Phys 476

GENERAL RELATIVITY

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Latest versions of all my course notes are available at **www.tcfraser.com**.

1 Introduction

1.1 History

The first lecture was a summary of astrophsyical history from around ∼200BC to today. I elected not to take notes as it was pretty standard stuff and a lot of slides. Sorry.

2 Tensor Formalism

At the core of General Relativity is the mathematics of differential geometry. Differential geometry requires the idea of tensors, a generalization of vectors and matricies and forms that can handle messy geometries and metrics.

Let V be a vector space of finite dimension. Any V is iosmorphic to $\mathbb{R}^{n+1}$ through the coefficients of a chosen basis. Let the basis of V be given by,

$$\{e_i\}_{i=0,\ldots,n}$$

Then any vector $v \in V$ is expressible by,

$$v = \sum_{i=0}^{n} v^i e_i$$

Where v^i are the i-th coefficients of the vector v with respect to the basis $\{e_i\}$.

2.1 Einstein Summation Rule

For convience let's provide a new, shorter notation for the vector v.

$$v^i e_i = v^0 e_0 + \ldots + v^n e_n = \sum_{i=0}^{n} v^i e_i$$

Effectively, we have just **dropped the summation sign**. The einstein summation rule is as follows:

If there are two identical indicies, 1 "up" and 1 "down", it means that a summation is secretly present, it's just be removed for convience. Note that the i in this case is *dummy index*.

$$v^i e_i = v^\alpha e_\alpha = v^j e_j$$

Here v^i are the components of vector $v \in V$ and are real numbers. $v^i \in \mathbb{R}, \forall i \in \{0, \ldots, n\}$.

Note v^i is called the vector v when i is the set $\{0, \ldots, n\}$, but can also be called the i-th component of v when i has a fixed value $i \in \{0, \ldots, n\}$.

2.2 Examples of Basis for V

The values of e_i or the i's themselves can take on many possible values.

- cartesian coordinates t, x, y, z
- spherical coordinates t, r, ϕ, θ
- etc.

Each of the above examples is the space $V = \mathbb{R}^4$ (with some bounds for spherical coordinates).

2.3 Dual Vector Space

The dual vector space of V denoted V^* is also iosmorphic to $\mathbb{R}^{n+1}$ and is built from the space of linear forms on V.

$$V^* = \{w : V \to \mathbb{R} \mid w(\alpha v_1 + \beta v_2) = \alpha w(v_1) + \beta w(v_2)\}$$

where $v_1, v_2 \in V$ and $\alpha, \beta \in \mathbb{R}$.
In Quantum Mechanics, the vectors are the bras and the elements of the dual space (called the covectors) are the kets.

We note,

$$\{f^i\}_{i=0,\ldots,n}$$

is the basis for V^* is defined by the kronecker symbol δ,

$$f^j(e_j) = \delta^j{}_i$$

$$\delta^j{}_i = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

An element in V^* is $w = w_i f^i$. w_i are the components of the covector w. Note that for a **finite dimensional vector space**,

$$V^{**} = V$$

2.4 Bilinear Maps

Introduce a bilinear map $B(v, w)$ where $B : V \times V \to \mathbb{R}$ where,

$$B(\alpha v_1 + \beta v_2, w) = \alpha B(v_1, w) + \beta B(v_2, w)$$

and the same for the other parameter w.

Examples include the inner product (otherwise known as the scale or dot product).
Bilinear forms are bilinear maps such that the following conditions are true:

- symmetric: $B(v, w) = B(w, v)$
- non-degenerated: $B(v, w) = 0 \quad \forall v \implies w = 0$

Playing with indicies,

$$\begin{aligned} B(v, w) &= B(v^\alpha e_\alpha, w^\beta e_\beta) \\ &= v^\alpha B(e_\alpha, w^\beta e_\beta) \quad \text{By linearity} \\ &= v^\alpha w^\beta B(e_\alpha, e_\beta) \quad \text{By linearity} \end{aligned}$$

A bilinear map used in this way provides a way to eliminate the headache of complicated cross sums. Define new notation,

$$B(e_\alpha, e_\beta) \equiv g_{\alpha\beta}$$

Where $g_{\alpha\beta}$ is a real number $\mathbb{R}$ whenever α and β are fixed.

$$B(v, w) = v^\alpha w^\beta g_{\alpha\beta} = v^\alpha g_{\alpha\beta} w^\beta = w^\beta g_{\alpha\beta} v^\alpha$$

All of the above terms are commutative because in the end, it represents a sum over all α, β.

$$B(v,w) = \underbrace{v^0 w^0 g_{00} + \ldots + v^2 w^3 g_{2,3} + \ldots + v^n w^n g_{nn}}_{(n+1)^2 \text{terms}}$$

2.5 Distance and Norms

To define a distance in a vector space, we can use norms. In this case, $g_{\alpha\beta}$ would be called the metric. The Euclidean metric (with respect to a cartesian basis) for example would be,

$$g_{\alpha\beta} = \begin{cases} 1 & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases}$$

We can also choose to enforce that the basis be orthonormal,

$$B(e_i, e_j) = \begin{cases} \pm 1 & i = j \\ 0 & i \neq j \end{cases}$$

Note that the potential for a negative norm means the notion of positive definiteness is no longer gauranteed.

2.6 Signatures of Metrics

We call the signature of the metric the number of $+1$'s and -1's appearing in g_{ij} when dealing with the orthonormal basis. Signature is denoted as:

$$(p,q) = \left(\underbrace{p}_{\text{postive}}, \underbrace{q}_{\text{negative}} \right)$$

For example,

- Euclidean metric: $(n+1, 0)$
- Minkowski metric: $(n, 1)$

Note the order of the signature is chosen to be (p, q) and not (q, p) by convention.

2.7 Covectors from Vectors

Note that v^i was called the vector and w_i was called the covector. This notation seems to indicate that conversion between V and V^* is notationally equivalent to raising and lowering the indicies.

We call the following opperation "Lowing the index using the metric".

$$\underbrace{v^\alpha}_{\text{components of vector}} \mapsto g_{\alpha\beta} v^\beta = \underbrace{v_\alpha}_{\text{components of covector}}$$

In use,

$$B(v,w) = v^\alpha g_{\alpha\beta} w^\beta = \underbrace{v_\beta}_{\text{bra}} \underbrace{w^\beta}_{\text{ket}}$$