Stat 433

STOCHASTIC PROCESSES

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1 DTMC

1.1 Review of Probability

A **random variable (r.v.)** X is a real valued function of the outcomes of a random experiment.

$$X : \Omega \to R$$

Where $\Omega = \{\omega_1, \omega_2, \ldots\}$ is the **sample space** corresponding to all possible outcomes ω_i. The outcomes can in principle be any objects (numbers, strings, etc.). We say that X maps each outcome ω to a real number $\omega \mapsto X(\omega) \in \mathbb{R}$.

A **stochastic process** is a family of random variables $\{X_t\}_{t\in T}$, defined on a common sample space Ω. T is referred to as the index set for the stochastic process which is often understood as time. The index set T can take a discrete spectrum,

$$T = \{0, 1, 2, \ldots\} \qquad \{X_n \mid n = 0, 1, 2, \ldots\}$$

Alternatively, T can take on a continuous spectrum,

$$T = \{t \mid t \geq 0\} = [0, \infty)$$

The **state space** S is the collection of all possible values of X_t's. It is important to understand the distinction of between sample space and state space. Additionally, the state space can either have discrete or continuous spectrum.

A question remains, *Why do we need the family of random variables to be defined on a common sample space?* The answer being that we would like to be able to discuss the joint behaviour of X_t's. If X_1 has domain Ω_1 and X_2 has domain Ω_2 (where $\Omega_1 \neq \Omega_2$), then one can *not* talk about common ideas of correlations and associations between X_1 and X_2. As such we assert that all members of a stochastic process share the same sample space domain Ω.

1.2 Discrete-time Markov Chain

A **discrete-time stochastic process** $\{X_n \mid n \in 0, 1, 2, \ldots\}$ is said to be a **Discrete-time Markov Chain (DTMC)** if the following conditions hold:

1. The state space is at most *countable*[1] (i.e. finite or countable).

$$S = \{0, 1, \ldots, k\} \quad \text{or} \quad S = \{0, 1, 2, \ldots\}$$

2. **Markov Property**: For any $n = 0, 1, 2, \ldots$,

$$P(X_{n+1} = x_{n+1} \mid X_n = x_n, X_{n-1} = x_{n-1}, \ldots, X_0 = x_0) = P(X_{n+1} = x_{n+1} \mid X_n = x_n)$$

We use capital letters X to denote the random variable and lower case letters x to denote a specific realization or valuation of X. The motivation of the Markov property is that future events $X_{n+1} = x_{n+1}$ are independent of past histories $\{X_i = x_i \mid i = 0, 1, \ldots, n-1\}$ given the immediate past state $X_n = x_n$. The intuition being that the future and the past are probabilistically independent.

Given the present, the future and the past are independent.

[1] Countable meaning there is a one-to-one mapping from the state space to the natural numbers.

1.3 Transition Probability

The **transition probability** from a state $i \in S$ at time n to state $j \in S$ (at time $n+1$) is given by,

$$P_{n,i,j} \equiv P(X_{n+1} = j \mid X_n = i) \qquad n = 0, 1, 2, \dots \tag{1.1}$$

In full generality, the transition probability could depend on time n but in this course we will restrict ourselves to transition probabilities that *do not* depend on time n ($P_{n,i,j} = P_{i,j}$). We say that the MC is **(time-)homogeneous** if this property holds. From now on, this will be our default setting.

The matrix of all transition probabilities $P = \{P_{i,j} \mid i, j \in S\}$ is called the **one-step transition (probability) matrix** for $\{X_n \mid n \in T\}$.

$$P = \begin{pmatrix} P_{00} & P_{01} & \cdots & P_{0j} & \cdots \\ P_{10} & P_{11} & \cdots & P_{1j} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \cdots \\ P_{i0} & P_{i1} & \cdots & P_{ij} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The one-step transition matrix P has the following properties,

$$P_{i,j} \geq 0 \tag{1.2}$$

$$\forall i : \sum_{j \in S} P_{ij} = 1 \tag{1.3}$$

The entries are non-negative because they represent probabilities and the row sums for P are always unitary.

The **n-step transition probability** is defined via the homogeneous property,

$$\forall i, j \in S : P_{ij}^{(n)} \equiv P(X_{n+m} = j \mid X_n = i) = P(X_n = j \mid X_0 = i)$$

Analogously, the **n-step transition matrix** is the matrix,

$$P^{(n)} = \left\{ P_{ij}^{(n)} \mid i, j \in S \right\}$$

Theorem 1. *There is a simple relation between the n-step transition matrix $P^{(n)}$ and the one step transition matrix P.*

$$P^{(n)} = P^{(n-1)} \cdot P = \underbrace{P \cdot P \cdot \cdots \cdot P}_{n} = P^n$$

Proof. Proof by induction:

$$P^{(1)} = P \quad \text{By definition.}$$

We also have $P^{(0)} = P^0 = \mathbb{I}$ is the identity matrix. We now assume $P^{(n)} = P^n$. Then $\forall i, j \in S$,

$$\begin{aligned} P_{ij}^{(n+1)} &= P(X_{n+1} = j \mid X_0 = i) \\ &= \sum_{k \in S} P(X_{n+1} = j, X_n = k \mid X_0 = i) \quad \text{Total probability} \\ &= \sum_{k \in S} \frac{P(X_{n+1} = j, X_n = k, X_0 = i)}{P(X_0 = i)} \\ &= \sum_{k \in S} \frac{P(X_{n+1} = j, X_n = k, X_0 = i)}{P(X_n = k, X_0 = i)} \frac{P(X_n = k, X_0 = i)}{P(X_0 = i)} \\ &= \sum_{k \in S} P(X_{n+1} = j \mid X_n = k, X_0 = i) \cdot P(X_n = k \mid X_0 = i) \quad \text{Conditional total probability} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k\in S} P(X_{n+1} = j \mid X_n = k) \cdot P(X_n = k \mid X_0 = i) \quad \text{Use Markov Property} \\
&= \sum_{k\in S} P_{kj} \cdot P_{ik}^{(n)} \quad \text{Matrix terms} \\
&= \left(P \cdot P^{(n)}\right)_{ij} \quad \text{Matrix product} \\
&= \left(P^{n+1}\right)_{ij} \quad \text{Inductive Hypothesis}
\end{aligned}
$$

There we have proved that $P^{(n+1)} = P^{n+1}$ and so we have a complete proof that $P^{(n)} = P^n$.

□

Corollary 2. *As a corollary, we have obtained that,*

$$P^{(n)} = P^{(m)} \cdot P^{(n-m)} \quad \forall 0 \le m \le n$$

Or equivalently the ***Chapman-Kolmogorov Equation*** *or simply C-K equation,*

$$P_{ij}^{(n)} = \sum_{k\in S} P_{ik}^{(m)} P_{kj}^{(n-m)} \quad \forall i, j \in S, \forall 0 \le m \le n \tag{1.4}$$

So far, we have only been discussing transition probabilities. We will now divert our attention to actual distributions for a stochastic process.

Let $\alpha_n = (\alpha_{n,0}, \alpha_{n,1}, \ldots)$ be the **probability distribution vector** for X_n at time n.

$$\alpha_{n,k} = P(X_n = k) \qquad \forall k \in S$$

Note that $\alpha_{n,k} \ge 0$ and $\sum_{k\in S} \alpha_{n,k} = 1$ and $n = 0, 1, 2, \ldots$. We also define the initial distribution α_0,

$$\alpha_0 = (P(X_0 = 0), P(X_0 = 1), \ldots)$$

Theorem 3. *The transition probability matrix reveals the following relationship between the distribution α_n at time n and the distribution α_0 at time 0,*

$$\alpha_n = \alpha_0 \cdot P^n \tag{1.5}$$

Proof. The proof eq. (1.5) is quite trivial:

$$
\begin{aligned}
\forall j \in S \quad \alpha_{n,j} &= P(X_n = j) \\
&= \sum_{i\in S} P(X_n = j \mid X_0 = i) \cdot P(X_0 = i) \\
&= \sum_{i\in S} \alpha_{0,i} \cdot P_{ij}^n \\
&= \alpha_{0,0} \cdot P_{0j}^n + \alpha_{0,1} \cdot P_{1j}^n + \ldots \\
&= (\alpha_0 \cdot P^n)_j
\end{aligned}
$$

□

More generally, for any $n = 1, 2, \ldots$ the finite dimensional distribution can be obtained from the following process iterative process,

$$
\begin{aligned}
&P(X_n = x_n, X_{n-1} = x_{n-1}, \ldots, X_0 = x_0) = \\
&\qquad P(X_0 = x_0) \cdot \\
&\qquad P(X_1 = x_1 \mid X_0 = x_0) \cdot \\
&\qquad P(X_2 = x_2 \mid X_1 = x_1, X_0 = x_0) \cdots
\end{aligned}
$$

$$P(X_n = x_n \mid X_{n-1} = x_{n-1}, \dots, X_0 = x_0)$$

But by the Markov condition, it must be that,

$$\begin{aligned}P(X_n = x_n, X_{n-1} = x_{n-1}, \dots, X_0 = x_0) = \\ &P(X_0 = x_0)\cdot \\ &P(X_1 = x_1 \mid X_0 = x_0)\cdot \\ &P(X_2 = x_2 \mid X_1 = x_1)\cdots \\ &P(X_n = x_n \mid X_{n-1} = x_{n-1})\end{aligned}$$

First recognize the first term on the RHS ($P(X_0 = x_0) = \alpha_{0,x_0}$), and also the remaining terms are transition probabilities as per eq. (1.1). Therefore it must be that,

$$P(X_n = x_n, X_{n-1} = x_{n-1}, \dots, X_0 = x_0) = \alpha_{0,x_0} P_{x_0 x_1} P_{x_1 x_2} \cdots P_{x_{n-1} x_n}$$

Even more generally, for $0 \le t_1 < t_2 < \cdots < t_n$,

$$P\big(X_{t_n} = x_{t_n}, X_{t_{n-1}} = x_{t_{n-1}}, \dots, X_{t_1} = x_{t_1}\big) = P(X_{t_1} = x_{t_1})\big(P^{t_2-t_1}\big)_{x_{t_1}x_{t_2}}\big(P^{t_3-t_2}\big)_{x_{t_2}x_{t_3}}\cdots\big(P^{t_n-t_{n-1}}\big)_{x_{t_{n-1}}x_{t_n}}$$

Since $P(X_{t_1} = x_{t_1}) = \alpha_{t_1 x_{t_1}} = \sum_{k\in S} \alpha_{0,k} P^{t_1}_{k,x_{t_1}}$,

$$\alpha_{t_1} = \alpha_0 \cdot P^{t_1}$$

This means the probabilistic properties of a DTMC are fully characterized by two things:

1. The initial distribution α_0
2. Transition matrix P

1.4 Classification of States

State j is **accessible** from state i (denoted $i \to j$) if there exists $n = 0, 1, \dots$ such that $P^{(n)}_{ij} > 0$. Intuitively, one can transition from state i to state j in finite steps n with positive probability. If i is also accessible from j, then we say i and j **communication**, denoted as $i \leftrightarrow j$.

$$i \leftrightarrow j \Leftrightarrow \exists m, n \ge 0, P^{(m)}_{ij} > 0, P^{(n)}_{ji} > 0$$

Theorem 4. *The binary communication relation "$\leftrightarrow$" is in fact a equivalence relation:*

- *Reflexivity* $i \leftrightarrow i$
- *Symmetry* $i \leftrightarrow j \implies j \leftrightarrow i$
- *Transitivity* $i \leftrightarrow j, j \leftrightarrow k \implies i \leftrightarrow k$

Proof. First, reflexivity is easy to prove by definition. Let $n = 0$ and recognize that $P^{(0)}_{ii}$ has a certain probability by definition,

$$P^{(0)}_{ii} = 1 \implies i \leftrightarrow i$$

Second, symmetry follows by definition,

$$P^{(m)}_{ij} > 0, P^{(n)}_{ji} > 0 \Leftrightarrow P^{(n)}_{ji} > 0, P^{(m)}_{ij} > 0$$

Third, transitivity can be proving by letting m and n be the unknown quantifiers:

$$\exists m \quad P^{(m)}_{ij} > 0, \exists n \quad P^{(n)}_{jk} > 0$$

Then by the CK equation eq. (1.4),

$$P_{ik}^{(m+n)} = \sum_{l \in S} P_{il}^{(m)} P_{lk}^{(n)}$$

Let $l = j$ be a single, fixed entry in the summation,

$$P_{ik}^{(m+n)} \geq P_{ij}^{(m)} P_{jk}^{(n)} > 0$$

Therefore we have that k is accessible from i ($i \to j$). Analogously we have that $i \to j$ therefore $i \leftrightarrow k$.

□

The communication equivalence relations then divides the state space S into different equivalence classes. That is, the states in one class comm with each other; the states in different classes do not comm. The equivalent classes form a *partition* of the state space S.

The family $\{S_1, S_2, \dots S_n\}$ is a **partition** of S if,

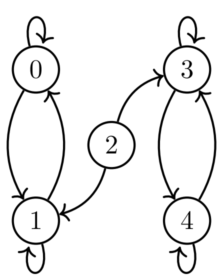
1. $S_i \subset S \mid \forall i \in 1, 2, \dots, n$
2. $S_i \cap S_j \neq \emptyset$ for all $i \neq j$
3. $\bigcup_i S_i = S$

We can find the equivalent classes by drawing a graph where the states in S are the nodes of the graph and a directed edge is placed going from i to j if j is accessible from i in one-step: $P_{ij} > 0$. Then identifying the the equivalent classes corresponds to identifying the loops of this graph within one step.

Example 1. As an example, consider the transition matrix P as follows.

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0.2 & 0.8 & 0 & 0 & 0 \\ 0.6 & 0.4 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.7 & 0.3 \\ 0 & 0 & 0 & 0.1 & 0.9 \end{pmatrix} \end{array}$$

The associated one-step accessibility graph is then,



Where the loops of $S = \{0, 1, 2, 3, 4\}$ form the following partition,

$$S_1 = \{0, 1\} \quad S_2 = \{2\} \quad S_3 = \{3, 4\}$$

These equivalent classes are useful for Markov chains because it allows one to separate the behaviour of the equivalence classes and study them individually. A MC which has only one equivalent class is called **irreducible**.

Furthermore, let us define the **period** of state i as,

$$d(i) = \gcd\{n \in \mathbb{Z}^+ \mid P_{ij}^n > 0\}$$

Additionally, if $P_{ii}^n = 0$ holds for all $n > 0$, we say that $d(i) = \infty$. If the period of i happens to be $d(i) = 1$ then the state i is said to be **aperiodic**. Alternatively, locus of steps that we can go back by are *co-prime*. A MC is called aperiodic if all its states S are aperiodic.

The period of a state is useful do to the following theorem,

Theorem 5. *The period of a state is a class property. If $i \leftrightarrow j$, then $d(i) = d(j)$.*

Proof. If $i = j$ we are already done. If $i \neq j$, since $i \leftrightarrow j$, then $\exists n, m$ such that,

$$P_{ij}^n > 0 \quad P_{ji}^m > 0$$

Then for any l such that $P_{jj}^l > 0$,

$$P_{ii}^{n+m+l} \geq P_{ij}^n P_{jj}^l P_{ji}^m \tag{1.6}$$

Because $P_{ij}^n P_{jj}^l P_{ji}^m$ happens to be a specific way for P_{ii}^{n+m+l} to occur. Since $i \leftrightarrow j$ and l was chosen carefully,

$$P_{ii}^{n+m+l} > 0$$

Moreover, we also have that,

$$P_{ii}^{n+m} \geq P_{ij}^n P_{ji}^m \tag{1.7}$$

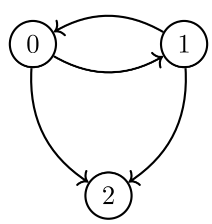
Since $d(i)$ divides both $n + m$ and $n + m + l$ by eqs. (1.7) and (1.6), then $d(i)$ also divides l. This holds for all l such that $P_{ii}^l > 0$. This implies that $d(i)$ is a common divisor of $\{l \mid P_{jj}^l > 0\}$ an thus $d(i)$ divides,

$$d(j) = \gcd\{l \mid P_{jj}^l > 0\}$$

By symmetry $d(j)$ divides $d(i)$. Therefore $d(i) = d(j)$.

□

Remark 1. It is important to note that $d(i) = k \not\Rightarrow P_{ii}^{(k)} > 0$. As a counterexample consider the following one step accessibility graph,



Evidently $P_{00} = 0$ but we have $d(0) = 1$ because $d(0) = \gcd\{2, 3, \ldots\}$.

Remark 2. If the MC is irreducible (having only one class) then all the states have the same period. In this case we ascribe the entire MC the period $d(i)$ where $i \in S$.