# Phys 476

## GENERAL RELATIVITY

### University of Waterloo

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Latest versions of all my course notes are available at **www.tcfraser.com**.

# 1 Introduction

## 1.1 History

The first lecture was a summary of astrophsyical history from around 200BC to today. I elected note to take notes as it was pretty standard stuff and a lot of slides.

# 2 Tensor Formalism

Let $V$ be a vector space of finite dimension. Any $V$ is iosmorphic to $\mathbb{R}^{n+1}$ through the coefficients of a chosen basis. Let the basis of $V$ be given by,

$$\{e_i\}_{i=0,\ldots,n}$$

Then any vector $v \in V$ is expressible by,

$$v = \sum_{i=0}^{n} v^i e_i$$

Where $v^i$ are the $i$-th coefficients of the vector $v$ with respect to the basis $\{e_i\}$.

## 2.1 Einstein Summation Rule

For convience let's provide a new, shorter notation for the vector $v$.

$$v^i e_i = v^0 e_0 + \ldots + v^n e_n = \sum_{i=0}^{n} v^i e_i$$

Effectively, we have just **dropped the summation sign**. The einstein summation rule is as follows:

If there are two identical indicies, 1 "up" and 1 "down", it means that a summation is secretly present, it's just be removed for convience. Note that the $i$ in this case is *dummy index*.

$$v^i e_i = v^\alpha e_\alpha = v^j e_j$$

Here $v^i$ are the components of vector $v \in V$ and are real numbers. $v^i \in \mathbb{R}, \forall i \in \{0, \ldots, n\}$.

Note $v^i$ is called the vector $v$ when $i$ is the set $\{0, \ldots, n\}$, but can also be called the $i$-th component of $v$ when $i$ has a fixed value $i \in \{0, \ldots, n\}$.

## 2.2 Examples of Basis for V

The values of $e_i$ or the $i$'s themselves can take on many possible values.

- cartesian coordinates $t, x, y, z$
- spherical coordinates $t, r, \phi, \theta$
- etc.

Each of the above examples is the space $V = \mathbb{R}^4$ (with some bounds for spherical coordinates).

## 2.3 Dual Vector Space

The dual vector space of $V$ denoted $V^*$ is also iosmorphic to $\mathbb{R}^{n+1}$ and is built from the space of linear forms on V.

$$V^* = \{w : V \to \mathbb{R} \mid w(\alpha v_1 + \beta v_2) = \alpha w(v_1) + \beta w(v_2)\}$$

where $v_1, v_2 \in V$ and $\alpha, \beta \in \mathbb{R}$.
In Quantum Mechanics, the vectors are the bras and the elements of the dual space (called the covectors) are the kets.

We note,

$$\left\{f^i\right\}_{i=0,\dots,n}$$

is the basis for $V^*$ is defined by the kronecker symbol $\delta$,

$$f^j(e_j) = \delta^j{}_i$$

$$\delta^j{}_i = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

An element in $V^*$ is $w = w_i f^i$. $w_i$ are the components of the covector $w$. Note that for a **finite dimensional vector space**,

$$V^{**} = V$$

## 2.4 Bilinear Maps

Introduce a bilinear map $B(v, w)$ where $B : V \times V \to \mathbb{R}$ where,

$$B(\alpha v_1 + \beta v_2, w) = \alpha B(v_1, w) + \beta B(v_2, w)$$

and the same for the other parameter $w$.

Examples include the inner product (otherwise known as the scale or dot product).
Bilinear forms are bilinear maps such that the following conditions are true:

- symmetric: $B(v, w) = B(w, v)$
- non-degenerated: $B(v, w) = 0 \quad \forall v \implies w = 0$

Playing with indicies,

$$\begin{aligned} B(v, w) &= B(v^\alpha e_\alpha, w^\beta e_\beta) \\ &= v^\alpha B(e_\alpha, w^\beta e_\beta) \quad \text{By linearity} \\ &= v^\alpha w^\beta B(e_\alpha, e_\beta) \quad \text{By linearity} \end{aligned}$$

A bilinear map used in this way provides a way to eliminate the headache of complicated cross sums. Define new notation,

$$B(e_\alpha, e_\beta) \equiv g_{\alpha\beta}$$

Where $g_{\alpha\beta}$ is a real number $\mathbb{R}$ whenever $\alpha$ and $\beta$ are fixed.

$$B(v, w) = v^\alpha w^\beta g_{\alpha\beta} = v^\alpha g_{\alpha\beta} w^\beta = w^\beta g_{\alpha\beta} v^\alpha$$

All of the above terms are commutative because in the end, it represents a sum over all $\alpha, \beta$.

$$B(v,w) = \underbrace{v^0 w^0 g_{00} + \ldots + v^2 w^3 g_{2,3} + \ldots + v^n w^n g_{nn}}_{(n+1)^2 \text{terms}}$$

## 2.5 Distance and Norms

To define a distance in a vector space, we can use norms. In this case, $g_{\alpha\beta}$ would be called the metric. The Euclidean metric (with respect to a cartesian basis) for example would be,

$$g_{\alpha\beta} = \begin{cases} 1 & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases}$$

We can also choose to enforce that the basis be orthonormal,

$$B(e_i, e_j) = \begin{cases} \pm 1 & i = j \\ 0 & i \neq j \end{cases}$$

Note that the potential for a negative norm means the notion of positive definiteness is no longer gauranteed.

## 2.6 Signatures of Metrics

We call the signature of the metric the number of $+1$'s and $-1$'s appearing in $g_{ij}$ when dealing with the orthonormal basis. Signature is denoted as:

$$(p,q) = \left( \underbrace{p}_{\text{postive}}, \underbrace{q}_{\text{negative}} \right)$$

For example,

- Euclidean metric: $(n+1, 0)$
- Minkowski metric: $(n, 1)$

Note the order of the signature is chosen to be $(p,q)$ and not $(q,p)$ by convention.

## 2.7 Covectors from Vectors

Note that $v^i$ was called the vector and $w_i$ was called the covector. This notation seems to indicate that conversion between $V$ and $V^*$ is notationally equivalent to raising and lowering the indicies.

We call the following opperation "Lowing the index using the metric".

$$\underbrace{v^\alpha}_{\text{components of vector}} \mapsto g_{\alpha\beta} v^\beta = \underbrace{v_\alpha}_{\text{components of covector}}$$

In use,

$$B(v,w) = v^\alpha g_{\alpha\beta} w^\beta = \underbrace{v_\beta}_{\text{bra}} \underbrace{w^\beta}_{\text{ket}}$$