Phys 442

ELECTRICITY & MAGNETISM 3

University of Waterloo

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Disclaimer

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Latest versions of all my course notes are available at **www.tcfraser.com/coursenotes**.

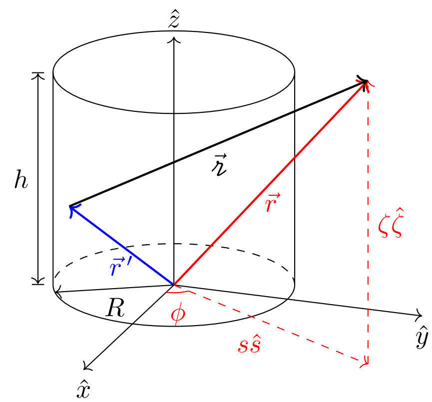
1 Coordinates and Symmetry

A clever choice of coordinates systems typically makes solving a problem considerably easier. Mathematically, this is due to *Noether's Theorem*. A typical three dimensional Lagrangian will have three dependent generalized coordinates $L = L(x, y, z) = L(s, \theta, \zeta) = \cdots$. However, if one can identify generalized coordinates q that make the Lagrangian invariant $\frac{\partial L}{\partial q} = 0$, then the *Euler-Lagrange* equations are considerably similar,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \implies \frac{\partial L}{\partial \dot{q}} = \text{const.} \implies L \propto \dot{q}$$

As such, the number of equations that remain to solved has been reduced.

2 First Assignment?



A1.1: Use cylindrical coordinates with ζ along the axis of the cable,

$$V(\zeta) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{C}} \frac{\mathrm{d}\rho}{\mathscr{r}}$$

Where $\vec{\mathscr{r}} = \vec{r} - \vec{r}'$, $\vec{r}'$ is the source point and $\vec{r}$ is the field point. The entire cylinder is the set of all source points $\vec{r}'$ that are contained inside $|\vec{r}'| \leq R$.

$$\vec{r} = \zeta\hat{\zeta}$$
$$\vec{r}' = s'\hat{s}' + \zeta'\hat{\zeta}$$

$$V(\zeta) = \frac{\rho}{4\pi\epsilon_0}\int_{\mathcal{C}} \frac{\mathrm{d}V}{|\vec{r} - \vec{r}'|}$$

Where $\mathrm{d}V = s\mathrm{d}s\mathrm{d}\theta\mathrm{d}\zeta$. One can then find the electric *field* by doing $\vec{E} = -\vec{\nabla}V = E_\zeta\hat{\zeta} = -\frac{\partial V}{\partial \zeta}\hat{\zeta}$
A1.2:
Between the two conductors, there will be a radial electric field $\vec{E} = E(s)\hat{s}$ and parallel magnetic field $\vec{B} = B(s)\hat{\zeta}$. Outside the two conductors, there will be no electric or magnetic field.

$$E_{\text{vac}}^{\parallel} = 0$$
$$E_{\text{vac}}^{\perp} = \frac{\sigma}{\epsilon_0}$$
$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

For part g), use Laplace's equation $\nabla^2 V = 0$. In cylindrical coordinates, Laplace's equation is,

$$\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) = 0$$

Cylindrical coordinates gives us the following symmetries $\frac{\partial V}{\partial \phi} = \frac{\partial V}{\partial \zeta} = 0$. Solving this system gives the potential in terms of s: $V(s) = \cdots$. Then the electric field can then be obtained via $\vec{E} = -\vec{\nabla} V$.

A1.3: Using cylindrical coordinates once again, the electric field is going to be radial outwards to the uniform charge density. For the uniform density cylinder, construct a Gaussian surface cylindrically around the cylinder. For the current density cylinder, the current density is the current per cross sectional area. Construct an Amperian loop,

$$\oint_{\mathcal{A}} \mathrm{d}\vec{\ell} \cdot \vec{B} = \mu I_{\text{enc}}$$

Part e), finding the vector potential,

$$A(\vec{r}) = \frac{\mu_0}{4\pi} \int_{\mathcal{C}} \mathrm{d}\tau' \frac{\vec{J}(\vec{r})}{\mathscr{r}}$$

Evidently, $\hat{s}$ and $\hat{s}'$ are in *different* directions. Solving such an equation yields,

$$A(s) = \frac{\mu_0}{4\pi} \int_0^{2\pi} \mathrm{d}\phi' \int_0^a s'\mathrm{d}s' \int_{-\infty}^{\infty} \mathrm{d}\zeta' \frac{J(s)}{\left|s\hat{s} - s'\hat{s}' - \zeta'\hat{\zeta}\right|}$$

Recognize the structure of the potential integral,

$$V(s) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{C}} \mathrm{d}\tau' \frac{\rho(r')}{\mathscr{r}} = \frac{\rho_0}{4\pi\epsilon_0} \int_{\mathcal{C}} \frac{\mathrm{d}\tau'}{\mathscr{r}}$$

Comparing to the vector potential, we have an equivalent integral (up to a constant).

$$A(s) = \frac{\mu_0 J_0}{4\pi} \int_{\mathcal{C}} \frac{\mathrm{d}\tau'}{\mathscr{r}}$$

For question f), use the definition of $\vec{B}$ in terms of $\vec{A}$,

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

Further, recall that if $\vec{E} = -\vec{\nabla} V$, then by Stoke's theorem for some loop $\mathcal{L}$,

$$V = -\int_{\mathcal{L}} \mathrm{d}\vec{\ell} \cdot \vec{E}$$

3 Conservation Laws

Beginning with one of Maxwell's equations,

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Taking the divergence of the above equation,

$$\cancelto{0}{\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{B}\right)} = \mu_0 \vec{\nabla} \cdot \vec{J} + \mu_0 \epsilon_0 \frac{\partial}{\partial t}\left(\vec{\nabla} \cdot \vec{E}\right)$$

Luckily, the divergence of a curl is always 0. Dividing by relevant constants we obtain the following conservation law,

$$0 = \vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} \tag{3.1}$$

This is a conservation of charge. It is a **local** conservation law because it holds for all points in space $\vec{r}$. Intuitively, is claims that the rate of charge of charge at a point is equal to the amount of current following in or out of the take point.

A2.1: Again using cylindrical coordinates $\vec{r} = s\hat{s} + \zeta\hat{\zeta}$. Let the current flow in such a way that the magnetic field points along the ζ-axis. Let $\mathcal{L}$ be an Amperian loop with one side at distance $|\vec{r}| \to \infty$,

$$\int_{\mathcal{L}} \mathrm{d}\vec{\ell} \cdot \vec{B} = \mu_0 I_{\text{enc}}$$

The same equation can be reused to calculate the vector potential for a Gaussian surface $\mathcal{S}$,

$$\int_{\mathcal{L}} \mathrm{d}\vec{\ell} \cdot \vec{A} = \int_{\mathcal{S}} \mathrm{d}\vec{a} \cdot \vec{B} = \Phi$$

Where Φ is the magnetic flux through $\mathcal{S}$. Furthermore, the energy required to set up a magnetic field is,

$$W = \frac{1}{2\mu_0} \int_{\mathcal{C}} \mathrm{d}\tau B^2 = \frac{1}{2} \int_{\mathcal{C}} \mathrm{d}\tau \vec{J} \cdot \vec{A} = \frac{1}{2} L I^2$$

Where L is the self-inductance of the solenoid.

4 Poynting's Theorem

First we begin with two of Maxwell's equations,

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{4.1}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{4.2}$$

Computing the inner product between eq. (4.1) and $\vec{B}$, and the inner product between eq. (4.2) and $\vec{E}$ and taking a difference,

$$\vec{B} \cdot \left(\vec{\nabla} \times \vec{E}\right) - \vec{E} \cdot \left(\vec{\nabla} \times \vec{B}\right) = -\frac{\partial}{\partial t} \left(\frac{\epsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2 \right) = \mu_0 \vec{E} \cdot \vec{J}$$

Letting $\frac{\epsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2$ be the **electromagnetic energy density** u, we have the following identity,

$$\vec{\nabla} \cdot \left(\vec{E} \times \vec{B}\right) = -\mu_0 \frac{\partial u}{\partial t} - \mu_0 \vec{E} \cdot \vec{J} \tag{4.3}$$

Physically eq. (4.3) corresponds to a conservation of energy. We refer to the term $\frac{1}{\mu_0}\left(\vec{E} \times \vec{B}\right)$ as the Poynting vector $\vec{S}$ as it determines the direction of electromagnetic radiation. The Poynting vector $\vec{S}$ represents the power density.

$$\vec{\nabla} \cdot \vec{S} + \frac{\partial u}{\partial t} + \vec{E} \cdot \vec{J} = 0 \tag{4.4}$$

Much like eq. (3.1), eq. (4.4) is a local conservation of *energy*. The only algebraic difference is the term $\vec{E} \cdot \vec{J}$. If there is a flowing charge $\vec{J}$ through an electric field $\vec{E}$, then there is work done on the charge. By Gauss's theorem, the energy leaving through a surface $\mathcal{S}$ per unit time is,

$$\int_{\mathcal{V}} \vec{\nabla} \cdot \vec{S} \mathrm{d}\tau = \oint_{\mathcal{S}} \mathrm{d}\vec{a} \cdot \vec{S}$$

and the E-M energy in the volume $\mathcal{V}$ is given by,

$$\int_{\mathcal{V}} \mathrm{d}\tau u$$

Where again, u is the electromagnetic energy density. If we integrate over eq. (4.4),

$$\int_{\mathcal{V}} \mathrm{d}\tau \left(\vec{\nabla} \cdot \vec{S} + \frac{\partial u}{\partial t} + \vec{E} \cdot \vec{J} \right) = \oint_{\mathcal{S}} \mathrm{d}\vec{a} \cdot \vec{S} + \int_{\mathcal{V}} \mathrm{d}\tau \frac{\partial u}{\partial t} + \int_{\mathcal{V}} \mathrm{d}\tau \vec{E} \cdot \vec{J}$$

Each term in eq. (4.4) has it's purpose illuminated. The final term $\int_{\mathcal{V}} \mathrm{d}\tau \vec{E} \cdot \vec{J}$ corresponds to the work done on moving charges $\vec{J}$ in the volume $\mathcal{V}$. In it important to note that there are no terms that corresponding to "magnetic work".

Consider the work done to move a charge q a displacement $\mathrm{d}\vec{\ell}$ by E-M forces,

$$\begin{aligned}
\mathrm{d}W &= \mathrm{d}\vec{\ell} \cdot \vec{F} \\
&= \mathrm{d}\vec{\ell} \cdot q\left(\vec{E} + \vec{v} \times \vec{B}\right) \\
&= \vec{v}\mathrm{d}t \cdot q\left(\vec{E} + \vec{v} \times \vec{B}\right) \\
&= q\mathrm{d}t\left(\vec{v} \cdot \vec{E}\right) + q\mathrm{d}t\,\underbrace{\left(\vec{v} \cdot \left\{\vec{v} \times \vec{B}\right\}\right)}_{=0} \\
&= q\mathrm{d}t\left(\vec{v} \cdot \vec{E}\right)
\end{aligned}$$

So for a continuous charge distribution we have that $\mathrm{d}q = \rho \mathrm{d}\tau$ and $\rho\vec{v} = \vec{J}$. Which means that the rate of work done on the charge ρ in the volume $\mathcal{V}$ (i.e. creating the current density $\vec{J}$) is,

$$\dot{W} = \int_{\mathcal{V}} \mathrm{d}\tau \vec{E} \cdot \vec{J}$$

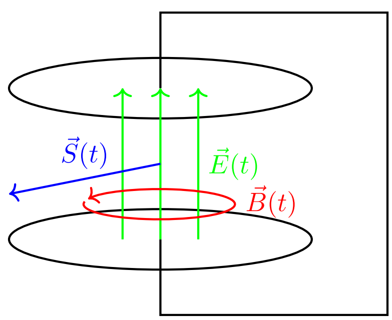
We can interpret this as the work done per unit time rearranging the charge in $\mathcal{V}$. One again eq. (4.4) is given by

$$\vec{\nabla} \cdot \vec{S} + \frac{\partial u}{\partial t} + \vec{E} \cdot \vec{J} = 0$$

With the following interpretations,

- $\vec{\nabla} \cdot \vec{S}$: amount of radiation energy leaving the point $\vec{r}$
- $\frac{\partial u}{\partial t}$: increase in E-M energy at the point $\vec{r}$
- $\vec{E} \cdot \vec{J}$: the amount of work done on charges at the point $\vec{r}$

As an illustrative example, consider a parallel plate capacitor with an electric field $\vec{E}$ between them.



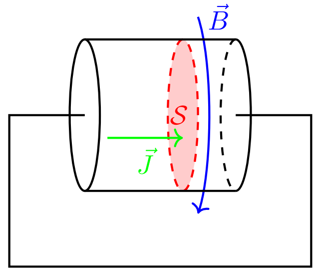
We ave that the magnetic field points in the $\hat{\phi}$ direction, $\vec{B} = V\hat{\phi}$. The electric field $\vec{E} = E\hat{\zeta}$, and Poynting vector are $\vec{S} = S\hat{s}$. We have that the radiation through the surface $\mathcal{S}$,

$$\int_{\mathcal{S}} \mathrm{d}\vec{a} \cdot \vec{S} = -(2\pi a h)S$$

Therefore $\frac{\partial U}{\partial t} = -(2\pi a h)S$ corresponding to the amount of energy flowing out of the capacitor and therefore,

$$U = \int_0^\infty \mathrm{d}t(-2\pi a h S) = \frac{1}{2}CV^2$$

Ex 8.1:



Inside the conductor the electric field moves parallel to its axis $\vec{E} = \frac{V_0}{\ell}\hat{\zeta}$. The magnetic field is then given by,

$$\vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial E}{\partial t} + \mu_0 \vec{J}$$

Integration over the surface $\mathcal{S}$,

$$\int_{\mathcal{S}} \mathrm{d}\vec{a} \cdot \left(\vec{\nabla} \times \vec{B}\right) = \mu_0 \int_{\mathcal{S}} \mathrm{d}\vec{a} \cdot \vec{J} = \mu_0 I_{\text{enc}}$$

Therefore computing this integral $\int \mathrm{d}\vec{\ell} \cdot \vec{B}$ yields,

$$\vec{B} = \frac{\mu_0 I}{2\pi a}\hat{\phi}$$

Moreover, the Poynting vector is given by,

$$\begin{aligned}\vec{S} &= \frac{1}{\mu_0}\vec{E} \times \vec{B} \\ &= \frac{1}{\mu_0}\frac{V_0}{\ell}\hat{\zeta} \times \frac{\mu_0 I}{2\pi a}\hat{\phi} \\ &= -\frac{V_0 I}{2\pi a \ell}\hat{s}\end{aligned}$$

Therefore the radiation flux,

$$\int_{\mathcal{S}} \mathrm{d}\vec{a} \cdot \vec{S} = -\frac{V_0 I}{2\pi a \ell}\int_{\mathcal{S}} \mathrm{d}a = -V_0 I$$

Which is exactly the amount of Joule heating for a current I though a wire with voltage V_0 across it. Using $V = IR$,

$$\int_{\mathcal{S}} \mathrm{d}\vec{a} \cdot \vec{S} = -I^2 R$$

Ex 8.2 (Griffiths Problem 8.13): A long thin solenoid of radius a has a time dependent current $I_s(t)$ flowing around it. Encircling the solenoid is a ring of radius b with current $I_r(t)$ ($b \gg a$) passing through it. The ring has resistance R. There is an induced electro-motive-force in the ring due to the solenoid,

$$\mathcal{E} = -\dot{\Phi}_S = -\frac{\partial}{\partial t}\left(\pi a^2 B_S\right)$$

Where $B_s = \mu_0 n I_s$. The EMF $\mathcal{E}$ must also equal $\mathcal{E} = I_r R$. Therefore,

$$I_r = -\frac{1}{R}\left(\mu_0 \pi a^2 n\right)\dot{I}_S$$

In order to calculate the electric and magnetic fields just outside solenoid, recognize that $\vec{B}_s = B_s(t)\hat{z}$ point along the axis of the solenoid. Similarly recognize that $\vec{E} = E\hat{\phi}$. Therefore the Poynting vector is given by,

$$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B}_s = \frac{1}{\mu_0}E B_s \hat{s} =?$$

We first need to calculate $\vec{E}$ and $\vec{B}_s$. The magnetic field is known to be $\vec{B} = \mu_0 n I_s \hat{z}$ on axis and $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$,

$$\int \mathrm{d}\vec{a}\cdot\vec{\nabla}\times\vec{E} = -\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}\vec{a}\cdot\vec{B}$$

$$\int \mathrm{d}\vec{\ell}\cdot\vec{E} = -\dot{\Phi} = 2\pi a E$$

Which gives,

$$\vec{E} = \frac{\dot{\Phi}}{2\pi a}\hat{\phi}$$

The magnetic field off axis and outside the solenoid due to the ring is given by,

$$\mathrm{d}\vec{B}_r(s) = \frac{\mu_0 I}{4\pi}\frac{\mathrm{d}\vec{\ell}\times\vec{\mathscr{r}}}{\mathscr{r}^2}$$

Where $\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$ and we take $\vec{r}\,' = b\hat{s}'$ and $\vec{r} = z\hat{z}$.

$$\vec{\mathscr{r}} = z\hat{z} - b\hat{s}'$$

We will take the infinitesimal loop to be $\mathrm{d}\vec{\ell} = b\mathrm{d}\phi'\hat{\phi}'$.

$$\begin{aligned}\mathrm{d}\vec{\ell}\times\vec{\mathscr{r}} &= \left(b\hat{\phi}'\mathrm{d}\hat{\phi}'\right)\times(z\hat{z} - b\hat{s}') \\ &= az\mathrm{d}\phi'\hat{s}' + b^2\mathrm{d}\phi'\hat{z}\end{aligned}$$

We integrating around the loop $\mathcal{L}$, all of the contributions in the $\hat{s}'$ directions will cancel out.

$$\int_{\mathcal{L}} \mathrm{d}\vec{\ell}\times\vec{\mathscr{r}} = \cdots$$

Thus,

$$\begin{aligned}\vec{B}_r &= \frac{\mu_0 I_r}{4\pi}\int \frac{b^2\mathrm{d}\phi'\hat{z}}{(z^2+b^2)^{3/2}} \\ &= \frac{\mu_0 I_r b^2 2\pi\hat{z}}{4\pi(z^2+b^2)^{3/2}} \\ &= \frac{\mu_0 b^2}{2(z^2+b^2)^{3/2}} I_r\hat{z}\end{aligned}$$

Therefore the Poynting vector points radial outward,

$$\begin{aligned}\vec{S} &= \frac{1}{\mu_0}\vec{E}_r\times\vec{B}_r \\ &= \frac{1}{\mu_0}\left(\frac{\pi a^2\mu_0 n\dot{I}_s}{2\pi a}\hat{\phi}\right)\times\left(\frac{\mu_0 b^2}{2(z^2+b^2)^{3/2}}I_r\hat{z}\right) \\ &= \frac{\mu_0}{4}an\dot{I}_s\frac{b^2}{(z^2+b^2)^{3/2}}I_r\hat{s}\end{aligned}$$

Now that the Poynting vector is known, one can calculate the power radiated from the system.

$$\begin{aligned}P &= \int \mathrm{d}\vec{a}\cdot\vec{S} \\ &= \frac{\mu_0}{4}an\dot{I}_s I_r b^2\int \mathrm{d}za\mathrm{d}\phi\hat{s}\cdot\frac{1}{(z^2+b^2)^{3/2}}\hat{s}\end{aligned}$$

$$\begin{aligned}
&= \frac{\mu_0}{4} an \dot{I}_s I_r b^2 (2\pi a) \int \mathrm{d}z \frac{1}{(z^2+b^2)^{3/2}} \\
&= \frac{\mu_0}{4} an \dot{I}_s I_r b^2 (2\pi a) \frac{2}{b^2} \quad \text{Integral Table} \\
&= \mu_0 \pi a^2 n \dot{I}_s I_r
\end{aligned}$$

But we know that $\mu_0 \pi a^2 n \dot{I}_s = -I_r R$. Therefore $P = -I_r^2 R$ as expected.
A2.2:
a,b) Answers in Griffiths.
c) Consider parallel metal strips with height h and width w where $h \ll w$. A current flows down one plate and up the other. The system will act as a capacitor. The magnetic field outside will be zero and non-negative inside.
d) Griffiths 8.1
A2.3:
Positive and negative charge build up on the surfaces between the capacitor. Of course, there will be a time varying current $I(t)$, electric field $\vec{E}(t)$ and magnetic field $\vec{B}(t)$.

5 Stress Energy Tensor

Last week we looked at conservation laws and we found,

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \quad \text{(charge)}$$

and,

$$\vec{\nabla} \cdot \vec{S} + \frac{\partial u}{\partial t} + \vec{J} \cdot \vec{E} = 0 \quad \text{(energy)}$$

This week we will continue with momentum and angular momentum and then we will examine the Maxwell stress tensor; the field equivalent for force in Newton's second law. But first we will look at momentum.

5.1 Momentum

Consider two charges $+q_1$ and $-q_2$ with velocities $\vec{v}_1$ and $\vec{v}_2$. The electric field at point 2 due to charge 1 will be denoted $\vec{E}_1$. Analogously for $\vec{B}_1$. The net force acting on charge q_2 is then,

$$\vec{F}_{2;E} = q_2 \vec{E}_1 \qquad \vec{F}_{2;B} = q_2 \vec{v}_2 \times \vec{B}_1$$

$$\vec{F}_{1;E} = q_1 \vec{E}_2 \qquad \vec{F}_{1;B} = q_1 \vec{v}_1 \times \vec{B}_2$$

One will notice that $\vec{F}_{1;B}$ and $\vec{F}_{2;B}$ are not equal and opposite forces like $\vec{F}_{1;E}$ and $\vec{F}_{2;E}$ are. What does this say about Newton's third law?

$$\sum \dot{\vec{p}}_i = \sum \vec{F}_{\text{net}}$$

We forgot about the fact that the electric and magnetic fields carry not only energy (via $\vec{S}$) but momentum as well. Recall that for photons,

$$E = hf = \hbar\omega$$

$$p = \frac{h}{\lambda} = \hbar k$$

Therefore we have that,

$$E = pc$$

Therefore knowing the energy density of the field gives you then momentum density of the field. The momentum density will be denoted $\vec{g}$.

$$\vec{g} = \frac{1}{c^2}\vec{S} = \mu_0 \epsilon_0 \vec{S} = \frac{1}{4\pi c}\vec{E} \times \vec{B}$$

The force per unit volume $\vec{f} = \Delta\vec{F}/\Delta\tau$ acting on a particle is given by the Lorrentz force.

$$\vec{f} = \rho\vec{E} + \vec{J} \times \vec{B}$$

Which when expanded out is,

$$\vec{f} = \epsilon_0\left(\vec{\nabla}\cdot\vec{E}\right)\vec{E} + \left(\frac{1}{\mu_0}\vec{\nabla}\times\vec{B} - \epsilon_0\frac{\partial\vec{E}}{\partial t}\right)\times\vec{B}$$

Todo (TC Fraser): Inject hand out Which after some algebra yields,

$$\vec{f} = \epsilon_0\left(\left(\vec{\nabla}\cdot\vec{E}\right)\vec{E} + \left(\vec{E}\cdot\vec{\nabla}\right)\vec{E}\right) + \frac{1}{\mu_0}\left(\left(\vec{\nabla}\cdot\vec{B}\right)\vec{B} + \left(\vec{B}\cdot\vec{\nabla}\right)\vec{B}\right) - \vec{\nabla}\left(\frac{\epsilon_0}{2}E^2 + \frac{1}{2\mu_0}B^2\right) - \mu_0\epsilon_0\frac{\partial\vec{S}}{\partial t} \tag{5.1}$$

We now introduce **Maxwell's stress energy tensor** T with components,

$$T_{ij} = \epsilon_0\left(E_iE_j - \frac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_0}\left(B_iB_j - \frac{1}{2}\delta_{ij}B^2\right) \tag{5.2}$$

Where $\vec{E} = \sum_i E_i\hat{e}_i$ and $\vec{B} = \sum_i B_i\hat{e}_i$. We now have that eq. (5.1) gives,

$$\vec{f} = \vec{\nabla}\cdot T - \epsilon_0\mu_0\frac{\partial\vec{S}}{\partial t}$$

As defined $\vec{F} = \int_{\mathcal{V}} \mathrm{d}\tau\vec{f}$ is the net mechanical force acting on the matter in a volume $\mathcal{V}$. Therefore,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\vec{p}_{\text{mech}} &= \int_{\mathcal{V}}\mathrm{d}\tau\vec{f} \\
&= \int_{\mathcal{V}}\mathrm{d}\tau\left(\vec{\nabla}\cdot T - \epsilon_0\mu_0\frac{\partial S}{\partial t}\right) \\
&= \oint_{\mathcal{S}}\mathrm{d}\vec{a}\cdot T - \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{V}}\mathrm{d}\tau\epsilon_0\mu_0\vec{S}
\end{aligned}$$

We usually define the second term here to be the momentum contained in the electromagnetic field,

$$\vec{p}_{\text{em}} = \int_{\mathcal{V}}\mathrm{d}\tau\epsilon_0\mu_0\vec{S}$$

Therefore the conservation of momentum is,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\vec{p}_{\text{mech}} + \vec{p}_{\text{em}}) = \oint_{\mathcal{S}}\mathrm{d}\vec{a}\cdot T \tag{5.3}$$

To draw intuition from continuum mechanics, the **Cauchy stress tensor** is a representation of the total forces acting on a chunk $\mathcal{V}$ of a material due to the neighboring pieces $\mathcal{N}(\mathcal{V})$. Each neighboring chunk $n(\mathcal{V})$ can exert parallel or shear forces on $\mathcal{V}$. This defines a matrix on force components on each face of $\mathcal{V}$. Let $\vec{f} = \sigma\cdot\mathrm{d}\vec{a}$ where $\vec{\sigma}$ is a rank 2 (3d) tensor. We call σ the Cauchy stress tensor such that,

$$\vec{f} = \sigma\cdot\mathrm{d}\vec{a}$$

The divergence of the Maxwell stress tensor is,

$$\frac{\partial}{\partial x_i}T_{ij} = \epsilon_0\left(\frac{\partial E_i}{\partial x_i}E_j + E_i\frac{\partial E_j}{\partial x_i} - \frac{1}{2}\delta_{ij}\frac{\partial E^2}{\partial x_i}\right) + \frac{1}{\mu_0}\left(\frac{\partial B_i}{\partial x_i}B_j + B_i\frac{\partial B_j}{\partial x_i} - \frac{1}{2}\delta_{ij}\frac{\partial B^2}{\partial x_i}\right)$$

Which in vector notation is much simplier Todo (TC Fraser): Verify this expression,

$$\vec{\nabla}\cdot T = \epsilon_0\left(\left(\vec{\nabla}\cdot\vec{E}\right)\vec{E} + \left(\vec{E}\cdot\vec{\nabla}\right)\vec{E}\right) - \frac{1}{2}\epsilon_0\vec{\nabla}\cdot\vec{E} + \frac{1}{\mu_0}\left(\left(\vec{\nabla}\cdot\vec{B}\right)\vec{B} + \left(\vec{B}\cdot\vec{\nabla}\right)\vec{B}\right) - \frac{1}{2}\epsilon_0\vec{\nabla}\cdot\vec{B}$$

While the force per unit volume is,

$$\vec{f} = \vec{\nabla} \cdot T - \epsilon_0 \mu_0 \frac{\partial \vec{S}}{\partial t}$$

Which can be integrated over a volume $\mathcal{V}$ in order to obtain the total force,

$$\begin{aligned}\int_{\mathcal{V}} \mathrm{d}\tau \vec{f} &= \int_{\mathcal{V}} \mathrm{d}\tau \vec{\nabla} \cdot T \\ &= \int_{\mathcal{V}} \mathrm{d}\tau \vec{\nabla} \cdot T - \mu_0 \epsilon_0 \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{V}} \mathrm{d}\tau \vec{S} \\ \frac{\mathrm{d}\vec{p}_{\text{mech}}}{\mathrm{d}t} &= \oint_{\mathcal{S}} \mathrm{d}\vec{a} \cdot T - \frac{\mathrm{d}}{\mathrm{d}t} \underbrace{\int_{\mathcal{V}} \mathrm{d}\tau \vec{g}}_{\vec{p}_{\text{em}}}\end{aligned}$$

Therefore we recover eq. (5.3) again,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\vec{p}_{\text{mech}} + \vec{p}_{\text{em}}) = \oint_{\mathcal{S}} \mathrm{d}\vec{a} \cdot T$$

This conservation of momentum equation can be interpreted as Newton's second law for E&M.

Ex 8.4: Two point charges a distance 2ℓ apart. Due to the rotational symmetry of the problem, we can exploit cylindrical coordinates $\vec{r} = s\hat{s} + z\hat{z}$ because no physical quantities can depend on ϕ.
a) The electric field in a plane ($\phi = 0$) can be obtained as follows. Let $\vec{r}\,'$ be the location of the source q and $\vec{r}$ be the field location. The origin is between the two identical charges. Let $\vec{E}_+$ be the electric field due to the charge in the $z > 0$ direction.

$$\vec{r}\,'_\pm = \pm \ell \hat{z}$$

And on the axis perpendicular to $\hat{z}$,

$$\vec{r} = s\hat{s}$$

Therefore,

$$\vec{\mathscr{r}} = \vec{r} - \vec{r}\,' = s\hat{s} - \ell\hat{z}$$

Which gives electric field,

$$\begin{aligned}\vec{E}_\pm &= \frac{q_\pm \vec{\mathscr{r}}_\pm}{4\pi\epsilon_0 \mathscr{r}_\pm^3} \\ &= \frac{q_\pm}{4\pi\epsilon_0} \frac{s\hat{s} \mp \ell\vec{z}}{(s^2 + \ell^2)^{3/2}}\end{aligned}$$

Therefore,

$$\vec{E}_{\{z=0\}} = \vec{E}_+ + \vec{E}_- = \frac{q_\pm}{4\pi\epsilon_0} \frac{2s\hat{s}}{(s^2 + \ell^2)^{3/2}}$$

Upon reflection, the direction of $\vec{E}_{\{z=0\}}$ could have only been in the $\hat{s}$ direction by symmetry.
b) Calculate the Maxwell Stress Tensor using eq. (5.2). Notice that $\vec{=}B$ and $\vec{E}_{\{z=0\}} = E(s)\hat{s}$,

$$\hat{s} = \hat{x}\cos\phi + \hat{y}\sin\phi$$

So in Cartesian coordinates,

$$\vec{E}_{\{z=0\}} = \frac{q_\pm}{4\pi\epsilon_0} \frac{2\sqrt{x^2 + y^2}(\hat{x}\cos\phi + \hat{y}\sin\phi)}{(x^2 + y^2 + \ell^2)^{3/2}}$$

The components of $\vec{E}$ are then,

$$E_1 = E_x = \frac{q_\pm}{4\pi\epsilon_0} \frac{2\sqrt{x^2 + y^2}\cos\phi}{(x^2 + y^2 + \ell^2)^{3/2}}$$

$$E_2 = E_y = \frac{q_\pm}{4\pi\epsilon_0}\frac{2\sqrt{x^2+y^2}\sin\phi}{\left(x^2+y^2+\ell^2\right)^{3/2}}$$
$$E_3 = E_z = 0$$

For convenience let,

$$E_0 = \frac{q_\pm}{4\pi\epsilon_0}\frac{2\sqrt{x^2+y^2}}{\left(x^2+y^2+\ell^2\right)^{3/2}} \tag{5.4}$$

Such that,

$$E_1 = E_0\cos\phi \quad E_2 = E_0\sin\phi$$

And also,

$$E^2 = E_0^2$$

Therefore the components of T are determined by eq. (5.2),

$$T = \epsilon_0 E_0^2\begin{pmatrix}\cos^2\phi - \frac{1}{2} & \sin\phi\cos\phi & 0\\ \cos\phi\sin\phi & \sin^2\phi - \frac{1}{2} & 0\\ 0 & 0 & -\frac{1}{2}\end{pmatrix}$$

Which by trig-identities becomes,

$$T = \frac{1}{2}\epsilon_0 E_0^2\begin{pmatrix}\cos(2\phi) & \sin(2\phi) & 0\\ \sin(2\phi) & -\cos(2\phi) & 0\\ 0 & 0 & -1\end{pmatrix}$$

c) Construct a closed hemisphere $\mathcal{H}$ above $z>0$ enclosing the charge q_+ but not q_-. Since the charges are not moving, we have that,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{p}_{\text{mech}} + \vec{p}_{\text{em}}\right) = \vec{0}$$

Therefore it must be,

$$\oint_{\mathcal{H}} \mathrm{d}\vec{a}\cdot T = 0$$

However, $\mathcal{S}$ does *not* lie in the same plane as the Maxwell stress tensor computed above. Instead, we can take the radius R of the hemisphere to be $R\to\infty$ such that the "hemisphere" becomes a flat plane with a central circular region. The net force acting on q_+ is,

$$\vec{F}_+ = \int_{\mathcal{V}}\mathrm{d}\tau\vec{f} = \int_{\mathcal{V}}\mathrm{d}\vec{a}\vec{\nabla}\cdot T - \epsilon_0\mu_0\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{V}}\mathrm{d}\tau\cancelto{0}{\vec{S}} = \int_{\mathcal{S}}\mathrm{d}\vec{a}\cdot T$$

Therefore,

$$\begin{aligned}\vec{F}_+ &= \int_{-\infty}^{\infty}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}y\frac{1}{2}\epsilon_0E_0^2\begin{pmatrix}0 & 0 & -1\end{pmatrix}\begin{pmatrix}\cos(2\phi) & \sin(2\phi) & 0\\ \sin(2\phi) & -\cos(2\phi) & 0\\ 0 & 0 & -1\end{pmatrix}\\ &= \int_{-\infty}^{\infty}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}y\frac{1}{2}\epsilon_0E_0^2\begin{pmatrix}0\\0\\1\end{pmatrix}\end{aligned}$$

Where $E_0 = E_0(x,y)$ is given by eq. (5.4).

$$\begin{aligned}\vec{F}_+ &= \frac{1}{2}\epsilon_0\hat{z}\int_{-\infty}^{\infty}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}yE_0^2\\ &= \frac{1}{2}2\pi\epsilon_0\left(\frac{q}{2\pi\epsilon_0}\right)^2\hat{z}\int_0^{\infty}\mathrm{d}s\frac{s\left(s^2\right)}{\left(s^2+\ell^2\right)^3}\end{aligned}$$

$$= \cdots$$
$$= \frac{q^2 \hat{z}}{4\pi\epsilon_0 (2\ell)^2}$$

Which is simply a result of Columb's law which was expected.

5.2 Method of Images

8.4:

$$\vec{F} = \int_{\mathcal{S}} \mathrm{d}\vec{a} \cdot T$$

A3.1:

$$\vec{E}_{\text{vac}} \cdot \hat{t} = 0$$
$$\int \mathrm{d}\vec{\ell} \cdot \vec{E} = 0$$
$$\int \mathrm{d}\vec{a} \cdot \vec{\nabla} \times \vec{E} = -\frac{\mathrm{d}}{\mathrm{d}t} \int \mathrm{d}\vec{a} \cdot \vec{B} = 0$$
$$V(x, y, z > 0) = V(z, y, z) \quad V(x, y, z < 0) = 0$$

The boundary condition,

$$\vec{E} \cdot \hat{n} = \frac{\sigma}{\epsilon_0}$$

Gives the potential at $z = 0^+$.
Recall that the energy stored in the electromagnetic field is given by,

$$W = \frac{\epsilon_0}{2} \int \mathrm{d}\tau E^2$$

A3.?: The electromagnetic momentum,

$$\vec{g} = \mu_0 \epsilon_0 \vec{S} = \epsilon_0 \left(\vec{E} \times \vec{B} \right)$$

The electric field is in the z direction $\vec{E} = E\hat{z}$ and the magnetic field is $\vec{B} = B\hat{\phi}$. Therefore $\vec{S} = -S\hat{s}$. Now consider a charge q and a magnetic dipole $\vec{m}$ near each other. The magnetic field generated by $\vec{m}$ and the electric field generated by q generate a joint $\vec{S}$ field. The $\vec{S}$ forms closed circles around the system, meaning no energy is moving in or out of the system. Because of this, $\vec{g}$ is non-zero and is rotational around the system indicating that there is angular momentum stored in the field. The angular momentum density is then,

$$\vec{\ell} = \vec{r} \times \vec{g}$$

The angular momentum comes from *resisting* the magnetic force in a radial direction when trying to bring the charge q toward the dipole $\vec{m}$.
Feynman Vol. 2 17-4: Consider a plastic disk that is free to rotate and with surface charge σ (generating field $\vec{E}$). Then place a solenoid in the center of the disk and turn it on, generating a magnetic field $\vec{B}$. This field induces an electric field $\vec{E}\,'$ in the disk because of Maxwell's law,

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Generating some torque $\vec{N}$, the total angular momentum in the disk is given by,

$$\vec{L}_{\text{mech}} = \int_0^t \mathrm{d}t' \vec{N}$$

Turning the solenoid on and off transfers angular momentum from the mechanical system to the field system.
A3.3: (Griffiths 8.4, 8.21): Solenoid with radius R.

6 Waves

Todo (TC Fraser): Missed a lecture

Now that we have derived electromagnetic waves from Maxwell's equations in a vacuum,

$$\nabla^2 \vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0$$
$$\nabla^2 \vec{B} - \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2} = 0$$

Today, we will start by examining the 1D wave equation,

$$v^2 \frac{\partial^2 f}{\partial z^2} - \frac{\partial^2 f}{\partial t^2} = 0 \tag{6.1}$$

Where v is the wave speed and $f(z,t)$ is the *displacement* of the *medium* from its equilibrium position. To solve this second order PDE, D'Alembert came up with the substitution,

$$q_{\pm} = z \pm vt$$

Which has the inversion,

$$\begin{aligned} z &= \frac{1}{2}(q_+ + q_-) \\ v &= \frac{1}{2v}(q_+ - q_-) \end{aligned} \tag{6.2}$$

This substitution has the following chain rule,

$$\frac{\partial}{\partial z} = \frac{\partial q_+}{\partial z}\frac{\partial}{\partial q_+} + \frac{\partial q_-}{\partial z}\frac{\partial}{\partial q_-}$$
$$\frac{\partial}{\partial t} = \frac{\partial q_+}{\partial t}\frac{\partial}{\partial q_+} + \frac{\partial q_-}{\partial t}\frac{\partial}{\partial q_-}$$

So that eq. (6.1) becomes,

$$\begin{aligned} \frac{\partial^2 f}{\partial z^2} &= \left(\frac{\partial}{\partial q_+} + \frac{\partial}{\partial q_-}\right)^2 f \\ &= \frac{\partial^2 f}{\partial q_+^2} + \frac{\partial^2 f}{\partial q_-^2} + 2\frac{\partial^2 f}{\partial q_+ \partial q_-} \end{aligned}$$

Similarly,

$$\frac{\partial^2 f}{\partial t^2} = v^2\left(\frac{\partial^2 f}{\partial q_+^2} + \frac{\partial^2 f}{\partial q_-^2} - 2\frac{\partial^2 f}{\partial q_+ \partial q_-}\right)$$

Therefore eq. (6.1) becomes,

$$\frac{\partial^2 f}{\partial q_+ \partial q_-} = 0$$

Which has the general solution of a separable function,

$$f(q_+, q_-) = f_+(q_+) + f_-(q_-)$$

Or in terms of eq. (6.2),

$$f(z,t) = f_+(z + vt) + f_-(z - vt)$$

We will usually write this as:

$$f(z,t) = f_+(kz + \omega t) + f_-(kz - \omega t) \tag{6.3}$$

Where $v = \omega/k$ and ω is the temporal frequency and k is the spatial frequency. We shall see that the 3D generalization is easy to digest,

$$f_{\pm}(k_z z + k_y y + k_x x \pm \omega t) = f_{\pm}\left(\vec{k}\cdot\vec{r} \pm \omega t\right)$$

What we will see is that the **wave vector** $\vec{k}$ points in the same direction as the Poynting vector $\vec{S}$.

Due to the linearity of the wave equations, $v^2 \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2}$ can be treated as linear operator $\hat{L}$ such that,

$$\hat{L}(\alpha f + \beta g) = \alpha \hat{L}(f) + \beta \hat{L}(g)$$

We can make use of Fourier and his friends so that we only have to solve the wave equation for one frequency (both spatial and temporal). Promote $f(z,t)$ to be a complex amplitude $\tilde{f}(z,t)$ and then decompose it using spectral analysis,

$$\tilde{f}(z,t) = \int_{-\infty}^{\infty} \mathrm{d}k e^{i(kz-\omega t)} f_k(t) = \mathcal{F}\big[\tilde{f}_k(t)\big](z,t) \tag{6.4}$$

Where $\tilde{f}_k(t)$ is a complex function of k and t which can be found using an inverse Fourier transform,

$$\tilde{f}_k(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}z e^{-i(kz-\omega t)} \tilde{f}(z,t)$$

To verify that this works, substitute in $\tilde{f}(z,t)$ using eq. (6.4),

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}z e^{-i(kz-\omega t)} \tilde{f}(z,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}z e^{-i(kz-\omega t)} \int_{-\infty}^{\infty} \mathrm{d}k' e^{i\left(k'z-\omega t\right)} \tilde{f}_{k'}(t) f_{k'}(t) \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k' \tilde{f}_{k'}(t) f_{k'}(t) \int_{-\infty}^{\infty} \mathrm{d}z e^{-i(kz-\omega t)} e^{i\left(k'z-\omega t\right)} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k' \tilde{f}_{k'}(t) f_{k'}(t) \int_{-\infty}^{\infty} \mathrm{d}z e^{-i\left(k-k'\right)z} \\
&= 2\pi\left(\frac{1}{2\pi}\right)\int_{-\infty}^{\infty} \mathrm{d}k' \tilde{f}_{k'}(t) f_{k'}(t)\delta(k-k') \\
&= \tilde{f}_k(t)
\end{aligned}$$

We write as shorthand,

$$\tilde{f}_k(t) = \mathcal{F}^{-1}\big[\tilde{f}(z,t)\big]_k(t)$$

Recovering the actual solution to the wave equation corresponds to taking the real part to $\tilde{f}$,

$$\Re\big[\tilde{f}(z,t)\big] = \Re\big[\big|\tilde{f}\big|e^{i\delta}\big] = \big|\tilde{f}\big|\cos(\delta)$$