Phys 476

GENERAL RELATIVITY

University of Waterloo

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Version: 1.0

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Latest versions of all my course notes are available at **www.tcfraser.com**.

1 Introduction

1.1 History

The first lecture was a summary of astrophsyical history from around ∼200BC to today. I elected not to take notes as it was pretty standard stuff and a lot of slides. Sorry.

2 Tensor Formalism

At the core of General Relativity is the mathematics of differential geometry. Differential geometry requires the idea of tensors, a generalization of vectors and matricies and forms that can handle messy geometries and metrics.

Let V be a vector space of finite dimension. Any V is isomorphic to $\mathbb{R}^{n+1}$ through the coefficients of a chosen basis. Let the basis of V be given by,

$$\{e_i\}_{i=0,\ldots,n}$$

Then any vector $v \in V$ is expressible by,

$$v = \sum_{i=0}^{n} v^i e_i$$

Where v^i are the i-th coefficients of the vector v with respect to the basis $\{e_i\}$.

2.1 Einstein Summation Rule

For convenience let's provide a new, shorter notation for the vector v.

$$v^i e_i = v^0 e_0 + \ldots + v^n e_n = \sum_{i=0}^{n} v^i e_i$$

Effectively, we have just **dropped the summation sign**. The einstein summation rule is as follows:

If there are two identical indices, 1 "up" and 1 "down", it means that a summation is secretly present, it's just be removed for convenience. Note that the i in this case is *dummy index*.

$$v^i e_i = v^\alpha e_\alpha = v^j e_j$$

Here v^i are the components of vector $v \in V$ and are real numbers. $v^i \in \mathbb{R}, \forall i \in \{0, \ldots, n\}$.

Note v^i is called the vector v when i is the set $\{0, \ldots, n\}$, but can also be called the i-th component of v when i has a fixed value $i \in \{0, \ldots, n\}$.

2.2 Examples of Basis for V

The values of e_i or the i's themselves can take on many possible values.

- Cartesian coordinates t, x, y, z
- spherical coordinates t, r, ϕ, θ
- etc.

Each of the above examples is the space $V = \mathbb{R}^4$ (with some bounds for spherical coordinates).

2.3 Dual Vector Space

The dual vector space of V denoted V^* is also isomorphic to $\mathbb{R}^{n+1}$ and is built from the space of linear forms on V.

$$V^* = \{w : V \to \mathbb{R} \mid w(\alpha v_1 + \beta v_2) = \alpha w(v_1) + \beta w(v_2)\}$$

where $v_1, v_2 \in V$ and $\alpha, \beta \in \mathbb{R}$.
In Quantum Mechanics, the vectors are the bras and the elements of the dual space (called the co-vectors) are the kets.

We note,

$$\left\{f^i\right\}_{i=0,\ldots,n}$$

is the basis for V^* is defined by the kronecker symbol δ,

$$f^j(e_j) = \delta^j{}_i$$

$$\delta^j{}_i = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

An element in V^* is $w = w_i f^i$. w_i are the components of the covector w. Note that for a **finite dimensional vector space**,

$$V^{**} = V$$

2.4 Bilinear Maps

Introduce a bilinear map $B(v, w)$ where $B : V \times V \to \mathbb{R}$ where,

$$B(\alpha v_1 + \beta v_2, w) = \alpha B(v_1, w) + \beta B(v_2, w)$$

and the same for the other parameter w.

Examples include the inner product (otherwise known as the scale or dot product).
Bilinear forms are bilinear maps such that the following conditions are true:

- symmetric: $B(v, w) = B(w, v)$
- non-degenerated: $B(v, w) = 0 \quad \forall v \implies w = 0$

Playing with indices,

$$\begin{aligned} B(v, w) &= B(v^\alpha e_\alpha, w^\beta e_\beta) \\ &= v^\alpha B(e_\alpha, w^\beta e_\beta) \quad \text{By linearity} \\ &= v^\alpha w^\beta B(e_\alpha, e_\beta) \quad \text{By linearity} \end{aligned}$$

A bilinear map used in this way provides a way to eliminate the headache of complicated cross sums. Define new notation,

$$B(e_\alpha, e_\beta) \equiv g_{\alpha\beta}$$

Where $g_{\alpha\beta}$ is a real number $\mathbb{R}$ because α and β are summed over.

$$B(v, w) = v^\alpha w^\beta g_{\alpha\beta} = v^\alpha g_{\alpha\beta} w^\beta = w^\beta g_{\alpha\beta} v^\alpha$$

All of the above terms are commutative because in the end, it represents a sum over all α, β.

$$B(v,w) = \underbrace{v^0 w^0 g_{00} + \ldots + v^2 w^3 g_{2,3} + \ldots + v^n w^n g_{nn}}_{(n+1)^2 \text{terms}}$$

2.5 Distance and Norms

To define a distance in a vector space, we can use norms. In this case, $g_{\alpha\beta}$ would be called the metric. The Euclidean metric (with respect to a cartesian basis) for example would be,

$$g_{\alpha\beta} = \begin{cases} 1 & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases}$$

We can also choose to enforce that the basis be orthonormal,

$$B(e_i, e_j) = \begin{cases} \pm 1 & i = j \\ 0 & i \neq j \end{cases}$$

Note that the potential for a negative norm means the notion of positive definiteness is no longer gauranteed.

2.6 Signatures of Metrics

We call the signature of the metric the number of $+1$'s and -1's appearing in g_{ij} when dealing with the orthonormal basis. Signature is denoted as:

$$(p,q) = \left(\underbrace{p}_{\text{postive}}, \underbrace{q}_{\text{negative}} \right)$$

For example,

- Euclidean metric: $(n+1, 0)$
- Minkowski metric: $(n, 1)$

Note the order of the signature is chosen to be (p,q) and not (q,p) by convention.

2.7 Co-vectors from Vectors

Note that v^i was called the vector and w_i was called the covector. This notation seems to indicate that conversion between V and V^* is notationally equivalent to raising and lowering the indices.

We call the following opperation “Lowering the index using the metric”.

$$\underbrace{v^\alpha}_{\text{components of vector}} \mapsto g_{\alpha\beta} v^\beta = \underbrace{v_\alpha}_{\text{components of covector}}$$

In use,

$$B(v,w) = v^\alpha g_{\alpha\beta} w^\beta = \underbrace{v_\beta}_{\text{bra}} \underbrace{w^\beta}_{\text{ket}}$$

2.8 Linear Map on V to V

$$M : V \to V$$

Where M is a matrix. An the map is equivalent to $v \to Mv \in V$. Some definition,

$$(Mv)^\alpha = \underbrace{M^\alpha{}_\beta}_{\text{Matrix(components)}} v^\beta$$

Note that $M^\alpha{}_\beta \in \mathbb{R}$ for α and β fixed. Example: The identity matrix is denoted $\delta^\alpha{}_\beta = \mathbb{I}$.

2.9 Scalar Product on Dual Space

Introduce a scalar product for the co-vectors w.

$$w, t \in V^*$$

$$w \cdot t = w_\alpha h^{\alpha\beta} t_\beta$$

Where $h^{\alpha\beta}$ is symmetric and non-degenerate.
So how is the scalar product between the dual and normal space related? Specifically how are $g_{\alpha\beta}$ and $h^{\alpha\beta}$ connected? Well,

$$\begin{aligned} v^\alpha g_{\alpha\beta} w^\beta &= v^\alpha w_\alpha \\ &= v_\gamma h^{\gamma\alpha} w_\alpha \\ &= v^\nu g_{\nu\gamma} h^{\gamma\alpha} w_\alpha \\ &= v^\nu g_{\nu\gamma} h^{\gamma\alpha} g_{\alpha\mu} w^\mu \end{aligned}$$

Since this is true for any v and w we require that,

$$h^{\gamma\alpha} g_{\alpha\mu} = \delta^\gamma{}_\mu$$

This means we say that the metric h is the inverse of the metric g. Convention on V^*: we denote the metric $g^{\alpha\beta}$ (the indices are "up").

2.10 Invariance of Scalar Product

Let us say we have a matrix $M : v \to \tilde{v} = Mv, w \to \tilde{w} = Mw$ and that M preserves the scalar product.

$$\tilde{v} \cdot \tilde{w} = v \cdot w \qquad \forall v, w$$

Examine,

$$M^\gamma{}_\alpha v^\alpha g_{\alpha\beta} M^\beta{}_\rho w^\rho = v^\alpha g_{\alpha\beta} w^\beta$$

Use communitivity and dummyness of indices to obtain,

$$v^\alpha M^\gamma{}_\alpha g_{\alpha\rho} M^\rho{}_\beta w^\beta = v^\alpha g_{\alpha\beta} w^\beta$$

Drop outer co-vectors v and w to get,

$$M^\gamma{}_\alpha g_{\alpha\rho} M^\rho{}_\beta = g_{\alpha\beta} \tag{2.1}$$

Note that this expression is consistent with the Einstein summation convention.

An example of an M on euclidean space could be a rotation matrix, or the identity.

When M satisfies 2.1, it is said to be orthogonal. It $\det(M) = 1$ then we say that M is *special*.

2.11 Trace of M

What is the trace of M?

$$\text{Tr}(M) = M^{\alpha}{}_{\alpha} = M^{0}{}_{0} + \ldots + M^{n}{}_{n}$$

This is just a notationally convention. It is the sum of the diagonal terms of M.

2.12 Tensor Product

A tensor product makes a linear map a multi-linear map.

Theorem:
Let E and F be 2 vector spaces (with finite dimensionality.)
$\exists$ a unique (!) set (up to isomorphism) $E \otimes F$ such that if f is a bilinear map $f : E \times F \to \mathbb{R}$ then $\exists$ a linear map $f^* : E \otimes F \to \mathbb{R}$ such that $f = f^* \circ \phi$ with

$$\begin{tikzcd} E \times F \arrow[d, "\phi"'] \arrow[dr, "f"] & \\ E \otimes F \arrow[r, "f^*"'] & \mathbb{R} \end{tikzcd}$$

Then we have,

$$\text{Lin}(E \otimes F, \mathbb{R}) \cong \text{Bin}(E \times F, \mathbb{R})$$
$$\text{Lin}(f^*, \mathbb{R}) \cong \text{Bin}(f, \mathbb{R})$$

where '$\cong$' is used to denote *isomorphic*.

Properties:
Basis for $E \otimes F$ is $e_\alpha \otimes g_\alpha$ where e_α is the basis for E and g_α is the basis for F. For $a \in \mathbb{R}$ and $t, v \in E$, $u, w \in F$,

- $\dim(E \otimes F) = \dim(E) \dim(F)$
- $a(v \otimes w) = (av) \otimes w = v \otimes (aw)$
- $(v + t) \otimes w = v \otimes w + t \otimes w$
- $v \otimes (w + u) = v \otimes w + v \otimes u$
- $a \otimes w = aw$
- $\mathbb{R} \otimes F = F$

Note that $V^* \otimes V^* \cong \text{Bin}(V \times V, \mathbb{R})$. To motivate this, let $f^\alpha \otimes f^\beta$ be the basis for $V^* \otimes V^*$, and then a general element in $V^* \otimes V^*$ is,

$$t = t_{\alpha\beta} f^\alpha \otimes f^\beta$$

Note that $t_{\alpha\beta}$ is just a set of numbers. Then the tensor product is expanded as follows,

$$\begin{aligned} t(v \otimes w) &= t(v^\alpha e_\alpha \otimes w^\beta e_\beta) \\ &= t_{\gamma\delta}(f^\gamma \otimes f^\delta)(v^\alpha e_\alpha \otimes w^\beta e_\beta) \\ &= t_{\gamma\delta} v^\alpha w^\beta (f^\gamma \otimes f^\delta)(e_\alpha \otimes e_\beta) \quad \text{By linearity} \end{aligned}$$

$$
\begin{aligned}
&= t_{\gamma\delta} v^\alpha w^\beta f^\gamma(e_\alpha) f^\delta(e_\beta) \quad \text{By foiling and definition of } f \\
&= t_{\gamma\delta} v^\alpha w^\beta \delta^\gamma \alpha \delta^\delta{}_\beta \\
&= t_{\gamma\delta} v^\gamma w^\beta \delta^\delta{}_\beta \quad \text{By sifting property of } \delta \\
&= t_{\gamma\delta} v^\gamma w^\delta \quad \text{By sifting property of } \delta \text{ again}
\end{aligned}
$$

Since $t(v \otimes w)$ is the tensor product $V^* \otimes V^*$ and $t_{\gamma\delta}$ is the components of the bilinear form, one can see the connection $V^* \otimes V^* \cong \mathrm{Bin}(V \times V, \mathbb{R})$.

Tensors allow one to write bilinear maps as linear maps. What about multi-linear maps?

Tensors:
A tensor of rank (k, l) is a multilinear map

$$\underbrace{V^* \times \cdots \times V^*}_{k} \times \underbrace{V \times \cdots \times V}_{l} \to \mathbb{R}$$

which transforms *well* under the change of basis of V and V^*.

Tensor	Rank
vectors	$(1,0)$
co-vectors	$(0,1)$
scalar	$(0,0)$
metric	$(0,2)$
inverse metric	$(2,0)$
matrix	$(1,1)$

The set of tensors of rank (k, l) is a vector space of dimension n^{k+l} (if V has dimension n). Checking with the examples above motivates this fact.

Using the basis $e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_k} \otimes f^{\beta_1} \otimes \cdots \otimes f^{\beta_k}$

$$T = T^{\alpha_1 \alpha_2 \cdots \alpha_k}{}_{\beta_1 \beta_2 \cdots \beta_l} e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_k} \otimes f^{\beta_1} \otimes \cdots \otimes f^{\beta_k}$$

For fixed α_i and β_i this is a real number in $\mathbb{R}$. These are the *components of the tensor*.

By abuse of notation we will call $T^{\alpha_1 \alpha_2 \cdots \alpha_k}{}_{\beta_1 \beta_2 \cdots \beta_l}$ the tensor.

We are talking about these transformations as change of basis of V and V^*. Examples:

- rotations (boost)
- change of coordinates from Cartesian to spherical, cylindrical, etc.

We can have a linear change of basis $\tilde{x}^\mu = A^\mu{}_\nu x^\nu$.

Example:

Cartesian	Polar
$e_1 = \vec{i}$	$\tilde{e}_1 = e_r$
$e_2 = \vec{j}$	$\tilde{e}_2 = e_\theta$

Example:

$$\tilde{e}_\alpha = \underbrace{\frac{\partial x^\nu}{\partial \tilde{x}^\alpha}}_{\text{Jacobian}} e_\nu = A^\nu{}_\alpha e_\nu$$

Note: *Up in the denominator means down on the original coordinates (LHS).*
For example,

$$\begin{array}{r|l} x^1 = x & \tilde{x}^1 = r \\ x^2 = y & \tilde{x}^2 = \theta \end{array}$$

$$\tilde{e}_1 = e_r = \frac{\partial x^1}{\partial \tilde{x}^1} e_1 + \frac{\partial x^2}{\partial \tilde{x}^1} e_2 = \cos\theta e_1 + \sin\theta e_1$$
$$\tilde{e}_2 = e_\theta = \frac{\partial x^1}{\partial \tilde{x}^2} e_1 + \frac{\partial x^2}{\partial \tilde{x}^2} e_2 = -r\sin\theta e_1 + r\cos\theta e_1$$

Vectors in multiple basis:

$$v = v^\nu e_\nu = \tilde{v}^\nu \tilde{e}_\nu$$

With conversion of basis given by,

$$\tilde{e}_\alpha = A^\nu{}_\alpha e_\nu$$

Thus substituting in,

$$v^\nu e_\nu = \tilde{v}^\alpha A^\nu{}_\alpha e_\nu \quad \text{Drop } e_\nu$$

$$v^\nu = \tilde{v}^\alpha A^\nu{}_\alpha$$

But with A as a Jacobian,

$$v^\nu = \frac{\partial x^\nu}{\partial \tilde{x}^\alpha} \tilde{v}^\alpha$$
$$\tilde{v}^\alpha = \frac{\partial \tilde{x}^\alpha}{\partial x^\nu} v^\nu$$

But what about the dual space?
By definition,

$$\tilde{f}^\beta(\tilde{e}_\nu) = \delta^\beta_\mu = \tilde{f}^\beta(A^\alpha{}_\nu e_\alpha) = A^\alpha{}_\nu \tilde{f}^\beta(e_\alpha)$$

Let $\tilde{f}^\beta(e_\alpha)$ be expressed as $\tilde{f}^\beta = B^\beta{}_\gamma f^\gamma$

$$\begin{aligned} \tilde{f}^\beta(\tilde{e}_\nu) &= A^\alpha{}_\nu B^\beta{}_\gamma f^\gamma(e_\alpha) \\ &= A^\alpha{}_\nu B^\beta{}_\gamma \delta^\gamma{}_\alpha \\ &= B^\beta{}_\gamma A^\gamma{}_\nu \\ &= \delta^\beta{}_\nu \end{aligned}$$

Thus B is the inverse of A.

What does transforming *well* mean? A tensor is transforming well if its components transform as

$$T^{\nu_1\nu_2\cdots\nu_k}{}_{\alpha_1\alpha_2\cdots\alpha_l} \to \frac{\partial \tilde{x}^{\nu_1}}{\partial x^{\beta_1}} \cdots \frac{\partial \tilde{x}^{\nu_k}}{\partial x^{\beta_k}} \frac{\partial x^{\gamma_1}}{\partial \tilde{x}^{\alpha_1}} \cdots \frac{\partial x^{\gamma_k}}{\partial \tilde{x}^{\alpha_k}} T^{\beta_1\beta_2\cdots\beta_k}{}_{\gamma_1\gamma_2\cdots\gamma_l} = \tilde{T}^{\nu_1\nu_2\cdots\nu_k}{}_{\alpha_1\alpha_2\cdots\alpha_l}$$

If you find something like $T^\alpha{}_\beta$, is it a tensor? **No! You must check if it transforms well.**

$$\frac{\partial}{\partial x^\nu} v^\alpha \quad \text{This is not a tensor.}$$

The derivative here prevents it from being well-formed. In the future we will define a derivative that allows a tensor to transform well.

2.13 Operations on Tensors

- Add (with matching rank): $T^{\alpha_1\alpha_2}{}_{\beta_1\beta_2} + C^{\alpha_1\alpha_2}{}_{\beta_1\beta_2}$.
- Contraction (partial trace): $\mathcal{T}(k,k) \to \mathcal{T}(k-1,k-1)$.
  - $T^{\alpha_1\cdots\alpha_i\cdots\alpha_k}{}_{\beta_1\cdots\beta_j\cdots\beta_l} \to T^{\alpha_1\cdots\alpha_i\cdots\alpha_k}{}_{\beta_1\cdots\alpha_j\cdots\beta_l}$
- "Outer" Product (Gluing together tensors)
  - $\mathcal{T}(k,l) \times \mathcal{T}(k',l') \to \mathcal{T}(k+k', l+l')$
  - $(T_1, T_2) \to T_1T_2$
  - $T_1T_2 \to T_1^{\nu_1\cdots\nu_k}{}_{\alpha_1\cdots\alpha_l} T_2^{\beta_1\cdots\beta_k}{}_{\gamma_1\cdots\gamma_l}$
  - **Example:** $(v^\alpha, w_\beta) \to v^\alpha \otimes w_\beta = v^\alpha w_\beta$. (In QM this is $|\phi\rangle\langle\varphi|$)

The metric $g_{\alpha\beta}$ can change the rank of a tensor. Recall a metric is rank $(0,2)$ is symmetric and is non-degenerate.

Example:
Changing from rank $(1,0)$ to rank $(0,1)$:

$$v^\alpha \to v_a = g_{\alpha\beta}v^\beta$$

Changing from rank $(2,2)$ to rank $(4,0)$:

$$C^{\alpha\beta}{}_{\gamma\delta} \to C_{\alpha\beta\gamma\delta} = g_{\alpha\rho}g_{\beta\eta}C^{\rho\eta}{}_{\gamma\delta}$$

Changing from rank $(2,2)$ to a different rank $(2,2)$:

$$C^{\alpha\beta}{}_{\gamma\delta} \to C^{\alpha}{}_{\beta}{}^{\gamma}{}_{\delta} = g_{\beta\rho}g^{\gamma\eta}C^{\alpha\rho}{}_{\eta\delta}$$

2.14 Facts About Tensors

Order Matters:
The order of indices that label a tensor is **very** important. It indicates the product space you are mapping *from* to $\mathbb{R}$.

$$\begin{aligned} C^{\alpha}{}_{\beta} : &\quad V^* \times V \to \mathbb{R} \\ C_{\alpha}{}^{\beta} : &\quad V \times V^* \to \mathbb{R} \\ C^{\beta}_{\alpha} : &\quad \text{Nothing. Don't do this.} \end{aligned}$$

Equality between tensors:
As tensors, indices must match:

Position of indices is matching: $C^{\alpha}{}_{\gamma}{}^{\delta} = T^{\alpha}{}_{\gamma}{}^{\delta}$

Position of indices is **not** matching: $C^{\alpha}{}_{\gamma}{}^{\delta} \neq T^{\alpha}{}_{\gamma\delta}$

But for fixed α, γ, δ, one can abuse the notation a bit:

$$C^{\alpha}{}_{\gamma}{}^{\delta} = T^{\alpha}{}_{\gamma\delta} \quad \text{Try to avoid this.}$$

2.15 Outer Product and Contraction

Example:
Outer Product: $M^{\alpha}{}_{\beta}M^{\gamma}{}_{\delta} = C^{\alpha}{}_{\beta}{}^{\gamma}{}_{\delta}$
Contraction: $M^{\alpha}{}_{\beta}M^{\beta}{}_{\delta} = C^{\alpha}{}_{\beta}{}^{\beta}{}_{\delta} = C^{\alpha}{}_{\gamma}$

Example:
Outer product and contraction: $C^{\alpha\beta}{}_{\gamma\delta}T^{\gamma\delta}{}_{\rho} = A^{\alpha\beta}{}_{\rho}$
This doesn't make sense: $C^{\alpha\beta}{}_{\gamma\gamma}T^{\gamma\delta}{}_{\rho} =??$

Note, when there is a "+" sign we can be "loose" with the indices. Here the dual indices **do not** indicate a summation. This acts as an abuse of notation, but is sometimes difficult to avoid.

$$C^{\alpha\gamma}T_{\gamma}{}^{\delta} + F_{\gamma}{}^{\delta}A^{\alpha\gamma}$$

2.16 Interpretation of Tensors

By looking at the indices, how can we interpret the physical meaning of the tensor object?

Tensor	Interpretation
v^{ν}	vector
v_{ν}	covector
$M^{\alpha}{}_{\beta}$	matrix (α rows, β columns)
$M^{\alpha}{}_{\alpha}$	contracted matrix (trace)
$M^{\alpha\gamma}{}_{\delta}$	matrix whose elements are vectors themselves ($\cdot^{\gamma}{}_{\delta}$ is the matrix)
$M^{\alpha\gamma}{}_{\delta}$	vector with matrix components (M^{α} is the vector)
$R^{\alpha\beta}{}_{\gamma\delta}$	matrix of matricies *

*For example, if $\dim V = 4$, $R^{\alpha\beta}{}_{\gamma\delta}$ has $4^4 = 256$ components. Note however, there can be many symmetries that reduce the number of unique components.

2.17 Symmetry of Tensor

We can always build a symmetric and antisymmetric part of a tensor $T^{\alpha\beta}$. Let's look at the case of 2 indices α, β:

Symmetric Part:

$$T_{(\alpha\beta)} = \frac{1}{2}\left(T_{\alpha\beta} + T_{\beta\alpha}\right)$$
$$T_{(\alpha\beta)} = T_{(\beta\alpha)}$$

Antisymmetric Part:

$$T_{[\alpha\beta]} = \frac{1}{2}\left(T_{\alpha\beta} - T_{\beta\alpha}\right)$$
$$T_{[\alpha\beta]} = -T_{[\beta\alpha]}$$

Note that for all tensors $T^{\alpha\beta} = T^{(\alpha\beta)} + T^{[\alpha\beta]}$. This acts as the decomposition into odd and even symmetries of the tensor.

For more indices:

$$T^{(\alpha\beta)}{}_{[\gamma\delta]} = \frac{1}{4}\left(T^{\alpha\beta}{}_{\gamma\delta} + T^{\beta\alpha}{}_{\gamma\delta} - T^{\alpha\beta}{}_{\delta\gamma} - T^{\beta\alpha}{}_{\delta\gamma}\right)$$

What does $T^{(\alpha\beta\gamma)}$ mean? For that we will need a permutation group.

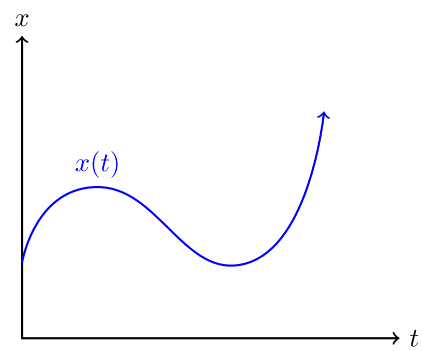
3 Physics Review

Moving away from tensors for a moment...

3.1 Newtonian Physics

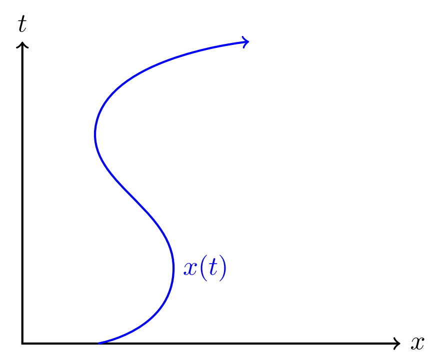
According to Galileo and Newton, we got the interpretation that both space and time is flat $\left(\mathbb{R}^3\right)$ and is absolute. More specifically, all clocks will have the same time if they are started/synced at some shared moment. It is built on cartesian coordinate system: $(\vec{x}, t)$. With this we say that an object is at position $\vec{x}$ at time t. In this context, coordinates are *outcomes of measurements*. In General Relativity, the notion of coordinates can be quite different.

Consider a particle (2d spacetime):



Typically, x is drawn as the ordinate (y-axis) and t as the abscissa (x-axis).

Spacetime diagram:
In a spacetime diagram, t is drawn as the ordinate.



If we begin to use light to probe the position of objects, we are going to run into some surprising results. We will have to abandon Newtonian Physics and switch to the domain of Special Relativity.

3.1.1 Newton's Dynamical Law & Inertial Observers

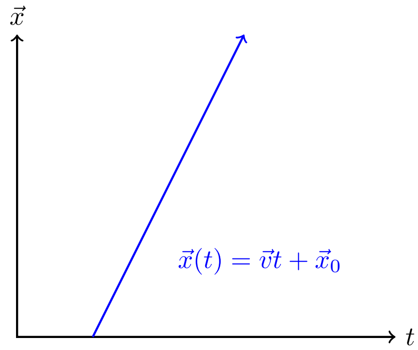
$$\vec{F} = m\vec{a}$$

Where $\vec{F}$ is the total force applied to the system, $\vec{a} = \ddot{\vec{x}} = \frac{\mathrm{d}^2\vec{x}}{\mathrm{d}t^2}$ and m is the inertial mass. For $\vec{x}$ is it convienent to use the Cartesian coordinate system.

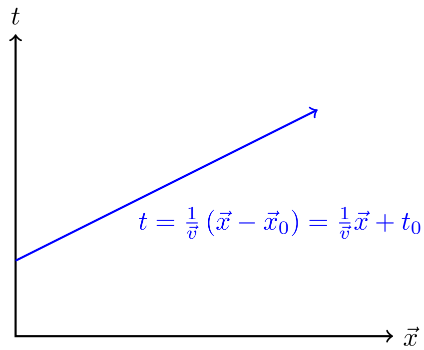
If $\vec{F} = \vec{0}$ then the dynamics becomes $\ddot{\vec{x}} = 0$ which yields solution,

$$\vec{x}(t) = \vec{v}t + \vec{x}_0$$

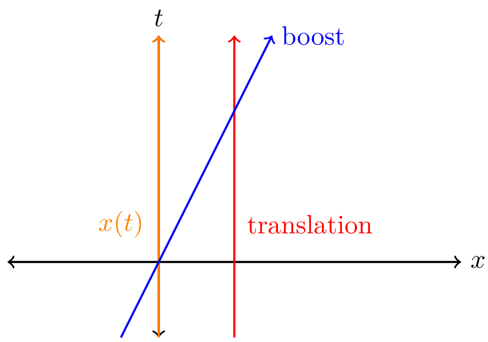
Where $\vec{x}_0$ is the initial condition and $\vec{v}$ is the velocity in the observer's frame. This solution describes a straight line.



However, consider this solution in a spacetime diagram,



Definition:
The class of frames (observers) for which the the dynamics of a system is $\ddot{\vec{x}} = \vec{0}$ are called an inertial observers.



Note that rotations are not visible in this diagram as there is only one 1 space dimension.
Transformations that relate inertial observers:

- translation: $\vec{x}(t) \to \vec{x}'(t) = \vec{x}(t) + \vec{a}$
- rotation: $\vec{x}(t) \to \vec{x}'(t) = R \cdot \vec{x}$
- Galilean boost: $\vec{x}(t) \to \vec{x}'(t) = -\vec{v}t + \vec{x}(t)$

For each of these transformations $\ddot{\vec{x}}' = \vec{0}$. We will now prove the set of all these transformation of $\vec{x}(t) \to \vec{x}'(t)$ form a group.

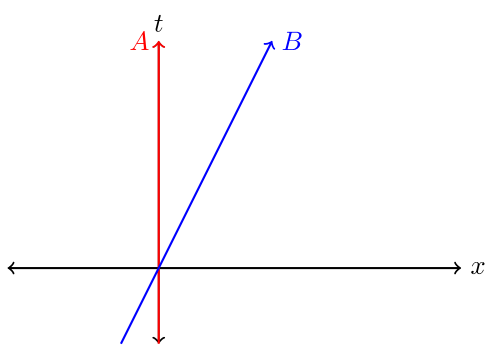
Groups:
A Group is a set G equipped with an associated product $(\cdot)$, a unit element and an inverse.

- $g_1 \cdot g_2 = g \in G, g_i \in G$
- $g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$
- $g \cdot 1 = 1 \cdot g = g$
- $g \cdot g^{-1} = g^{-1} \cdot g = 1$
- In general, $g_1 \cdot g_2 \neq g_2 \cdot g_1$.
- An abelian group is one where $g_1 \cdot g_2 = g_2 \cdot g_1$.

Upon careful examinations of translations and rotation of space $\mathbb{R}^n$, both translations and rotations form a group. What about Galilean boosts?

Consider person A (Alice) standing on the ground and person B (Bob) in a rocket traveling with velocity $\vec{v}_1$ with respect to B. Give person B a ball in the rocket and let him/her kick it with velocity $\vec{v}_2$ with respect to B. What is the velocity of the ball with respect to person A? Switching between the perspectives of the system is equivalent to performing a Galilean Boost.

Matrix Representation of a Group:



The boost $B^{\alpha}{}_{\gamma}$ is given by,

$$B^{\alpha}{}_{\gamma} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -v_1 & 1 & 0 & 0 \\ -v_2 & 0 & 1 & 0 \\ -v_3 & 0 & 0 & 1 \end{bmatrix}$$

Person A (sitting on the ground) is given by,

$$A \sim \begin{bmatrix} t \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Their product is given by,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -v_1 & 1 & 0 & 0 \\ -v_2 & 0 & 1 & 0 \\ -v_3 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ -v_1 t \\ -v_2 t \\ -v_3 t \end{bmatrix}$$

This is the trajectory of Alice with respect to Bob parametrized with time t. What about Bob's perspective under this linear map?

$$\begin{bmatrix} t \\ v_1 t \\ v_2 t \\ v_3 t \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -v_1 & 1 & 0 & 0 \\ -v_2 & 0 & 1 & 0 \\ -v_3 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} t \\ -v_1 t + v_1 t \\ -v_2 t + v_2 t \\ -v_3 t + v_3 t \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus Galilean boosts form an abelian group. When we move to the regime of Special Relativity, we will see that boosts no longer form an abelian group.

3.2 The Relativity Principle

Two inertial observers moving with constant velocity cannot be distinguished by any physical experiment.

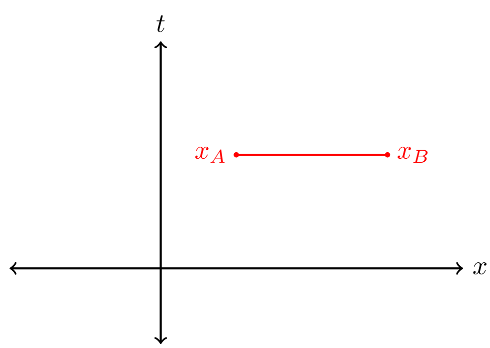
Or alternatively,

Inertial frames are equivalent in terms of the description of physical phenomena.

This is most easily observed when sitting on a train next to another train. When your train is moving, it is unclear whether or not your train is moving or the other one is.

Inertial frames are systems with $\ddot{\vec{x}} = \vec{0}$ equipped with rods and clocks for measurements.

What happens to the notion of spatial length when you change inertial frames? Nothing should change due to the Relativity Principle, the lengths should remain the same.



In one frame,

$$|\vec{x}_B - \vec{x}_A|^2 = \ell^2$$

Perform a Galilean boost,

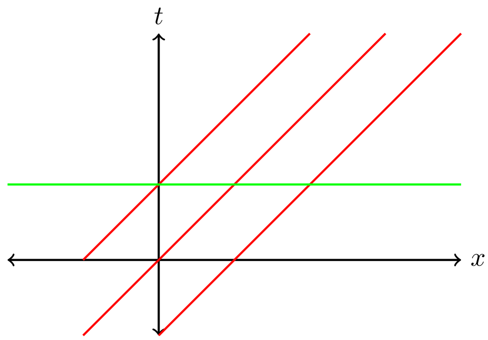
$$\vec{x}'_{A,B} = -\vec{v}t + \vec{x}_{A,B}$$

$$|\vec{x}'_B - \vec{x}'_A|^2 = \ell'^2$$

But it must be that,

$$\ell' = \ell$$

What about time? Galilean transformation leave time invariant because time is absolute in this Newtonian regime. The notion of simultaneity is the same for any inertial observer.



Red lines are stationary observers, and intersection with red lines indicate simultaneous events.

Math Perspective:
Are Galilean transformations the most general transformations between inertial observers? Use axioms?

Physics Perspective:

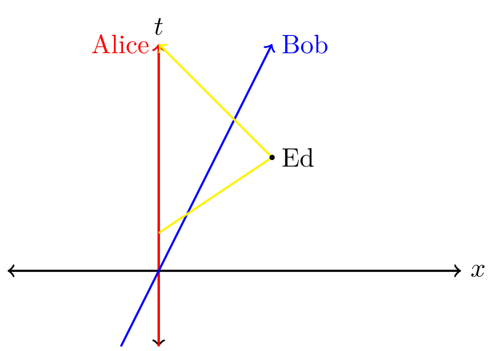
- Maxwell's equations do not transform well under Galilean transforms. Lorentz found the Lorentz transformations that allow Maxwell's equations to transform well.
- Michelson-Morley experiment: reveals that speed of light is invariant under the change of frame.
  - $v_1 + v_2 = v_3$ This is **not** the case if $v_2 = c$
  - $v_1 + c = c$ What why???
  - Under the assumption that light is a wave in the ether. Results suggest that the ether is not measurable.

3.3 Lorentz Transformations

Let's use light to measure objects in two frames; specifically let's determine the position using light.

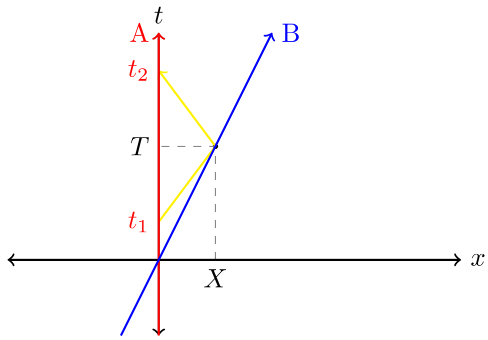
Assumptions:

- The speed of light is the same in any frame.
- Relativity Principle
- Bob will move at velocity $v < c$
  - If an observer is moving $v > c$, their position can't be measured using light
- 2d for simplicity
- Set $c = 1$, $x = ct + x_0 = t + x_0$



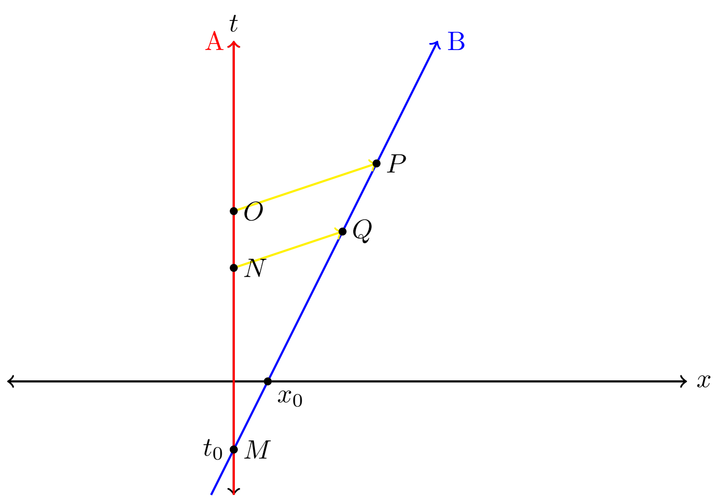
Let's measure the coordinates of Ed in Alice's frame and Bob's frame and the map relating the times using light to measure positions.
1) Using light to measure position,

$$x = \frac{1}{2c}(t_2 - t_1)$$
$$T = t_1 + \frac{1}{2}(t_2 - t_1) = \frac{1}{2}(t_2 + t_1)$$



2) Determine how the difference of times of reception and emission are related?

$$\Delta t_A = ON \quad \Delta t_B = QP$$
$$\frac{MN}{OM} = \frac{MP}{QM}$$
$$\frac{MN}{MN + \Delta t_A} = \frac{MP}{MP + \Delta t_B}$$
$$\frac{\Delta t_A}{MN} = \frac{\Delta t_B}{MP}$$
$$\Delta t_A \propto \Delta t_B$$



$$\Delta t_A = f^{-1}(v, c, x_0, y_0)\Delta t_B$$

By translational invariance, f cannot depend on x_0 or t_0. Therefore,

$$\Delta t_A = f^{-1}(v, c)\Delta t_B$$

For convenience, we will find f defined as,

$$\Delta t_B = f(v,c)\Delta t_A$$

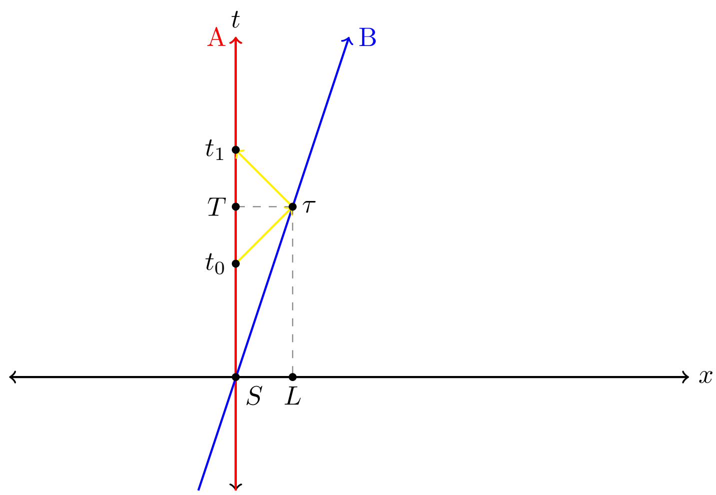
By similar analysis of a pair of light rays emanating from bob,

$$\Delta \tilde{t}_A = \tilde{f}(v,\tilde{c})\Delta \tilde{t}_B = \tilde{f}(v,c)\Delta \tilde{t}_B$$

This uses the assumption $\tilde{c} = c$. Furthermore by the relativity principle, we must have $\tilde{f} = f$. Otherwise, the two inertial frames (A, B) would be distinguishable through $f, \tilde{f}$. In conclusion:

$$\Delta t_{\text{received}} = f(v,c)\Delta t_{\text{emitted}}$$

So what is the form of f? Let us assume that there is a synchronization between the two frames so that calculations become easier.



Let's define:

$$\begin{aligned}
\Delta t_A &= S \to t_0 = t_0 \\
\Delta \tilde{t}_A &= S \to t_1 = t_1 \\
\Delta t_B &= S \to \tau = \tau = \Delta \tilde{t}_B
\end{aligned}$$

Therefore $\tau = f(v,c)t_0$ with,

$$t_1 = \Delta \tilde{t}_A = f(v,c)\Delta \tilde{t}_B = f^2(v,c)t_0$$

Thus we have from radar measurements,

$$\begin{aligned}
L &= \frac{1}{2}c\left(t_1 - t_0\right) = \frac{1}{2}c\left(f^2(v,c) - 1\right)t_0 \\
T &= \frac{1}{2}\left(t_1 + t_0\right) = \frac{1}{2}c\left(f^2(v,c) + 1\right)t_0
\end{aligned}$$

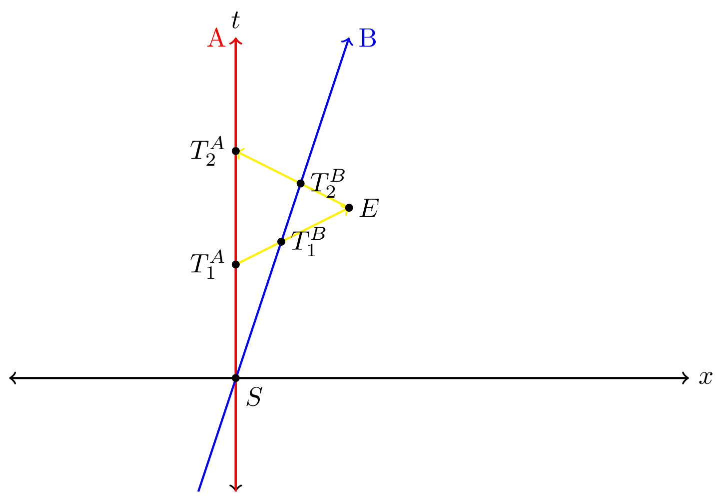
The ratio is given by,

$$\frac{L}{T} = v = c\frac{f^2(v,c) - 1}{f^2(v,c) + 1}$$

Inverting this expression (using $-c < v < c$) yields,

$$f(v,c) = \left(\frac{1+\frac{v}{c}}{1-\frac{v}{c}}\right)^{1/2}$$

4) Ed's coordinate from A's and B's perspective.



Let E be identified in two ways,

Alice's Perspective	(x_A^E, t_A^E)
Bob's Perspective	(x_B^E, t_B^E)

Therefore,

$$x_A^E = \frac{1}{2}c\left(T_2^A - T_1^A\right)$$

$$t_A^E = \frac{1}{2}\left(T_2^A + T_1^A\right)$$

$$x_B^E = \frac{1}{2}c\left(T_2^B - T_1^B\right)$$

$$t_B^E = \frac{1}{2}\left(T_2^B + T_1^B\right)$$

We know that $T_1^B = fT_1^A$ and $T_2^A = fT_2^B$ hence we can get the relation between (x_A^E, t_A^E) and (x_B^E, t_B^E).

$$x_B^E = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left(x_A^E - vt_A^E\right)$$

$$t_B^E = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left(t_A^E - \frac{v}{c^2}x_A^E\right)$$

For convenience we can relabel,

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

In matrix form,

$$\begin{bmatrix} T' \\ X' \end{bmatrix} = \underbrace{\gamma \begin{bmatrix} 1 & \frac{-v}{c^2} \\ -v & 1 \end{bmatrix}}_{\text{Lorentz Boost}} \begin{bmatrix} T \\ X \end{bmatrix}$$

Notice in the limit that $v << c$, the Lorentz boost becomes equivalent to the Galilean boost discussed earlier.

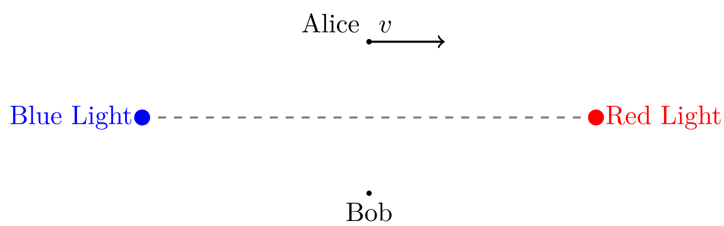
3.3.1 Consequences of Lorentz Transformations

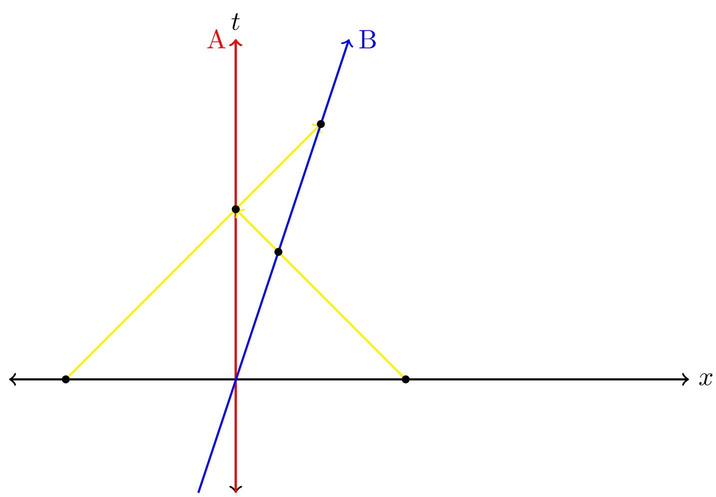
1. From Alice's perspective, Bob's time axis is,
   - $x^B = 0 = \gamma(x_A - vt_A) \implies x = vt$
2. From Alice's perspective, Bob's spatial slice is,
   - $T^B = 0 = \gamma\left(t_A - \frac{v}{c^2}\right)x_A \implies t = \frac{v}{c^2}x$
   - The slope of this line in Alice's perspective is the **inverse** (with $c \to 1$) of the slope of Bob's time axis

Again, notice that Galilean simultaneity is recovered in the limit that $v \to v \ll c$. Our new theory is still consistent with our old theories.

3.4 Length & Time

3.4.1 Einstein's Train & Simultaneity





Therefore, simultaneity is **relative**.

3.4.2 Length

What about spatial length? Alice has a ruler of length $x_2 - x_1 = \ell$. In Bob's frame,

$$x_i = \Bigg(\underbrace{x'_i + vt'}_{\text{Bob's coords}}\Bigg)\gamma \quad i = 1, 2$$

Thus,

$$x'_2 - x'_1 = (x_2 - x_1)\frac{1}{\gamma} \implies \ell' = \frac{\ell}{\gamma} \quad \text{Length contraction.}$$

If Bob has a ruler of length $\ell' = x_2' - x_1'$,

$$x_i' = \left(\underbrace{x_i + vt}_{\text{Alice's coords}} \right) \gamma \quad i = 1, 2$$

$$x_2 - x_1 = \ell = \frac{1}{\gamma}\ell' \quad \text{Length contraction.}$$

3.4.3 Time

In a similar manner, we have time dilation,

$$t_2' - t_1' = \gamma \left(t_2 - t_1\right) \quad \text{With } \gamma > 1.$$

3.4.4 Invariant Length & Minkowski Metric

Can we construct/define a notion of length that is invariant under these transformations? Namely,

$$\begin{bmatrix} \gamma & -\gamma \frac{v}{c^2} \\ -\gamma v & \gamma \end{bmatrix} = \Lambda^{\alpha}{}_{\beta}$$

Let's introduce a metric, $g_{\alpha\beta}$ that is of course symmetric and non non-degenerate. Therefore,

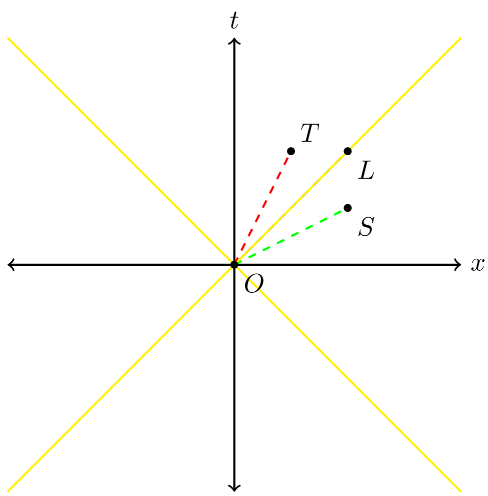
$$\Lambda^{\alpha}{}_{\beta} g_{\alpha\gamma} \Lambda^{\gamma}{}_{\delta} = g_{\beta\delta}$$

Reminder: $v \cdot w = v' \cdot w'$ with $v'^{\alpha} = \Lambda^{\alpha}{}_{\gamma} v^{\gamma}$. The solution here for 1d motion:

$$g_{\alpha\beta} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

These are called *Minkowski Metrics*. The difference between these two is a matter of convention. We will select the convention,

$$g_{\alpha\beta} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$



Let us define,

$$\left|\vec{OT}\right|^2 < 0 \quad \text{“time like”}$$

$$\left|\vec{OL}\right|^2 = 0 \quad \text{“light like”}$$
$$\left|\vec{OS}\right|^2 > 0 \quad \text{“space like”}$$

We say that a vector v^α is time-like if $v^\alpha \eta_{\alpha\beta} v^\beta \equiv |v|^2 < 0$, light-like if $v^\alpha \eta_{\alpha\beta} v^\beta \equiv |v|^2 = 0$ and space-like if $v^\alpha \eta_{\alpha\beta} v^\beta \equiv |v|^2 > 0$.

What are inertial observers in the relativistic regime? They are *still* given by a *straight lines*. What are the set of transformations relating inertial observers (4d) ?

- rotations: R_x, R_y, R_z

$$\to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- boost: B_x, B_y, B_z

$$B_z \to \begin{bmatrix} \gamma & -\gamma\frac{v}{c^2} & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore,

$$\det B_z = \gamma^2 - \gamma^2 \frac{v^2}{c^2} = \gamma^2 \left(1 - \frac{v^2}{c^2}\right) = 1$$

Let us relabel $\gamma^2 = \cosh^2\eta$ and $\gamma^2 \frac{v^2}{c^2} = \sinh^2\eta$,

$$B_z \to \begin{bmatrix} \cosh\eta & -\frac{\sinh\eta}{c^2} & 0 & 0 \\ -\sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- translation: $\vec{x} \to \vec{x} + \vec{a}$; $t \to t + b$
  - $x^\nu \to x^\nu + a^\nu$

All of these together form the Poincaré group.

3.4.5 Poincaré Group

Galilean boosts:

$$\vec{v}_1 + \vec{v}_2 = \vec{v}_3$$

Lorentzian boosts:

$$\overrightarrow{v_1 \oplus v_2} = \frac{1}{1 + \frac{\vec{v}_1 \cdot \vec{v}_2}{c^2}} \left(\vec{v}_1 + \frac{1}{\gamma_1}\vec{v}_2 + \frac{1}{c}\frac{\gamma_1}{1-\gamma_1}\left(\vec{v}_1 \cdot \vec{v}_2\right)\vec{v}_2\right)$$

Note these aren't associative,

$$\overrightarrow{\overrightarrow{v_1 \oplus v_2} \oplus v_3} \neq \overrightarrow{v_1 \oplus \overrightarrow{v_2 \oplus v_3}}$$

However it $\vec{v}_1$ is parallel to $\vec{v}_2$ then,

$$\overrightarrow{v_1 \oplus v_2} = \frac{\vec{v}_1 + \vec{v}_2}{1 + \frac{\vec{v}_1 \cdot \vec{v}_2}{c^2}}$$

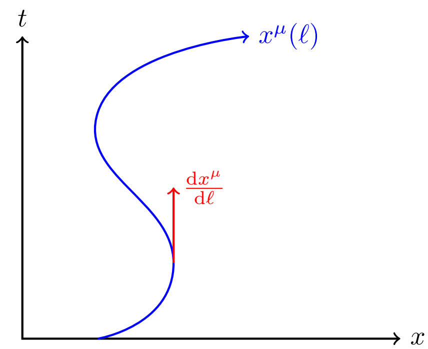
Which forces $\left|\overrightarrow{v_1 \oplus v_2}\right|^2 = c^2$ whenever either $\vec{v}_1$ or $\vec{v}_2$ has $\left|\vec{v}_i\right|^2 = c^2$.

3.5 Flat Spacetime

Spacetime is $\mathbb{R}^4$ and we will consider the Minkowski metric $\eta_{\alpha\beta}$. In Cartesian coordinates,

$$x^\mu = \begin{bmatrix} ct & x & y & z \end{bmatrix}$$

Consider a curve in spacetime $x^\nu(\ell)$ where ℓ is a curvilinear parameter that parametrizes the curve. The tangent vector is given by $\frac{\mathrm{d}x^\mu}{\mathrm{d}\ell}$.



If the tangent $\frac{\mathrm{d}x^\mu}{\mathrm{d}\ell}$ is always time-like, then the curve is called time-like. Analogously for space-like and light-like curves. We can then define an arclength of the curve.

$$\tau \neq \int \sqrt{\frac{\mathrm{d}x^\mu}{\mathrm{d}\ell} n_{\mu\nu} \frac{\mathrm{d}x^\nu}{\mathrm{d}\ell}} \mathrm{d}\ell$$

This expression does not work because the metric $\eta_{\mu\nu}$ is not positive definite. Therefore the length could be complex. To combat this, introduce an absolute magnitude,

$$\tau = \int \sqrt{\left|\frac{\mathrm{d}x^\mu}{\mathrm{d}\ell} n_{\mu\nu} \frac{\mathrm{d}x^\nu}{\mathrm{d}\ell}\right|} \mathrm{d}\ell$$

Which suggests,

$$\frac{\mathrm{d}\tau}{\mathrm{d}\ell} = \sqrt{\left|\frac{\mathrm{d}x^\mu}{\mathrm{d}\ell} n_{\mu\nu} \frac{\mathrm{d}x^\nu}{\mathrm{d}\ell}\right|}$$

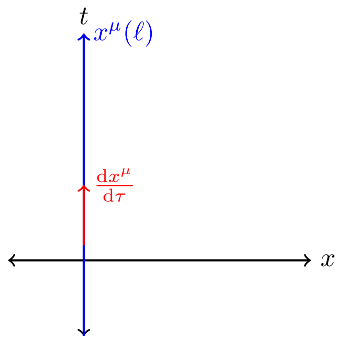
Which implies

$$\mathrm{d}\tau^2 = |\mathrm{d}x^\mu n_{\mu\nu} \mathrm{d}x^\nu|$$

Now consider using τ as the curvilinear parameter. So that the tangent is defined as,

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$$

What does this encapsulate? Consider an observer at rest:



Here, the notion of arclength coincides with time.

$$\mathrm{d}\tau^2 = \mathrm{d}t^2$$

Considering a time-like curve, the proper length or arclength is interpreted as the time as measured by a clock carried by the observer having the curve as his/her spacetime trajectory (world-line).

Parametrization of the world line of an observer at rest ($\vec{x} = \vec{0}$).

$$x^0(\tau) = \tau \quad \text{and} \quad \vec{x} = \vec{0}$$

Which gives,

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \delta^\mu{}_0 \neq \delta^0{}_\mu \quad \text{Be careful to matching indices.}$$

Labeling $\delta^\mu{}_0 = V^\alpha$,

$$V^\alpha \eta_{\alpha\beta} V^\beta = \delta^\alpha{}_0 \eta_{\alpha\beta} \delta^\beta{}_0 = \eta_{00} = -1 < 0$$

Therefore, V^α is time-like. However in general,

$$\begin{aligned}
V^\mu \eta_{\mu\beta} V^\beta &= \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \eta_{\mu\alpha} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} \\
&= \frac{\mathrm{d}x^\mu \eta_{\mu\alpha} \mathrm{d}x^\alpha}{\mathrm{d}\tau^2} \\
&= \frac{\mathrm{d}s^2}{\mathrm{d}\tau^2} \quad \mathrm{d}s^2 \text{ is the line element.} \\
&= -1 \quad \text{Since } \mathrm{d}\tau^2 = \left|\mathrm{d}s^2\right| = -\mathrm{d}s^2 \text{ (this vector is time-like).}
\end{aligned}$$

We can can also write (with $c = 1$),

$$\begin{aligned}
\mathrm{d}\tau^2 &= \mathrm{d}t^2 - \mathrm{d}\vec{x}^2 \\
&= \mathrm{d}t^2 \left(1 - \frac{\mathrm{d}\vec{x}^2}{\mathrm{d}t^2}\right) \\
&= \mathrm{d}t^2 \left(1 - \vec{v}^2\right)
\end{aligned}$$

Which when rearranged yields (noting the negative is ignored because we are considering time moving forward),

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{\sqrt{1 - \vec{v}^2}} = \gamma$$

Noting that,

$$v^\mu = \begin{bmatrix} \frac{\mathrm{d}t}{\mathrm{d}\tau}c \\ \frac{\mathrm{d}\vec{x}}{\mathrm{d}\tau} \end{bmatrix} = \begin{bmatrix} \gamma c \\ \frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} \end{bmatrix} = \begin{bmatrix} \gamma c \\ \gamma \vec{v} \end{bmatrix}$$

Noting that $\vec{v}$ is a velocity in 3d space while v^μ is a relativistic velocity which is a 4d object. We have shown that,

$$v^\mu v_\mu = -1$$

In the Galilean regime, $\vec{v} \in \mathbb{R}^3$ which allows one to use the vector space structure of $\mathbb{R}^3$ to add velocities,

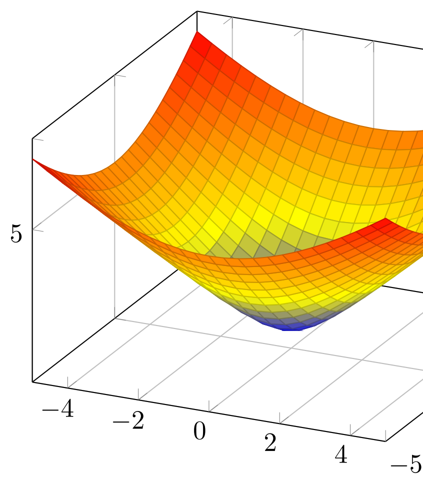
$$\vec{v}_1 + \vec{v}_2 = \vec{v}_{12}$$

But for relativistic velocities $\vec{v} \in \mathbb{H}^3$ hyperboloid.

$$\left\{ v^\mu \in \mathbb{R}^4 \mid v^\mu v_\mu = -1 \right\}$$

This is somewhat surprising, bur there exists a constraint that reduces the dimensionality.

$$v^\mu v_\mu = -1 = -\left(v^t\right)^2 + (v^x)^2 + (v^y)^2 + (v^z)^2$$



This new space $\mathbb{H}^3$ allows for the addition of velocities,

$$\overrightarrow{v_1 \oplus v_2} = \vec{v}_{12}$$

Particle Considerations:

- The world-line of a material particle is a time-like curve
- The world-line of photons (massless) is a light-like curve
- The world-line of a tachyon ($|v| > c$) is space-like

Relativistic acceleration is given by,

$$a^\mu = \frac{\mathrm{d}v^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2}$$

For inertial observer, $a^\mu = 0^\mu$.
Relativistic momentum is given for a material particle with mass m,

$$p^\mu = mv^\mu \implies p^\mu p_\mu = -m^2$$

Which has the property,

$$p^\mu = \begin{bmatrix} E = m\gamma \\ \vec{p} = m\gamma\vec{v} \end{bmatrix}$$

Examine the space term,

$$E = m\gamma = m\frac{1}{\sqrt{1-v^2}} = m + \underbrace{\frac{1}{2}mv^2}_{\text{kinetic energy}} + \cdots \quad \text{Taylor Series}$$

And also rest,

$$E^2 = m^2c^4$$

3.6 Relativistic Dynamics

What about the dynamics in the relativistic regime? Recall in the non-relativistic regime, Newton's law is given by,

$$m_i\vec{a} = \vec{F}_{\text{tot}}$$

Where m_i is the inertial mass. How can we modify this equation to the relativistic regime. m_i has no need to change, $\vec{a}$ becomes the 4d spacetime vector $a^\mu = \frac{\mathrm{d}^2x^\nu}{\mathrm{d}\tau^2}$ and force becomes,

$$m_ia^\mu = F^\mu$$

3.6.1 Lorentz Force

Consider the Lorentz force,

$$\vec{F} = q\left(\vec{E} + \vec{v}\times\vec{B}\right)$$

Where q is the charge of the particle, $\vec{E}$ is the electric field, $\vec{v}$ is the velocity of the particle and $\vec{B}$ is the magnetic field of the particle. In the relativistic regime,

$$F^\mu = qF^{\alpha\beta}V_\beta$$

Where $F^{\alpha\beta}$ is known as the Maxwell tensor that is anti-symmetric,

$$F^{\alpha\beta} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -E_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{bmatrix}$$

And V_β is the relativistic velocity with,

$$V_\beta = \eta_{\alpha\beta}V^\alpha$$

3.6.2 Gravity

What about the relativistic behavior of gravity?

$$\vec{F} = -m_g\vec{\nabla}\phi = -\frac{Gm_gM\hat{r}}{r^2}$$

Where m_g is gravitational mass, ϕ is the gravitational potential, G is Newton's constant, and M is the mass of the system that generates the gravitational force. For a mass density ρ,

$$M = \iiint_V \rho\mathrm{d}V$$

Given a source with mass density ρ we have ϕ given by the Poisson Equation,

$$\vec{\nabla}\vec{\nabla}\phi = \Delta\phi = 4\pi G\rho$$

Where Δ is given by,

$$\Delta = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$$

Gravity tells me how the matter source propagate,

$$\vec{F}_g = m_i\vec{a} \quad \text{gravity} \implies \text{matter} \tag{3.1}$$

While matter tells gravity how to behave through the Poisson Equation,

$$\Delta\phi = 4\pi G\rho \quad \text{matter} \implies \text{gravity} \tag{3.2}$$

Here you can see the dual nature between equations (3.1) and (3.2). How can we generalize these equations to the relativistic regime. What we will see is that (3.1) become the geodesic equations, while (3.2) become the Einstein field equations.

Let's find the relativistic version of (3.2).

$$\Delta\phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi$$

Here Δ is the Laplacian. The relativistic notation is given by,

$$\Box\phi \equiv \eta^{\alpha\beta}\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial x^\beta}\phi = \left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \Delta\right)\phi$$

Where $\Box$ is called the *D'Alembertian*. What about the RHS of (3.2)? $4, \pi, G$ are all constants, so all that remains is ρ. Notice that $\rho \sim M/V$ is a mass over a volume. As we have seen above mass M is like an energy. Furthermore, boosts affect the volume V.

$$p^\mu = \begin{bmatrix} E \\ \vec{p} \end{bmatrix}$$

Now perform a boost on p^μ ($p^\mu \to \tilde{p}^\mu$) using $\Lambda^\alpha{}_\beta$,

$$\Lambda^\alpha{}_\beta = \begin{bmatrix} \gamma & \frac{\gamma v}{c^2} & 0 & 0 \\ \gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.3}$$

As you can see, the energy terms will get mixed up with the momentum terms.

Consider a *perfect fluid* (set of particles) characterized by,

- velocity v
- mass density ρ
- pressure p

A perfect field is called *dust* is $p = 0$. Then $\rho = M/V$ where the total mass M is given by,

$$M = n \cdot m$$

Where n is the number of particles and m is the mass of each particle. Therefore,

$$\rho = \frac{M}{V} = \frac{nm}{V} = \frac{n}{V}m$$

Now the term n/V represents a number density of particles. What is the relativistic version of each of these two terms (n/V and m). Evident m is going to be generalized using the four-momentum p^μ. But what about n/V?

$$N \equiv \frac{n}{V} = \text{ number density}$$

In the co-moving frame (i.e. the frame at which the fluid is at rest) we have, $N = n/V$. Consider a rectangular prism in 3d space with sides $\Delta x, \Delta y, \Delta z$. Then the total volume is $\Delta x \Delta y \Delta z$. Now consider a frame that is not co-moving. Consider this frame moving at velocity $\vec{v} = v_x \hat{x}$. What happens to Δx? Length contraction decreases the width of the box.

$$\Delta x \to \Delta \bar{x} = \frac{1}{\gamma}\Delta x$$

What happens to N? Well since the total number of particles n remains constant, the number density increases,

$$N \to \bar{N} = \frac{n}{\Delta \bar{V}} = \gamma \frac{n}{\Delta x \Delta y \Delta z}$$

Can we see a flux as a number density (i.e. a number of particles crossing an area per unit area of time)? In the co-moving frame, we are moving with the particles so there is no flux. However in the non co-moving frame where $\vec{v} = v_x \hat{x}$, consider a slice along the $\bar{x}$ axis ($\bar{y}\bar{z}$-plane). What is the flux of particles flowing through this slice?

$$\#\text{ particles crossing slice } = \bar{N} \cdot V = \bar{N} \underbrace{\Delta \bar{x}}_{v\Delta \bar{t}} \underbrace{\Delta \bar{y} \Delta \bar{z}}_{\Delta A}$$

The flux then is given by,

$$\text{Flux } = \frac{\#}{\Delta \bar{t} \Delta \bar{A}}$$

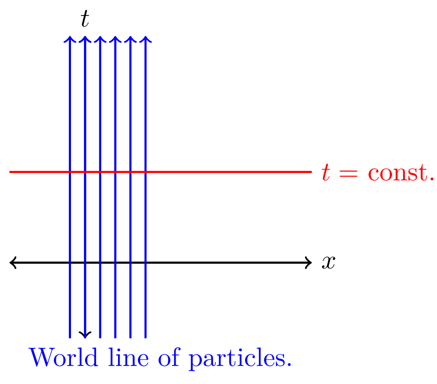
Where $\Delta \bar{t} \Delta \bar{A}$ can be thought of as a *spacetime volume* in 3d. So flux is a number density defined with respect to a spacetime 3d volume. Therefore,

$$\text{Flux } = \bar{N} v = v \gamma N$$

In summary,

	Co-moving Frame	Frame (v_x)
# density	$N = \frac{m}{V}$	$\bar{N} = \frac{n}{\bar{V}} = \gamma \frac{m}{V} = \gamma N$
Flux	0	$F_x = \gamma N v_x$

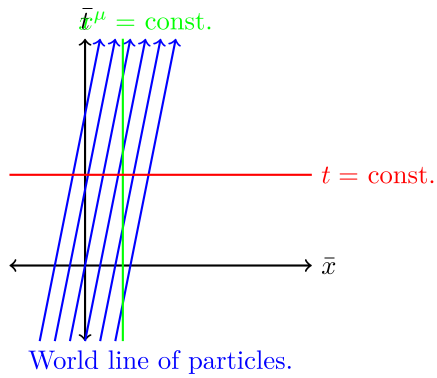
The question becomes, can we see a flux as a number density? In the comoving frame we have:



Thus this represents some sort of flux through a fixed time.

$$\#\text{density} = \frac{\#\text{particles}}{\Delta V} = \frac{\#\text{particles}}{\Delta x \Delta A}$$

Whereas in the non co-moving frame the particles moving at speed v_x:



The relevant notion of relativistic number density is he notion of flux through the x^μ = const. We will introduce the "#-flux" vector N^μ. Where are rest,

$$N^\mu = \begin{bmatrix} N \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

And under a boost such as equation (3.3) where $v = v_x$,

$$N^\mu = \begin{bmatrix} \gamma N \\ \gamma N v_x \\ 0 \\ 0 \end{bmatrix} = N \begin{bmatrix} \gamma \\ \gamma v_x \\ 0 \\ 0 \end{bmatrix} = N v^\mu$$

As a result, the magnitude of N^μ is given by,

$$N^\mu N_\mu = N^2 v^\mu v_\mu = -N^2$$

Now consider the mass density discussed above $\rho = M/V$. At rest we have $\rho = M/V$ with $E = M$ $(c = 1)$. Under a boost with v_x,

$$\begin{bmatrix} E = m \\ 0 \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} \gamma m = \bar{E} \\ m\gamma v \\ 0 \\ 0 \end{bmatrix}$$

Furthermore the volume under the boost becomes $V \to \bar{V} = \frac{1}{\gamma} V$. So $\rho = M/V$ at rest becomes under a boost,

$$\bar{\rho} = \gamma^2 \rho$$

This is similar to the velocity terms but instead we have a γ^2 term instead of γ. Note that density is given by,

$$\rho = \frac{M}{V} = \frac{n}{V} m \tag{3.4}$$

Where m is the mass of a single particle that should be considered the mass given by the four-momentum p^α. The number density should be given by the four-number density N^β. Using this interpretation, (3.4) is expressed under the relativistic regime as,

$$\rho \to N^\alpha \otimes p^\beta = Nv^\alpha \otimes mv^\beta = mNv^\alpha \otimes v^\beta = \rho v^\alpha \otimes v^\beta$$

This new quantity $\rho v^\alpha \otimes v^\beta$ will be called $T^{\alpha\beta}$. At rest,

$$v^\alpha = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Which gives $T^{00} = \rho$ and $T^{\alpha\beta} = 0$ whenever $\alpha \neq 0$ or $\alpha \neq 0$. While in frame (v_x),

$$v^\alpha = \begin{bmatrix} \gamma \\ v_x \\ 0 \\ 0 \end{bmatrix}$$

Which gives,

$$T^{00} = \gamma^2 \rho$$

So why does this γ^2 term keep showing up? Since T^{00} is a tensor with two components the γ induced by the boost affects each of the components, namely M and V. The boost affects the tensor for each index,

$$T^{\alpha\beta} \underbrace{\to}_{\text{boost}} \Lambda^\alpha{}_\gamma \Lambda^\beta{}_\delta T^{\gamma\delta} = T^{\bar{\alpha}\beta}$$

This tensor $T^{\alpha\beta}$ is called the *stress energy tensor* and will act as a source of gravity.

- T^{00}: energy density
- T^{0i}: energy flux across i-th surface $(i = 1, 2, 3)$
- T^{i0}: momentum density $(i = 1, 2, 3)$
- T^{ij}: flux of i-th momentum through j-th surface $(i, j = 1, 2, 3)$
  - this is known as the stress tensor that appears when particles have interactions with one another.

3.6.3 Properties of Stress Energy Tensor

symmetric: $T^{\alpha\beta} = T^{\beta\alpha}$

Here is a "proof" using dimensional arguments.

$$\begin{aligned} T^{0i} &= \text{energy flux} \\ &= \text{density of energy} \times \text{speed of flow} \\ &= \text{density of mass} \times \text{speed of flow} \\ &= \text{density of momentum} \\ &= T^{i0} \end{aligned}$$

conservation: $\frac{\partial}{\partial x^\beta} T^{\alpha\beta} = 0$

To demonstrate this conservation, this of conservation as "what goes in and out of a box encodes the variations of what's inside the box."
Consider a cube of side length ℓ aligned to a Cartesian coordinate system. Focusing on energy,

$$\ell^3 \frac{\partial}{\partial t} T^{00} \quad \text{variation of what's inside the box} \tag{3.5}$$

The rate of flow of energy of through each of the 6 faces of the cube is given by,

$$\ell^2 T^{0x}(x=0) - \ell^2 T^{0x}(x=\ell) + \ell^2 T^{0y}(y=0) - \ell^2 T^{0y}(y=\ell) + \ell^2 T^{0z}(z=0) - \ell^2 T^{0z}(z=\ell) \tag{3.6}$$

Therefore by conservation (3.5) must equal (3.6),

$$\ell^3 \frac{\partial}{\partial t} T^{00} = \ell^2 \left(T^{0x}(x=0) - T^{0x}(x=\ell) + T^{0y}(y=0) - T^{0y}(y=\ell) + T^{0z}(z=0) - T^{0z}(z=\ell) \right)$$

Or more cleanly, dividing by ℓ^3 and considering the limit as $\ell \to 0$,

$$\frac{\partial}{\partial t} T^{00} = \lim_{\ell \to 0} \frac{1}{\ell} \sum_{i=1}^{3} \left(T^{0i}(x^i = 0) - T^{0i}(x^i = \ell) \right)$$

Using the definition of partial derivatives,

$$\frac{\partial}{\partial t} T^{00} = -\frac{\partial}{\partial x} T^{0x} - \frac{\partial}{\partial y} T^{0y} - \frac{\partial}{\partial z} T^{0z}$$

This then implies,

$$\frac{\partial}{\partial x^\alpha} T^{\alpha x} = 0$$

Now what can we propose about gravity? Nordstrom in 1907 proposed to Einstein the idea of taking,

$$\Delta \phi = 4\pi G \rho$$

And replacing this with,

$$\Box \phi = 4\pi G T^{\alpha}{}_{\beta}$$

This proposed theory is incorrect because the LHS is linear by the RHS is not. Here, gravity doesn't *gravitate*.
In 1912, after hard work, Nordstrom proposed,

$$\frac{\Box \phi}{\phi} = 4\pi G \tilde{T}^{\alpha}{}_{\beta}$$

This theory is self consistent by it can be shown that under this theory, light doesn't bend with gravity. This is been proven to be true in our universe so it must also be wrong.