Stat 433

STOCHASTIC PROCESSES

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Version: 1.0

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1 Review

If $X \perp\!\!\!\perp Y$ then $\mathsf{Cov}(X,Y) = 0$ and,

$$\mathsf{Var}(X+Y) = \mathsf{Var}(X) + \mathsf{Var}(Y)$$

We see that independence implies uncorrelated, but uncorrelation does not imply independence.

Remark 1. We have that,

$$\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y) \tag{1.1}$$

If $X \perp\!\!\!\perp Y$ then we also have that,

$$\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y) \tag{1.2}$$

and that,

$$\mathsf{Var}(XY) = \mathsf{Var}(X)\mathsf{Var}(Y) \tag{1.3}$$

It is important to remember that the first result (eq. (1.1)) and the other two results (eqs. (1.2) and (1.3)) have a very different natures. The first is a consequence of the linearity in the definition of expectation and holds unconditionally. However eqs. (1.2) and (1.3) require that $X \perp\!\!\!\perp Y$. As such it is more appropriate to consider eqs. (1.2) and (1.3) as properties of independence rather than the properties of expectation and variance.

1.1 Indicator

A r.v. $\mathbf{1}$ is called an **indicator** for an event A if,

$$\mathbf{1}_A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}$$

The most important property of the indicator random variable is that the expectation of $\mathbf{1}_A$ is the same as the probability of the event A,

$$\mathbb{E}(\mathbf{1}_A) = P(A)$$

Proof. Since $\mathbf{1}_A$ is a Bernoulli random variable, the proof is easy. Consider,

$$\begin{aligned} P(\mathbf{1}_A = 1) &= P(\{\omega : \mathbf{1}_A(\omega) = 1\}) \\ &= P(\{\omega : \omega \in A\}) \\ &= P(A) \end{aligned}$$

Therefore the expectation of $\mathbf{1}_A$ must be,

$$\mathbb{E}(\mathbf{1}_A) = 1 \cdot P(\mathbf{1}_A = 1) + 0 \cdot P(\mathbf{1}_A = 0) = P(\mathbf{1}_A = 1) = P(A)$$

□

Example 1. We see $\mathbf{1}_A$ is just a Bernoulli random variable,

$$\mathbf{1}_A \sim \mathsf{Ber}(P(A))$$

Example 2. Let $X \sim \mathsf{Bin}(n,p)$; X is the number of successes in n Bernoulli trials, each with a probability p of success.

$$X = \mathbf{1}_1 + \cdots + \mathbf{1}_n \tag{1.4}$$

Where $\{\mathbf{1}_1, \ldots, \mathbf{1}_n\}$ are indicators for independent events. $\mathbf{1}_i = 1$ if the i-th trial is a success and $\mathbf{1}_i = 0$ if the i-th trial is a failure. Hence, I_i are **iid** (independent and identically distributed) r.v.s. It is known that the expectation of X is given by,

$$\mathbb{E}(X) = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$$

However eq. (1.4) yields the following approach,

$$\begin{aligned}\mathbb{E}(X) &= \mathbb{E}(\mathbf{1}_1 + \cdots + \mathbf{1}_n) \\ &= \mathbb{E}(\mathbf{1}_1) + \cdots + \mathbb{E}(\mathbf{1}_n) \\ &= n\mathbb{E}(\mathbf{1}_1) \\ &= np\end{aligned}$$

Moreover,

$$\begin{aligned}\mathsf{Var}(X) &= \mathsf{Var}(\mathbf{1}_1 + \cdots + \mathbf{1}_n) \\ &= \mathsf{Var}(\mathbf{1}_1) + \cdots + \mathsf{Var}(\mathbf{1}_n) \quad \text{Independence} \\ &= n\mathsf{Var}(\mathbf{1}_1) \\ &= np(1-p)\end{aligned}$$

The variance $\mathsf{Var}(I_1)$ is given by,

$$\mathsf{Var}(I_1) = \mathbb{E}\left(I_1^2\right) - (\mathbb{E}(I_1))^2$$

But notice that $I_1^2 = I_1$ is idempotent. Therefore,

$$\mathsf{Var}(I_1) = p - p^2 = p(1-p)$$

Example 3. Let X be a r.v. taking values in non-negative integers $\{0, 1, 2 \ldots\}$. Then we find that the expectation of X is given by,

$$\mathbb{E}(X) = \sum_{n=0}^{\infty} P(X > n)$$

Note that,

$$X = \sum_{n=0}^{\infty} \mathbf{1}_n$$

Where notationally $\mathbf{1}_n \equiv \mathbf{1}_{\{X>n\}}$. The intuition being that if $X = 3$, then $X = 1 + 1 + 1$ since $X = \underbrace{\mathbf{1}_0 + \mathbf{1}_1 + \mathbf{1}_2}_{3} + \underbrace{\mathbf{1}_3}_{0} + \ldots$.

$$\begin{aligned}\mathbb{E}(X) &= \mathbb{E}\left(\sum_{n=0}^{\infty} \mathbf{1}_n\right) \\ &= \sum_{n=0}^{\infty} \mathbb{E}(\mathbf{1}_n) \quad \text{Fubini's Theorem} \\ &= \sum_{n=0}^{\infty} P(X > n)\end{aligned}$$

Example 4. In particular let $X \sim \mathsf{Geo}(p)$ where $\mathbb{E}(X) = \sum_{k=0}^{\infty} k(1-p)^{k-1}p$. More easily we have seen that $P(X > n) = (1-p)^n$. Therefore by the geometric series,

$$\mathbb{E}(X) = \sum_{n=0}^{\infty} P(X > n) = \sum_{n=0}^{\infty} (1-p)^n = \frac{1}{1-(1-p)} = \frac{1}{p}$$

1.2 Moment Generating Function

Definition 1. Let X be a r.v. Then the function,

$$M(t) = \mathbb{E}\left(e^{tX}\right) \tag{1.5}$$

is called the **moment generating function (mgf)** if X if the expectation exists for all $t \in (-h, h)$ for some $h > 0$.

Remark 2. The moment generating function M is not always defined. It is important to check the existence of the expectation.
To compensate this, the latter condition in definition 1 is necessary because the expectation $\mathbb{E}(e^{tX})$ might not always exist for some t. Also notice that $M(0) = 1$ always.

We will now discuss some important properties of the moment generating function.

Theorem 1. *The moment generating function generates moments. For $t = 0$,*

$$M(0) = 1$$

Also,

$$M^{(k)}(0) \equiv \frac{\mathrm{d}^k}{\mathrm{d}t^k} M(t) \mid_{t=0}$$

Has the nice property,

$$M^{(k)}(0) = \mathbb{E}(X^k)$$

Proof. Evidently,

$$M(0) = \mathbb{E}(e^{0 \cdot X}) = \mathbb{E}(1) = 1$$

Moreover,

$$\begin{aligned}
M^{(k)}(0) &= \frac{\mathrm{d}^k}{\mathrm{d}t^k} M(t) \mid_{t=0} \\
&= \frac{\mathrm{d}^k}{\mathrm{d}t^k} \mathbb{E}(e^{tX}) \mid_{t=0} \\
&= \mathbb{E}\left(\frac{\mathrm{d}^k}{\mathrm{d}t^k} e^{tX} \mid_{t=0} \right) \quad \text{Dominant convergence theorem.} \\
&= \mathbb{E}\left(X \frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}} e^{tX} \mid_{t=0} \right) \\
&= \cdots \\
&= \mathbb{E}(X^k e^{tX} \mid_{t=0}) \\
&= \mathbb{E}(X^k)
\end{aligned}$$

□

As a result Taylor series gives,

$$\begin{aligned}
M(t) &= \sum_{k=0}^{\infty} \frac{M^{(k)}(0)}{k!} t^k \\
&= \sum_{k=0}^{\infty} \frac{\mathbb{E}(X^k)}{k!} t^k
\end{aligned}$$

Which is a method that can be used to obtain the moment of a mgf.

Theorem 2. *Let $X \perp\!\!\!\perp Y$ with mgfs M_x and M_y be respective mgfs. Let M_{X+Y} be the mgf of $X + Y$. Then,*

$$M_{X+Y} = M_X M_Y$$

Proof.

$$\begin{aligned}
M_{X+Y}(t) &= \mathbb{E}\left(e^{t(X+Y)}\right) \\
&= \mathbb{E}(e^{tX} e^{tY}) \\
&= \mathbb{E}(e^{tX}) \mathbb{E}(e^{tY}) \quad \text{Independence} \\
&= M_X(t) M_Y(t)
\end{aligned}$$

□

Theorem 3. *The moment generating function completely determines the distribution of a r.v.*

$$M_X(t) = M_Y(t) \quad \forall t \in (-h, h)$$

For some $h > 0$, then

$$X \overset{d}{=} Y$$

Which denotes that the random variables have the same distribution.

How can the moment generating function help?

Example 5. Let $X \sim \mathsf{Poi}(\lambda_1)$ and $Y \sim \mathsf{Poi}(\lambda_2)$ where $X \perp\!\!\!\perp Y$. Find the distribution of $X + Y$. To answer this, first derive the moment generating function of a Poisson distribution.

$$\begin{aligned}
M_X(t) &= \mathbb{E}\left(e^{tX}\right) \\
&= \sum_{n=0}^{\infty} e^{tn} P(X = n) \\
&= \sum_{n=0}^{\infty} e^{tn} \frac{\lambda_1^n}{n!} e^{-\lambda_1} \\
&= e^{-\lambda_1} \sum_{n=0}^{\infty} \frac{(e^t \lambda_1)^n}{n!} \\
&= e^{-\lambda_1} e^{\left(e^t \lambda_1\right)} \quad \text{Taylor series} \\
&= e^{\lambda_1\left(e^t - 1\right)}
\end{aligned}$$

Likewise, $M_Y(t) = e^{\lambda_2\left(e^t-1\right)}$. Therefore since $X \perp\!\!\!\perp Y$,

$$M_{X+Y}(t) = M_X(t) M_Y(t) = e^{(\lambda_1+\lambda_2)\left(e^t-1\right)}$$

Therefore by theorem 3, the distribution of $X + Y$ is the same distribution as $\mathsf{Poi}(\lambda_1 + \lambda_2)$.

In general, if $X_1, X_2, \ldots, X_n$ are independent and $X_i \sim \mathsf{Poi}(\lambda_i)$, then,

$$\sum_{i=1}^{n} X_i \sim \mathsf{Poi}\left(\sum_{i=1}^{n} \lambda_i\right)$$

Definition 2. Moreover, we define the **joint moment generating function (jmgf)** for X, Y random variables to be,

$$M(t_1, t_2) = \mathbb{E}\left(e^{t_1 X + t_2 Y}\right)$$

Provided that the expectation exists for $t_1 \in (-h_1, h_1)$ and $t_2 \in (-h_2, h_2)$ for $h_1, h_2 > 0$.

2 DTMC

2.1 Review of Probability

A **random variable (r.v.)** X is a real valued function of the outcomes of a random experiment.

$$X : \Omega \to R$$

Where $\Omega = \{\omega_1, \omega_2, \ldots\}$ is the **sample space** corresponding to all possible outcomes ω_i. The outcomes can in principle be any objects (numbers, strings, etc.). We say that X maps each outcome ω to a real number $\omega \mapsto X(\omega) \in \mathbb{R}$.

A **stochastic process** is a family of random variables $\{X_t\}_{t\in T}$, defined on a common sample space Ω. T is referred to as the index set for the stochastic process which is often understood as time. The index set T can take a discrete spectrum,

$$T = \{0, 1, 2, \ldots\} \qquad \{X_n \mid n = 0, 1, 2, \ldots\}$$

Alternatively, T can take on a continuous spectrum,

$$T = \{t \mid t \geq 0\} = [0, \infty)$$

The **state space** S is the collection of all possible values of X_t's. It is important to understand the distinction of between sample space and state space. Additionally, the state space can either have discrete or continuous spectrum.

A question remains, *Why do we need the family of random variables to be defined on a common sample space?* The answer being that we would like to be able to discuss the joint behaviour of X_t's. If X_1 has domain Ω_1 and X_2 has domain Ω_2 (where $\Omega_1 \neq \Omega_2$), then one can *not* talk about common ideas of correlations and associations between X_1 and X_2. As such we assert that all members of a stochastic process share the same sample space domain Ω.

2.2 Discrete-time Markov Chain

A **discrete-time stochastic process** $\{X_n \mid n \in 0, 1, 2, \ldots\}$ is said to be a **Discrete-time Markov Chain (DTMC)** if the following conditions hold:

1. The state space is at most *countable*[1] (i.e. finite or countable).

$$S = \{0, 1, \ldots, k\} \quad \text{or} \quad S = \{0, 1, 2, \ldots\}$$

2. **Markov Property**: For any $n = 0, 1, 2, \ldots$,

$$P(X_{n+1} = x_{n+1} \mid X_n = x_n, X_{n-1} = x_{n-1}, \ldots, X_0 = x_0) = P(X_{n+1} = x_{n+1} \mid X_n = x_n)$$

We use capital letters X to denote the random variable and lower case letters x to denote a specific realization or valuation of X. The motivation of the Markov property is that future events $X_{n+1} = x_{n+1}$ are independent of past histories $\{X_i = x_i \mid i = 0, 1, \ldots, n-1\}$ given the immediate past state $X_n = x_n$. The intuition being that the future and the past are probabilistically independent.

Given the present, the future and the past are independent.

2.3 Transition Probability

The **transition probability** from a state $i \in S$ at time n to state $j \in S$ (at time $n+1$) is given by,

$$P_{n,i,j} \equiv P(X_{n+1} = j \mid X_n = i) \qquad n = 0, 1, 2, \ldots \tag{2.1}$$

In full generality, the transition probability could depend on time n but in this course we will restrict ourselves to transition probabilities that *do not* depend on time n ($P_{n,i,j} = P_{i,j}$). We say that the MC is **(time-)homogeneous** if this property holds. From now on, this will be our default setting.

The matrix of all transition probabilities $P = \{P_{i,j} \mid i, j \in S\}$ is called the **one-step transition (probability) matrix** for $\{X_n \mid n \in T\}$.

$$P = \begin{pmatrix} P_{00} & P_{01} & \cdots & P_{0j} & \cdots \\ P_{10} & P_{11} & \cdots & P_{1j} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \cdots \\ P_{i0} & P_{i1} & \cdots & P_{ij} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

[1]Countable meaning there is a one-to-one mapping from the state space to the natural numbers.

The one-step transition matrix P has the following properties,

$$P_{i,j} \geq 0 \tag{2.2}$$

$$\forall i : \sum_{j \in S} P_{ij} = 1 \tag{2.3}$$

The entries are non-negative because they represent probabilities and the row sums for P are always unitary.

The **n-step transition probability** is defined via the homogeneous property,

$$\forall i, j \in S : P_{ij}^{(n)} \equiv P(X_{n+m} = j \mid X_n = i) = P(X_n = j \mid X_0 = i)$$

Analogously, the **n-step transition matrix** is the matrix,

$$P^{(n)} = \left\{ P_{ij}^{(n)} \mid i, j \in S \right\}$$

Theorem 4. *There is a simple relation between the n-step transition matrix $P^{(n)}$ and the one step transition matrix P.*

$$P^{(n)} = P^{(n-1)} \cdot P = \underbrace{P \cdot P \cdot \dots \cdot P}_{n} = P^n$$

Proof. Proof by induction:

$$P^{(1)} = P \quad \text{By definition.}$$

We also have $P^{(0)} = P^0 = \mathbf{1}$ is the identity matrix. We now assume $P^{(n)} = P^n$. Then $\forall i, j \in S$,

$$\begin{aligned}
P_{ij}^{(n+1)} &= P(X_{n+1} = j \mid X_0 = i) \\
&= \sum_{k \in S} P(X_{n+1} = j, X_n = k \mid X_0 = i) \quad \text{Total probability} \\
&= \sum_{k \in S} \frac{P(X_{n+1} = j, X_n = k, X_0 = i)}{P(X_0 = i)} \\
&= \sum_{k \in S} \frac{P(X_{n+1} = j, X_n = k, X_0 = i)}{P(X_n = k, X_0 = i)} \frac{P(X_n = k, X_0 = i)}{P(X_0 = i)} \\
&= \sum_{k \in S} P(X_{n+1} = j \mid X_n = k, X_0 = i) \cdot P(X_n = k \mid X_0 = i) \quad \text{Conditional total probability} \\
&= \sum_{k \in S} P(X_{n+1} = j \mid X_n = k) \cdot P(X_n = k \mid X_0 = i) \quad \text{Use Markov Property} \\
&= \sum_{k \in S} P_{kj} \cdot P_{ik}^{(n)} \quad \text{Matrix terms} \\
&= \left(P \cdot P^{(n)} \right)_{ij} \quad \text{Matrix product} \\
&= \left(P^{n+1} \right)_{ij} \quad \text{Inductive Hypothesis}
\end{aligned}$$

There we have proved that $P^{(n+1)} = P^{n+1}$ and so we have a complete proof that $P^{(n)} = P^n$.

□

Corollary 5. *As a corollary, we have obtained that,*

$$P^{(n)} = P^{(m)} \cdot P^{(n-m)} \quad \forall 0 \leq m \leq n$$

*Or equivalently the **Chapman-Kolmogorov Equation** or simply C-K equation,*

$$P_{ij}^{(n)} = \sum_{k \in S} P_{ik}^{(m)} P_{kj}^{(n-m)} \quad \forall i, j \in S, \forall 0 \leq m \leq n \tag{2.4}$$

So far, we have only been discussing transition probabilities. We will now divert our attention to actual distributions for a stochastic process.

Let $\alpha_n = (\alpha_{n,0}, \alpha_{n,1}, \dots)$ be the **probability distribution vector** for X_n at time n.

$$\alpha_{n,k} = P(X_n = k) \qquad \forall k \in S$$

Note that $\alpha_{n,k} \geq 0$ and $\sum_{k\in S} \alpha_{n,k} = 1$ and $n = 0, 1, 2, \dots$. We also define the initial distribution α_0,

$$\alpha_0 = (P(X_0 = 0), P(X_0 = 1), \dots)$$

Theorem 6. *The transition probability matrix reveals the following relationship between the distribution α_n at time n and the distribution α_0 at time 0,*

$$\alpha_n = \alpha_0 \cdot P^n \tag{2.5}$$

Proof. The proof eq. (2.5) is quite trivial:

$$\begin{aligned}
\forall j \in S \quad \alpha_{n,j} &= P(X_n = j) \\
&= \sum_{i\in S} P(X_n = j \mid X_0 = i) \cdot P(X_0 = i) \\
&= \sum_{i\in S} \alpha_{0,i} \cdot P_{ij}^n \\
&= \alpha_{0,0} \cdot P_{0j}^n + \alpha_{0,1} \cdot P_{1j}^n + \dots \\
&= (\alpha_0 \cdot P^n)_j
\end{aligned}$$

□

More generally, for any $n = 1, 2, \dots$ the finite dimensional distribution can be obtained from the following process iterative process,

$$\begin{aligned}
&P(X_n = x_n, X_{n-1} = x_{n-1}, \dots, X_0 = x_0) = \\
&\qquad P(X_0 = x_0) \cdot \\
&\qquad P(X_1 = x_1 \mid X_0 = x_0) \cdot \\
&\qquad P(X_2 = x_2 \mid X_1 = x_1, X_0 = x_0) \cdots \\
&\qquad P(X_n = x_n \mid X_{n-1} = x_{n-1}, \dots, X_0 = x_0)
\end{aligned}$$

But by the Markov condition, it must be that,

$$\begin{aligned}
&P(X_n = x_n, X_{n-1} = x_{n-1}, \dots, X_0 = x_0) = \\
&\qquad P(X_0 = x_0) \cdot \\
&\qquad P(X_1 = x_1 \mid X_0 = x_0) \cdot \\
&\qquad P(X_2 = x_2 \mid X_1 = x_1) \cdots \\
&\qquad P(X_n = x_n \mid X_{n-1} = x_{n-1})
\end{aligned}$$

First recognize the first term on the RHS ($P(X_0 = x_0) = \alpha_{0,x_0}$), and also the remaining terms are transition probabilities as per eq. (2.1). Therefore it must be that,

$$P(X_n = x_n, X_{n-1} = x_{n-1}, \dots, X_0 = x_0) = \alpha_{0,x_0} P_{x_0 x_1} P_{x_1 x_2} \cdots P_{x_{n-1} x_n}$$

Even more generally, for $0 \leq t_1 < t_2 < \dots < t_n$,

$$P\big(X_{t_n} = x_{t_n}, X_{t_{n-1}} = x_{t_{n-1}}, \dots, X_{t_1} = x_{t_1}\big) = P(X_{t_1} = x_{t_1}) \big(P^{t_2 - t_1}\big)_{x_{t_1} x_{t_2}} \big(P^{t_3 - t_2}\big)_{x_{t_2} x_{t_3}} \cdots \big(P^{t_n - t_{n-1}}\big)_{x_{t_{n-1}} x_{t_n}}$$

Since $P(X_{t_1} = x_{t_1}) = \alpha_{t_1 x_{t_1}} = \sum_{k\in S} \alpha_{0,k} P_{k,x_{t_1}}^{t_1}$,

$$\alpha_{t_1} = \alpha_0 \cdot P^{t_1}$$

This means the probabilistic properties of a DTMC are fully characterized by two things:

1. The initial distribution α_0
2. Transition matrix P

2.4 Classification of States

State j is **accessible** from state i (denoted $i \to j$) if there exists $n = 0, 1, \dots$ such that $P_{ij}^{(n)} > 0$. Intuitively, one can transition from state i to state j in finite steps n with positive probability. If i is also accessible from j, then we say i and j **communicate**, denoted as $i \leftrightarrow j$.

$$i \leftrightarrow j \Leftrightarrow \exists m, n \geq 0, P_{ij}^{(m)} > 0, P_{ji}^{(n)} > 0$$

Theorem 7. *The binary communication relation "$\leftrightarrow$" is in fact a equivalence relation:*

- *Reflexivity* $i \leftrightarrow i$
- *Symmetry* $i \leftrightarrow j \implies j \leftrightarrow i$
- *Transitivity* $i \leftrightarrow j, j \leftrightarrow k \implies i \leftrightarrow k$

Proof. First, reflexivity is easy to prove by definition. Let $n = 0$ and recognize that $P_{ii}^{(0)}$ has a certain probability by definition,

$$P_{ii}^{(0)} = 1 \implies i \leftrightarrow i$$

Second, symmetry follows by definition,

$$P_{ij}^{(m)} > 0, P_{ji}^{(n)} > 0 \Leftrightarrow P_{ji}^{(n)} > 0, P_{ij}^{(m)} > 0$$

Third, transitivity can be proving by letting m and n be the unknown quantifiers:

$$\exists m \quad P_{ij}^{(m)} > 0, \exists n \quad P_{jk}^{(n)} > 0$$

Then by the CK equation eq. (2.4),

$$P_{ik}^{(m+n)} = \sum_{l \in S} P_{il}^{(m)} P_{lk}^{(n)}$$

Let $l = j$ be a single, fixed entry in the summation,

$$P_{ik}^{(m+n)} \geq P_{ij}^{(m)} P_{jk}^{(n)} > 0$$

Therefore we have that k is accessible from i ($i \to j$). Analogously we have that $i \to j$ therefore $i \leftrightarrow k$. □

The communication equivalence relations then divides the state space S into different equivalence classes. That is, the states in one class comm with each other; the states in different classes do not comm. The equivalent classes form a *partition* of the state space S.

The family $\{S_1, S_2, \dots S_n\}$ is a **partition** of S if,

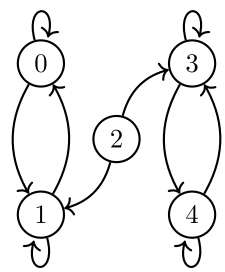
1. $S_i \subset S \mid \forall i \in 1, 2, \dots, n$
2. $S_i \cap S_j \neq \emptyset$ for all $i \neq j$
3. $\bigcup_i S_i = S$

We can find the equivalent classes by drawing a graph where the states in S are the nodes of the graph and a directed edge is placed going from i to j if j is accessible from i in one-step: $P_{ij} > 0$. Then identifying the the equivalent classes corresponds to identifying the loops of this graph within one step.

Example 6. As an example, consider the transition matrix P as follows.

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0.2 & 0.8 & 0 & 0 & 0 \\ 0.6 & 0.4 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.7 & 0.3 \\ 0 & 0 & 0 & 0.1 & 0.9 \end{pmatrix} \end{array}$$

The associated one-step accessibility graph is then,



Where the loops of $S = \{0, 1, 2, 3, 4\}$ form the following partition,

$$S_1 = \{0, 1\} \quad S_2 = \{2\} \quad S_3 = \{3, 4\}$$

These equivalent classes are useful for Markov chains because it allows one to separate the behaviour of the equivalence classes and study them individually. A MC which has only one equivalent class is called **irreducible**.

Furthermore, let us define the **period** of state i as,

$$d(i) = \gcd\left\{n \in \mathbb{Z}^+ \mid P_{ij}^n > 0\right\}$$

Additionally, if $P_{ii}^n = 0$ holds for all $n > 0$, we say that $d(i) = \infty$. If the period of i happens to be $d(i) = 1$ then the state i is said to be **aperiodic**. Alternatively, locus of steps that we can go back by are *co-prime*. A MC is called aperiodic if all its states S are aperiodic.

The period of a state is useful do to the following theorem,

Theorem 8. *The period of a state is a class property. If $i \leftrightarrow j$, then $d(i) = d(j)$.*

Proof. If $i = j$ we are already done. If $i \neq j$, since $i \leftrightarrow j$, then $\exists n, m$ such that,

$$P_{ij}^n > 0 \quad P_{ji}^m > 0$$

Then for any l such that $P_{jj}^l > 0$,

$$P_{ii}^{n+m+l} \geq P_{ij}^n P_{jj}^l P_{ji}^m \tag{2.6}$$

Because $P_{ij}^n P_{jj}^l P_{ji}^m$ happens to be a specific way for P_{ii}^{n+m+l} to occur. Since $i \leftrightarrow j$ and l was chosen carefully,

$$P_{ii}^{n+m+l} > 0$$

Moreover, we also have that,

$$P_{ii}^{n+m} \geq P_{ij}^n P_{ji}^m \tag{2.7}$$

Since $d(i)$ divides both $n + m$ and $n + m + l$ by eqs. (2.7) and (2.6), then $d(i)$ also divides l. This holds for all l such that $P_{ii}^l > 0$. This implies that $d(i)$ is a common divisor of $\left\{l \mid P_{jj}^l > 0\right\}$ an thus $d(i)$ divides,

$$d(j) = \gcd\left\{l \mid P_{jj}^l > 0\right\}$$

By symmetry $d(j)$ divides $d(i)$. Therefore $d(i) = d(j)$.

$\square$

Remark 3. It is important to note that $d(i) = k \not\Rightarrow P_{ii}^{(k)} > 0$. As a counterexample consider the following one step accessibility graph,

$$\begin{tikzcd} 0 \arrow[r, bend right] \arrow[dr, bend right] & 1 \arrow[l, bend right] \arrow[d] \\ & 2 \end{tikzcd}$$

Evidently $P_{00} = 0$ but we have $d(0) = 1$ because $d(0) = \gcd\{2, 3, \ldots\}$.

Remark 4. If the MC is irreducible (having only one class) then all the states have the same period. In this case we ascribe the entire MC the period $d(i)$ for some representative $i \in S$.

2.5 Recurrence and Transience

For $n \in \mathbb{Z}^+$ define,

$$f_{ij}^{(n)} = P(X_n = j, X_{n-1} \neq j, \cdots, X_1 \neq j \mid X_O = i) \quad \forall i, j \in S$$

Intuitively, $f_{ij}^{(n)}$ is the probability that X visits state j at time n for the first time since $X_0 = i$. A looming question: What is the relation between $f_{ij}^{(n)}$ and $P_{ij}^{(n)}$? First notice that,

$$P_{ij}^{(n)} \geq f_{ij}^{(n)}$$

These reads: the probability that X visits j at time n is more larger that the probability that X visits j at time n provided it did not visit j prior. A more detailed equality is the following,

$$P_{ij}^{(n)} = \sum_{k=1}^{n} f_{ij}^{(k)} P_{jj}^{(n-k)} \tag{2.8}$$

Expanded out gives,

$$P_{ij}^{(n)} = f_{ij}^{(n)} + \sum_{k=1}^{n-1} f_{ij}^{(k)} P_{jj}^{(n-k)}$$

Proof.

$$\begin{aligned}
P_{ij}^{(n)} &= P(X_n = j \mid X_0 = i) \\
&= \sum_{k=1}^{n} P(X_n = j, X \text{ first visits } j \text{ at time } k \mid X_0 = i) \\
&= \sum_{k=1}^{n} P(X_n = j, \mid X \text{ first visits } j \text{ at time } k, X_0 = i) \cdot P(X \text{ first visits } j \text{ at time } k \mid X_0 = i) \\
&= \sum_{k=1}^{n} P(X_n = j, \mid X_k = j, X_{k-1} \neq j, \ldots, X_1 \neq j, X_0 = i) \cdot P(X_k = j, X_{k-1} \neq j, \ldots, X_1 \neq j \mid X_0 = i) \\
&= \sum_{k=1}^{n} P(X_n = j, \mid X_k = j, X_{k-1} \neq j, \ldots, X_1 \neq j, X_0 = i) \cdot f_{ij}^{(k)} \\
&= \sum_{k=1}^{n} P(X_n = j, \mid X_k = j) \cdot f_{ij}^{(k)} \quad \text{Markov Condition} \\
&= \sum_{k=1}^{n} P_{jj}^{(n-k)} \cdot f_{ij}^{(k)}
\end{aligned}$$

□

In fact eq. (2.8) defines a recurrence relation to compute $f_{ij}^{(n)}$ from $f_{ij}^{(k)}$ where $k < n$,

$$f_{ij}^{(n)} = P_{ij}^{(n)} - \sum_{k=1}^{n-1} f_{ij}^{(k)} P_{jj}^{(n+k)}$$

We now define f_{ij} *without* the superscript to be,

$$f_{ij} = \sum_{n=1}^{\infty} f_{ij}^{(n)}$$

The probability that X will *ever* reach state $j \in S$ provided it started at i ($f_{ij} \leq 1$). Whether or not f_{ij} is certain or not defines the following two properties.

A state i is called **transient** if $f_{ii} < 1$; and **recurrent** if $f_{ii} = 1$. Intuitively, f_{ii} is the probability the MC returns to state i given it started in state i. If i is transient, then there is a non-negative probability that the MC does not return to i and if $f_{ii} = 1$ then the MC always returns to state i.

Another way to characterize recurrence and transience: Define M_i to be the total number of times the MC (re)visits i after time 0. In more mathematical terms,

$$M_i = \sum_{n=1}^{\infty} \mathbf{1}_{[X_n = i]}$$

Where $\mathbf{1}_{[X_n=i]}$ is the indicator defined by,

$$\mathbf{1}_{[X_n=i]} = \begin{cases} 1 & X_n = i \\ 0 & X_n \neq i \end{cases}$$

If $f_{ii} < 1$ we have that the probability of visiting state i k times is given by,

$$P(M_i = k \mid X_0 = i) = \underbrace{f_{ii} \cdot f_{ii} \cdots f_{ii}}_{k} \underbrace{(1 - f_{ii})}_{\text{never return}}$$

Where $(1 - f_{ii})$ is necessary because it guarantees that we never return to state i more that k times. Given $X_0 = i$, M_i follows a geometric distribution with parameter $(1 - f_{ii})$. Thus,

$$\mathbb{E}(M_i \mid X_0 = i) = \frac{f_{ii}}{1 - f_{ii}} < \infty$$

Therefore if i is transient, there a finite number revisits are expected. In contrast if $f_{ii} = 1$ we have that,

$$\mathbb{E}(M_i \mid X_0 = i) = \lim_{f_{ii} \to 1} \frac{f_{ii}}{1 - f_{ii}} \to \infty$$

Alternatively, we can look at $\mathbb{E}(M_i \mid X_0 = i)$ as,

$$\mathbb{E}(M_i \mid X_0 = i) = \sum_{k=1}^{\infty} P(M_i \geq k \mid X_0 = i) \tag{2.9}$$

The proof of eq. (2.9) is left as an exercise to the reader. Clearly if $f_{ii} = 1$,

$$P(M_i \geq k \mid X_0 = i) = {f_{ii}}^k = 1 \quad \forall k \tag{2.10}$$

Therefore,

$$\mathbb{E}(M_i \mid X_0 = i) = \sum_{k=1}^{\infty} 1 = \infty$$

Theorem 9. *Therefore i is recurrent if and only if $P(M_i \geq k \mid X_0 = i) = \infty$ and i is transient if and only if only if $P(M_i \geq k \mid X_0 = i) < \infty$.*

Remark 5. We actually also have that i is recurrent if and only if $M_i = \infty$. This can be seen from eq. (2.10). Since $P(M_i \geq k \mid X_0 = i)$ is strictly positive for all k, then $M_i = \infty$. Analogously, we have that i is transient if and only if $M_i < \infty$.

Yet *another* way to characterize recurrence and transience is much more tractable. First,

Theorem 10. *The expectation of the indicator is given by $\mathbb{E}(\mathbf{1}_A) = P(A)$ for any event A.*

Therefore,

$$\begin{aligned}
\mathbb{E}(M_i \mid X_0 = i) &= \mathbb{E}\left(\sum_{n=1}^{\infty} \mathbf{1}_{[X_n = i]} \mid X_0 = i\right) \\
&= \sum_{n=1}^{\infty} \mathbb{E}\big(\mathbf{1}_{[X_n = i]} \mid X_0 = i\big) \quad \text{Fubini's Theorem} \\
&= \sum_{n=1}^{\infty} P(X_n = i \mid X_0 = i) \\
&= \sum_{n=1}^{\infty} P_{ii}^{(n)}
\end{aligned}$$

Thus i is recurrent if and only if $\sum_{n=1}^{\infty} P_{ii}^{(n)} = \infty$ and i is transient if and only if $\sum_{n=1}^{\infty} P_{ii}^{(n)} < \infty$.

Theorem 11. *Recurrence/transience are class properties. If $i \leftrightarrow j$ and i is recurrent, then j is recurrent.*

Proof. Since $i \leftrightarrow j$, $\exists m, n \geq 0$ such that,

$$P_{ij}^{(m)} > 0 \qquad P_{ji}^{(n)} > 0$$

We now what to show that $\sum_{s=1}^{\infty} P_{jj}^{(s)}$ is infinite,

$$\sum_{s=1}^{\infty} P_{jj}^{(s)} \geq \sum_{s=n+m+1}^{\infty} P_{jj}^{(s)}$$

Now exchange of variables $l = s - n - m$,

$$\sum_{s=1}^{\infty} P_{jj}^{(s)} \geq \sum_{l=1}^{\infty} P_{jj}^{(n+l+m)}$$

Then by the eq. (2.4),

$$\sum_{l=1}^{\infty} P_{jj}^{(n+l+m)} \geq \sum_{l=1}^{\infty} P_{ji}^{(n)} P_{ii}^{(l)} P_{ij}^{(m)} = P_{ji}^{(n)} P_{ij}^{(m)} \left\{\sum_{l=1}^{\infty} P_{ii}^{(l)}\right\}$$

But since i is recurrent, $\sum_{l=1}^{\infty} P_{ii}^{(l)} = \infty$. Also, $P_{ji}^{(n)} P_{ij}^{(m)} > 0$ by the choice of m, n. Therefore $\sum_{l=1}^{\infty} P_{jj}^{(n+l+m)} = \infty$ and thus $\sum_{s=1}^{\infty} P_{jj}^{(s)} = \infty$. Therefore j is also recurrent. □

Corollary 12. *If $i \leftrightarrow j$ and i is transient, then j is transient.*

As a result, if we know that if a MC is irreducible (admitting only one class), then either all states are transient or they are all recurrent. Also, it is *impossible* for all states to be transient if the state space S is finite. If all states are transient then each state $i \in S$ has a time k that is the *last* visit time for all states, this is impossible because $P_{ij} \neq 0$ for at least some choice $i, j \in S$.

Theorem 13. *If i is recurrent, and i does not communicate with j, then $P_{ij} = 0$.*

Proof. Proof by contradiction. Assume that $P_{ij} > 0$. Since i and j do not communicate, then either j is not accessible from i or vice versa. But if $P_{ij} > 0$ then j is accessible from i. It must be that i is not accessible from j. Recall that f_{ii} is the probability that the MC ever revisits the state i given the starting state was i. Therefore $1 - f_{ii}$ is the probability that the MC never revisits state i.

$$f_{ii} \leq 1 - P_{ij} < 1$$

This inequality holds because if $X_1 = j$ then the MC never revisits i (i is not accessible from j). But there are other ways it never revisits i. Therefore,

$$P(X_1 = j \mid X_0 = i) = P_{ij} \leq P(\text{MC never revisits } i \mid X_0 = i)$$

But if $f_{ii} < 1$, then i is not recurrent; it is transient. Therefore the assumption that $P_{ij} > 0$ is wrong; $P_{ij} = 0$. □