Phys 475

COSMOLOGY

University of Waterloo

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Table Of Contents

Page

1 Introduction **4**
1.1 History of Cosmology . . . 4
1.2 Studying the Universe as a Whole . . . 4
1.2.1 Optical . . . 4
1.2.2 Other Wavelengths . . . 6
1.2.3 Matter in The Universe . . . 7
1.3 Hubble Law . . . 8

2 Newtonian Cosmology **9**
2.1 Solving the Friedmann Equation . . . 10

3 Geometry **10**
3.1 Summary of General Relativity . . . 11
3.2 Einstein's Equations and Cosmology . . . 13
3.3 Review . . . 13

4 Redshift & Expansion **13**

5 Equations of State **14**

6 Making Measurements **15**

7 Densities **17**
7.1 Age of The Universe . . . 18

8 Geodesics **19**

Disclaimer

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1 Introduction

1.1 History of Cosmology

The first lecture consisted of everyone introducing themselves and then a brief summary of historical cosmology from Copernicus, to Kepler, Newton, and Einstein. The Copernican principle demonstrated that the earth is not special; Kepler's Laws revealed that the motion of the planets can be described by mathematical tools; Newton's laws unified physical properties observed on earth to those observed in the night sky. Finally, Einstein's equivalence principle further illuminated the equivalence between different observers. All of these observations and discovers have progressed us to the understanding we have today. The *Cosmological Principle* is as follows,

At large scales the universe is homogeneous and isotropic.
Equivalently, all observers see the same thing.

However, there are two important caveats. First, the Cosmological Principle holds on very large scales (typically 6×10^{22} m). Second, the Cosmological Principle holds for space but *not* time. This latter caveat was not fully accepted until after Einstein. Einstein was under the motivation that the Universe was static and unchanging because of his unification of space and time (i.e. the homogeneity of space *should* imply the homogeneity of time). However there was an observation that disagrees with this idea. **Olbers' Paradox** concerns itself with the issue of the darkness of the night sky. If the universe is homogeneous and isotropic, then in every direction one can look in the night sky, there should be a star at some distance away. In dual statement: no point in the night sky should be dark; hence the paradox. The resolution to Olbers' paradox is that the universe must not be infinite.

More rigorously, let the solid angle of an object a distance r away with radius R be $\pi R^2/r^2$. Therefore the total solid angle for all stars should be,

$$\sum_i \frac{\pi R_i^2}{r_i^2} = n_* \int_0^{r_{\max}} 4\pi r^2 \mathrm{d}r \frac{\pi R_*^2}{r^2} \propto r_{\max} R_*^2 \to \infty$$

In 1922, Hubble discovered the cosmic expansion of the universe which in turn implies the *Big Bang*; following the "linear" expansion *backward* in time, then at some point everything needs to be allocated at a singular point.

Remark: In general, there does not seem to be a clear distinction between cosmology and astrophysics. For clarity, we will consider cosmology to be the evolution of the universe as a *whole*. Of course there will be many exceptions to this focus, when we temporarily divert our attention to high energy particle physics, general relativity and other areas of physics.

1.2 Studying the Universe as a Whole

To study the paradigm of Cosmology, we will have to study the stuff that composes it. We can learn about the universe as a whole in many ways. For example, most of our observations are via the electromagnetic spectrum (γ/X-ray, UV, optical, IR, μ-waves, radio). Moreover we have the ability to probe the universe through neutrinos, cosmic rays and more recently (due to the work of the LIGO observatory), gravitational waves.

1.2.1 Optical

The building blocks of the visible/optical universe are:

- Stars

  - Mass: $M \approx M_\odot \approx 2 \times 10^{30}\,\mathrm{kg}$
  - Distance: $D \gtrsim \mathrm{pc} \approx 3\,\mathrm{lyr} \approx 3 \times 10^{16}\,\mathrm{m} \approx 2 \times 10^5\,\mathrm{AU}$

- Galaxies
  - Number of stars: $N \approx 1 \times 10^{11}$
  - Mass: $M \approx N \cdot M_\odot$
  - Radius: $R \approx 100\,\mathrm{kpc}$

- Globular Clusters
  - Number of stars: $N \approx 1 \times 10^8$

Of course, the Milky Way (the galaxy we live in) is observable in the visible spectrum with the naked eye. The Milky Way is a relatively flat disk but appears to us as a thin band due to our location inside it. The approximate thickness of the Milky Way is 300 pc and our distance to the center is roughly 8.5 kpc. The central (roughly spherical) bulge in the center of the Milky Way is on the order of $1\,\mathrm{kpc} - 2\,\mathrm{kpc}$. The orbital velocity of the solar system in the Milky Way is $v \approx 220\,\mathrm{km\,s^{-1}}$. The total orbital period is then,

$$t = \frac{2\pi r}{v} \approx 2 \times 10^8\,\mathrm{yr}$$

While the age of the universe is $t_{\mathrm{univ}} \approx 1.4 \times 10^{12}\,\mathrm{yr}$.

Moreover, there are other galaxies in the Universe other than the Milky Way; the most famous one being the Andromeda Galaxy (M31). The 'M' stands for Messier. In the 20th century, people started making measurements of the distance to the nearest galaxies (which at the time were unknown objects) and discovered they were much farther than previously thought $d \gtrsim 770\,\mathrm{kpc}$.

Galaxies come in two distinct types: some are elliptical and some are disks (spirals). To characterize a given galaxy, it depends on the size of the central bulge. Disk galaxies (bluer, older) are very ordered and have collated orbits while elliptical galaxies (reder, younger) are almost entirely bulges.

A collection of galaxies can form larger structures themselves. The Andromeda and Milky Way galaxies themselves are members of a larger structure known as the **Local Group** bound together by gravity. Other members of the local group include the Large Magellanic Cloud and various small satellites to the Milky Way. A galaxy group typically has on the order of a few dozens of large objects.

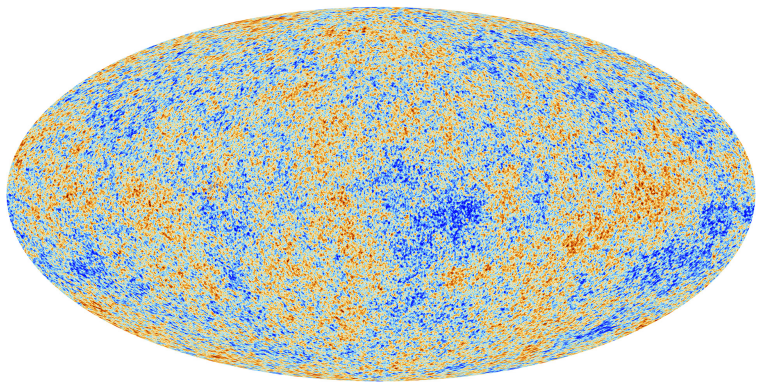
Continuing to larger and larger scales, galaxies can form clusters. Some examples include the Virgo cluster, or the Coma Cluster. A galaxy cluster typically has roughly thousands of members and has galaxy speeds $v_{\mathrm{gal}} \approx 1000\,\mathrm{km\,s^{-1}}$. Galaxy clusters are the largest known bound structures in the universe with a size on the order of 1 Mpc.

Even larger, there are super clusters or voids. These objects are large regions where there are very many galaxies or very few galaxies and are on the order of $d \approx 10\,\mathrm{Mpc}$. At larger scales, the universe is homogeneous. The approximate deviation in density is on the order of $n_{\mathrm{gal}} \approx (4-5) \times \langle n_{\mathrm{gal}} \rangle$. There is an assortment of names for these objects including:

- Filaments
- Fingers of God
- Great wall
- Matchstick man

1.2.2 Other Wavelengths

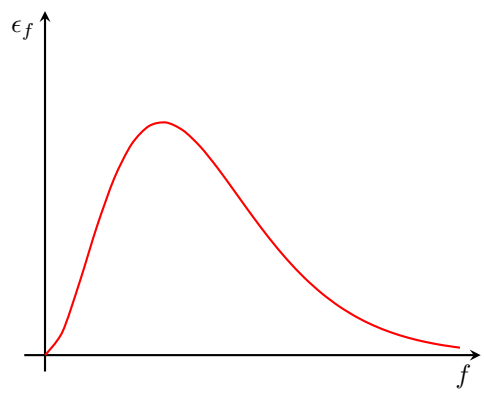
Looking at the Universe in other wavelengths (γ/X-ray, UV, IR, μ-waves, radio) allows one to view features that are not visible in the optical spectrum. To see the universe in shorter wavelengths like γ/X-ray, UV, we have had to build telescopes outside the earth's atmosphere. Although many things can be learned from making observations in each spectrum, the μ-waves spectrum is of particular importance to cosmology.



The **Cosmic Microwave Background (CMB)** was first discovered by Penzias & Wilson in 1965 by accident. If the universe is expanding, then at some point in the past hot media would have emitted radiation and cooled to much lower temperatures. The CMB is a isotropic source of radiation that is at a temperature,

$$T_{\text{CMB}} = 2.725(1)\,\text{K}$$

Which was first measured in 1992 by COBE-FIRAS. Recall the blackbody spectrum,



Where ϵ_f is the energy of radiation per unit volume per frequency,

$$\frac{\mathrm{d}E}{\mathrm{d}V\mathrm{d}f} = \epsilon_f = \frac{8\pi h f^3}{c^3\left[e^{hf/kT} - 1\right]}$$

Which under appropriate normalization matches the CMB extremely well; making it our most accurate indication for big-bang cosmology. However, there are slight anisotropies in the CMB. The variation in temperature of the CMB is extremely small (discovered COBE-DMR 1992),

$$\frac{\delta T}{T} \approx 1 \times 10^{-5}$$

For these discoveries, John Marher and G. Smoot shared the Nobel prize in 2006. Since these observations the WMAP (2001) and Planck (2013) projects have obtained higher and higher resolution images of the CMB.

In addition to μ-waves, the infrared (IR) spectrum is also important for cosmology. Dust absorbs more and more light at higher and higher frequencies. As such, lower wavelengths are important to be able to see through cosmic dust. As a quick summary, here are some infrared surveys,

- 2MASS (2μ, neat IR) stars
- IRAS (80's) ($> 20\mu$, far IR) dust
- Spitzer (new)

Moreover X-rays are much more energetic than optical light rays,

$$E_{\text{optical}} \approx \text{eV} \approx 1.6 \times 10^{-19}\,\text{J} \qquad E_{\text{X}} \approx \text{keV}$$

The typical temperature of an X-ray is $T_{\text{X}} \approx 1 \times 10^7\,\text{K}$ which is great for studying clusters of galaxies because most of a clusters mass is in plasma and $T_{\text{plasma}} \approx T_{\text{X}}$. Clusters are excellent cosmic labs because they are so dense. In fact, our first evidence for dark matter came from studying galaxy clusters via X-rays.

Radiowaves are also useful for cosmology because the transition of an electron in a neutral hydrogen atom from spin up to spin down emits weak energy emissions around 21 cm. The precision of radiowaves allows one to measure speeds of cosmic bodies by examining shifts in the 21 cm spectrum. The CHIME project is an up-and-coming radiowave telescope.

1.2.3 Matter in The Universe

Most of the matter in the universe is made up of particles (with the potential notable exception of dark matter). In truth, the modern perspective on matter is an emergent phenomena of quantum *fields*. From Einstein, we have that,

$$E^2 = p^2c^2 + m^2c^4$$

Where E is the energy of the particle, p is the momentum and m is the rest mass. If $p \ll mc$ we can take the non-relativistic limit $v = \frac{p}{m} \ll c$,

$$E = \sqrt{p^2c^2 + m^2c^4} = mc^2 + \frac{1}{2}\frac{p^2}{m} + \mathcal{O}\left(mc^2\frac{p^4}{m^4c^4}\right)$$

Clearly $\frac{1}{2}\frac{p^2}{m}$ refers to the standard classical kinetic energy. Higher order corrections to these formulas typically are on the order of $\mathcal{O}(c^{-2})$. On the other hand, for relativistic particles we have that $E \approx pc$ when $v \approx c$. To retain generality, we define the velocity of a particle to be,

$$\vec{v} = \frac{\vec{p}c^2}{E}$$

Here is a breakdown of the contents of the universe:
Cosmic Content:

- %4 – baryons/leptons (protons $m_{\text{p}}c^2 \approx$ GeV and electrons $m_{\text{e}}c^2 \approx 0.5\,\text{MeV}$)
- %0.004 – radiation (photons $m_\gamma c^2 = 0$ and neutrinos $m_\nu c^2 \approx 0.1\,\text{eV}$)
- %25 – dark matter (clumpy)
- %70 – dark energy (smooth)

The main interactions in the universe are due to photons scattering off of free electrons (**Thomas Scattering**) and also weakly interacting neutrinos. We have relative densities,

$$\rho_\gamma \approx \rho_\nu \approx 10^{-3}\rho_b$$

$$n_\gamma \approx n_\nu \approx 10^9 n_b$$

1.3 Hubble Law

The Hubble law (discovered by Hubble) shows that nebulae at further and further distances are moving *away* from us faster and faster. Hubble measured distances (using variable stars) and velocities of spiral nebulae using their Doppler shift (galaxies) using and obtained the following result:

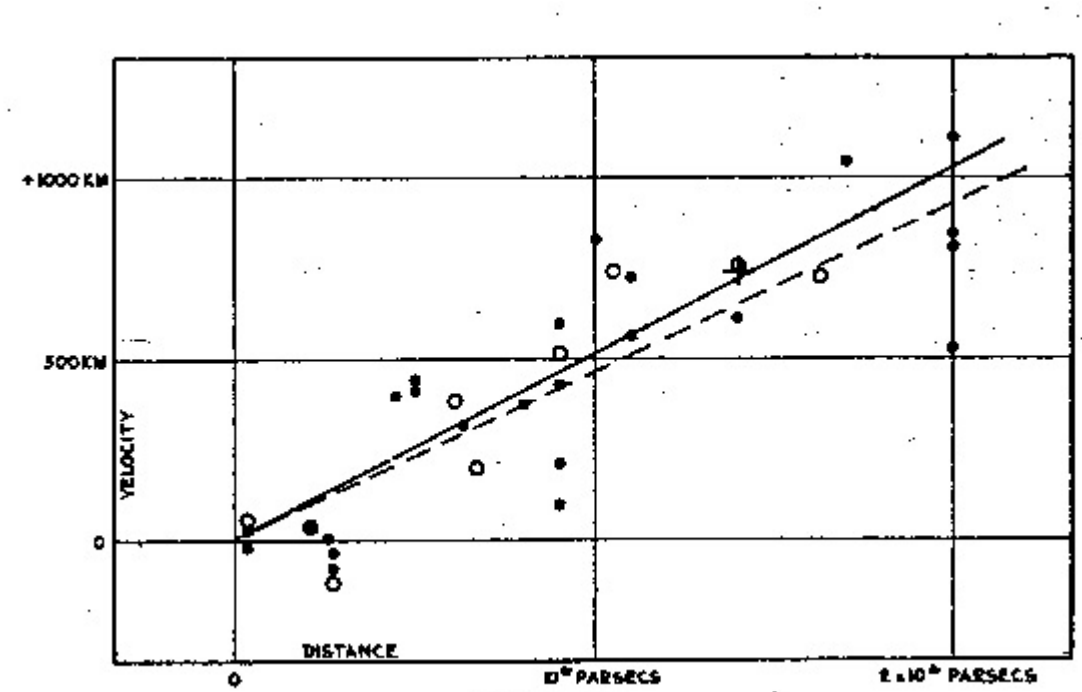
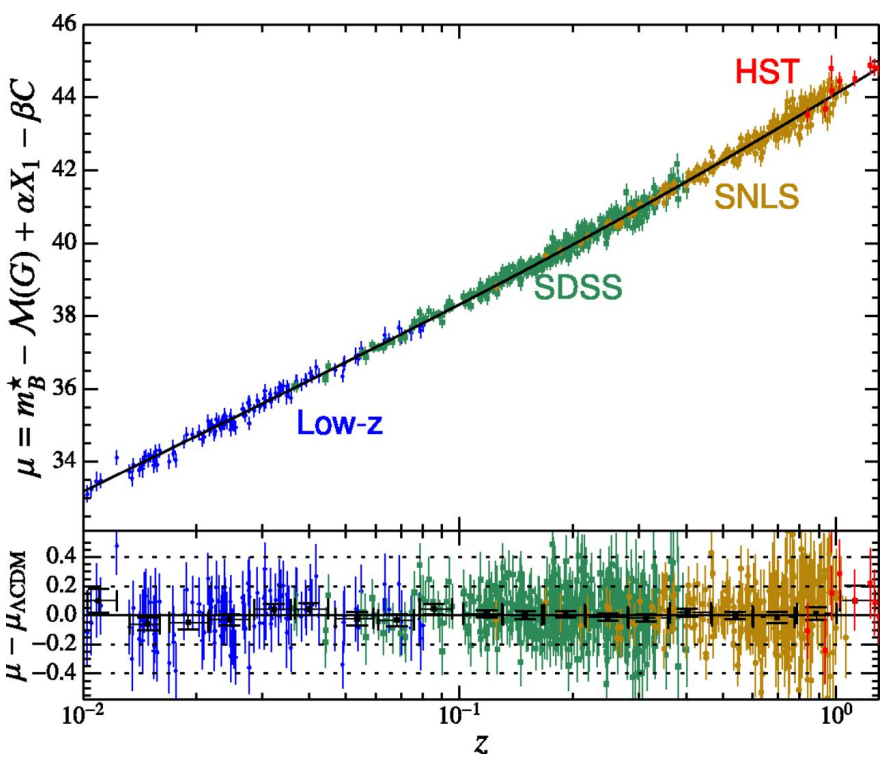


FIGURE 1

More recent measurements are depicted below:



$$v_r = Hr$$

Where r is the distance to the object, v_r is the radial velocity of that object and H is a constant known as **Hubble's Constant**. Hubble's constant has numerical value,

$$H = 100 \cdot h \mathrm{km\,s^{-1}\,Mpc^{-1}}$$

Where is a dimensionless number measured to an accuracy $h = 0.68(2)$. To measure velocities, Hubble used the Doppler shift,

$$1 + z = \frac{\lambda_{\text{obs}}}{\lambda_{\text{emit}}} = \frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}$$

Where z is called the **Redshift** and v is the radial velocity of the object.[1] As a first order approximation in small v,

$$1 + z = 1 + \frac{v}{c} + \mathcal{O}\left[\left(\frac{v}{c}\right)^2\right]$$

In the other limiting case, as $v \to c$, we have that $\lambda_{\text{obs}}/\lambda_{\text{emit}} \to \infty$ which makes $z \to \infty$.

To characterize deviations from Hubble's law, we define the peculiar velocity to be,

$$\vec{v}_p = \vec{v} - H\vec{r}$$

Where it is typical to have $\left\langle v_p^2 \right\rangle^{1/2} \approx 400\,\mathrm{km\,s^{-1}}$. Hubble's law was one of the first indications for the Big-Bang. It fact the time since the big bang is approximately,

$$t_{\text{big bang}} = \frac{1}{H} \approx 14\,\mathrm{Gyr}$$

Hubble's law gives us a picture of the kinematics of the universe.

2 Newtonian Cosmology

Now that we have a rough idea of the contents and kinematics of our universe, we shall start to study the dynamics of the universe. In doing so, we will approach an obstacle; as we look at farther and farther objects, Hubble's law dictates that these objects are moving faster and faster. At some point these objects are moving *faster* than the speed of light. As such, in order to do justice to Cosmology, one should study it through the lens of general relativity.

Despite this, we can take a shortcut ($\Phi_N = -GM/r$).

$$|\Phi_N|, v^2 \ll c^2 \quad \text{vs.} \quad |\Phi_N|, v^2 \approx c^2$$

To study local properties of the universe, it is sufficient to use Newtonian physics. Of course, global properties require a relativistic treatment. As such we will first focus on Newtonian Cosmology.

Consider two spherical[2] masses M and m and the force between them,

$$F = \frac{GMm}{r^2} \tag{2.1}$$

As well as the potential energy,

$$V = m\Phi_N = -\frac{GMm}{r}$$

Another thing of note is that masses are always positive $m \geq 0$. To contrast Newton, the most important equation in Cosmology is the **Friedmann equation**. If the universe is homogeneous we can pick a generic

[1] In fact, in a homogeneous universe, the only motion is radial.
[2] Objects that are not spherical exhibit deviations from eq. (2.1) known as **tidal forces**.

point to be the origin for the universe and define a ball of radius r (growing at rate $\dot{r}$) around this point enclosing a mass $M = \frac{4}{3}\pi r^3 \rho$. By conservation of energy,

$$\begin{aligned} U &= T + V = \text{const.} \\ &= \frac{1}{2}m\dot{r}^2 - \frac{GMm}{r} \\ &= \frac{1}{2}m\dot{r}^2 - \frac{4}{3}\pi\rho G m r^2 \end{aligned} \tag{2.2}$$

Next define a **co-moving** coordinate $\vec{x}$ such that $\vec{r}(\vec{x}, t) = a(t)\vec{x}$. For each particle at $\vec{x}$, as the universe expands, $\vec{x}$ follows the location of the particle. The physical coordinate of the particle is $\vec{r}$ and the relevant dimensionless scale factor is $a(t)$. These concepts hold for all "origins" provided that the universe is homogeneous and isotropic. Substituting the co-moving coordinate into eq. (2.2) yields the Friedmann equation,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_b - \frac{kc^2}{a^2} \tag{2.3}$$

Where k is defined such that,

$$kc^2 = -\frac{2U(x)}{mx^2}$$

In order for the universe to remain homogeneous, it must be that k does not depend on x. Therefore k is a constant in a homogeneous universe. By dimensional analysis on eq. (2.3) it can be seen that k has dimensions length^{-2}. In general relativity k becomes the curvature of the space. Intuitively, a homogeneous universe has the same curvature everywhere.

One misconception associated with the expansion of the universe is that *everything* in the universe is expanding. However, object that are close enough can be bound together by interactions in various fields. As such, the expansion of the universe has little to no observable affect on local objects like planets or people or galaxies.

2.1 Solving the Friedmann Equation

Solving the Friedmann equation corresponds to solving eq. (2.3) together with the **fluid equation**. The fluid equation comes from the first law of thermodynamics ($\mathrm{d}E + p\mathrm{d}V = T\mathrm{d}S = 0$), which translates to,

$$E = \pi a^3 x^3 \rho c^2 \qquad V = \frac{4}{3}\pi a^3 x^3$$

Which gives the continuity equation,

$$\dot{\rho} - \frac{3\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right) = 0 \tag{2.4}$$

Where the pressure $P(\rho)$ is determined by the equation of state. Combining eq. (2.4) and eq. (2.3) gives,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right)$$

Which is a relativistic manifestation of Newton's law. The difference being the term term involving pressure P.

3 Geometry

Geometry	k
Hyperbolic	-1
Flat	0
Geometry	+1

Todo (TC Fraser): Clean up this lecture

$$\ell = \begin{cases} \phi R \sin\left(\frac{r}{R}\right) & k = +1 \\ \phi r & k = 0 \\ \phi R \sinh\left(\frac{r}{R}\right) & k = -1 \end{cases}$$

Introduce the spacial components of the **Friedmann Robertson Walker (FRW) metric**,

$$\mathrm{d}s^2 = a^2(t)\left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right)\right) \tag{3.1}$$

There is a choice in dimensionality among the terms involved. Typically, $a(t)$ is taken to be a dimensionless quantity such that r and k have units. In doing so, we can make the scaling choice of,

$$a(t_{\text{now}}) = a(t_0) = a_0 \equiv 1$$

Where t_0 is used to denote the present time. Therefore r has units of length and k necessarily has units of length^{-2}. Considering the different possibilities for k ($\{k < 0, k = 0, k > 0\}$), observational data has not been able to distinguish the curvature k of our universe. Experiments suggest it is close to zero.

Recall that for an isotropic expansion, we need $a(t)$ on the "outside" of eq. (3.1) and so r, θ, ϕ (or x, y, z in flat space) are commoving coordinates. We will also see that galaxies have r, θ, ϕ approximately fixed (in a homogeneous universe).

What is the velocity of a galaxy? Obviously velocity is the derivative of a distance. However there are many different distances to consider in cosmology. One option is to define the distance between two objects at a fixed time called **proper time**. To calculate proper time take $\mathrm{d}s^2$ at fixed time t and integrating over $\mathrm{d}r$ and setting $r = 0$ to be the reference galaxy. Integrating along a radial line $\mathrm{d}\phi = \mathrm{d}\theta = 0$. Thus,

$$d_{\text{prop}} = a(t) \underbrace{\int_0^r \frac{\mathrm{d}r'^2}{\sqrt{1-kr'^2}}}_{\chi}$$

Note that for a galaxy, if r is constant then χ is constant as well. The **proper velocity** in this case is of course,

$$\dot{d}_{\text{prop}} = \dot{a}\chi = \dot{a}\frac{d_{\text{prop}}}{d} = \left(\frac{\dot{a}}{a}\right) d_{\text{prop}}$$

Which is identical to Hubble's law,

$$\{\dot{r} = H_0 r\} \sim \left\{\dot{d}_{\text{prop}} = \left(\frac{\dot{a}}{a}\right) d_{\text{prop}}\right\}$$

Therefore it could be that Hubble's constant is a function of time,

$$H(t) = \frac{\dot{a}}{a} \tag{3.2}$$

3.1 Summary of General Relativity

Without delving into the rigor and formalism of General Relativity, we summarize some of the ideas behind the foundations of GR.

The **equivalence principle** is best introduced via an analogy to electrostatics. In electrostatics, the force between two charges Q, q is,

$$F = \frac{1}{4\pi\epsilon_0}\frac{Qq}{r^2}$$

Therefore the acceleration on the smaller charge q with mass m is,

$$a = \frac{Qq}{4\pi\epsilon_0 m r^2} = \frac{Q}{4\pi\epsilon_0 r^2}\left(\frac{q}{m}\right)$$

Quite characteristically, the acceleration depends on the ratio q/m. Therefore an electron and proton will have a very different acceleration when nearby the same charge Q. In contrast, gravity behaves *differently*. In gravity,

$$F = \frac{GMm}{r^2} = ma$$

Therefore,

$$a = \frac{GM}{r^2}\left(\frac{m}{m}\right) = \frac{GM}{r^2}$$

Therefore any object (regardless of its mass) has the same acceleration; hinting at the fact that gravity is *special*. The equivalence of m_{grav} and m_{inert} then suggests the famous elevator thought experiment. Because of the mass equivalence, no experiment can distinguish the difference between an experiment on earth (in a uniform gravitational field) and a uniform acceleration (like in a rocket). Therefore, if the light from a laser in a rocket appears to bend downward in a rocket (due to the acceleration) it must also be the case that light bends downward toward earth! The resolution between this thought experiment and Fermat's optical principle is that space itself is bending in a gravitational field. The light in a gravitational field follows "straight" **geodesics**. Freely-falling objects (particles, photons) follow geodesics in space-time.

There is an important caveat to the equivalence principle. If the rocketships (elevators) has size comparable to the size of the earth, then the body will experience forces in different directions across its extent in a gravitational field but *not* when uniformly accelerating. These forces are known as **tidal forces**. So in Gr space time must behave like special relativity on small scales. In SR the spacetime line element is written as,

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$$

Photons taking at speed c has the defining property that,

$$\mathrm{d}s_\gamma^2 = 0 \tag{3.3}$$

Which can be stated as: *photons travel on null geodesics*. Therefore the **full FRW metric** must be,

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + a^2(t)\left(\frac{\mathrm{d}r^2}{1-kr^2}\right) + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right) \tag{3.4}$$

To solve for the equations of motion or geodesics of photons, one simply needs to solve eq. (3.4) under the constraint that eq. (3.3). In summary the following comparison holds,

Special Relativity	General Relativity
Special observers are inertial	Freely falling
Light follows null geodesics	same
Particles move in straight lines	Particles follow geodesics
Empty universe / No curvature	Not empty universe / Curvature

Note that the "flat" universe ($k = 0$) is spatially flat, but curved in space-time.

In summary, general relativity is composed of a Lorentzian space-time,

$$\Delta s^2 = -(c\Delta t)^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$$

And the notion that gravity is consequence of the curvature of space-time. The particulars of how matter and curvature are harmonious is embodied in Einstein's Equations. As a high level summary, Einstein's equations dictate the following relationship.

$$\text{curvature} = \left(\frac{8\pi G}{c^2}\right) \cdot \text{energy-momentum-pressure density}$$

3.2 Einstein's Equations and Cosmology

In cosmology (and under the cosmological principle), there is a unique solution to Einstein's equations. It is the FRW metric previously derived above in eq. (3.4)

$$\Delta s^2 = -c^2\Delta t^2 + a^2(t)\left(\frac{\Delta r^2}{1-kr^2}\right) + r^2\left(\Delta\theta^2 + \sin^2\theta\Delta\phi^2\right)$$

Where k determines the geometry of the spacial parts of the universe,

k	Geometry	Boundary
-1	Hyperbolic	Open
0	Flat	Open
+1	Geometry	Closed

As far as we can tell, $k \approx 0$. Another possibility is that the universe has a non-trivial topology; albeit no observable evidence suggests this.
The Einstein equation relates functions of the metric ($\mathrm{d}s^2 = \cdots$) to the energy content of the universe. Typically, the Einstein equations are 10 non-linear equations. When applied to a homogeneous universe (as is considered in cosmology), the Einstein equations reduce to 2 equations. One of which is the Friedmann equation,

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3}\rho$$

Where kc^3/a^2 is related to the energy content but here it is related to the curvature as well (same k as in FRW metric).

3.3 Review

As a review we have the following set of cosmological equations. The Friedmann equation:

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho \tag{3.5}$$

Where the Hubble parameter is,

$$H = \frac{\dot{a}}{a}$$

The Fluid equation:

$$\dot{\rho} + 3H(\rho + p) = 0 \tag{3.6}$$

Combining eqs. (3.5) and (3.6) we arrive at,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right)$$

4 Redshift & Expansion

So far we have looked at Doppler shift as a source for redshift. Of course redshift is also caused by an expanding universe. At small velocities we have,

$$\frac{\Delta\lambda}{\lambda} = \frac{v}{c} = \frac{H\Delta r}{c}$$

Where Δr is a physical local distance $\Delta r = c\Delta t$,

$$\frac{\Delta\lambda}{\lambda} = \frac{\dot{a}\Delta t}{a} = \frac{\Delta a}{a}$$

Integration gives us that,

$$\lambda \propto a$$

Intuitively, the length of waves in the universe scales with the size of the universe.

$$1 + z = \frac{\lambda_{\text{obs}}}{\lambda_{\text{emit}}} = \frac{a(t_{\text{obs}})}{a(t_{\text{emit}})}$$

Where t_{obs} is typically taken to be today. The quantity is z is known as redshift. As an example, the farthest galaxy that we can see has a redshift of approximately $z \approx 7 - 8$. In contrast, the redshift of the cosmic microwave background is at $z \approx 1100$.

5 Equations of State

All "materials" in the universe have different equations of state $P(\rho)$. Matter or dust has effectively a pressure of zero ($P = 0$). This is because the actual pressure is much less that the characteristic pressure of ρc^2, meaning we can omit it from the fluid equation,

$$\dot{\rho} = \frac{3\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right) \approx \frac{3\dot{a}}{a}(\rho)$$

Therefore for dust and matter,

$$\frac{\dot{\rho}}{\rho} = 3\frac{\dot{a}}{a} \implies \rho = \frac{\rho_a}{a^3}$$

Plugging this into the Friedmann equation gives us that,

$$\dot{a}^2 = \frac{8\pi G\rho_a}{3a} \implies a = \left(\frac{t}{t_0}\right)^{2/3}$$

By convention, we take t_0 to be today and $a(t_0) = 1$ when making observations. We have that $1 + z = \frac{1}{a}$. Therefore in a matter dominated universe, a scales as $t^{2/3}$. Therefore Hubble's parameter scales as,

$$H = \frac{\dot{a}}{a} = \frac{2}{3t}$$

However for radiation, we have that $P = \frac{1}{3}\rho c^2$ (this holds for photons and neutrinos). A similar analysis yields the relationship,

$$\rho = \frac{\rho_a}{a^4}$$

Plugging into the Friedmann equation once again gives,

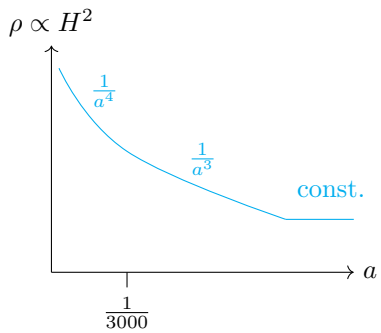
$$\dot{a}^2 = \frac{8\pi G\rho_a}{3a} \implies a = \left(\frac{t}{t_0}\right)^{1/2}$$

Which indicates that $H = 1/2t$.

Additionally, we can incorporate dark energy (cosmological constant fluid) to obtain,

$$P = -\rho c^2 \implies \rho = \rho_* \implies a = e^{Ht}$$

Our discovery of the acceleration of the expansion of the universe indicates that there is a dark energy contribution to the expansion of the universe. Depending on the composition of the universe, there are different behaviors for the expansion of the universe. Initially, (when a is smaller) radiation is dominating. Latter matter will dominate and then finally dark energy.



The number density of particles scales as,

$$n = \frac{n_0}{a^3}$$

While the number density or energy density of photons is given by,

$$E_\gamma = \frac{hc}{\lambda} \propto \frac{1}{a}$$

Looking at the Friedmann equation we have that,

$$\underbrace{\frac{1}{2}\dot{a}^2}_{\text{K.E.}} - \underbrace{\frac{4\pi G}{3}\rho a^2}_{\text{P.E.}} = -\frac{k}{2} = E_{\text{tot}}$$

6 Making Measurements

It is very remarkable (and somewhat bizarre) that we can describe the large scale structure of the universe. This is a unique feature of cosmology.

The question remains, how exactly are we able to measure cosmological features. As an examine, we measure Hubble's constant H_0 by using redshift z and distance r.

$$z = \frac{v}{c} = \frac{H_0 r}{c} \implies H_0 = \frac{zc}{r}$$

Measuring Hubble's constant is as old as cosmology itself. Nonetheless, there are a few subtleties associated with measuring H_0. As it turns out, it is very each to measure the redshift z by comparing the spectrum of distance bodies to those in the laboratory. Additionally, Hubble's law is not exact.

$$z = \frac{H_0 r}{c} \pm \delta z = \frac{H_0 r}{c} \pm \frac{v_{\text{pec}}}{c}$$

Where v_{pec} are peculiar velocities that dominate at small distances. Typical peculiar velocities are roughly $v_{\text{pec}} \approx 600\,\text{km}\,\text{s}^{-1}$. Measuring an accurate Hubble's constant H_0 requires making measurements on very large scales.

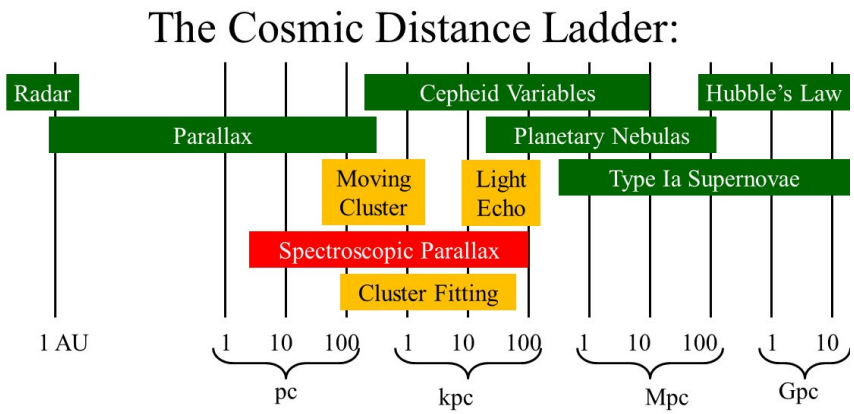
The harder part of measuring Hubble's constant is the measurement of distances r; there is no perfect way of measuring distance. Typically, distances are measured using parallaxes of distance bodies. In the past Hipparcos was the main survey of parallaxes of distance bodies. More recently, the GAIA telescope was launched.

A better technique for measuring distances is to make use of **standard candles**.

$$F = \frac{L_s}{4\pi r^2}$$

Where the luminosity L_s can be determined based on some standard process or model and the flux F is easily measured with instruments near earth. The best standard candles that are used today are variable Cepheids

stars and type 1a supernovae. We say that these processes are standardizable. All of these methods of measuring distance form a cosmic distance ladder that can be used to perform calibrations and consistency checks.



Our most precise measurements today are as follows.

Method	H_0	Source
Supernovae	$73.24(174)\,\mathrm{km/s}$	Riess *et al.* 2016
CMB	$66.93(62)\,\mathrm{km/s}$	Planck 2015

Evidently there is some tension associated with these measurements, namely that it is possible that supernovae are not as standardizable as most scientists assume. It is also possible that there is new physics that we do not understand yet.

As was mentioned previously, some works define h such that,

$$H = 100\,\mathrm{km\,s^{-1}\,Mpc^{-1}} \cdot h$$

Again recalling the Friedmann equation (eq. (2.3)),

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$$

We can define the **critical density** $\rho_c(t)$ to be,

$$\rho_c(t) = \frac{3H^2(t)}{8\pi G}$$

To be the density if $\rho = \rho_c$. Cosmologists define **density parameters**,

$$\Omega_i(t) = \frac{\rho_i(t)}{\rho_c(t)}$$

Where i can be any material in the universe (radiation, matter, dark energy). The density parameters sum to,

$$\Omega_{\mathrm{tot}} = \sum_i \Omega_i = 1 + \frac{k}{a^2 H^2}$$

Which can be used to measure the value of k. If Ω_{tot} is 1, then it must be that $k = 0$. Today we obtain,

$$\rho_c(t_0) = 1.88 \times 10^{-26}\,\mathrm{kg/m^3} \cdot h^2$$

$$= 2.78 \cdot h^{-1} \cdot \frac{1 \times 10^{11} M_\odot}{(h^{-1}\text{Mpc})^3}$$

Each of these density parameters is a function of time $\Omega_i = \Omega_i(t)$. We define for today,

$$\Omega_{i0} = \Omega_i(t_0)$$

Some authors defined Ω_k such that,

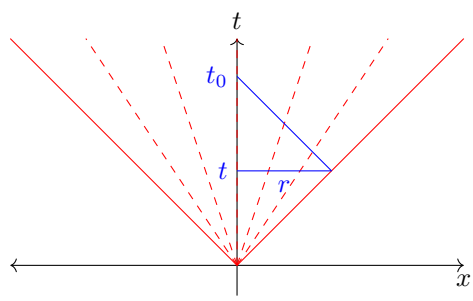
$$\Omega_k = \frac{-k}{a^2 H^2} \implies \Omega_{\text{tot}} + \Omega_k = 1$$

Where observations today indicate,

$$|\Omega_k| \lesssim 0.01 \tag{6.1}$$

7 Densities

Consider a flat spacetime that has expanded since the bigbang.



Due to the geometry of this universe (r is not a co-moving distance),

$$c(t_0 - t) = r \qquad v = \frac{r}{t}$$

Therefore,

$$c\left(t_0 - \frac{r}{v}\right) = r$$

Which gives,

$$v = \frac{r}{t_0 - \frac{r}{c}} \implies \frac{c}{v} = \frac{ct_0}{r} - 1$$

Therefore the redshift can be computed,

$$1 + z = \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}} = \cdots = \left(1 - \frac{2r}{ct_0}\right)^{-1/2} \approx 1 + \frac{r}{ct_0} + \frac{3}{8}\frac{4r^2}{c^2 t_0^2}$$

We define the **deceleration parameter** q,

$$q = -\frac{\ddot{a}}{aH^2} = -\frac{\ddot{a}a}{\dot{a}^2} = \frac{\Omega_m}{2} + \Omega_r - \Omega_\Lambda$$

The deceleration parameter acts as a sub-leading correction to the expansion of the universe. When it was observed that experimentally $q < 0$, one have to conclude that $\Omega_\Lambda > \frac{\Omega_m}{2} + \Omega_r$. In fact in 1999 (Nobel Prize in 2011) dark energy was discovered with value,

$$\Omega_\Lambda \approx 0.7$$

Combining these observations with measurements from the CMB (namely eq. (6.1)), one obtains the density of matter in the universe,

$$\Omega_m \approx 0.3$$

With decomposition,

$$\Omega_{\text{DM}} + \Omega_{\text{Baryon}} = \Omega_m \qquad \Omega_{\text{DM}} \approx 0.26 \qquad \Omega_{\text{Baryon}} \approx 0.04$$

The main distinction between dark matter and dark energy is that dark matter is pressure-less but dark energy has an measurable negative pressure.

What the Heck is Λ? Einstein's equations are as follows,

$$\Lambda g_{\mu\nu} + G_{\mu\nu} = \frac{8\pi G}{c^2} T_{\mu\nu} \tag{7.1}$$

Where $G_{\mu\nu}$ encodes the curvature, $g_{\mu\nu}$ is the metric and $T_{\mu\nu}$ is the energy-momentum tensor. Einstein first introduced the cosmological constant Λ in order to enforce a static universe. Nonetheless, today we appreciate it as the source of our dynamic universe. It is also possible to move $\Lambda g_{\mu\nu}$ to the RHS of eq. (7.1) and treat it as a form of matter,

$$\rho_\Lambda = +\frac{\Lambda}{8\pi G} \qquad P_\Lambda = -\frac{\Lambda c^2}{8\pi G}$$

Where $P_\Lambda = -\rho_\Lambda c^2$ and $\Omega_\Lambda = \frac{\Lambda}{3H^2}$. It is convenient to think of Λ as contributions that are due to the vacuum of empty space. Observationally we measure,

$$\Omega_\Lambda \approx 0.7$$

Nonetheless in Quantum Mechanics, the predicted contributions due to vacuum fluctuations are,

$$\Omega_\Lambda \approx 1 \times 10^{60} - 1 \times 10^{120}$$

This is known as the **cosmological constant problem**. It is important to also understand that,

$$w_Q = -\frac{P_Q}{\rho_Q c^2} = \begin{cases} -1 & \text{for } \Lambda \\ -1.01(5) & \text{Planck 2015} \end{cases}$$

More accurate measurements of w_Q could possibly point to the origin of the cosmological constant problem. Other approaches to fixing the CCP would be to discovering a new theory for gravity.

7.1 Age of The Universe

One of the earliest indications of the presence of Λ was from measuring the age of the universe. From our current measurement of H_0 we have that,

$$t_H = \frac{1}{H_0} = 9.77 h^{-1}\,\text{Gyr} \approx 14\,\text{Gyr}$$

However if $\Omega_m = 1$,

$$a \propto t^{2/3} \implies H_0 = \frac{2}{3t_0} \implies t_0 = \frac{2}{3} t_H \approx 9.3\,\text{Gyr}$$

Which would not have been a problem until measurements on the ages of globular clusters indicated ages of roughly $t \approx 10\,\text{Gyr} - 13\,\text{Gyr}$.

8 Geodesics

So far we have talked about the Friedmann equations and how to solve them for various density contributions. This gives us $a(t)$. Alternatively one could measure distances to galaxies to determine how $a(t)$ changes with time. Unfortunately, the timescales for doing this accurately are very large.

Alternatively, we can make use of the FRW metric,

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2\right]$$

The distance between two points (say A, B) can be obtained by integrating $\mathrm{d}s$,

$$\mathcal{S} = \int_A^B \mathrm{d}s$$

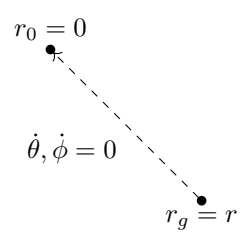
If $\mathcal{S}$ is real, then the distance is space-like and the two locations can not communicate. Alternatively if $\mathcal{S}$ is imaginary then we say that the two locations are time-like separated. Finally, a light ray has null distance.

$\mathcal{S}$	Separation	Communication
Real	Space-Like	Not Possible
Imaginary	Time-Like	Possible
0	Null	–

Geodesics can be found by extremizing $\mathcal{S}$ (i.e. either maximizing or minimizing). In particular, time-like geodesics maximum the time contributions to distance while space-like geodesics minimize $\mathcal{S}$. We define the **proper time** of a geodesic to be τ such that,

$$-c^2\mathrm{d}\tau^2 = \mathrm{d}s^2$$

In cosmology we are interested in the light that travels to us from distant galaxies. Let $r = r_g$ be the distance to a distant galaxy.



Since the light ray has no angular deviation, the FRW metric reduces to,

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + \frac{a^2\mathrm{d}r^2}{1-kr^2}$$

Solving this metric gives,

$$\frac{c\mathrm{d}t}{a(t)} = -\frac{\mathrm{d}r}{\sqrt{1-kr^2}}$$

$$c\int_t^{t_0}\frac{\mathrm{d}t'}{a(t')} = \int_0^r \frac{\mathrm{d}r'}{\sqrt{1-kr^2}} = \begin{cases} r & k=0 \\ \frac{1}{\sqrt{k}}\sin^{-1}\left(\sqrt{k}r\right) & k>0 \\ \frac{1}{\sqrt{-k}}\sinh^{-1}\left(\sqrt{-k}r\right) & k<0 \end{cases}$$

Solving the time integral is impossible without knowing the behaviour of $a(t)$. Recall that for $\Lambda = k = 0$ we have that,

$$a(t) = \left(\frac{t}{t_0}\right)^{2/3}$$

This gives us,

$$r = c\int_t^{t_0}\left(\frac{t'}{t_0}\right)^{-2/3}\mathrm{d}t' = 3ct_0$$

It is important to note that r is not a physical distance. Instead it is just a co-moving coordinate. As such, it should be alarming that $r/t_0 = 3c > c$ seems to be faster than the speed of light.

Next consider a detector that aims to measure light coming from a distance galaxy. Let this detector subtend a solid angle of $\mathrm{d}\Omega$. Then the distance to the galaxy can be measured using,

$$\mathrm{d}_L^2 = \frac{L}{4\pi F}$$

Where L is the luminosity of the object ([E]/[T]) while F is the observed flux on the detector ([E]/[T][Area]). In flat space d_L corresponds to a regular distance. However in a curved spacetime, this is not generally the case. The flux observed can be directly computed from the number of photons emitted.

$$F = \frac{N_\gamma E_{0,\gamma}}{\Delta t_0 a_0^2 r^2 \mathrm{d}\Omega}$$

Where as the number of photons emitted can be determined from the luminosity,

$$N_\gamma = \frac{L\Delta t}{E_\gamma}\frac{\mathrm{d}\Omega}{4\pi}$$

Combining these results yields (and that $\Delta t_0/\Delta t = E_\gamma/E_{0,\gamma} = 1+z$),

$$F = \left(\frac{E_{0,\gamma}}{E_\gamma}\right)\left(\frac{\Delta t}{\Delta t_0}\right)\frac{L}{4\pi(a_0 r)^2}$$

Therefore d_L in general is,

$$d_L = a_0 r(1+z)$$

Relating this to the acceleration parameter,

$$a(t) = a_0 + \dot{a}(t-t_0) + \frac{1}{2}\ddot{a}(t-t_0)^2$$

$$q = -\frac{a\ddot{a}}{\dot{a}^2}$$

Therefore for $k = 0$,

$$r(z) = \int \frac{c\mathrm{d}t}{a(t)}$$

We recover the Hubble law Todo (TC Fraser): This might not be correct,

$$H_0 d_L = z + \left(\frac{1}{2} - q\right)z^2 + \mathcal{O}(z^3)$$

This can be used to measure Λ,

$$q = \frac{\Omega_m}{2} - \Omega_\Lambda$$

Alternatively we can measure distances via angular size,

$$d_A = ar = \frac{r}{1+z}$$

There are two ways of measuring distances in cosmology d_A, d_L.