot E^2 + \frac{1}{\mu_0} \left((\vec{\nabla} \cdot \vec{B}) \vec{B} + (\vec{B} \cdot \vec{\nabla}) \vec{B} \right) - \frac{1}{2} \epsilon_0 \vec{\nabla} \cdot B^2$$

While the force per unit volume is,

$$\vec{f} = \vec{\nabla} \cdot \vec{T} - \epsilon_0 \mu_0 \frac{\partial \vec{S}}{\partial t}$$

Which can be integrated over a volume \mathcal{V} in order to obtain the total force,

$$\begin{aligned} \int_{\mathcal{V}} d\tau \vec{f} &= \int_{\mathcal{V}} d\tau \vec{\nabla} \cdot \vec{T} \\ &= \int_{\mathcal{V}} d\tau \vec{\nabla} \cdot \vec{T} - \mu_0 \epsilon_0 \frac{d}{dt} \int_{\mathcal{V}} d\tau \vec{S} \\ \frac{d\vec{p}_{\text{mech}}}{dt} &= \oint_S d\vec{a} \cdot \vec{T} - \frac{d}{dt} \underbrace{\int_{\mathcal{V}} d\tau \vec{g}}_{\vec{p}_{\text{em}}} \end{aligned}$$

Therefore we recover eq. (5.3) again,

$$\frac{d}{dt}(\vec{p}_{\text{mech}} + \vec{p}_{\text{em}}) = \oint_S d\vec{a} \cdot \vec{T}$$

This conservation of momentum equation can be interpreted as Newton's second law for E&M.

Ex 8.4: Two point charges a distance 2ℓ apart. Due to the rotational symmetry of the problem, we can exploit cylindrical coordinates $\vec{r} = s\hat{s} + z\hat{z}$ because no physical quantities can depend on ϕ .

a) The electric field in a plane ($\phi = 0$) can be obtained as follows. Let \vec{r}' be the location of the source q and \vec{r} be the field location. The origin is between the two identical charges. Let \vec{E}_+ be the electric field due to the charge in the $z > 0$ direction.

$$\vec{r}'_{\pm} = \pm \ell \hat{z}$$

And on the axis perpendicular to \hat{z} ,

$$\vec{r} = s\hat{s}$$

Therefore,

$$\vec{r} = \vec{r} - \vec{r}' = s\hat{s} - \ell\hat{z}$$

Which gives electric field,

$$\begin{aligned} \vec{E}_{\pm} &= \frac{q_{\pm} \vec{r}_{\pm}}{4\pi\epsilon_0 r_{\pm}^3} \\ &= \frac{q_{\pm}}{4\pi\epsilon_0} \frac{s\hat{s} \mp \ell\hat{z}}{(s^2 + \ell^2)^{3/2}} \end{aligned}$$

Therefore,

$$\vec{E}_{\{z=0\}} = \vec{E}_+ + \vec{E}_- = \frac{q_{\pm}}{4\pi\epsilon_0} \frac{2s\hat{s}}{(s^2 + \ell^2)^{3/2}}$$

Upon reflection, the direction of $\vec{E}_{\{z=0\}}$ could have only been in the \hat{s} direction by symmetry.

b) Calculate the Maxwell Stress Tensor using eq. (5.2). Notice that $\vec{E} = E\hat{s}$ and $\vec{E}_{\{z=0\}} = E(s)\hat{s}$,

$$\hat{s} = \hat{x} \cos \phi + \hat{y} \sin \phi$$

So in Cartesian coordinates,

$$\vec{E}_{\{z=0\}} = \frac{q_{\pm}}{4\pi\epsilon_0} \frac{2\sqrt{x^2 + y^2}(\hat{x} \cos \phi + \hat{y} \sin \phi)}{(x^2 + y^2 + \ell^2)^{3/2}}$$

The components of \vec{E} are then,

$$E_1 = E_x = \frac{q_{\pm}}{4\pi\epsilon_0} \frac{2\sqrt{x^2 + y^2} \cos \phi}{(x^2 + y^2 + \ell^2)^{3/2}}$$

$$E_2 = E_y = \frac{q_{\pm}}{4\pi\epsilon_0} \frac{2\sqrt{x^2 + y^2} \sin \phi}{(x^2 + y^2 + \ell^2)^{3/2}}$$

$$E_3 = E_z = 0$$

For convenience let,

$$E_0 = \frac{q_{\pm}}{4\pi\epsilon_0} \frac{2\sqrt{x^2 + y^2}}{(x^2 + y^2 + \ell^2)^{3/2}} \quad (5.4)$$

Such that,

$$E_1 = E_0 \cos \phi \quad E_2 = E_0 \sin \phi$$

And also,

$$E^2 = E_0^2$$

Therefore the components of T are determined by eq. (5.2),

$$T = \epsilon_0 E_0^2 \begin{pmatrix} \cos^2 \phi - \frac{1}{2} & \sin \phi \cos \phi & 0 \\ \cos \phi \sin \phi & \sin^2 \phi - \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}$$

Which by trig-identities becomes,

$$T = \frac{1}{2} \epsilon_0 E_0^2 \begin{pmatrix} \cos(2\phi) & \sin(2\phi) & 0 \\ \sin(2\phi) & -\cos(2\phi) & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

c) Construct a closed hemisphere \mathcal{H} above $z > 0$ enclosing the charge q_+ but not q_- . Since the charges are not moving, we have that,

$$\frac{d}{dt}(\vec{p}_{\text{mech}} + \vec{p}_{\text{em}}) = \vec{0}$$

Therefore it must be,

$$\oint_{\mathcal{H}} d\vec{a} \cdot T = 0$$

However, \mathcal{S} does *not* lie in the same plane as the Maxwell stress tensor computed above. Instead, we can take the radius R of the hemisphere to be $R \rightarrow \infty$ such that the “hemisphere” becomes a flat plane with a central circular region. The net force acting on q_+ is,

$$\vec{F}_+ = \int_{\mathcal{V}} d\tau \vec{f} = \int_{\mathcal{V}} d\vec{a} \vec{\nabla} \cdot T - \epsilon_0 \mu_0 \frac{d}{dt} \int_{\mathcal{V}} d\tau \vec{\mathcal{S}}^0 = \int_{\mathcal{S}} d\vec{a} \cdot T$$

Therefore,

$$\begin{aligned} \vec{F}_+ &= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \frac{1}{2} \epsilon_0 E_0^2 \begin{pmatrix} 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \cos(2\phi) & \sin(2\phi) & 0 \\ \sin(2\phi) & -\cos(2\phi) & 0 \\ 0 & 0 & -1 \end{pmatrix} \\ &= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \frac{1}{2} \epsilon_0 E_0^2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{aligned}$$

Where $E_0 = E_0(x, y)$ is given by eq. (5.4).

$$\begin{aligned} \vec{F}_+ &= \frac{1}{2} \epsilon_0 \hat{z} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy E_0^2 \\ &= \frac{1}{2} 2\pi \epsilon_0 \left(\frac{q}{2\pi \epsilon_0} \right)^2 \hat{z} \int_0^{\infty} ds \frac{s(s^2)}{(s^2 + \ell^2)^3} \end{aligned}$$

$$\begin{aligned} &= \dots \\ &= \frac{q^2 \hat{z}}{4\pi\epsilon_0(2\ell)^2} \end{aligned}$$

Which is simply a result of Columb's law which was expected.