Phys 442

ELECTRICITY & MAGNETISM 3

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Version: 1.0

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Disclaimer

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Latest versions of all my course notes are available at **www.tcfraser.com/coursenotes**.

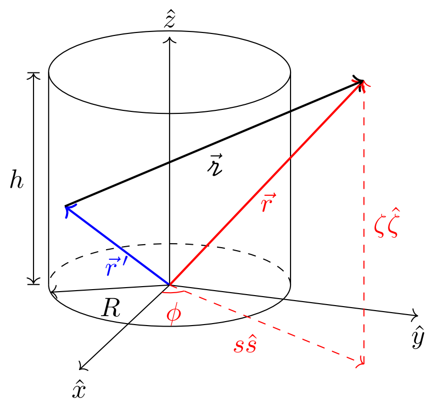
1 Coordinates and Symmetry

A clever choice of coordinates systems typically makes solving a problem considerably easier. Mathematically, this is due to *Noether's Theorem*. A typical three dimensional Lagrangian will have three dependent generalized coordinates $L = L(x, y, z) = L(s, \theta, \zeta) = \cdots$. However, if one can identify generalized coordinates q that make the Lagrangian invariant $\frac{\partial L}{\partial q} = 0$, then the *Euler-Lagrange* equations are considerably similar,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \implies \frac{\partial L}{\partial \dot{q}} = \text{const.} \implies L \propto \dot{q}$$

As such, the number of equations that remain to solved has been reduced.

2 First Assignment?



A1.1: Use cylindrical coordinates with ζ along the axis of the cable,

$$V(\zeta) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{C}} \frac{\mathrm{d}\rho}{\imath}$$

Where $\vec{\imath} = \vec{r} - \vec{r}'$, $\vec{r}'$ is the source point and $\vec{r}$ is the field point. The entire cylinder is the set of all source points $\vec{r}'$ that are contained inside $|\vec{r}'| \le R$.

$$\vec{r} = \zeta\hat{\zeta}$$
$$\vec{r}' = s'\hat{s}' + \zeta'\hat{\zeta}$$

$$V(\zeta) = \frac{\rho}{4\pi\epsilon_0}\int_{\mathcal{C}} \frac{\mathrm{d}V}{|\vec{r} - \vec{r}'|}$$

Where $\mathrm{d}V = s\mathrm{d}s\mathrm{d}\theta\mathrm{d}\zeta$. One can then find the electric *field* by doing $\vec{E} = -\vec{\nabla}V = E_\zeta\hat{\zeta} = -\frac{\partial V}{\partial \zeta}\hat{\zeta}$

A1.2:

Between the two conductors, there will be a radial electric field $\vec{E} = E(s)\hat{s}$ and parallel magnetic field $\vec{B} = B(s)\hat{\zeta}$. Outside the two conductors, there will be no electric or magnetic field.

$$E^{\parallel}_{\text{vac}} = 0$$
$$E^{\perp}_{\text{vac}} = \frac{\sigma}{\epsilon_0}$$
$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$$

For part g), use Laplace's equation $\nabla^2 V = 0$. In cylindrical coordinates, Laplace's equation is,

$$\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) = 0$$

Cylindrical coordinates gives us the following symmetries $\frac{\partial V}{\partial \phi} = \frac{\partial V}{\partial \zeta} = 0$. Solving this system gives the potential in terms of s: $V(s) = \cdots$. Then the electric field can then be obtained via $\vec{E} = -\vec{\nabla} V$.
A1.3: Using cylindrical coordinates once again, the electric field is going to be radial outwards to the uniform charge density. For the uniform density cylinder, construct a Gaussian surface cylindrically around the cylinder. For the current density cylinder, the current density is the current per cross sectional area. Construct an Amperian loop,

$$\oint_{\mathcal{A}} \mathrm{d}\vec{\ell} \cdot \vec{B} = \mu I_{\text{enc}}$$

Part e), finding the vector potential,

$$A(\vec{r}) = \frac{\mu_0}{4\pi} \int_{\mathcal{C}} \mathrm{d}\tau' \frac{\vec{J}(\vec{r})}{\mathcal{r}}$$

Evidently, $\hat{s}$ and $\hat{s}'$ are in *different* directions. Solving such an equation yields,

$$A(s) = \frac{\mu_0}{4\pi} \int_0^{2\pi} \mathrm{d}\phi' \int_0^a s' \mathrm{d}s' \int_{-\infty}^{\infty} \mathrm{d}\zeta' \frac{J(s)}{\left| s\hat{s} - s'\hat{s}' - \zeta'\hat{\zeta} \right|}$$

Recognize the structure of the potential integral,

$$V(s) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{C}} \mathrm{d}\tau' \frac{\rho(r')}{\mathcal{r}} = \frac{\rho_0}{4\pi\epsilon_0} \int_{\mathcal{C}} \frac{\mathrm{d}\tau'}{\mathcal{r}}$$

Comparing to the vector potential, we have an equivalent integral (up to a constant).

$$A(s) = \frac{\mu_0 J_0}{4\pi} \int_{\mathcal{C}} \frac{\mathrm{d}\tau'}{\mathcal{r}}$$

For question f), use the definition of $\vec{B}$ in terms of $\vec{A}$,

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

Further, recall that if $\vec{E} = -\vec{\nabla} V$, then by Stoke's theorem for some loop $\mathcal{L}$,

$$V = -\int_{\mathcal{L}} \mathrm{d}\vec{\ell} \cdot \vec{E}$$

3 Conservation Laws

Beginning with one of Maxwell's equations,

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Taking the divergence of the above equation,

$$\cancelto{0}{\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{B} \right)} = \mu_0 \vec{\nabla} \cdot \vec{J} + \mu_0 \epsilon_0 \frac{\partial}{\partial t} \left(\vec{\nabla} \cdot \vec{E} \right)$$

Luckily, the divergence of a curl is always 0. Dividing by relevant constants we obtain the following conservation law,

$$0 = \vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} \tag{3.1}$$

This is a conservation of charge. It is a **local** conservation law because it holds for all points in space $\vec{r}$. Intuitively, is claims that the rate of charge of charge at a point is equal to the amount of current following in or out of the take point.

A2.1: Again using cylindrical coordinates $\vec{r} = s\hat{s} + \zeta\hat{\zeta}$. Let the current flow in such a way that the magnetic field points along the ζ-axis. Let $\mathcal{L}$ be an Amperian loop with one side at distance $|\vec{r}| \to \infty$,

$$\int_{\mathcal{L}} \mathrm{d}\vec{\ell} \cdot \vec{B} = \mu_0 I_{\text{enc}}$$

The same equation can be reused to calculate the vector potential for a Gaussian surface $\mathcal{S}$,

$$\int_{\mathcal{L}} \mathrm{d}\vec{\ell} \cdot \vec{A} = \int_{\mathcal{S}} \mathrm{d}\vec{a} \cdot \vec{B} = \Phi$$

Where Φ is the magnetic flux through $\mathcal{S}$. Furthermore, the energy required to set up a magnetic field is,

$$W = \frac{1}{2\mu_0} \int_{\mathcal{C}} \mathrm{d}\tau B^2 = \frac{1}{2} \int_{\mathcal{C}} \mathrm{d}\tau \vec{J} \cdot \vec{A} = \frac{1}{2} L I^2$$

Where L is the self-inductance of the solenoid.

4 Poynting's Theorem

First we begin with two of Maxwell's equations,

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{4.1}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{4.2}$$

Computing the inner product between eq. (4.1) and $\vec{B}$, and the inner product between eq. (4.2) and $\vec{E}$ and taking a difference,

$$\vec{B} \cdot \left(\vec{\nabla} \times \vec{E}\right) - \vec{E} \cdot \left(\vec{\nabla} \times \vec{B}\right) = -\frac{\partial}{\partial t}\left(\frac{\epsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2\right) = \mu_0 \vec{E} \cdot \vec{J}$$

Letting $\frac{\epsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2$ be the **electromagnetic energy density** u, we have the following identity,

$$\vec{\nabla} \cdot \left(\vec{E} \times \vec{B}\right) = -\mu_0 \frac{\partial u}{\partial t} - \mu_0 \vec{E} \cdot \vec{J} \tag{4.3}$$

Physically eq. (4.3) corresponds to a conservation of energy. We refer to the term $\frac{1}{\mu_0}\left(\vec{E} \times \vec{B}\right)$ as the Poynting vector $\vec{S}$ as it determines the direction of electromagnetic radiation. The Poynting vector $\vec{S}$ represents the power density.

$$\vec{\nabla} \cdot \vec{S} + \frac{\partial u}{\partial t} + \vec{E} \cdot \vec{J} = 0 \tag{4.4}$$

Much like eq. (3.1), eq. (4.4) is a local conservation of *energy*. The only algebraic difference is the term $\vec{E} \cdot \vec{J}$. If there is a flowing charge $\vec{J}$ through an electric field $\vec{E}$, then there is work done on the charge. By Gauss's theorem, the energy leaving through a surface $\mathcal{S}$ per unit time is,

$$\int_{\mathcal{V}} \vec{\nabla} \cdot \vec{S} \mathrm{d}\tau = \oint_{\mathcal{S}} \mathrm{d}\vec{a} \cdot \vec{S}$$

and the E-M energy in the volume $\mathcal{V}$ is given by,

$$\int_{\mathcal{V}} \mathrm{d}\tau u$$

Where again, u is the electromagnetic energy density. If we integrate over eq. (4.4),

$$\int_{\mathcal{V}} \mathrm{d}\tau \left(\vec{\nabla} \cdot \vec{S} + \frac{\partial u}{\partial t} + \vec{E} \cdot \vec{J}\right) = \oint_{\mathcal{S}} \mathrm{d}\vec{a} \cdot \vec{S} + \int_{\mathcal{V}} \mathrm{d}\tau \frac{\partial u}{\partial t} + \int_{\mathcal{V}} \mathrm{d}\tau \vec{E} \cdot \vec{J}$$

Each term in eq. (4.4) has it's purpose illuminated. The final term $\int_{\mathcal{V}} \mathrm{d}\tau \vec{E} \cdot \vec{J}$ corresponds to the work done on moving charges $\vec{J}$ in the volume $\mathcal{V}$. In it important to note that there are no terms that corresponding to "magnetic work".

Consider the work done to move a charge q a displacement $\mathrm{d}\vec{\ell}$ by E-M forces,

$$\begin{aligned}
\mathrm{d}W &= \mathrm{d}\vec{\ell} \cdot \vec{F} \\
&= \mathrm{d}\vec{\ell} \cdot q\left(\vec{E} + \vec{v} \times \vec{B}\right) \\
&= \vec{v}\mathrm{d}t \cdot q\left(\vec{E} + \vec{v} \times \vec{B}\right) \\
&= q\mathrm{d}t\left(\vec{v} \cdot \vec{E}\right) + q\mathrm{d}t \underbrace{\left(\vec{v} \cdot \left\{\vec{v} \times \vec{B}\right\}\right)}_{=0} \\
&= q\mathrm{d}t\left(\vec{v} \cdot \vec{E}\right)
\end{aligned}$$

So for a continuous charge distribution we have that $\mathrm{d}q = \rho \mathrm{d}\tau$ and $\rho\vec{v} = \vec{J}$. Which means that the rate of work done on the charge ρ in the volume $\mathcal{V}$ (i.e. creating the current density $\vec{J}$) is,

$$\dot{W} = \int_{\mathcal{V}} \mathrm{d}\tau \vec{E} \cdot \vec{J}$$

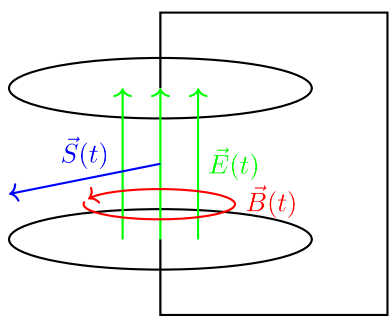
We can interpret this as the work done per unit time rearranging the charge in $\mathcal{V}$. One again eq. (4.4) is given by

$$\vec{\nabla} \cdot \vec{S} + \frac{\partial u}{\partial t} + \vec{E} \cdot \vec{J} = 0$$

With the following interpretations,

- $\vec{\nabla} \cdot \vec{S}$: amount of radiation energy leaving the point $\vec{r}$
- $\frac{\partial u}{\partial t}$: increase in E-M energy at the point $\vec{r}$
- $\vec{E} \cdot \vec{J}$: the amount of work done on charges at the point $\vec{r}$

As an illustrative example, consider a parallel plate capacitor with an electric field $\vec{E}$ between them.



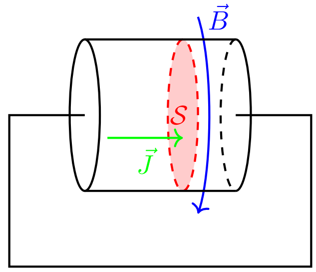
We ave that the magnetic field points in the $\hat{\phi}$ direction, $\vec{B} = V\hat{\phi}$. The electric field $\vec{E} = E\hat{\zeta}$, and Poynting vector are $\vec{S} = S\hat{s}$. We have that the radiation through the surface $\mathcal{S}$,

$$\int_{\mathcal{S}} \mathrm{d}\vec{a} \cdot \vec{S} = -(2\pi a h)S$$

Therefore $\frac{\partial U}{\partial t} = -(2\pi a h)S$ corresponding to the amount of energy flowing out of the capacitor and therefore,

$$U = \int_0^\infty \mathrm{d}t(-2\pi a h S) = \frac{1}{2}CV^2$$

Ex 8.1:



Inside the conductor the electric field moves parallel to its axis $\vec{E} = \frac{V_0}{\ell}\hat{\zeta}$. The magnetic field is then given by,

$$\vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial E}{\partial t} + \mu_0 \vec{J}$$

Integration over the surface $\mathcal{S}$,

$$\int_{\mathcal{S}} \mathrm{d}\vec{a} \cdot \left(\vec{\nabla} \times \vec{B}\right) = \mu_0 \int_{\mathcal{S}} \mathrm{d}\vec{a} \cdot \vec{J} = \mu_0 I_{\text{enc}}$$

Therefore computing this integral $\int \mathrm{d}\vec{\ell} \cdot \vec{B}$ yields,

$$\vec{B} = \frac{\mu_0 I}{2\pi a}\hat{\phi}$$

Moreover, the Poynting vector is given by,

$$\begin{aligned}\vec{S} &= \frac{1}{\mu_0}\vec{E} \times \vec{B} \\ &= \frac{1}{\mu_0}\frac{V_0}{\ell}\hat{\zeta} \times \frac{\mu_0 I}{2\pi a}\hat{\phi} \\ &= -\frac{V_0 I}{2\pi a \ell}\hat{s}\end{aligned}$$

Therefore the radiation flux,

$$\int_{\mathcal{S}} \mathrm{d}\vec{a} \cdot \vec{S} = -\frac{V_0 I}{2\pi a \ell}\int_{\mathcal{S}} \mathrm{d}a = -V_0 I$$

Which is exactly the amount of Joule heating for a current I though a wire with voltage V_0 across it. Using $V = IR$,

$$\int_{\mathcal{S}} \mathrm{d}\vec{a} \cdot \vec{S} = -I^2 R$$

Ex 8.2 (Griffiths Problem 8.13): A long thin solenoid of radius a has a time dependent current $I_s(t)$ flowing around it. Encircling the solenoid is a ring of radius b with current $I_r(t)$ $(b \gg a)$ passing through it. The ring has resistance R. There is an induced electro-motive-force in the ring due to the solenoid,

$$\mathcal{E} = -\dot{\Phi}_S = -\frac{\partial}{\partial t}\left(\pi a^2 B_S\right)$$

Where $B_s = \mu_0 n I_s$. The EMF $\mathcal{E}$ must also equal $\mathcal{E} = I_r R$. Therefore,

$$I_r = -\frac{1}{R}\left(\mu_0 \pi a^2 n\right)\dot{I}_S$$

In order to calculate the electric and magnetic fields just outside solenoid, recognize that $\vec{B}_s = B_s(t)\hat{z}$ point along the axis of the solenoid. Similarly recognize that $\vec{E} = E\hat{\phi}$. Therefore the Poynting vector is given by,

$$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B}_s = \frac{1}{\mu_0}E B_s \hat{s} = ?$$

We first need to calculate $\vec{E}$ and $\vec{B}_s$. The magnetic field is known to be $\vec{B} = \mu_0 n I_s \hat{z}$ on axis and $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$,

$$\int \mathrm{d}\vec{a}\cdot\vec{\nabla}\times\vec{E} = -\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}\vec{a}\cdot\vec{B}$$

$$\int \mathrm{d}\vec{\ell}\cdot\vec{E} = -\dot{\Phi} = 2\pi a E$$

Which gives,

$$\vec{E} = \frac{\dot{\Phi}}{2\pi a}\hat{\phi}$$

The magnetic field off axis and outside the solenoid due to the ring is given by,

$$\mathrm{d}\vec{B}_r(s) = \frac{\mu_0 I}{4\pi}\frac{\mathrm{d}\vec{\ell}\times\vec{\mathscr{r}}}{\mathscr{r}^2}$$

Where $\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$ and we take $\vec{r}\,' = b\hat{s}'$ and $\vec{r} = z\hat{z}$.

$$\vec{\mathscr{r}} = z\hat{z} - b\hat{s}'$$

We will take the infinitesimal loop to be $\mathrm{d}\vec{\ell} = b\mathrm{d}\phi'\hat{\phi}'$.

$$\begin{aligned}\mathrm{d}\vec{\ell}\times\vec{\mathscr{r}} &= \left(b\hat{\phi}'\mathrm{d}\hat{\phi}'\right)\times(z\hat{z} - b\hat{s}') \\ &= az\mathrm{d}\phi'\hat{s}' + b^2\mathrm{d}\phi'\hat{z}\end{aligned}$$

We integrating around the loop $\mathcal{L}$, all of the contributions in the $\hat{s}'$ directions will cancel out.

$$\int_{\mathcal{L}} \mathrm{d}\vec{\ell}\times\vec{\mathscr{r}} = \cdots$$

Thus,

$$\begin{aligned}\vec{B}_r &= \frac{\mu_0 I_r}{4\pi}\int\frac{b^2\mathrm{d}\phi'\hat{z}}{(z^2+b^2)^{3/2}} \\ &= \frac{\mu_0 I_r b^2 2\pi\hat{z}}{4\pi(z^2+b^2)^{3/2}} \\ &= \frac{\mu_0 b^2}{2(z^2+b^2)^{3/2}}I_r\hat{z}\end{aligned}$$

Therefore the Poynting vector points radial outward,

$$\begin{aligned}\vec{S} &= \frac{1}{\mu_0}\vec{E}_r\times\vec{B}_r \\ &= \frac{1}{\mu_0}\left(\frac{\pi a^2\mu_0 n\dot{I}_s}{2\pi a}\hat{\phi}\right)\times\left(\frac{\mu_0 b^2}{2(z^2+b^2)^{3/2}}I_r\hat{z}\right) \\ &= \frac{\mu_0}{4}an\dot{I}_s\frac{b^2}{(z^2+b^2)^{3/2}}I_r\hat{s}\end{aligned}$$

Now that the Poynting vector is known, one can calculate the power radiated from the system.

$$\begin{aligned}P &= \int\mathrm{d}\vec{a}\cdot\vec{S} \\ &= \frac{\mu_0}{4}an\dot{I}_s I_r b^2\int\mathrm{d}za\mathrm{d}\phi\hat{s}\cdot\frac{1}{(z^2+b^2)^{3/2}}\hat{s}\end{aligned}$$

$$
\begin{aligned}
&= \frac{\mu_0}{4} a n \dot{I}_s I_r b^2 (2\pi a) \int \mathrm{d}z \frac{1}{(z^2+b^2)^{3/2}} \\
&= \frac{\mu_0}{4} a n \dot{I}_s I_r b^2 (2\pi a) \frac{2}{b^2} \quad \text{Integral Table} \\
&= \mu_0 \pi a^2 n \dot{I}_s I_r
\end{aligned}
$$

But we know that $\mu_0 \pi a^2 n \dot{I}_s = -I_r R$. Therefore $P = -I_r^2 R$ as expected.

A2.2:

a,b) Answers in Griffiths.

c) Consider parallel metal strips with height h and width w where $h \ll w$. A current flows down one plate and up the other. The system will act as a capacitor. The magnetic field outside will be zero and non-negative inside.

d) Griffiths 8.1

A2.3:

Positive and negative charge build up on the surfaces between the capacitor. Of course, there will be a time varying current $I(t)$, electric field $\vec{E}(t)$ and magnetic field $\vec{B}(t)$.

5 Stress Energy Tensor

Last week we looked at conservation laws and we found,

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \quad \text{(charge)}$$

and,

$$\vec{\nabla} \cdot \vec{S} + \frac{\partial u}{\partial t} + \vec{J} \cdot \vec{E} = 0 \quad \text{(energy)}$$

This week we will continue with momentum and angular momentum and then we will examine the Maxwell stress tensor; the field equivalent for force in Newton's second law. But first we will look at momentum.

5.1 Momentum

Consider two charges $+q_1$ and $-q_2$ with velocities $\vec{v}_1$ and $\vec{v}_2$. The electric field at point 2 due to charge 1 will be denoted $\vec{E}_1$. Analogously for $\vec{B}_1$. The net force acting on charge q_2 is then,

$$\vec{F}_{2;E} = q_2 \vec{E}_1 \qquad \vec{F}_{2;B} = q_2 \vec{v}_2 \times \vec{B}_1$$

$$\vec{F}_{1;E} = q_1 \vec{E}_2 \qquad \vec{F}_{1;B} = q_1 \vec{v}_1 \times \vec{B}_2$$

One will notice that $\vec{F}_{1;B}$ and $\vec{F}_{2;B}$ are not equal and opposite forces like $\vec{F}_{1;E}$ and $\vec{F}_{2;E}$ are. What does this say about Newton's third law?

$$\sum \dot{\vec{p}}_i = \sum \vec{F}_{\text{net}}$$

We forgot about the fact that the electric and magnetic fields carry not only energy (via $\vec{S}$) but momentum as well. Recall that for photons,

$$E = hf = \hbar\omega$$

$$p = \frac{h}{\lambda} = \hbar k$$

Therefore we have that,

$$E = pc$$

Therefore knowing the energy density of the field gives you then momentum density of the field. The momentum density will be denoted $\vec{g}$.

$$\vec{g} = \frac{1}{c^2} \vec{S} = \mu_0 \epsilon_0 \vec{S} = \frac{1}{4\pi c} \vec{E} \times \vec{B}$$

The force per unit volume $\vec{f} = \Delta \vec{F} / \Delta \tau$ acting on a particle is given by the Lorrentz force.

$$\vec{f} = \rho \vec{E} + \vec{J} \times \vec{B}$$

Which when expanded out is,

$$\vec{f} = \epsilon_0 \left(\vec{\nabla} \cdot \vec{E} \right) \vec{E} + \left(\frac{1}{\mu_0} \vec{\nabla} \times \vec{B} - \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \times \vec{B}$$

Todo (TC Fraser): Inject hand out Which after some algebra yields,

$$\vec{f} = \epsilon_0 \left(\left(\vec{\nabla} \cdot \vec{E} \right) \vec{E} + \left(\vec{E} \cdot \vec{\nabla} \right) \vec{E} \right) + \frac{1}{\mu_0} \left(\left(\vec{\nabla} \cdot \vec{B} \right) \vec{B} + \left(\vec{B} \cdot \vec{\nabla} \right) \vec{B} \right) - \vec{\nabla} \left(\frac{\epsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2 \right) - \mu_0 \epsilon_0 \frac{\partial \vec{S}}{\partial t} \tag{5.1}$$

We now introduce **Maxwell's stress energy tensor** T with components,

$$T_{ij} = \epsilon_0 \left(E_i E_j - \frac{1}{2} \delta_{ij} E^2 \right) + \frac{1}{\mu_0} \left(B_i B_j - \frac{1}{2} \delta_{ij} B^2 \right) \tag{5.2}$$

Where $\vec{E} = \sum_i E_i \hat{e}_i$ and $\vec{B} = \sum_i B_i \hat{e}_i$. We now have that eq. (5.1) gives,

$$\vec{f} = \vec{\nabla} \cdot T - \epsilon_0 \mu_0 \frac{\partial \vec{S}}{\partial t}$$

As defined $\vec{F} = \int_{\mathcal{V}} \mathrm{d}\tau \vec{f}$ is the net mechanical force acting on the matter in a volume $\mathcal{V}$. Therefore,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \vec{p}_{\mathrm{mech}} &= \int_{\mathcal{V}} \mathrm{d}\tau \vec{f} \\
&= \int_{\mathcal{V}} \mathrm{d}\tau \left(\vec{\nabla} \cdot T - \epsilon_0 \mu_0 \frac{\partial S}{\partial t} \right) \\
&= \oint_{\mathcal{S}} \mathrm{d}\vec{a} \cdot T - \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{V}} \mathrm{d}\tau \epsilon_0 \mu_0 \vec{S}
\end{aligned}$$

We usually define the second term here to be the momentum contained in the electromagnetic field,

$$\vec{p}_{\mathrm{em}} = \int_{\mathcal{V}} \mathrm{d}\tau \epsilon_0 \mu_0 \vec{S}$$

Therefore the conservation of momentum is,

$$\frac{\mathrm{d}}{\mathrm{d}t} (\vec{p}_{\mathrm{mech}} + \vec{p}_{\mathrm{em}}) = \oint_{\mathcal{S}} \mathrm{d}\vec{a} \cdot T \tag{5.3}$$

To draw intuition from continuum mechanics, the **Cauchy stress tensor** is a representation of the total forces acting on a chunk $\mathcal{V}$ of a material due to the neighboring pieces $\mathcal{N}(\mathcal{V})$. Each neighboring chunk $n(\mathcal{V})$ can exert parallel or shear forces on $\mathcal{V}$. This defines a matrix on force components on each face of $\mathcal{V}$. Let $\vec{f} = \sigma \cdot \mathrm{d}\vec{a}$ where $\vec{\sigma}$ is a rank 2 (3d) tensor. We call σ the Cauchy stress tensor such that,

$$\vec{f} = \sigma \cdot \mathrm{d}\vec{a}$$

The divergence of the Maxwell stress tensor is,

$$\frac{\partial}{\partial x_i} T_{ij} = \epsilon_0 \left(\frac{\partial E_i}{\partial x_i} E_j + E_i \frac{\partial E_j}{\partial x_i} - \frac{1}{2} \delta_{ij} \frac{\partial E^2}{\partial x_i} \right) + \frac{1}{\mu_0} \left(\frac{\partial B_i}{\partial x_i} B_j + B_i \frac{\partial B_j}{\partial x_i} - \frac{1}{2} \delta_{ij} \frac{\partial B^2}{\partial x_i} \right)$$

Which in vector notation is much simplier Todo (TC Fraser): Verify this expression,

$$\vec{\nabla} \cdot T = \epsilon_0 \left(\left(\vec{\nabla} \cdot \vec{E} \right) \vec{E} + \left(\vec{E} \cdot \vec{\nabla} \right) \vec{E} \right) - \frac{1}{2} \epsilon_0 \vec{\nabla} \cdot \vec{E} + \frac{1}{\mu_0} \left(\left(\vec{\nabla} \cdot \vec{B} \right) \vec{B} + \left(\vec{B} \cdot \vec{\nabla} \right) \vec{B} \right) - \frac{1}{2} \epsilon_0 \vec{\nabla} \cdot \vec{B}$$

While the force per unit volume is,

$$\vec{f} = \vec{\nabla} \cdot T - \epsilon_0 \mu_0 \frac{\partial \vec{S}}{\partial t}$$

Which can be integrated over a volume $\mathcal{V}$ in order to obtain the total force,

$$\begin{aligned}
\int_{\mathcal{V}} \mathrm{d}\tau \vec{f} &= \int_{\mathcal{V}} \mathrm{d}\tau \vec{\nabla} \cdot T \\
&= \int_{\mathcal{V}} \mathrm{d}\tau \vec{\nabla} \cdot T - \mu_0 \epsilon_0 \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{V}} \mathrm{d}\tau \vec{S} \\
\frac{\mathrm{d}\vec{p}_{\text{mech}}}{\mathrm{d}t} &= \oint_{\mathcal{S}} \mathrm{d}\vec{a} \cdot T - \frac{\mathrm{d}}{\mathrm{d}t} \underbrace{\int_{\mathcal{V}} \mathrm{d}\tau \vec{g}}_{\vec{p}_{\text{em}}}
\end{aligned}$$

Therefore we recover eq. (5.3) again,

$$\frac{\mathrm{d}}{\mathrm{d}t} (\vec{p}_{\text{mech}} + \vec{p}_{\text{em}}) = \oint_{\mathcal{S}} \mathrm{d}\vec{a} \cdot T$$

This conservation of momentum equation can be interpreted as Newton's second law for E&M.

Ex 8.4: Two point charges a distance 2ℓ apart. Due to the rotational symmetry of the problem, we can exploit cylindrical coordinates $\vec{r} = s\hat{s} + z\hat{z}$ because no physical quantities can depend on ϕ.
a) The electric field in a plane ($\phi = 0$) can be obtained as follows. Let $\vec{r}\,'$ be the location of the source q and $\vec{r}$ be the field location. The origin is between the two identical charges. Let $\vec{E}_+$ be the electric field due to the charge in the $z > 0$ direction.

$$\vec{r}\,'_\pm = \pm \ell \hat{z}$$

And on the axis perpendicular to $\hat{z}$,

$$\vec{r} = s\hat{s}$$

Therefore,

$$\vec{\mathscr{r}} = \vec{r} - \vec{r}\,' = s\hat{s} - \ell\hat{z}$$

Which gives electric field,

$$\begin{aligned}
\vec{E}_\pm &= \frac{q_\pm \vec{\mathscr{r}}_\pm}{4\pi\epsilon_0 \mathscr{r}_\pm^3} \\
&= \frac{q_\pm}{4\pi\epsilon_0} \frac{s\hat{s} \mp \ell\vec{z}}{(s^2 + \ell^2)^{3/2}}
\end{aligned}$$

Therefore,

$$\vec{E}_{\{z=0\}} = \vec{E}_+ + \vec{E}_- = \frac{q_\pm}{4\pi\epsilon_0} \frac{2s\hat{s}}{(s^2 + \ell^2)^{3/2}}$$

Upon reflection, the direction of $\vec{E}_{\{z=0\}}$ could have only been in the $\hat{s}$ direction by symmetry.
b) Calculate the Maxwell Stress Tensor using eq. (5.2). Notice that $\vec{=}B$ and $\vec{E}_{\{z=0\}} = E(s)\hat{s}$,

$$\hat{s} = \hat{x} \cos\phi + \hat{y} \sin\phi$$

So in Cartesian coordinates,

$$\vec{E}_{\{z=0\}} = \frac{q_\pm}{4\pi\epsilon_0} \frac{2\sqrt{x^2 + y^2}(\hat{x}\cos\phi + \hat{y}\sin\phi)}{(x^2 + y^2 + \ell^2)^{3/2}}$$

The components of $\vec{E}$ are then,

$$E_1 = E_x = \frac{q_\pm}{4\pi\epsilon_0} \frac{2\sqrt{x^2 + y^2}\cos\phi}{(x^2 + y^2 + \ell^2)^{3/2}}$$

$$E_2 = E_y = \frac{q_\pm}{4\pi\epsilon_0} \frac{2\sqrt{x^2+y^2}\sin\phi}{\left(x^2+y^2+\ell^2\right)^{3/2}}$$
$$E_3 = E_z = 0$$

For convenience let,

$$E_0 = \frac{q_\pm}{4\pi\epsilon_0} \frac{2\sqrt{x^2+y^2}}{\left(x^2+y^2+\ell^2\right)^{3/2}} \tag{5.4}$$

Such that,

$$E_1 = E_0\cos\phi \quad E_2 = E_0 \sin\phi$$

And also,

$$E^2 = E_0^2$$

Therefore the components of T are determined by eq. (5.2),

$$T = \epsilon_0 E_0^2 \begin{pmatrix} \cos^2\phi - \frac{1}{2} & \sin\phi\cos\phi & 0 \\ \cos\phi\sin\phi & \sin^2\phi - \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}$$

Which by trig-identities becomes,

$$T = \frac{1}{2}\epsilon_0 E_0^2 \begin{pmatrix} \cos(2\phi) & \sin(2\phi) & 0 \\ \sin(2\phi) & -\cos(2\phi) & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

c) Construct a closed hemisphere $\mathcal{H}$ above $z > 0$ enclosing the charge q_+ but not q_-. Since the charges are not moving, we have that,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{p}_{\mathrm{mech}} + \vec{p}_{\mathrm{em}}\right) = \vec{0}$$

Therefore it must be,

$$\oint_{\mathcal{H}} \mathrm{d}\vec{a} \cdot T = 0$$

However, $\mathcal{S}$ does *not* lie in the same plane as the Maxwell stress tensor computed above. Instead, we can take the radius R of the hemisphere to be $R \to \infty$ such that the "hemisphere" becomes a flat plane with a central circular region. The net force acting on q_+ is,

$$\vec{F}_+ = \int_{\mathcal{V}} \mathrm{d}\tau \vec{f} = \int_{\mathcal{V}} \mathrm{d}\vec{a}\vec{\nabla} \cdot T - \epsilon_0\mu_0 \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{V}} \mathrm{d}\tau \cancelto{0}{\vec{S}} = \int_{\mathcal{S}} \mathrm{d}\vec{a} \cdot T$$

Therefore,

$$\begin{aligned}\vec{F}_+ &= \int\limits_{-\infty}^{\infty} \mathrm{d}x \int\limits_{-\infty}^{\infty} \mathrm{d}y \frac{1}{2}\epsilon_0 E_0^2 \begin{pmatrix} 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \cos(2\phi) & \sin(2\phi) & 0 \\ \sin(2\phi) & -\cos(2\phi) & 0 \\ 0 & 0 & -1 \end{pmatrix} \\ &= \int\limits_{-\infty}^{\infty} \mathrm{d}x \int\limits_{-\infty}^{\infty} \mathrm{d}y \frac{1}{2}\epsilon_0 E_0^2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\end{aligned}$$

Where $E_0 = E_0(x, y)$ is given by eq. (5.4).

$$\begin{aligned}\vec{F}_+ &= \frac{1}{2}\epsilon_0 \hat{z} \int\limits_{-\infty}^{\infty} \mathrm{d}x \int\limits_{-\infty}^{\infty} \mathrm{d}y E_0^2 \\ &= \frac{1}{2} 2\pi\epsilon_0 \left(\frac{q}{2\pi\epsilon_0}\right)^2 \hat{z} \int\limits_{0}^{\infty} \mathrm{d}s \frac{s\left(s^2\right)}{\left(s^2+\ell^2\right)^3}\end{aligned}$$

$$= \cdots$$
$$= \frac{q^2 \hat{z}}{4\pi\epsilon_0 (2\ell)^2}$$

Which is simply a result of Coulomb's law which was expected.

5.2 Method of Images

8.4:

$$\vec{F} = \int_{\mathcal{S}} \mathrm{d}\vec{a} \cdot T$$

A3.1:

$$\vec{E}_{\text{vac}} \cdot \hat{t} = 0$$
$$\int \mathrm{d}\vec{\ell} \cdot \vec{E} = 0$$
$$\int \mathrm{d}\vec{a} \cdot \vec{\nabla} \times \vec{E} = -\frac{\mathrm{d}}{\mathrm{d}t} \int \mathrm{d}\vec{a} \cdot \vec{B} = 0$$
$$V(x, y, z > 0) = V(z, y, z) \quad V(x, y, z < 0) = 0$$

The boundary condition,

$$\vec{E} \cdot \hat{n} = \frac{\sigma}{\epsilon_0}$$

Gives the potential at $z = 0^+$.
Recall that the energy stored in the electromagnetic field is given by,

$$W = \frac{\epsilon_0}{2} \int \mathrm{d}\tau E^2$$

A3.?: The electromagnetic momentum,

$$\vec{g} = \mu_0 \epsilon_0 \vec{S} = \epsilon_0 \left(\vec{E} \times \vec{B} \right)$$

The electric field is in the z direction $\vec{E} = E\hat{z}$ and the magnetic field is $\vec{B} = B\hat{\phi}$. Therefore $\vec{S} = -S\hat{s}$. Now consider a charge q and a magnetic dipole $\vec{m}$ near each other. The magnetic field generated by $\vec{m}$ and the electric field generated by q generate a joint $\vec{S}$ field. The $\vec{S}$ forms closed circles around the system, meaning no energy is moving in or out of the system. Because of this, $\vec{g}$ is non-zero and is rotational around the system indicating that there is angular momentum stored in the field. The angular momentum density is then,

$$\vec{\ell} = \vec{r} \times \vec{g}$$

The angular momentum comes from *resisting* the magnetic force in a radial direction when trying to bring the charge q toward the dipole $\vec{m}$.
Feynman Vol. 2 17-4: Consider a plastic disk that is free to rotate and with surface charge σ (generating field $\vec{E}$). Then place a solenoid in the center of the disk and turn it on, generating a magnetic field $\vec{B}$. This field induces an electric field $\vec{E}'$ in the disk because of Maxwell's law,

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Generating some torque $\vec{N}$, the total angular momentum in the disk is given by,

$$\vec{L}_{\text{mech}} = \int_0^t \mathrm{d}t' \vec{N}$$

Turning the solenoid on and off transfers angular momentum from the mechanical system to the field system.
A3.3: (Griffiths 8.4, 8.21): Solenoid with radius R.

6 Waves

Todo (TC Fraser): Missed a lecture

Now that we have derived electromagnetic waves from Maxwell's equations in a vacuum,

$$\nabla^2 \vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0$$
$$\nabla^2 \vec{B} - \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2} = 0$$

Today, we will start by examining the 1D wave equation,

$$v^2 \frac{\partial^2 f}{\partial z^2} - \frac{\partial^2 f}{\partial t^2} = 0 \tag{6.1}$$

Where v is the wave speed and $f(z,t)$ is the *displacement* of the *medium* from its equilibrium position. To solve this second order PDE, D'Alembert came up with the substitution,

$$q_\pm = z \pm vt$$

Which has the inversion,

$$\begin{aligned} z &= \frac{1}{2}(q_+ + q_-) \\ v &= \frac{1}{2v}(q_+ - q_-) \end{aligned} \tag{6.2}$$

This substitution has the following chain rule,

$$\frac{\partial}{\partial z} = \frac{\partial q_+}{\partial z}\frac{\partial}{\partial q_+} + \frac{\partial q_-}{\partial z}\frac{\partial}{\partial q_-}$$
$$\frac{\partial}{\partial t} = \frac{\partial q_+}{\partial t}\frac{\partial}{\partial q_+} + \frac{\partial q_-}{\partial t}\frac{\partial}{\partial q_-}$$

So that eq. (6.1) becomes,

$$\begin{aligned} \frac{\partial^2 f}{\partial z^2} &= \left(\frac{\partial}{\partial q_+} + \frac{\partial}{\partial q_-}\right)^2 f \\ &= \frac{\partial^2 f}{\partial q_+^2} + \frac{\partial^2 f}{\partial q_-^2} + 2\frac{\partial^2 f}{\partial q_+ \partial q_-} \end{aligned}$$

Similarly,

$$\frac{\partial^2 f}{\partial t^2} = v^2\left(\frac{\partial^2 f}{\partial q_+^2} + \frac{\partial^2 f}{\partial q_-^2} - 2\frac{\partial^2 f}{\partial q_+ \partial q_-}\right)$$

Therefore eq. (6.1) becomes,

$$\frac{\partial^2 f}{\partial q_+ \partial q_-} = 0$$

Which has the general solution of a separable function,

$$f(q_+, q_-) = f_+(q_+) + f_-(q_-)$$

Or in terms of eq. (6.2),

$$f(z,t) = f_+(z + vt) + f_-(z - vt)$$

We will usually write this as:

$$f(z,t) = f_+(kz + \omega t) + f_-(kz - \omega t) \tag{6.3}$$

Where $v = \omega/k$ and ω is the temporal frequency and k is the spatial frequency. We shall see that the 3D generalization is easy to digest,

$$f_\pm(k_z z + k_y y + k_x x \pm \omega t) = f_\pm\left(\vec{k}\cdot\vec{r} \pm \omega t\right)$$

What we will see is that the **wave vector** $\vec{k}$ points in the same direction as the Poynting vector $\vec{S}$.

Due to the linearity of the wave equations, $v^2\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2}$ can be treated as linear operator $\hat{L}$ such that,

$$\hat{L}(\alpha f + \beta g) = \alpha\hat{L}(f) + \beta\hat{L}(g)$$

We can make use of Fourier and his friends so that we only have to solve the wave equation for one frequency (both spatial and temporal). Promote $f(z,t)$ to be a complex amplitude $\tilde{f}(z,t)$ and then decompose it using spectral analysis,

$$\tilde{f}(z,t) = \int_{-\infty}^{\infty} \mathrm{d}k e^{i(kz-\omega t)} f_k(t) = \mathcal{F}\left[\tilde{f}_k(t)\right](z,t) \tag{6.4}$$

Where $\tilde{f}_k(t)$ is a complex function of k and t which can be found using an inverse Fourier transform,

$$\tilde{f}_k(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}z e^{-i(kz-\omega t)}\tilde{f}(z,t)$$

To verify that this works, substitute in $\tilde{f}(z,t)$ using eq. (6.4),

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}z e^{-i(kz-\omega t)}\tilde{f}(z,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}z e^{-i(kz-\omega t)} \int_{-\infty}^{\infty} \mathrm{d}k' e^{i\left(k'z-\omega t\right)}\tilde{f}_{k'}(t) f_{k'}(t) \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k' \tilde{f}_{k'}(t) f_{k'}(t) \int_{-\infty}^{\infty} \mathrm{d}z e^{-i(kz-\omega t)} e^{i\left(k'z-\omega t\right)} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k' \tilde{f}_{k'}(t) f_{k'}(t) \int_{-\infty}^{\infty} \mathrm{d}z e^{-i\left(k-k'\right)z} \\
&= 2\pi\left(\frac{1}{2\pi}\right)\int_{-\infty}^{\infty} \mathrm{d}k' \tilde{f}_{k'}(t) f_{k'}(t)\delta(k-k') \\
&= \tilde{f}_k(t)
\end{aligned}$$

We write as shorthand,

$$\tilde{f}_k(t) = \mathcal{F}^{-1}\left[\tilde{f}(z,t)\right]_k(t)$$

Recovering the actual solution to the wave equation corresponds to taking the real part to $\tilde{f}$,

$$\Re\left[\tilde{f}(z,t)\right] = \Re\left[\left|\tilde{f}\right|e^{i\delta}\right] = \left|\tilde{f}\right|\cos(\delta)$$

Ex. 9.1: a)

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

Therefore,

$$\begin{aligned}
y_1(x,t) &= A_1\cos(kx-\omega t)\cos S_1 - A_1\sin(kx-\omega t)\sin S_1 \\
y_2(x,t) &= A_2\cos(kx-\omega t)\cos S_2 - A_2\sin(kx-\omega t)\sin S_2
\end{aligned}$$

Combining yields,

b)

$$y(x,t) = \underbrace{(A_1\cos S_1 + A_2\cos S_2)}_{a}\cos(kx-\omega t) - \underbrace{(A_1\sin S_1 + A_2\sin S_2)}_{b}\sin(kx-\omega t)$$

c)

$$y(x,t) = \sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}}\cos(kx-\omega t) - \frac{b}{\sqrt{a^2+b^2}}\sin(kx-\omega t)\right)$$

d)

$$y(x,t) = A\cos(kx - \omega t + \delta)$$

Where,

$$A = \sqrt{a^2 + b^2} \quad \delta = \tan^{-1}\left(\frac{b}{a}\right)$$

e)

$$y(x,t) = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos(\delta_1 - \delta_2)}\sin\left(kx - \omega t + \tan^{-1}\left(\frac{A_1\sin\delta_1 + A_2\sin\delta_2}{A_1\cos\delta_1 + A_2\cos\delta_2}\right)\right)$$

f)

$$y_1 = \Re\left\{A_1 e^{i\delta_1} e^{i(kx-\omega t)}\right\}$$
$$y_2 = \Re\left\{A_2 e^{i\delta_2} e^{i(kx-\omega t)}\right\}$$

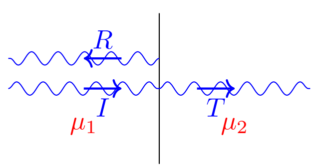
Abusing notation a little bit,

$$y = y_1 + y_2 = \Re\left\{\left(A_1 e^{i\delta_1} + A_2 e^{i\delta_2}\right) e^{i(kx-\omega t)}\right\}$$

$$\begin{aligned} A^2 &= \left(A_1 e^{i\delta_1} + A_2 e^{i\delta_2}\right)\left(A_1 e^{-i\delta_1} + A_2 e^{-i\delta_2}\right) \\ &= \cdots \\ &= A_1^2 + A_2^2 + 2A_1A_2\cos(\delta_1 - \delta_2) \end{aligned}$$

$$\begin{aligned} \tan\delta &= \frac{\Im\{A_1 e^{i\delta_1} + A_2 e^{i\delta_2}\}}{\Re\{A_1 e^{i\delta_1} + A_2 e^{i\delta_2}\}} \\ &= \cdots \\ &= \frac{A_1\sin\delta_1 + A_2\sin\delta_2}{A_1\cos\delta_1 + A_2\cos\delta_2} \end{aligned}$$

Ex. 9.2:



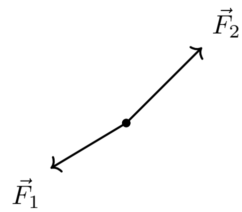
$$y_I(x,t) = A_I\sin(k_1(x - v_1 t))$$
$$y_R(x,t) = A_R\sin(k_1(x + v_1 t))$$
$$y_T(x,t) = A_T\sin(k_2(x - v_2 t))$$

a)

$$\omega_1 = \omega_2 \implies k_1 v_1 = k_2 v_2$$

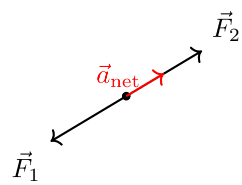
This means that the "knot" at the interface is a "good" knot – i.e. the string at either side of the interface ($x = 0$) move with the same frequency – i.e. the strings are connected.

b)



$$m\vec{a}_{\text{net}} = \vec{F}_1 + \vec{F}_2$$

If the slopes are the same then $\vec{F}_1 \parallel \vec{F}_2$ and so $\vec{a}_{\text{net}}$ is also parallel.



By the principle of superposition the equations for the waves in the first medium can be simply added together,

$$y_1(x,t) = y_I(x,t) + y_R(x,t)$$
$$y_2(x,t) = y_T(x,t)$$

Introduce the constraints of a good string,

$$y_1(x=0,t) = y_2(x=0,t)$$

Therefore,

$$A_I \sin(-k_1 v_1 t) + A_R \sin(k_1 v_1 t) = A_T \sin(-k_2 v_2 t)$$

Therefore,

$$-A_I + A_R = -A_T$$

Equivalently,

$$A_I = A_R + A_T$$

d) The next constraint imposes,

$$\frac{\partial}{\partial x} y_1(x=0,t) = \frac{\partial}{\partial x} y_2(x=0,t)$$
$$k_1 A_I \cos(-k_1 v_1 t) + k_1 A_R \cos(k_1 v_1 t) = k_2 A_T \cos(-k_2 v_2 t)$$

Thus,

$$k_1 A_I + k_2 A_R = k_2 A_T$$

Now solve for the reflected and transmitted A_R, A_T in terms of the given incidence amplitude. So,

$$A_R = \frac{v_1 - v_2}{v_1 + v_2} A_I$$

$$A_T = \frac{2v_2}{v_1 + v_2} A_I$$

e) If $v_2 > v_1$ then $A_R \propto -(A_I)$.

7 Polarization

So far, we have only considered 1D waves. However, the electric and magnetic fields are vector quantities. As we shall see shortly, EM radiation has field components that are perpendicular to the direction of propagation. We can thus write a traveling wave as

$$\tilde{\vec{f}}(z,t) = \tilde{f}_k \hat{n} e^{i(kz-\omega t)}$$

Where $e^{i(kz-\omega t)}$ acts as the traveling wave, $\hat{n}$ is the polarization and $\tilde{f}_k$ is the amplitude and phase. Note that $\tilde{f}_k$ is not a function of position $\vec{r}$ and time t. The possible polarizations of this plane wave are $\hat{x}, \hat{y}$ or a linear combination thereof. An important linear combination is,

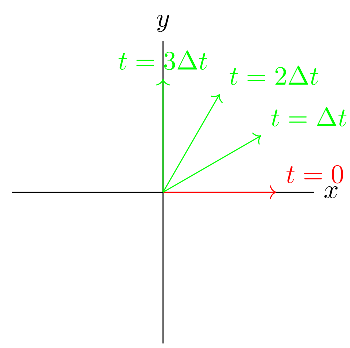
$$\hat{n}_\pm = \hat{x} \pm i\hat{y} = \hat{x} \pm e^{i\pi/2}\hat{y}$$

So that,

$$\hat{n}_\pm e^{i(kz-wt)} = \hat{x}e^{i(kz-wt)} \pm \hat{y}e^{i(kz-wt+\pi/2)}$$

Which makes the total wave,

$$\begin{aligned}\Re\left\{\tilde{f}_k \hat{n}_\pm e^{i(kz-wt)}\right\} &= \Re\left\{\left|\tilde{f}_k\right| e^{i\delta} \hat{n}_\pm e^{i(kz-wt)}\right\} \\ &= \left|\tilde{f}_k\right| \Re\{(\hat{x}+i\hat{y})(\cos(kz-wt+\delta)+i\sin(kz-wt+\delta))\} \\ &= \left|\tilde{f}_k\right| (\hat{x}\cos(kz-wt+\delta) - \hat{y}\sin(kz-wt+\delta))\end{aligned}$$



7.1 EM Plane Waves

While this is sufficient for a generic wave equation however, for those satisfying Maxwell's Equations there are further conditions to impose. In particular, for a plane wave traveling in the z-direction we have,

$$\tilde{\vec{E}}(z,t) = \tilde{\vec{E}}_0 e^{i(kz-\omega t)}$$

With derivatives,

$$\frac{\partial}{\partial x}\tilde{\vec{E}} = \frac{\partial}{\partial y}\tilde{\vec{E}} = \vec{0}$$

$$\frac{\partial}{\partial z}\tilde{\vec{E}} = ik\tilde{\vec{E}}$$

$$\frac{\partial}{\partial t}\tilde{\vec{E}} = -i\omega\tilde{\vec{E}}$$

We will write,

$$\tilde{\vec{E}}_0 = \hat{x}\tilde{E}_{0,x} + \hat{y}\tilde{E}_{0,y} + \hat{z}\tilde{E}_{0,z}$$

And similarly for the magnetic field,

$$\tilde{\vec{B}}_0 = \hat{x}\tilde{B}_{0,x} + \hat{y}\tilde{B}_{0,y} + \hat{z}\tilde{B}_{0,z}$$

Note that this is not the same as the electric field itself,

$$\tilde{E}_x = \tilde{\vec{E}} \cdot \hat{x} = \tilde{\vec{E}}_0 \cdot \hat{x} e^{i(kz-\omega t)} = \tilde{E}_{0,x} e^{i(kz-\omega t)}$$

Next make use of Gauss's law (in vacuum),

$$0 = \vec{\nabla} \cdot \vec{E} = \tilde{E}_{0,z} ike^{i(kz-\omega t)} \implies \tilde{E}_{0,z} = 0$$

Similarly for $\vec{B}$,

$$\tilde{B}_{0,z} = 0$$

So that EM waves are *transverse* and $\tilde{\vec{E}}_0$ only has components in the $\hat{x}$ and $\hat{y}$ directions.

Moreover make use of Faraday's law,

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Which gives,

$$-ik\hat{x}\tilde{E}_y + ik\hat{y}\tilde{E}_x = i\omega\left(\hat{x}\tilde{B}_x + \hat{y}\tilde{B}_y + \hat{z}\tilde{B}_z\right)$$

Therefore matching components,

$$\begin{aligned} -ik\tilde{E}_y &= i\omega\tilde{B}_x \\ ik\tilde{E}_x &= i\omega\tilde{B}_y \\ 0 &= i\omega\tilde{B}_z \end{aligned}$$

Which makes,

$$\begin{aligned} \tilde{E}_y &= -\frac{\omega}{k}\tilde{B}_x = -c\tilde{B}_x \\ \tilde{E}_x &= +\frac{\omega}{k}\tilde{B}_y = +c\tilde{B}_y \end{aligned}$$

And so $\tilde{\vec{E}}$ and $\tilde{\vec{B}}$ have the same phase but are in the different direction. The set $\left\{\tilde{\vec{E}}, \tilde{\vec{B}}, \hat{z}\right\}$ for a right-hand set. Recall that $\mu_0\vec{S} = \vec{E} \times \vec{B}$. We can generalize this to a plane wave traveling in an arbitrary direction by introducing the **wave vector** $\vec{k} = k_x\hat{x} + k_y\hat{y} + k_z\hat{z}$. Also,

$$\vec{k} = k\hat{k} = \sqrt{k_x^2 + k_y^2 + k_z^2}\hat{k}$$

The recurring relation $kz - \omega t$ becomes,

$$\vec{k} \cdot \vec{r} - \omega t$$

Monochrome plane waves in vacuum traveling in the $\hat{k}$ direction are,

$$\tilde{\vec{E}}(\vec{r}, t) = \tilde{\vec{E}}_0 e^{i\left(\vec{k}\cdot\vec{r} - \omega t\right)}$$

$$c\tilde{\vec{B}}(\vec{r}, t) = \hat{k} \times \tilde{\vec{E}}(\vec{r}, t)$$

7.2 Energy and Momentum

$$\begin{aligned} u_{\text{EM}} &= \frac{1}{2\mu_0}\left(\epsilon_0\mu_0 E^2 + B^2\right) \\ &= \frac{\epsilon_0}{2}\left(E^2 + c^2B^2\right) \\ &= \epsilon_0 E^2 \\ &= \epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta) \end{aligned}$$

$$\begin{aligned}
\vec{S} &= \frac{1}{\mu_0}\vec{E}\times\vec{B} \\
&= c\epsilon_0 E^2 \hat{z} \\
&= c u_{\text{EM}} \hat{z}
\end{aligned}$$