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MATH 237

Calculus III

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Table of Contents

Abstract

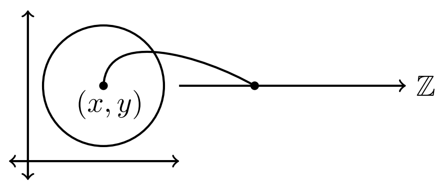
These notes are intended as a resource for myself; past, present, or future students of this course, and anyone interested in the material. The goal is to provide an end-to-end resource that covers all material discussed in the course displayed in an organized manner. If you spot any errors or would like to contribute, please contact me directly.

1 Function of Several Variables

So far we've primarily seen functions of one variable. For example, some function f usually has one input x that outputs a unique result $f(x)$. Now, we'll look at functions g that take some (x,y) and produce a unique output $f(x,y,z)$. In general,

$$(x_1, \cdots, x_n) \longrightarrow f \longrightarrow f(x_1, \cdots, x_n)$$

Definition 1.1. A **scalar function** is a function $f(x,y)$ is a rule which assigns each ordered pair (x,y) in the set $D \subseteq \mathbb{R}^2$ a unique real number $z = f(x,y)$. D is called the **domain** of f. The set $\{f(x,y)|(x,y) \in D\}$ is called the **range**.



There is a similar definition for $f(x_1, \cdots, x_n)$. (See course notes)

Example 1.1. Consider the function $f(x,y) = 3x + 4y + 5$. Then $f(1,1) = 3 + 4 + 5 = 12$. The domain $D = \mathbb{R}^2$ and the range is $\mathbb{R}$.

Example 1.2. Wind chill index with T temperature C° and v wind speed in km/hr.

$$W(T,v) = 13.12 + 0.6215T - 11.37v^{0.16} + 0.3965Tv^{0.16}$$

Thus for temperature $T = -10$ C° and $v = 20$ we find the wind chill to be -18.

2 Interpretations of $f(x,y)$

Geometrical Consider a function $f : \mathbb{R}^2 \to \mathbb{R}$, this can be understood as a point on a plane consisting of the x and y axis, that maps to a single value on the z plane. Visually, you can see the function f turn points on a plane to a surface relative to z.

Physical Some examples include the temperature at (x,y) like in Example 2. Others include calculating the density of 3-dimensional object at the point (x,y), or the pressure, etcetera.

Example 2.1. Consider the function $f(x,y) = 16 - x^2 - 4y^2$. The domain is all of $\mathbb{R}^2$ and the range is $\mathbb{R}_{\geq 16}$ since having domain restricted to $\mathbb{R}^2$ means the squared terms will always be greater than 0.

Definition 2.1 (level curve)**.** The **level curves** of $f(x,y)$ are the curves in $\mathbb{R}^2$ with the equation $f(x,y) = k$ where k is a constant in the range of f.

Returning to our example, $16-x^2-4y^2=k$ is the equation for our level curves. Rearranging gives $x^2+4y^2=16-k$ for $k\in\mathbb{R}_{\leq 16}$; this is a family of ellipses.
$k=0 \implies x^2+4y^2=16$
$k=4 \implies x^2+4y^2=12$
$k=8 \implies x^2+4y^2=8$
$k=12 \implies x^2+4y^2=4$
$k=16 \implies x^2+4y^2=0$

Refer to **Graph 1.1**

Our surface looks like $z=16-x^2-4y^23$

Refer to **Graph 1.2**

Example 2.2. $f(x,y)=x^2-y^2$. The Level curves: $x^2-y^2=k$ is a family of hyperbolas. Note that $k=0 \implies x^2=y^2 \implies y=\pm x$. Additionally, as an aside note that $y=\pm\sqrt{x^2-k}\approx\pm\sqrt{x^2}$ for large x.

Refer to **Graph 1.3, 1.4**

Now we plot the **cross-sections** of $z=f(x,y)$ using $z=x^2-y^2$. (here he plots two regular 2d graphs, one for y-z, one for x-z using $z=c^2-y^2$ and $z=x^2-d^2$ for various values of c and d.
Finally the surface can be drawn, it looks like a "Saddle Surface".

Refer to **Graph 1.5**

Example 2.3. $z=f(x,y)=\sqrt{4-x^2-y^2}$. The domain is $\{(x,y)|x^2+y^2\leq 4\}$ which is just a circle of radius 2 centered at the origin. The range is $0\leq z\leq 2$.
The level curves are described by $\sqrt{4-x^2-y^2}=k$ for $k\in[0,2] \implies x^2+y^2=4-k^2$ (circles).
The cross-sections can be examined so

$$x=c \implies z=\sqrt{4-c^2-y^2} \implies y^2+z^2=4-c^2$$

with $z\geq 0$ (semicircles). It is often the case that x and y have symmetry, such that replacing x with y in the last step would produce the same cross-sections with the x axis swapped with the y. The surfaces of this graph look like a hemisphere. It's a sphere of radius R centered at origin: $x^2+y^2+z^2=R^2$ with $R=2$.

Refer to **Graph 1.6**

Definition 2.2. A **quadratic surface** is $Ax^2+By^2+Cz^2+Dxy+Ey^2+Fx^2+Gx+Hy+Iz+J=0$

Example 2.4. Sketch $\frac{x^2}{4}+y^2-\frac{z^2}{4}=1$ (implicity defined surface). The level curves where $z=k$ are

$$\frac{x^2}{4}+y^2-\frac{k^2}{4}=1 \implies \frac{x^2}{4}+y^2=1+\frac{k^2}{4}$$

The cross-sections where $x=c$ refer to

$$y^2-\frac{z^2}{4}=1-\frac{c^2}{4}$$

Refer to **Graphs 1.7 - 1.9**

The generalized term for level curves is **level sets**, so for example the level sets of $f(x,y,z)=k$ are a family of surfaces (level surfaces).

3 Limits

Notation 3.1. We'll represent a point in $\mathbb{R}^n$ as $\vec{x}$. For example in $\mathbb{R}^2$, $\vec{x} = (x^{(1)}, x^{(2)})$ and in general

$$\vec{x} = (x^{(1)}, \cdots, x^{(n)})$$

Recall that in first year we looked at limits for $x \in \mathbb{R}$, so there were only two sides to a limit to check for consistency. They were $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$. For functions of multiple variables, there are infinitely many directions to check for this consistency.

Definition 3.1 (neighbourhood)**.** A **neighbourhood** of $\vec{a} \in \mathbb{R}^2$ of radius $r > 0$ is a subset of $\mathbb{R}^2$ defined by

$$N_r(\vec{a}) = \{\vec{x} \in \mathbb{R}^2 \mid ||\vec{x} - \vec{a}|| < r\}$$

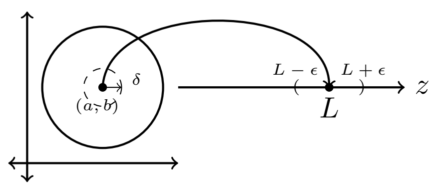
For simplicity we will mostly deal with functions of the form $f : \mathbb{R}^2 \to \mathbb{R}$, thus our limit definition will be defined for these too.

Definition 3.2. Suppose $f(x, y)$ is defined in some neighbourhood of $\vec{a}$, except possibly at $\vec{a}$. If for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$||\vec{x} - \vec{a}|| < \delta \implies |f(\vec{x}) - L| < \epsilon$$

then we say that "the limit as $\vec{x}$ approaches $\vec{a}$ exists and equals L" and we write

$$\lim_{(\vec{x}\to\vec{a})} f(x) = L$$



Proving that a limit does not exist
To prove that **a limit does not exist**, the key idea is to show that different limits are reached along different paths.

Example 3.1. Show that the following limit does not exist

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$$

Look at the value of f along straight lines through the origin as $(x, y) \to (0, 0)$.

$$\lim_{x\to 0} f(x, mx) = \lim_{x\to 0} \frac{1 - m^2}{1 + m^2}$$

Since this depends on m, the limit does not exist.

Example 3.2.

$$\lim_{\vec{x}\to\vec{0}} \frac{2x^2 y^{\frac{1}{3}}}{x^3 + y}$$

Trying $y = mx$ still simplifies to a term containing x thus goes to 0. Thus, we try a cubic function of x; $y = x^3$. So,

$$lim_{x\to 0} f(x, x^3) = \lim_{x\to 0} \frac{2x^2(x^3)^{\frac{1}{3}}}{x^3 + x^3} = 1$$

Since this limit is not unique for different curved approaching the same point, this limit does not exist.

Example 3.3. Show that

$$\lim_{(x,y)\to(1,0)} \frac{x^2 - y - 1}{x + y - 1}$$

does not exist. We could try $y = m(x - 1)$, or even easier, we will try the vertical line $x = 1$:

$$\lim_{y\to 0} f(1, y) = \lim_{y\to 0} \frac{1 - y - 1}{1 + y - 1} = -1$$

Similarly along the horizontal line $y = 0$ we get

$$\lim_{x\to 1} f(x, 1) = \lim_{x\to 1} \frac{x^2 - 1}{x - 1} = \lim_{x\to 1} \frac{(x - 1)(x + 1)}{x - 1} = 2$$

Since the limits are different, we have shown that the limit does not exist.

Proving that a limit does exist

Theorem 3.1 (squeeze theorem)**.** If there exists a $B(x, y)$ such that

$$|f(x, y) - L| \leq B(x, y)$$

for all (x, y) in some neighbourhood of (a, b) except possibly at (a, b) and

$$\lim_{(x,y)\to(a,b)} B(x, y) = 0 \quad \text{then} \quad \lim_{(x,y)\to(a,b)} f(x, y) = L$$

Proof. Since $\lim_{(x,y)\to(a,b)} B(x, y) = 0$, there exists a $\delta > 0$ such that $|B(x, y) - 0| < \epsilon$ whenever $||(x, y) - (a, b)|| < \delta$ for any $\epsilon > 0$ by the definition of a limit. Then, $|f(x, y) - L| \leq |B(x, y)| < \epsilon$ whenever $||(x, y) - (a, b)|| < \delta$. So by the definition of a limit,

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

□

Example 3.4. Show that

$$\lim_{(x,y)\to(0,0)} \frac{2x^2 y}{x^2 + y^2} = 0.$$

First we consider

$$\begin{aligned} |f(x, y) - L| &= \left| \frac{2x^2 y}{x^2 + y^2} - 0 \right| \\ &= \frac{2x^2|y|}{x^2 + y^2} \\ &\leq \frac{(2x^2 + 2y^2)|y|}{x^2 + y^2} \\ &= 2|y| \end{aligned}$$

So we choose $B(x, y) = 2|y|$. Now, clearly we see that

$$\lim_{(x,y)\to(0,0)} 2|y| = 0$$

Now, by the squeeze theorem (since $0 \leq f(x, y) \leq B(x, y)$)

$$\lim_{(x,y)\to(0,0)} \frac{2x^2 y}{x^2 + y^2} = 0$$

Example 3.5. Evaluate

$$f(x, y) = \lim_{(x,y)\to(0,0)} \frac{x^4 + x^2 + y^4 + y^2}{x^2 + y^2}$$

or show that it does not exist. First we try to prove that it does not exist, so we will test the lines $y = mx$. Then,

$$f(x, mx) = \frac{x^4 + x^2 + (mx)^4 + (mx)^2}{x^2 + (mx)^2} = \frac{x^2((1 + m^2) + x^2(1 + m^4))}{x^2(1 + m^2)} = 1 + x^2 \frac{1 + m^4}{1 + m^2}$$

So, $\lim_{x\to 0} f(x, mx) = 1 + 0 = 1$, thus the limit might be 1. Next, we try to show that the limit is 1; we try to find a function to use in conjuction with the squeeze theorem. Consider,

$$\begin{aligned} \left| \frac{x^4 + x^2 + y^4 + y^2}{x^2 + y^2} - 1 \right| &= \left| \frac{x^4 + x^2 + y^4 + y^2 - (x^2 + y^2)}{x^2 + y^2} \right| \\ &= \frac{x^4 + y^4}{x^2 + y^2} \end{aligned}$$

Note that

$$(x^2 + y^2)^2 = x^4 + y^4 + 2x^2y^2 \implies x^4 + y^4 = (x^2 + y^2)^2 - 2x^2y^2 \leq (x^2 + y^2)^2$$

So,

$$\begin{aligned} |f(x, y) - 1| &= \frac{x^4 + y^4}{x^2 + y^2} \\ &\leq \frac{(x^2 + y^2)^2}{x^2 + y^2} \\ &= x^2 + y^2 \\ &\to 0 \quad \text{as } (x, y) \to (0, 0) \end{aligned}$$

Remark 3.1. A few comments on inequality tricks (pg. 8 of course notes):

$$|x| = \sqrt{x^2} \leq \sqrt{x^2 + y^2}$$

$$\begin{aligned} (|x| - |y|)^2 &\geq 0 \\ x^2 - 2|x||y| + y^2 &\geq 0 \\ 2|x||y| &\leq x^2 + y^2 \end{aligned}$$

And some Limit theorems (pg. 10 of course notes):

$$\text{e.g., } \lim_{x\to 0} f(x) + g(x) = \lim_{x\to 0} f(x) + \lim_{x\to 0} g(x)$$

4 Continuous Functions

Definition 4.1 (continuous)**.** A function $f(x,y)$ is **continuous** at (a,b) if and only if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

and f is **continuous** on $D \subseteq \mathbb{R}^2$ if it is continuous at every point in D.

Example 4.1. Let $f(x,y) = \frac{x^4+x^2+y^4+y^2}{x^2+y^2}$, $(x,y) \neq (0,0)$.
Earlier we showed that

$$\lim_{(x,y)\to(0,0)} f(x,y) = 1$$

If we define $f(0,0) = 1$, then f would be continuous at $(0,0)$.

Example 4.2. Let

$$f(x,y) = \begin{cases} \frac{\sin(x^2+2y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ k & \text{if } (x,y) = (0,0) \end{cases}$$

Can we define k to make f continuous at $(0,0)$?
This requires showing that the following limit exists

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+2y^2)}{x^2+y^2}$$

Consider the lin $y = 0$, then we find that the limit goes to 1, and for the line $x = 0$ we fine the line goes to 2, therefore the limit does not exists. So, no such k exists to satisfy the function's continuity.

Definition 4.2 (continuity theorem)**.** Let f and g be continuous at (a,b). Then

(1) $f + g$ and fg are continuous at (a,b).

(2) $\frac{f}{g}$ is continuous at (a,b) for $g(a,b) \neq 0$.

Definition 4.3 (composition of functions)**.** Suppose $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$, then

$$(f \circ g)(x,y) = f(g(x,y))$$

Definition 4.4 (Continuity of Composition Theorem)**.** Let f be a function of one variable and g be a function of two variables. Then, if g is continuous at (a,b) and f is continuous at $g(a,b)$ then $f \circ g$ is continuous at (a,b).

Example 4.3. Consider $f(x,y) = \frac{y\sin x - \cos y}{x^2+y^2}$, since f is quotient of two continuous functions, by the continuity theorem the function f is continuous at all points in $\mathbb{R}^2$ except $(0,0)$ (since $x^2+y^2 = 0$ at $(0,0)$).

Example 4.4. Consider the function $e^{x^3-\sin(xy)}$. Clearly it is continuous everywhere by the continuity theorem.

Example 4.5. Evaluate

$$\lim_{(x,y)\to(0,0)} \frac{e^{x^2} + \ln(2+y^2+x^4)}{(x-1)^2+y^4}$$

Plugging in $(0,0)$ gives us the result, $1+\ln(2)$; which we can do since the function is continuous at $(0,0)$ by the continuity theorems.

Example 4.6. Determine where the following function is continuous:

$$f(x,y) = \begin{cases} \frac{e^{xy}-1}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

For $(x,y) \neq (0,0)$, f is continuous by the continuity theorems. Now, does

$$\lim_{(x,y)\to(0,0)} \frac{e^{xy}-1}{x^2+y^2} = 0$$

Checking along $x=0$ we find the limit to be 0, and checking along $y=x$ we find the limit to be

$$\begin{aligned} \lim_{x\to 0} \frac{e^{x^2}-1}{2x^2} &= \lim_{x\to 0} \frac{2xe^{x^2}}{4x} && \text{(L'hopital's Rule)} \\ &= \frac{1}{2} \\ &\neq 0 \end{aligned}$$

So, f is continuous for all $(x,y) \neq (0,0)$, but it is not continuous at the origin.

Example 4.7. Determine where

$$f(x,y) = \begin{cases} \frac{x^4y^6}{x^6+y^{12}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

is continuous.
At $(x,y) \neq (0,0)$ f is continuous by continuity theorems. Now we check whether

$$\lim_{(x,y)\to(0,0)} \frac{x^4y^6}{x^6+y^{12}} = f(0,0) = 0$$

We attempt to show the correctness of this limit by squeeze theorem. We use the following rewrite

$$\left| \frac{x^4y^6}{x^6+y^{12}} - 0 \right| = \frac{(x^6)^{\frac{2}{3}}(y^{12})^{\frac{1}{2}}}{x^6+y^{12}}$$

Then clearly we can show that

$$\frac{(x^6)^{\frac{2}{3}}(y^{12})^{\frac{1}{2}}}{x^6+y^{12}} \leq \frac{(x^6+y^{12})^{\frac{2}{3}}(y^{12}+x^6)^{\frac{1}{2}}}{x^6+y^{12}}$$

So,

$$\left| \frac{x^4y^6}{x^6+y^{12}} - 0 \right| \leq \frac{(x^6+y^{12})^{\frac{7}{6}}}{x^6+y^{12}} = (x^6+y^{12})^{\frac{1}{6}} \longrightarrow 0 \text{ as } (x,y) \to (0,0)$$

Therefore by squeeze theorem,

$$\lim_{(x,y)\to(0,0)} \frac{x^4y^6}{x^6+y^{12}} = 0$$

So, f is continuous at $(0,0)$ by inspection.

5 The Linear Approximation

Partial Derivatives We now look at different ways to differentiate a function $f : \mathbb{R}^2 \to \mathbb{R}$. We consider two ways

1. Hold y fixed, differentiate with respect to x.
$$\frac{\partial f}{\partial x}$$
2. Hold x fixed, differentiate with respect to y.
$$\frac{\partial f}{\partial y}$$

Example 5.1. Differentiate $f(x, y) = y^2 \sin(xy)$.

$$\frac{\partial f}{\partial x}(x, y) = y^2 \cos(xy) \cdot y$$
$$\frac{\partial f}{\partial y}(x, y) = 2y \sin(xy) + y^2 \cos(xy) \cdot x$$

We simply treat the non-dependent variables as constants when differentiating. Now, evaluating at a point (e.g., $(1, \pi)$) we get

$$\frac{\partial f}{\partial x}(1, \pi) = \pi^2 \cos(\pi) \cdot \pi = -\pi^3$$
$$\frac{\partial f}{\partial y}(1, \pi) = 2\pi \sin(\pi) + \pi^2 \cos(\pi) = -\pi^2$$

Notation 5.1. In this course we will use the following notation:

$$\frac{\partial f}{\partial x} = f_x$$

for any variable x.

Definition 5.1 (partial derivative)**.** The **partial derivatives** of $f(x, y)$ at (a, b) are

$$\frac{\partial f}{\partial x}(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h}$$
$$\frac{\partial f}{\partial y}(a, b) = \lim_{h \to 0} \frac{f(a, b + h) - f(a, b)}{h}$$

provided that the limit exists.

Note. This definition can of course be expanded to $f : \mathbb{R}^n \to \mathbb{R}$.

Example 5.2. Consider the function

$$f(x, y) = \begin{cases} \frac{x^3+y^4}{x^2+y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Find $f_x(0, 0)$ and $f_y(0, 0)$.

We use the formal defintion of a partial derivative,

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{h^3+0}{h^2+0}-0}{h} = 1$$
$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,0+h)-f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{0+x^4}{0+h^2}-0}{h} = 0$$

Note. You must use the definition when the usual rules for differentiation don't apply.

Example 5.3. Review Example 2 on page 29 of the course notes.

Example 5.4. The volume of an ideal gas in cm^3 with pressure in atm and temperature in K is

$$V = \frac{82.06T}{P}$$

Find the rate of change of volume with respect to temperarture and with respect to pressure when $T = 300$ K and $P = 5$ atm.

$$\frac{\partial V}{\partial T} = \frac{82.06}{P} \approx 16.41\ \frac{\text{cm}^3}{\text{K}}$$
$$\frac{\partial V}{\partial P} = -82.06TP^{-2} \approx -984.72\ \frac{\text{cm}^3}{\text{atm}}$$

Notation 5.2. Second partial derivatives can be written as

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx} = D_1 D_1 f = D_1^2 f$$

with the operator notation,

$$D_1 = \frac{\partial}{\partial x}, D_2 = \frac{\partial}{\partial yz}$$

So taking derivatives with respect to different variables can be shown as,

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy} = D_2 D_1 f$$

and

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx} = D_1 D_2 f$$

Example 5.5. Find the second partials of $f(x,y) = x^2 e^{-xy}$. The first are,

$$\frac{\partial f}{\partial x} = 2xe^{-xy} - x^2 y e^{-xy}$$

$$\frac{\partial f}{\partial y} = -x^3 e^{-xy}$$

Then the seconds are,

$$f_{xx} = \frac{\partial}{\partial x}\left(2xe^{-xy} - x^2 y e^{-xy}\right) = 2e^{-xy} - 4xye^{-xy} + x^2y^2e^{-xy}$$

$$f_{xy} = -3x^2 e^{-xy} + x^3 y e^{-xy}$$

$$f_{yx} = 3x^2e^{-xy} + x^3ye^{-xy}$$
$$f_{yy} = x^4e^{-xy}$$

Note that the mixed partials are equivalent.

Theorem 5.1 (Clairaut's Theorem)**.** If f_x, f_y, f_{xy}, and f_{yx} are all defined in some neighbourhood of $f_{xy}(x, y)$, and $f_{yx}(x, y)$ are continuous at (a, b) then
$$f_{xy}(a, b) = f_{yx}(a, b)$$

Definition 5.2 (Hessian Matrix)**.**
$$Hf(x, y) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

Remark 5.1. There are 2^n n-partial-derivatives of a function $f(x, y)$. (e.g., 4 possible second derivates of $f(x, y)$)

Definition 5.3 (class)**.** $f \in C^k$ "f is of class C^k" means that the k^{th} partial derivatives of f are continuous. For example, if f is of class C^2 then $f_{xx}, f_{xy}, f_{yx}, f_{yy}$ are continuous, so $f_{xy} = f_{yx}$ by Clairaut's Theorem.

Definition 5.4 (tangent plane)**.** The geometric interpreation of f_x and f_y is a **tangent plane**. For example, consider the surface of $f(x, y) = x^2 + y^2$, then $f_x = 2x$ and $f_y = 2y$. For the point $(0, 1, f(0, 1)) = (0, 1, 1)$ we see that $f_x(0, 1) = 0$ and $f_y(0, 1) = 2$. Graphically these can be seen as the slopes of the cross sections through that point $(0, 1, 1)$. In general,
$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$
is the equation of the tangent plane to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$.

Example 5.6. Find the equation of the tangent plane to $z = f(x, y) = \frac{xy}{x^2+y^2}$ at $(x, y) = (1, 2)$.
First, $f(1, 2) = \frac{2}{5}$ so,
$$f_x(1, 2) = \frac{x^2 + y^2)y - 2x^2y}{(x^2 + y^2)^2} = \frac{6}{25}$$
and $f_y(1, 2) = \frac{1}{25}$ so the equation of the tangent plane is
$$z = \frac{2}{5} + \frac{6}{25}(x - 1 + \frac{1}{25}(y - 2)$$

Definition 5.5 (linearization)**.** The **linearization** of $f(x, y)$ at (a, b) is
$$L_{(a,b)}(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

Definition 5.6 (linear approximation)**.** The **linear approximation** is
$$f(x, y) \approx L_{(a,b)}(x, y) \text{ near } (a, b)$$

Example 5.7. Approximate $(0.99)^2 + (1.98)^2$.
Let $f(x, y) = x^2 + y^2$, then we use the linear approximation near $(1, 2)$.
$$\begin{aligned} L_{(1,2)}(x, y) &= f(1, 2) + f_x(1, 2)(x - 1) + f_y(1, 2)(y - 2) \\ &= 5 + 2(x - 1) + 4(y - 2) \end{aligned}$$

So the linear approximation is $5 + 2(x - 1) + 4(y - 2)$ near $(1, 2)$. With $x = 0.99, y = 1.98$ we get
$$(0.99)^2 + (1.98)^2 \approx 5 + 2(-0.01) + 4(-0.02) = 4.9$$

So, $(0.99)^2 + (1.98)^2 \approx 4.9$. The actual value is 4.9005 - pretty good! This is a great party trick.

Definition 5.7 (increment form)**.**

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$
$$f(x,y) - f(a,b) = \Delta f \approx \underbrace{(f_x(a,b), f_y(a,b))}_{\substack{\text{"gradient of } f\text{"} \\ \nabla f(a,b)}} \bullet \underbrace{(x-a, y-b)}_{\vec{x}-\vec{a} \text{ or } \Delta\vec{x}}$$

So,

$$\Delta f \approx \nabla f(a,b) \bullet \Delta\vec{x}$$

Example 5.8.

$$f(x,y) = \sin(xy) + y^3$$

Then, $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (y\cos(xy), x\cos(xy) + 3y^2)$, so $\nabla f(\pi, 1) = (-1, 3-\pi)$.

Remark 5.2. In higher dimensions we can use the same idea, for example

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right)$$

Then,

$$f(\vec{x}) \approx f(\vec{a}) + \nabla f(\vec{a}) \bullet (\vec{x} - \vec{a})$$

where $\vec{x} = (x_1, x_2, \ldots, x_n)$, then $\vec{a} = (a_2, a_2, \ldots, a_n)$,

$$\Delta f \approx \nabla f(\vec{a}) \bullet (\vec{x} - \vec{a})$$

Example 5.9. Let $h(x,y,z) = xyz$ and $\vec{a} = (1,2,3)$. What is the approximate change in h if x increases by 0.01, y increases by 0.02, and z decreases by 0.01?

We use the increment form:

$$\Delta h \approx \nabla h(1,2,3) \bullet (\Delta x, \Delta y, \Delta z)$$

where

$$\begin{aligned} \nabla h &= (h_x, h_y, h_z) \\ &= (yz, xz, xy) \\ \nabla h(1,2,3) &= (6,3,2) \end{aligned}$$

So, $\Delta h \approx (6,3,2) \bullet (0.01, 0.02, -0.01) = 0.1$.

6 Differentiable Functions

Example 6.1. Suppose we're trapped on a desert island and we know that

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

and we want to approximate $f(0.1, 0.1)$, how do we do it? (assuming we don't know how to plug in numbers) The linear approximation is

$$\begin{aligned} f(x, y) &\approx f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) \quad \text{near } (0,0) \\ &= 0 + 0(x-0) + 0(y-0) \\ &= 0 \end{aligned}$$

So, $f(x, y) \approx 0$ near $(0, 0)$. Thus, $f(0.1, 0.1) \approx 0$. The linear approximation gives a **terrible** approximation in this case.

Note. $f_x(0,0)$ and $f_y(0,0)$ exist, but the function itself is not even continuous (try taking the limit for $y = x$) at (0,0). Clearly we need a better definition of differentiability. We will look at the accuracy of the linear approximation.

So how can we define differentiability?

For single variable functions,

$$f(x) \approx \underbrace{f(a) + f'(a)(x-a)}_{L_a(x)}$$

More precisely, $f(x) = L_a(x) + R_{1,a}(x)$

Theorem 6.1. If $f'(a)$ exists then

$$\lim_{x \to a} \frac{|R_{1,a}(x)|}{|x-a|} = 0 \qquad (1)$$

Proof.

$$\begin{aligned} \frac{|R_{1,a}(x)|}{|x-a|} &= \frac{|f(x) - (f(a) + f'(a)(x-a))|}{|x-a|} \\ &= \left| \frac{f(x) - f(a)}{x-a} - f'(a) \right| \longrightarrow |f'(a) - f'(a)| = 0 \quad \text{as } x \longrightarrow a \text{ by definition} \end{aligned}$$

We interpret this as the error in the linear approximation approaches zero **faster** than the distance $|x - a|$.

$\square$

Definition 6.1 (differentiable)**.** For a function of two variables $f(x, y)$, we say that f is **differentiable** at a point (a, b) if there is a linear function $L(x, y) = f(a, b) + c(x-a) + d(y-b)$ for $c, d \in \mathbb{R}$ such that

$$\lim_{(x,y) \to (a,b)} \frac{|R_{1,(a,b)}(x, y)|}{||(x, y) - (a, b)||} = 0$$

where

$$R_{1,(a,b)}(x, y) = f(x, y) - L(x, y)$$

Theorem 6.2. If $f(x, y)$ is differentiable at (a, b) with linear function $L(x, y)$, then $L(x, y)$ is the linearization of f at (a, b). That is,

$$c = \frac{\partial f}{\partial x}(a, b) \quad \text{and} \quad d = \frac{\partial f}{\partial y}(a, b)$$

Proof. f is differentiable at (a, b) means

$$\lim_{(x,y) \to (a,b)} \frac{|R_{1,(a,b)}(x, y)|}{||(x, y) - (a, b)||} = 0$$

So the limit along any path is 0. Along $y = b$ we have

$$\begin{aligned}
\lim_{x\to a} \frac{|f(x,b) - (f(a,b) + c(x-a))|}{\sqrt{(x-a)^2}} &= 0 \\
\lim_{x\to a} \left| \frac{f(x,b) - f(a,b)}{x-a} - \frac{c(x-a)}{x-a} \right| &= 0 \\
\left| \frac{\partial f}{\partial x}(a,b) - c \right| &= 0 \\
\frac{\partial f}{\partial x}(a,b) &= c
\end{aligned}$$

Similarly, approach along $x = a$ to get $d = \frac{\partial f}{\partial y}(a,b)$.
We conclude that f is differentiable at (a,b) if

$$\lim_{(x,y)\to(a,b)} \frac{|R_{1,(a,b)}(x,y)|}{||(x,y)-(a,b)||} = 0$$

where $R_{1,(a,b)}(x,y) = f(x,y) - L_{(a,b)}(x,y)|$. So the error goes to 0 faster than the distance from (x,y) to (a,b). $\square$

Example 6.2. Show that $f(x,y) = x^2 + y^2$ is differentiable at $(1,0)$.
The linearization is

$$\begin{aligned}
L_{(1,0)}(x,y) &= f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0) \\
&= 1 + 2(x-1)
\end{aligned}$$

then

$$\begin{aligned}
\frac{|R_{1,(a,b)}(x,y)|}{||(x,y)-(a,b)||} &= \frac{|f(x,y) - (1+2(x-1))|}{\sqrt{(x-1)^2+y^2}} \\
&= \frac{|x^2 - 2x + 1 + y^2|}{\sqrt{(x-1)^2+y^2}} \\
&= \frac{(x-1)^2+y^2}{\sqrt{(x-1)^2+y^2}} \\
&= \sqrt{(x-1)^2+y^2} \longrightarrow 0 \quad \text{as } (x,y) \longrightarrow (1,0)
\end{aligned}$$

Example 6.3. Show that $f(x,y) = \sqrt{|xy|}$ is continuous and / or differentiable at $(0,0)$. Since

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$$

the function is continuous. Now we check if it is differentiable, using the definition, so we look at

$$\frac{|R_{1,(0,0)}(x,y)|}{\sqrt{x^2+y^2}}$$

where $f(0,0) = 0$ and

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

and by symmetry, $f_y(0,0) = 0$ so

$$L_{(0,0)}(x,y) = 0 + 0x + 0y = 0$$

so

$$\begin{aligned} R_{1,(0,0)}(x,y) &= f(x,y) - L_{(0,0)}(x,y) \\ &= \sqrt{|xy|} - 0 \end{aligned}$$

and

$$\frac{|R_{1,(0,0)}(x,y)|}{\sqrt{x^2+y^2}} = \frac{\sqrt{|xy|}}{\sqrt{x^2+y^2}}$$

we we want to determine the limit for this quotient as (x,y) goes to $(0,0)$. Try along the line $y = x$ and we find that

$$\lim_{x\to 0} \frac{\sqrt{|x^2|}}{\sqrt{2x^2}} = \frac{1}{\sqrt{2}}$$

so f is **not** differentiable. The surface $z = \sqrt{|xy|}$ is not smooth at $(0,0)$! The tangent plane doesn't exist. That is, the equation of the tangent plane $z = L_{(a,b)}(x,y)$ is only valid if f is differentiable at (a,b).

Theorem 6.3. If $f(x,y)$ is differentiable at (a,b), then it is continuous at (a,b).

Proof. Observe that

$$R_{1,(a,b)}(x,y) = \frac{R_{1,(a,b)}(x,y)}{||(x,y)-(a,b)||}||(x,y)-(a,b)|| \longrightarrow 0 \text{ as } (x,y) \longrightarrow (a,b)$$

Also

$$f(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

and clearly $(x-a)$ and $(y-b)$ clearly go to 0 as $(x,y) \to (a,b)$ so

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

so f is continuous by definition. □

Note. The contrapositive states that if f is discontinuous at (a,b), then f is not differentiable at (a,b).

Theorem 6.4. If $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous at $\vec{a} \in \mathbb{R}^2$, then f is differentiable at $\vec{a}$.

Proof. f_x and f_y are continuous, so they exist and so does $L_{\vec{a}}(\vec{x})$. So we'd like to show that

$$\lim_{\vec{x}\to\vec{a}} \frac{R_{1,\vec{a}}(\vec{x})}{||\vec{x}-\vec{a}||} = 0$$

we have

$$\begin{aligned} R_{1,\vec{a}}(\vec{x}) &= f(x,y) - f(a,b) - f_x(a,b)(x-a) - f_y(a,b)(y-b) \\ &= \underbrace{f(x,y) - f(a,y)}_{g(x)-g(a)} + \underbrace{f(a,y) - f(a,b)}_{h(y)-h(b)} - f_x(a,b)(x-a) - f_y(a,b)(y-b) \quad (1) \end{aligned}$$

Recall the Mean Value Theorem (single variable case) states that there exists a $c \in (x, a)$ such that

$$g'(c) = \frac{g(x) - g(a)}{x - a} \qquad \left(\text{i.e.,} f_x(c, y) = \frac{f(x, y) - f(a, y)}{x - a}\right) \quad (2)$$

Similarly,

$$h'(d) = \frac{h(y) - h(b)}{y - b} \qquad \left(\text{i.e.,} f_y(d, y) = \frac{f(a, y) - f(a, b)}{y - b}\right) \quad (3)$$

Substitute (2) and (3) into (1) and

$$\begin{aligned} R_{a,\vec{a}}(\vec{x}) &= f_x(c, y)(x - a) + f_y(a, d)(y - b) - f_x(a, b)(x - a) - f_y(a, b)(y - b) \\ &= \underbrace{[f_x(c, y) - f_x(a, b)]}_{A}(x - a) + \underbrace{[f_y(a, d) - f_y(a, b)]}_{B}(y - b) \end{aligned}$$

Now,

$$\begin{aligned} \frac{|R_{1,\vec{a}}(\vec{x})|}{||\vec{x} - \vec{a}||} &= \frac{|A(x - a) + B(y - b)|}{\sqrt{(x - a)^2 + (y - b)^2}} \\ &\le \frac{|A||x - a| + |B||y - b|}{\sqrt{(x - a)^2 + (y - b)^2}} && \text{(Triangle Inequality)} \\ &\le \frac{|A|\sqrt{(x - a)^2 + (y - b)^2} + |B|\sqrt{(x - a)^2 + (y - b)^2}}{\sqrt{(x - a)^2 + (y - b)^2}} \\ &= |A| + |B| \end{aligned}$$

As $\vec{x} \to \vec{a}$, we have that $c \to a$ (since c is between x and a) and $d \to b$. Then,

$$\lim_{\vec{x} \to \vec{a}} f_x(c, y) = f_x(a, b)$$

by the continuitity of f_x at (a, b). Similarly,

$$\lim_{\vec{x} \to \vec{a}} f_y(a, d) = f_y(a, b)$$

Therefore $A \to 0$ and $B \to 0$ as $\vec{x} \to \vec{a}$. That is, $|A| + |B| \to 0$. So,

$$\frac{|R_{1,\vec{a}}(\vec{x})|}{||\vec{x} - \vec{a}||} \longrightarrow 0 \quad \text{as } \vec{x} \to \vec{a}$$

□

Example 6.4. Consider $f(x, y) = \ln\left(\frac{y}{x}\right)$, then the domain of f is $D = \{(x, y)|xy > 0\}$, then

$$\frac{\partial f}{\partial x} = \frac{x}{y}(-yx^{-2}) = \frac{-1}{x} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{x}{y}\left(\frac{1}{x}\right) = \frac{1}{y}$$

Clearly $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous on D, so f is differentiable on D.

Example 6.5. Consider $f(x,y)=\sqrt{x^2+y^2}$, then the domain is $\mathbb{R}^2$. Then

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{1}{2}(x^2+y^2)^{\frac{-1}{2}}(2x) \\ &= \frac{x}{\sqrt{x^2+y^2}} && \text{(for } (x,y)\neq(0,0)\text{)} \\ \frac{\partial f}{\partial y} &= \frac{y}{\sqrt{x^2+y^2}}\end{aligned}$$

which are continuous for all $(x,y)\neq(0,0)$ by continuity theorems, so f is differentiable for all $(x,y)\neq(0,0)$ (at least) by theorem 6.4.

7 Chain Rule

We'll start with a review of the 1-dimensional chain rule:

$$\frac{d}{dt}f(x(t)) = f'(t(x(t)))x'(t) \quad \text{or} \quad \frac{df}{dt}=\frac{df}{dx}\frac{dx}{dt}$$

Also recall from parametric / vector curves that $\vec{x}(t)=(x(t),y(t))$, and $\vec{x}'(t)=(x'(t),y'(t))$.
Chain Rule for $f(x(t),y(t))$

Consider a friendly duck with position $(x(t),y(t))$, and $f(x,y)$ is the temperature of the pond at (x,y). Let $T(t)=f(x(t),y(t))$ be the temperature that the duck feels at time t. Find $T'(t)$.

Over some time interval Δt,

$$\begin{aligned}\Delta x &= x(t+\Delta t)-x(t)\\ \Delta y &= y(t+\Delta t)-y(t)\end{aligned}$$

By linear approximation (increment form)

$$\Delta f \approx \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y \implies \frac{\Delta f}{\Delta t} \approx \frac{\partial f}{\partial x}\frac{\Delta x}{\Delta t}+\frac{\partial f}{\partial y}\frac{\Delta y}{\Delta t}$$

Now let $\Delta t \longrightarrow 0$,

$$\underbrace{\frac{d}{dt}f(x(t),y(t))}_{T'(t)} = \frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}$$

Note that this is hardly a proof, because we're just approximating.

Theorem 7.1 (chain rule)**.** Let $G(t)=f(x(t),y(t))$. Let $x(t_0)=a$, $y(t_0)=b$. If f is differentiable at (a,b) and $x'(t_0)$ and $y'(t_0)$ exist, then $G'(t_0)$ exists and $G'(t_0)=f_x(a,b)x'(t_0)+f_y(a,b)y'(t_0)$.

Proof. See page 55 - 56 of the course notes. □

We often write

$$\frac{d}{dt}f(x(t),y(t)) = \frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}$$

which can be viewed as a tree diagram as such:

$$\begin{array}{ccc} & f & \\ / & & \backslash \\ x & & y \\ | & & | \\ t & & t \end{array}$$

or just

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

and in vector form:

$$\frac{d}{dt}f(x(t), y(t)) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \bullet \left(\frac{dx}{dt}, \frac{dy}{dt}\right) = \nabla f \bullet \vec{x}'(t)$$

and this idea can be extrapolated in n dimensions.

Example 7.1. The temperature of the pond at (x, y) is

$$T(x, y) = \frac{4}{x^2 + y^2 + 1}$$

and the path of the duck is $\vec{x} = (t, t^2)$, what is the rate of change of the temperature at $t = 2$?

We use the chain rule.

$$\frac{d}{dt}T(x(t), y(t)) = \frac{\partial T}{\partial x}\left((x(2), y(2)\right)\left.\frac{dx}{dt}\right|_{t=2} + \frac{\partial T}{\partial y}\left(x(2), y(2)\right)\left.\frac{dy}{dt}\right|_{t=2}$$

Now we have $x(2) = 2$ and $y(2) = 2^2 = 4$ and $\frac{dx}{dt} = 1$, $\frac{dy}{dt} = 2t = 4$ when $t = 2$. Also, (and note that when $t = 2$ we have $(x, y) = (2, 4)$)

$$\begin{aligned} \frac{\partial T}{\partial x} &= -4(x^2 + y^2 + 1)^{-2}2x \\ &= \frac{-8x}{(x^2 + y^2 + 1)^2} \\ &= \frac{-16}{441} \\ \frac{\partial T}{\partial y} &= \frac{-8y}{(x^2 + y^2 + 1)} \\ &= \frac{-32}{441} \end{aligned}$$

So the rate of change is $\left(\frac{-16}{441}\right)(1) + \left(\frac{-32}{441}\right)(4) = \frac{-16}{49}$ so the duck is cooling down!

Example 7.2. Let $g(t) = f(\underbrace{te^t}_{x}, \underbrace{t^2 + 2t - 1}_{y})$. If $\nabla f(0, -1) = (3, 4)$, find $g'(0)$.

Note that when $t = 0$ we have $x = 0e^0 = 0$ and $y = 0 + 0 - 1 = -1$. By chain rule,

$$\begin{aligned} g'(t) &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \\ &= \nabla f \bullet \left(\frac{dx}{dt}, \frac{dy}{dt}\right) \end{aligned}$$

when $t = 0$, $\frac{dx}{dt} = (e^t + te^t)\big|_{t=0}$ and $\frac{dy}{dt} = (2t+2)\big|_{t=0} = 2$ so

$$\begin{aligned} g'(0) &= \nabla f(x(0), y(0)) \bullet (1, 2) \\ &= (3, 4) \bullet (1, 2) \\ &= 11 \end{aligned}$$

Note. We assumed that f is differentiable at $(0, -1)$. Often we'll assume that $f \in C^1$ which implies that f (by definition of class) is differentiable.

The chain rule for $g(u, v) = f(x(u, v), y(u, v))$

$$\begin{array}{ccccccc} & & & f & & & \\ & & / & & \backslash & & \\ & x & & & & y & \\ / & & \backslash & & / & & \backslash \\ u & & v & & u & & v \end{array}$$

$$\begin{aligned} \frac{\partial g}{\partial u} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} \\ \frac{\partial g}{\partial v} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} \end{aligned}$$

Example 7.3. $f(x, y) = 2x^2 - \sin y$ where $x(u, v) = u + v$, $y(u, v) = u - v^2$. Find $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ (this is an abuse of notation).

$$\begin{aligned} \frac{\partial f}{\partial u} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} \\ &= 4x \cdot 1 + (-\cos y) \cdot 1 \\ &= 4(u + v) - \cos(u - v^2) \end{aligned}$$

Remark 7.1. Direct substitution yields

$$\begin{aligned} f(x(u, v), y(u, v)) &= 2(u + v)^2 - \sin(u - v^2) \\ &= 4(u + v) \cdot 1 - \cos(u - v^2) \cdot 1 \end{aligned}$$

And finally

$$\begin{aligned}\frac{\partial f}{\partial v} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} \\ &= 4x \cdot 1 + (-\cos y)\cdot(-2v) \\ &= 4(u+v) + 2v\cos(u-v^2)\end{aligned}$$

Example 7.4. Suppose $\nabla T(1,\sqrt{3}) = (-1,1)$ and

$$x = x(r,\theta) = r\cos\theta \quad \text{and} \quad y = y(r,\theta) = r\sin\theta$$

Find $\frac{\partial T}{\partial \theta}$ at $(x,y) = (1,\sqrt{3})$.

By chain rule,

$$\begin{tikzcd}[column sep=small] & & & T \arrow[dll, dash] \arrow[drr, dash] & & & \\ & x \arrow[dl, dash] \arrow[dr, dash] & & & & y \arrow[dl, dash] \arrow[dr, dash] & \\ r & & \theta & & r & & \theta \end{tikzcd}$$

First note that when $x = 1$, $y = \sqrt{3}$ we have $1 = r\cos\theta$ and $\sqrt{3} = r\sin\theta$. This implies that

$$1^2 + \sqrt{3}^2 = r^2(\cos^2\theta + \sin^2\theta) \implies r^2 = 4 \implies r = 2$$

Then,

$$\frac{\sqrt{3}}{1} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta \implies \theta = \frac{\pi}{3}$$

so

$$\frac{\partial x}{\partial \theta} = -r\sin\theta = -2\sin\frac{\pi}{3} = -\sqrt{3}$$

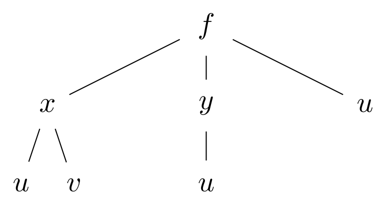
$$\frac{\partial y}{\partial \theta} = r\cos\theta = 2\cos\frac{\pi}{3} = 1$$

$$\begin{aligned}\frac{\partial T}{\partial \theta} &= \frac{\partial T}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial T}{\partial y}\frac{\partial y}{\partial \theta} \\ &= \nabla T \bullet \left(\frac{\partial x}{\partial \theta}, \frac{\partial y}{\partial \theta}\right) \\ &= \nabla T(1,\sqrt{3}) \bullet \left(-\sqrt{3}, 1\right) \\ &= \sqrt{3} + 1\end{aligned}$$

There is also a quick shortcut in this example, where we could have calculated $-r\sin\theta$ by taking $\sqrt{3} = r\sin\theta$ and multiplying by -1 to get that $\frac{\partial x}{\partial \theta} = -\sqrt{3}$.

Example 7.5.

$$\frac{\partial}{\partial u}\left(\underbrace{f\left(x(u,v),y(u),u\right)}_{g(u,v)}\right)$$



Then,

$$\frac{\partial g}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{dy}{du} + \frac{\partial f}{\partial u}$$

Remark 7.2. In Fluid Mechanics, $T(x,y;t) =$ "temperature at (x,y)" at time t. The temperature of a particle at a position $(x(t),y(t))$ is $\nabla T = \left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}\right)$

$$\hat{T}(t) = T(x(t),y(t),t) \implies \hat{T}'(t) = \frac{\partial T}{\partial x}x'(t) + \frac{\partial T}{\partial y}y'(t) + \frac{\partial T}{\partial t} = \nabla T \bullet (x',y') + \frac{\partial T}{\partial t}$$

Example 7.6. $f(x,y) = g(u,v)$ where $u = xe^{xy}$ and $v = x + y^2$. Find $\frac{\partial^2 f}{\partial x \partial y}$.

First note that $u_x = e^{xy} + xye^{xy}$ and $u_y = x^2e^{xy}$ and $v_x = 1$ and $v_y = 2y$. Then by chain rule

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial g}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial g}{\partial v}\frac{\partial v}{\partial y}\\ &= x^2e^{xy}\frac{\partial g}{\partial u} + 2y\frac{\partial g}{\partial v}\end{aligned}$$

Now take $\frac{\partial}{\partial x}$:

$$\begin{aligned}\frac{\partial^2 f}{\partial x\partial y} &= \frac{\partial}{\partial x}\left(x^2e^{xy}\frac{\partial g}{\partial u} + 2y\frac{\partial g}{\partial v}\right)\\ &= 2xe^{xy}\frac{\partial g}{\partial u} + x^2(ye^{xy})\frac{\partial g}{\partial u} + x^2e^{xy}\frac{\partial}{\partial x}\left(\frac{\partial g}{\partial u}\right) + 2y\frac{\partial}{\partial x}\left(\frac{\partial g}{\partial v}\right) && \text{(product rule)}\\ &= \text{(1st two terms)} + x^2e^{xy}\underbrace{\left(\frac{\partial^2 g}{\partial u^2}\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial g}{\partial u}\right)\frac{\partial v}{\partial x}\right)}_{\text{by chain rule}} + 2y\left(\frac{\partial^2 g}{\partial u}\partial v\frac{\partial u}{\partial x} + \frac{\partial^2 g}{\partial v^2}\frac{\partial v}{\partial x}\right)\end{aligned}$$

Example 7.7. $z = f(x,y), f \in C^2$ where $x = r\theta^2$, $y = r^2\theta$. Find z_{rr}.

$$\begin{aligned}
z_r &= \frac{\partial}{\partial r}\left(f(x,y)\right)\\
&= f_x x_r + f_y y_r\\
&= \theta^2 f_x + 2r\theta f_y\\
z_{rr} &= \frac{\partial}{\partial r}\left(\theta^2 f_x + 2r\theta f_y\right)\\
&= \theta^2 \frac{\partial}{\partial r}\left(f_x\right) + 2\theta f_y + 2r\theta \frac{\partial}{\partial r}\left(f_y\right)\\
&= \theta^2 \cdot \left(f_{xx}x_r + f_{xy}y_r\right) + 2\theta f_y + 2r\theta\left(f_{yx}x_r + f_{yy}y_r\right)\\
&= \theta^4 f_{xx} + 2r\theta^3 f_{xy} + 2\theta f_y + 2r\theta^3 f_{yx} + 4r^2\theta^2 f_{yy}\\
&= \theta^4 f_{xx} + 4r\theta^3 f_{xy} + 4r^2\theta^2 f_{yy} + 2\theta f_y && \text{(Clairaut's Theorem)}
\end{aligned}$$

8 Directional Derivatives

The motivation for this section is **Mountain Climbing**! This section has many graphs, so there will be many gaps in this PDF, check the course notes. Recall for now that $\frac{\partial f}{\partial x}(a,b)$ is the rate of chance of f as x increases (walks east). $\frac{\partial f}{\partial y}$ is the same thing but is the slope of the mountain as you walk north. We have a few questions, what abour arbitrary directions? What about the direction of the maximum rate of change?

Given $\vec{a} \in \mathbb{R}^2$, and a unit vector $\hat{u}$, a line through $\vec{a}$ with direction $\hat{u}$ has equation

$$\vec{x} = \vec{a} + s\hat{u}, \qquad s \in \mathbb{R}$$

Definition 8.1 (directional derivative)**.** The **directional derivative** of f at a point $\vec{a} = (a,b)$, in the direction of unit vector $\hat{u}$ is

$$D_{\hat{u}}f(\vec{a}) = \frac{d}{ds}f(\vec{a} + s\hat{u})\Big|_{s=0}$$

Note that we're assuming that this derivative of f exists.

Theorem 8.1. If f is differentiable at $\vec{a}$ then $D_{\hat{u}}f(\vec{a}) = \nabla f(\vec{a}) \bullet \hat{u}$ where $\hat{u}$ is a unit vector.

Proof. By definition,

$$\begin{aligned}
D_{\hat{u}}f(\vec{a}) &= \frac{d}{ds}f(\vec{a} + s\hat{u})\Big|_{s=0}\\
&= \frac{d}{ds}f(a + su_1, b + su_2)\Big|_{s=0} && (\hat{u} = (u_1, u_2))\\
&= \left(\frac{\partial f}{\partial x}\left(\vec{a} + s\hat{u}\right)\frac{dx}{ds} + \frac{\partial f}{\partial y}\left(\vec{a} + s\hat{u}\right)\frac{dy}{ds}\right)\Big|_{s=0}\\
&= \frac{\partial f}{\partial x}(\vec{a})u_1 + \frac{\partial f}{\partial y}(\vec{a})u_2\\
&= \nabla f(\vec{a}) \bullet \hat{u}
\end{aligned}$$

$\square$

- Note that this generalizes for a function $f(x_1, \ldots, x_n)$ in the expected way.
- If $\hat{u} = (1, 0)$ then we can see
$$D_{\hat{u}} f(\vec{a}) = \nabla f(\vec{a}) \bullet (1, 0) = \frac{\partial f}{\partial x}(\vec{a})$$
Similarly for $\hat{u} = (0, 1)$ we get $D_{\hat{u}} f(\vec{a}) = \frac{\partial f}{\partial y}(\vec{a})$.
- Note that $\hat{u}$ **must** be a unit vector; if it is not, you must normalize it. That is
$$\hat{u} = \frac{\vec{u}}{||\vec{u}||}$$
- Lastly, if f is **not** differentiable at the point $\vec{a}$, then you ust use the definition.

Example 8.1. Consider $f(x, y) = \frac{x}{x^2+y^2}$. Calculate the rate of change of f at the point $(2, 0)$ in the direction $(1, 1)$.

First we normalize the direction vector. So,
$$\hat{u} = \frac{(1, 1)}{\sqrt{1+1}} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$
Note that
$$\nabla f = \left(\frac{x^2 + y^2 - 2x^2}{(x^2 + y^2)^2}, \frac{-2xy}{(x^2 + y^2)^2}\right)$$
By continuity theorems clearly both partial derivatives are continuous which implies that f is differentiable at $(2, 0)$. By theorem 6.4,
$$\begin{aligned} D_{\hat{u}} f(2, 0) &= \nabla f(2, 0) \bullet \hat{u} \\ &= \left(\frac{-1}{4}, 0\right) \bullet \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \\ &= \frac{1}{4\sqrt{2}} \\ &= \frac{-\sqrt{2}}{8} \end{aligned}$$

Example 8.2. Find the directional derivative of $f(x, y, z) = x^2 \cos z + e^y$ in the direction $(-1, 1, -1)$ at $(1, \ln 2, 0)$.

First note that we normalize $\hat{u}$,
$$\hat{u} = \frac{(-1, 1, -1)}{\sqrt{(-1)^2 + 1^2 + (-1)^2}} = \left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$$
Then
$$\begin{aligned} \nabla f &= (f_x, f_y, f_z) \\ &= (2x \cos z, e^y, -x^2 \sin z) \end{aligned}$$

These are *clearly* continuous, therefore f is differentiable by theorem 6.4. So,

$$D_{\hat{u}}f(1, \ln 2, 0) = \nabla f(1, \ln 2, 0) \bullet \hat{u} = (2, 2, 0) \bullet \left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right) = 0$$

Quick comment; $D_{\hat{u}}$ represents the rate of change of f with respect to distance.

Example 8.3. Suppose $f(x, y)$ is equal to the height at (x, y). Then that would imply that $D_{\hat{u}}f(\vec{a})$ is the slope of the surface in direction $\hat{u}$. For a more detailed explanation with a *very nice* picture, read the course notes, page 74. Anyway, so $T(x, y)$ is the temperature at (x, y) then

$$D_{\hat{u}}T(\vec{a}) = \text{ rate of change of temperature per unit distance when moving in } \hat{u} \text{ direction}$$

So now we're wondering, which direction $\hat{u}$ is the best to achieve a maximum rate of change of temperature (warm up fastest). Well the answer is

$$\begin{aligned} D_{\hat{u}}f(\vec{a}) &= \nabla f(\vec{a}) \bullet \hat{u} \\ &= ||\nabla f(\vec{a})||||\hat{u}|| \cos\theta \\ &= ||\nabla f(\vec{a})|| \cos\theta \end{aligned}$$

To maximize, $D_{\hat{u}}f(\vec{a})$, choose $\hat{u}$ such that $\theta = 0$. Thus,

$$D_{\hat{u}}f(\vec{a}) = \underbrace{||\nabla f(\vec{a})||}_{\text{max rate of change}}$$

$\hat{u}$ is in the same direction as $\nabla f(\vec{a})$. Also a fun thing to note, when is $D_{\hat{u}}f(\vec{a}) = 0$? Well, when $\theta = \frac{\pi}{2}$ that is, $\hat{u}$ is perpendicular to $\nabla f(\vec{a})$. Additionally, $D_{\hat{u}}f(\vec{a})$ is "minimized" when $\theta = \pi$, so $D_{\hat{u}}f(\vec{a}) = \underbrace{||\nabla f(\vec{a})|| \cdot (-1)}_{\text{min}}$, so $\hat{u}$ in the direction of $-\nabla f(\vec{a})$.

Note. All of this is **assuming f is differentiable**.

Theorem 8.2. If $f(x, y)$ is differentiable at (a, b) and $\nabla f(a, b) \neq (0, 0)$, then $\nabla f(a, b)$ is **orthogonal** to the level curve $f(x, y) = k$ at (a, b).

We now present a sketchy proof. (well not really because I have no time to LaTeX this sketch, check the course notes).

Proof. Imagine some level curve with (a, b) as a point and equation $f(x, y) = k$. Parameterize the curve by $x = x(t)$ and $y = y(t)$. Then $f(x(t), y(t)) = k$. Take $\frac{d}{dt}$,

$$f_x \frac{dx}{dt} + f_y \frac{dy}{dt} = 0$$

by chain rule implies that

$$\begin{aligned} (f_x, f_y) \bullet (x', y') &= 0 \\ \nabla f \bullet \underbrace{(x', y')}_{\text{tangent vector}} &= 0 \end{aligned}$$

□

This idea generalizes to $\mathbb{R}^3$. $\nabla f(a,b,c)$ is orthogonal to the level surface $f(x,y,z)=k$ through (a,b,c). Thus this gives us another way to find a tangent plane. Imagine a surface with a point $\vec{a}$ on it and a tangent plane to this point, with $\vec{x}$ a point in this plane. Then $\nabla f(\vec{a})$ is orgonal to $\vec{x}-\vec{a}$. The equation of the tangent plane to a surface $f(x,y,z)=k$ at $\vec{a}=(a,b,c)$ is

$$\nabla f(\vec{a}) \bullet (\vec{x}-\vec{a}) = 0$$

Example 8.4. Consider $f(x,y,z)=x^2+y^2+z^2$. The level surfaces are $x^2+y^2+z^2=k$ (spheres of radius $\sqrt{k}$), for example at $\vec{a}=(2,1,1)$, $f(2,1,1)=6$, so the level surface through $\vec{a}$ is $x^2+y^2+z^2=6$. Then, $\nabla f=(2x,2y,2z)$ and $\nabla f(2,1,1)=(4,2,2)$. So the equatio of the tangent plane is

$$\nabla f(2,1,1)\bullet(\vec{x}-(2,1,1))=0 \implies (4,2,2)\bullet(x-2,y-1,z-1)=0$$

which implies that

$$z=6-2x-y$$

Example 8.5. Find the equation of the tangent plane to the surface $z=\sqrt{x^2+y^2}$ at $(1,1,\sqrt{2})$.

Solution 1. Let $f(x,y)=\sqrt{x^2+y^2}$ then the equation of the tangent plane is $z=f(1,1)+f_x(1,1)(x-1)+f_y(1,1)(y-1)$.

Solution 2. Rewrite $z=\sqrt{x^2+y^2}$ to $\sqrt{x^2+y^2}-z=0$, and call $\sqrt{x^2+y^2}-z=g(x,y,z)$. So the equation of the tangent plane is $\nabla g(1,1,\sqrt{2})\bullet(x-1,y-1,z-\sqrt{2})=0$.

Solution 3. $z=\sqrt{x^2+y^2} \implies \underbrace{z^2=x^2+y^2}_{h(x,y,z)}=0$ so $\nabla h=(2x,2y,-2z)$ so the tangent plane is found from $\nabla h(1,1,\sqrt{2})\bullet(x-1,y-1,z-\sqrt{2})=0$ which implies

$$z=\frac{1}{\sqrt{2}}x+\frac{1}{\sqrt{2}}y$$

9 Taylor Polynomials

We previously studies taylor polynomials of one variable, in particular

$$f(x)=\underbrace{\underbrace{f(a)+f'(a)(x-a)}_{P_1(x))}+\frac{f''(a)}{2!}(x-a)^2}_{P_2(x)}+\cdots \text{ so } f\in C^\infty$$

What about $f(x,y)$?

$$f(x,y)=f(a,b)+f_x(a,b)(x-a)+f_y(a,b)(y-b)+\frac{1}{2!}[f_{xx}(a,b)(x-a)^2+2f_{xy}(a,b)(x-a)(y-b)+f_{yy}(a,b)(y-b)^2]+\cdots$$

The idea is that $f(x,y)\approx P_{2,(a,b)}(x,y)$ near $(x,y)=(a,b)$ with better accuracy than the linearization

$$L_{(a,b)}(x,y)=P_{1,(a,b)}(x,y)$$

Example 9.1. $f(x,y)=x^2y-2xy+y^2$, find $P_{2,(1,2)}(x,y)$.

Concise way to calculate derivatives:

$$\nabla f = (2xy - 2y, x^2 - 2x + 2y)$$
$$\nabla f(1,2) = (0,3)$$

Recall the Hessian Matrix,

$$Hf(x,y) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

Then,

$$Hf(1,2) = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$$

So,

$$P_{2,(1,2)}(x,y) = 2 + 0(x-1) + 3(y-2) + \frac{1}{2!}\left[4(x-1)^2 + 2(0)(x-1)(y-2) + 2(y-2)^2\right]$$

Then we have that

$$\underbrace{x^2y - 2xy + y^2}_{f(x,y)} \approx 2 + 3(y-2) + 2(x-1)^2 + (y-2)^2 \qquad \text{near } (x,y) = (1,2)$$

So, for example, $f(1.1, 2.1) \approx 2+3(0.1)+2(0.1)^2+0.1^2 = 2.33$, the actual value is 2.331 and the linear approximation is 2.3.

Question: In general, what is the error in the approximation $f(x,y) \approx P_{1,(a,b)}(x,y)$?

Recall from single-variable functions that

$$f(x) = \underbrace{f(a) + f'(a)(x-a)}_{P_{1,a}(x)} + \underbrace{\frac{1}{2!}f''(c)(x-a)^2}_{R_{1,a}(x)}$$

for some c between x and a, by Taylor's Theorem. The idea is that $f(x) \approx f(a) + f'(a)(x-a)$ with |error|= $\left|\frac{1}{2!}f''(c)(x-a)^2\right| \leq \ldots$.

Theorem 9.1 (taylor's formula / theorem)**.** If $f \in C^2$ in some neighbourhood of $\vec{a} = (a,b)$, then there exists a $\vec{c} = (c,d)$ on the line segment joining $\vec{a}$ to $\vec{x} = (x,y)$ such that

$$f(\vec{x}) = f(\vec{a}) + f_x(\vec{a})(x-a) + f_y(\vec{a})(y-b) + R_{1,\vec{a}}(\vec{x})$$

where

$$R_{1,\vec{a}}(\vec{x}) = \frac{1}{2!}\left[f_{xx}(\vec{c})(x-a)^2 + 2f_{xy}(\vec{c})(x-a)(y-b) + f_{yy}(\vec{c})(y-b)^2\right]$$

Proof. Parameterize the line segment with $L(t) = (a,b) + t((x,y) - (a,b)) = (a,b) + t(\underbrace{x-a}_{h}, \underbrace{y-b}_{k})$, with $0 \leq t \leq 1$.

The idea is to reduce the problem to one variable, t.

Define $g(t) = f(L(t)) = f(a+th, b+tk)$ and now we compute derivatives:

$$g'(t) = f_x(a+th, b+tk)\cdot h + f_y(a+th, b+tk)\cdot k \qquad (1) \text{ by chain rule}$$
$$g''(t) = [f_xx(a+th, t+tk)h + f_{xy}(a+th, b+tk)k]h + [f_{yx}(a+th, b+tk)h + f_{yy}(a+th, b+tk)k]k$$

by chain rule again ($f \in C^2 \implies f_x, f_y$ differentiable). Then,

$$g''(t) = f_{xx}(a+th, b+tk)h^2 + 2f_{xy}(a+th, b+tk)hk + f_{yy}(a+th, b+tk)k^2 \qquad (2)$$

(since by Clairaut's Theorem, $f_x = f_y$). Now by single variable Taylor's Theorem, $g(1) = g(0) + g'(0)(1-0) + \frac{1}{2!}g''(x)(1-0)^2$ for some $c \in [0,1]$, which implies

$$f(L(1)) = f(L(0)) + f_x(a,b)h + f_y(a,b)k + \frac{1}{2!}\left[f_{xx}(a+ch, b+ck)h^2 + 2f_{xy}(a+ch, b+ck)hk + f_{yy}(a+ch, b+ck)k^2\right]$$
$$f(xy) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \frac{1}{2!}\left[f_{xx}(\vec{c},\vec{d})(x-a)^2 + 2f_{xy}(\vec{c},\vec{d})(x-a)(y-b) + f_{yy}(\vec{c},\vec{d})(y-b)^2\right]$$

where $\vec{c} = a + ch$ and $\vec{d} = b + ck$ □

Example 9.2. Suppose $f(xy) = e^{x^2+y^2}$, show that the error in the linear approximation at $(1,1)$ is at most $5e^2[(x-1)^2 + (y-1)^2]$ if $0 \le x \le 1, 0 \le y \le 1$.

Use taylor's formula / theorem, first we compute

$$\begin{aligned}
f_x &= 2xe^{x^2+y^2} \\
f_y &= 2ye^{x^2+y^2} \\
f_{xx} &= (2+4x^2)e^{x^2+y^2} \\
f_{xy} &= 4xye^{x^2+y^2} \\
f_{yy} &= (2+4y^2)e^{x^2+y^2}
\end{aligned}$$

Notice that

$$\begin{aligned}
|f_{xx}| &\le (2+4)e^{1+1} = 6e^2 \\
|f_{xy}| &\le 4e^2 \\
|f_{yy}| &\le 6e^2
\end{aligned}$$

By taylor's formula / theorem, $e^{x^2+y^2} \approx f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$ with remainder $R_{1,(1,1)}(x,y) = \frac{1}{2!}\left[f_{xx}(\vec{c})(x-1)^2 + 2f_{xy}(\vec{c})(x-1)(y-1) + f_{yy}(\vec{c})(y-1)^2\right]$ then,

$$\begin{aligned}
|R_{1,(1,1)}(x,y)| &\le \frac{1}{2!}[6e^2(x-1)^2 + 2\cdot 4e^2|x-1||y-1| + 6e^2(y-1)^2] \\
&= e^2[3(x-1)^2 + 4|x-1||y-1| + 3(y-1)^2] \\
&\le e^2[3(x-1)^2 + 2((x-1)^2 + (y-1)^2) + 3(y-1)^2] \text{by } (*) \\
&= 5e^2[(x-1)^2 + (y-1)^2]
\end{aligned}$$

Note.

$$2|A||B| \le A^2 + B^2 \implies (|A| - |B|)^2 \ge 0 \qquad (*)$$

Example 9.3. Consider $f(x,y) = \frac{1}{2x-3y}$. Show that $f(x,y) \approx L_{(2,-1)}(x,y)$ with $|error| \le 15[(x-2)^2 + (y+1)^2]$ if $x \ge 2$ and $y \le 1$.

By taylor's formula / theorem, the remainder is:

$$|R_{1,(2,-1)}(x,y)| = \frac{1}{2!}[f_{xx}(\vec{c})(x-2)^2 + 2f_{xy}(\vec{x})(x-2)(y+1) + f_{yy}(\vec{c})(y+1)^2]$$
$$\leq \frac{1}{2!}\left(|f_{xx}(\vec{c})(x-2)^2 + 2|f_{xy}(\vec{c})||x-2||y+1\right.$$

Note $(\vec{c}) = (c,d)$ is between $(2,-1)$ and (x,y). Compute partials (left as an exercise).

$$f_{xx} = \frac{8}{|2x-3y|^3} \qquad f_{xy} = \frac{-12}{|2x-3y|^3} \qquad f_{yy} = \frac{18}{|2x-3y|^3}$$

So $|f_{xx}(c,d)| = \frac{8}{|2c-3d|^3} \leq \frac{8}{(1)^2}$ by inspection (where $c \geq 2$, $d \leq 1$)

Similarly, $f_{xy}(\vec{x})| \leq 12$, $|f_{yy}(\vec{c})| \leq 8$. So,

$$\begin{aligned}
|R_1| &\leq \frac{1}{2!}(8(x-2)^2 + 2\cdot 12|x-2)||y+1| + 18(y+1)^2) \\
&\leq \frac{1}{2!}(8(x-2)^2 + 12x-2)||y+1| + 18(y+1)^2) \\
&= 10(x-2)^2 + 15(y+1)^2 \\
&\leq 15[(x-2)^2 + (y+1)^2]
\end{aligned}$$

Comment: In general, if $|f(x,y) - P_{1,(a,b)}(x,y) \leq M||\vec{x}-\vec{a}||^2$ for M a constant. Joe West thinks this is a good midterm problem.

Some Generalizations

For the k-th order Taylor polynomial $P_{k,(a,b)}(x,y)$: Define the differential operator

$$D = [(x-a)D_1 + (y-b)D_2]f(a,b) = (x-a)\frac{\partial f}{\partial x}(a,b) + (y-b)\frac{\partial f}{\partial y}(a,b) \qquad D_1 = \frac{\partial}{\partial x}, D_2 = \frac{\partial}{\partial y}$$

Then,

$$\begin{aligned}
P_{1(a,b)}(x,y) &= f(a,b) + [(x-a)D_1 + (y-b)D_2]f(a,b) \\
P_{2(a,b)}(x,y) &= P_{1(a,b)}(x,y) + \frac{1}{2!}[(x-a)D_1 + (y-b)D_2]^2 f(a,b) \\
&= P_{1(a,b)}(x,y) + \frac{1}{2!}[(x-a)^2D_1^2 + 2(x-a)(y-b)D_2D_1(y-b)^2D_2^2]f(a,b) \\
&\vdots \\
P_{n(a,b)}(x,y) &= P_{(n-1)(a,b)}(x,y) + \frac{1}{n!}[(x-a)D_1 + (y-b)D_2]^n f(a,b)
\end{aligned}$$

The **Remainder** is

$$R_{k(a,b)} = f(x,y) - P_{k(a,b)}(x,y) = \frac{1}{(k+1)!}[(x-a)D_1 + (y-b)D_2]^{k+1}f(c,d)$$

where (c,d) are between (a,b) and (x,y). From Taylor's Theorem, $f \in C^{k+1}$.

For $f(x_1, x_2, \ldots, x_n)$ centered at $(a_1, a_2, \ldots, a_n)$, use the differential operator

$$\begin{aligned}(x_1 - a_1)D_1 + (x_2 - a_2) + \cdots + (x_n - a_n)D_n \\ &= (x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n) \bullet (D_1, D_2, \ldots, D_n) \\ &= (\vec{x} - \vec{a}) \bullet \nabla\end{aligned}$$

10 Critical Points

Definition 10.1 (local maximum)**.** A point (a, b) is a **local max** of f is $f(a, b) \geq f(x, y)$ for all (x, y) in some neighbourhood of (a, b).

Definition 10.2 (local minimum)**.** A point (a, b) is a **local max** of f is $f(a, b) \leq f(x, y)$ for all (x, y) in some neighbourhood of (a, b).

Theorem 10.1. If (a, b) is local maximum or a local minimum of f then

$$\frac{\partial f}{\partial x}(a, b) = 0 \text{ or DNE} \qquad \text{and} \qquad \frac{\partial f}{\partial y}(a, b) = 0 \text{ or DNE} \qquad (*)$$

Proof. Proof by page number, 93. □

Definition 10.3 (critical point)**.** If a point (a, b) satisfies $(*)$ then we call (a, b) a **critical point** of f.

Definition 10.4 (saddle point)**.** A critical point that is neither a local maximum nor a local minimum is called a **saddle point** of f.

Example 10.1. Consider the paraboloid $z = x^2 + y^2$, then there is a local minimum at $(0, 0)$.

$$f_x = 2x = 0$$

$$f_y = -2y = 0$$

which implies that $(x, y) = (0, 0)$ is a critical point.

10.1 Finding Critical Points

Example 10.2. Suppose we have $f(x, y) = y^3 + x^2 - 6xy + 3x + 6y$. Find the critical points.

First we find where $f_x = 0$ or DNE and $f_y = 0$ or DNE. Well,

$$\begin{aligned} f_x &= 2x - 6y + 3 = 0 \\ f_y &= 3y^2 - 6x + 6 = 0 \end{aligned}$$

Then rearrange the first equation to get $x = \frac{3-6y}{2}$ and then plug into second equation to get $3y^2 - 6\left(\frac{3-6y}{2}\right) + 3 = 0$ which implies that $y = 1$ or $y = 5$. So, when $y = 1$, $x = \frac{6(1)-3}{2} = \frac{3}{2}$ and when $y = 5$ we have $x = \frac{6(5)-3}{2} = \frac{27}{2}$. Therefore, the critical points are

$$\left(\frac{3}{2}, 1\right) \qquad \text{and} \qquad \left(\frac{27}{2}, 5\right)$$

Example 10.3. Let $f(x,y) = 6xy^2 - 2x^3 - 3y^4$. Then

$$f_x = 6y^2 - 6x^2 = 0 \implies (y-x)(y+x) = 0 \quad (1)$$
$$f_y = 12xy - 12y^3 = 0 \implies y(x-y^2) = 0 \quad (2)$$

So Equation (2) implies $y = 0$ or $x = y^2$. So,
Case 1 ($y = 0$): Then $(0-x)(0+z) = 0$ by (1) which implies $x = 0$. Therefore one critical point is $(0,0)$.
Case 2 ($x = y^2$): Then

$$(y+y^2)(y-y^2) = 0 \implies y^2(1+y)(1-y) = 0 \implies y = 0, -1, 1$$

Then since $x = y^2$, $y = 0 \implies x = 0$, $y = -1 \implies x = (-1)^2 = 1$ and $y = 1 \implies x = 1^2 = 1$. Therefore we have that $(1,-1)$ and $(1,1)$ are also critical points. We conclud that the critical points are

$$(0,0) \qquad \text{and} \qquad (1,-1) \qquad \text{and} \qquad (1,1)$$

10.2 Classifying Critical Points

If $f \in C^2$ and (a,b) is a critical point of f then

$$f(x,y) \approx f(a,b) + \cancelto{0}{f_x(a,b)(x-a)} + \cancelto{0}{f_y(a,b)(y-b)} + \frac{1}{2}\left(f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2\right)$$

Definition 10.5 (quadratic form)**.** A function of the form $Q(u,v) = a_{11}u^2 + 2a_{12}uv + a_{22}v^2$ is called a **quadratic form** on $\mathbb{R}^2$. In matrix form,

$$Q(u,v) = (u,v)\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

Definition 10.6 (positive definite)**.** If $Q(u,v) > 0$ for all $(u,v) \neq (0,0)$ then Q is called a **positive definite**

Definition 10.7 (negative definite)**.** If $Q(u,v) < 0$ for all $(u,v) \neq (0,0)$ then Q is called a **negative definite**

Definition 10.8 (indefinite)**.** If $Q(u,v) > 0$ for some (u,v) and $Q(u,v) < 0$ for some other (u,v), then Q is called **indefinite**.

Definition 10.9 (semidefinite)**.** If Q is neither positive definite, negative definite, nor indefinite then Q is called **semidefinite**. For example, $Q > 0$ for some (u,v), and $Q = 0$ for some other $(u,v) \neq (0,0)$.

Example 10.4. Suppose $Q(x,y) = 3x^2 + y^2 > 0$ for all $(x,y) \neq (0,0)$, then Q is positive definite.

Example 10.5. Suppose $Q(x,y) = 3x^2 + 5xy + y^2$. Note $Q(x,0) = 3x^2 > 0$ and $Q(1,-1) = 3 - 5 + 1 < 0$ then by definition, Q is indefinite.

Comment: The terminology also applies to the corresponding matrix. For eample, in Example 10.4, we say that

$$\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$

is positive definite. Also, in Example 10.5, we say that

$$\begin{pmatrix} 3 & \frac{5}{2} \\ \frac{5}{2} & 1 \end{pmatrix}$$

is indefinite.

Theorem 10.2. Let

$$Q(x,y) = a_{11}x^2 + 2a_{12}xy + a_{22}y^2 = (x,y)\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

and $D = \det\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}^2$. Then,

(i) Q is positive definite iff $D > 0$ and $a_{11} > 0$.

(ii) Q is negative definite iff $D > 0$ and $a_{11} < 0$.

(iii) Q is indefinite iff $D < 0$.

If $D = 0$, Q is called **degenerate**, and is classified as semidefinite.

Definition 10.10 (determinant)**.** The **determinant** of a 2×2 matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is

$$\det(M) = ad - bc$$

Back to Critical Points

Near a critical point (a,b),

$$f(x,y) - f(a,b) \approx \frac{1}{2}\left[f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2\right] = ...$$

assuming that $f \in C^2$. Well we the $\frac{1}{2}$ doesn't matter much, and let's let $(x-a) = u$ and $(y-b) = v$, then

$$... = \frac{1}{2}(u,v)\begin{pmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{xy}(a,b) & f_{yy}(a,b) \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

so this is why the Hessian Matrix is useful, before this it was basically just a fancy name.

Theorem 10.3 (second-derivative test)**.** If $f \in C^2$ in some neighbourhood of critical point (a,b) then

(i) If $Hf(a,b)$ is positive definite, (a,b) is a local minimum point.

(ii) If $Hf(a,b)$ is negative definite, (a,b) is a local maximum point.

(iii) If $Hf(a,b)$ is indefinite, then (a,b) is neither a saddle point.

Note. The semidefinite case must be looked at seperately.

Proof. See proof in Section 9.3 of course notes (not required for course). □

Example 10.6. Classify critical points of $f(x,y) = 6xy^2 - 2x^3 - 3y^4$. From Example 10.3, we already found the critical points to be $(0,0)$, $(1,-1)$, and $(1,1)$. The Hessian Matrix is

$$Hf(x,y) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} = \begin{pmatrix} -12x & 12y \\ 12y & 12x - 36y^2 \end{pmatrix}$$

So,

$$Hf(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

which has determinant $D = 0$. So $Hf(0,0)$ is semidefinite, that is $(0,0)$ is a degenerate critical point and must be looked at seperately (Later).

$$Hf(1,1) = \begin{pmatrix} -12 & 12 \\ 12 & -24 \end{pmatrix} \implies \det(Hf(1,1)) = (-12)(-24) - 12^2 > 0$$

Also, the first entry is -12 which is less than 0, so $Hf(1,1)$ is negative definite (by theorem 10.2), and $(1,1)$ is therefore a local maximum point.

$$Hf(1,-1) = \begin{pmatrix} -12 & -12 \\ -12 & -24 \end{pmatrix} \implies \det(Hf(1,-1)) = (12)(24) - 12^2 > 0$$

since the first first entry is -12 which is less than 0, at $(1,-1)$ we also have a local maximum.

Back to $(0,0)$

$f(x,y) = 6xy^2 - 2x^3 - 3y^4$, then the question is whether $f(x,y) > 0$ or $f(x,y) < 0$ near $(0,0)$. Well, $f(x,0) = -2x^3$, which means that, by definition $(0,0)$ is a saddle point

Example 10.7. Consider $f(x,y) = x^2y - 2xy^2 + 3xy + 4$, find and classify the critical points.

First,

$$f_x = 2xy - 2y^2 + 3y = y(2x - 2y + 3) = 0 \qquad (1)$$
$$f_y = x^2 - 4xy + 3x = x(x - 4y + 3) = 0 \qquad (2)$$

(2) implies that $x = 0$ or $x - 4y + 3 = 0$. If $x = 0$ then equation (1) implies $y(-2y + 3) = 0$ which implies $y = 0$ or $\frac{3}{2}$.

Now, if $x - 4y + 3 = 0$, then $x = 4y - 3$, and equation (1) implies $y(2(4y - 3) - 2y + 3) = 0$ implies $y = 0, \frac{1}{2}$. Then, $x = -3$ or -1 respectively. Therefore the critical points are

$$(0,0), \left(0, \frac{3}{2}\right), (-3,0), \left(-1, \frac{1}{2}\right)$$

Next, the Hessian Matrix is

$$HF(x,y) = \begin{pmatrix} 2y & 2x - 4y + 3 \\ 2x - 4y + 3 & -4x \end{pmatrix}$$

Then

$$|Hf(0,0)| = \begin{vmatrix} 0 & 3 \\ 3 & 0 \end{vmatrix} = -9 < 0$$

so (0,0) is a saddle point. And,

$$|Hf(-3,0)| = \begin{vmatrix} 0 & -3 \\ -3 & 12 \end{vmatrix} = -9 < 0$$

so (-3,0) is a saddle point. And,

$$\left|Hf\left(0,\frac{3}{2}\right)\right| = \begin{vmatrix} 3 & -3 \\ -3 & 0 \end{vmatrix} = -9 < 0$$

is another saddle point. Finally,

$$\left|Hf\left(-1,\frac{1}{2}\right)\right| = \begin{vmatrix} 1 & -1 \\ -1 & 4 \end{vmatrix} = 4 - 1 = 3 > 0$$

So $Hf\left(-1,\frac{1}{2}\right)$ is positive definite and $\left(-1,\frac{1}{2}\right)$ a local minimum point.

Remark 10.1.

- If (a,b) is not a critical point, then $\nabla f(a,b) \neq (0,0)$ (assuming it exists). (a,b) is called a **regular point** of f and the level curves are smooth around these points
- If $f \in C^2$ and (a,b) is a non-degenerate (that is $|Hf(a,b)| \neq 0$), then $f(x,y) \approx P_{2,(a,b)}(x,y)$. The level curves of P_2 will be approximately equal to the level curves of f.

11 Extreme Values

Given $f(x,y)$, how can we find the maximum and minimum values on $S \subset \mathbb{R}^2$?

Definition 11.1 (abolute maximum point)**.** If $f(\vec{a}) \geq f(\vec{x}) \forall \vec{x} \in S$ then $\vec{a}$ is an **absolute maximum point**; the value $f(\vec{a})$ is the **absolute maximum value of** f.

Definition 11.2 (abolute minimum point)**.** If $f(\vec{a}) \leq f(\vec{x}) \forall \vec{x} \in S$ then $\vec{a}$ is an **absolute minimum point**; the value $f(\vec{a})$ is the **absolute minimum value of** f.

Recall, to maximize a single variable function $f(x)$ on $[a,b]$, check critical points and end points. (Extreme Value Theorem)

Definition 11.3 (boundary point)**.** A point $\vec{a}$ is a boundary point if every neighbourhood around $\vec{a}$ contains at least one point in S and one point not in S.

Definition 11.4 (bounded)**.** $S \subseteq \mathbb{R}^2$ is **bounded** means it can be contained in some **??**.

Definition 11.5 (closed)**.** $S \subset \mathbb{R}^2$ is **closed** means that it contains all of its boundary points.

If $S \subset \mathbb{R}^2$ is **closed** and **bounded** and f is continuous on S, then $\exists \vec{c}_1, \vec{c}_2 \in S$ such that

$$f(\vec{c}_1) \leq f(\vec{x}) \leq f(\vec{c}_2) \quad \text{for all } \vec{x} \in S$$

Proof. Beyond scope. □

Algorithm for Extreme Values

1. Evaluate f at all critical points in S.
2. Find the maximum and minimum values of f on the boundary of S.
3. Choose the points from step 1. and step 2. which give the maximum and minimum values.

Example 11.1. Find the abolute maximum point and abolute minimum point of $f(x, y) = xy - x - y + 3$ on a triangular region contained by the points $(0, 4), (0, 0)$ and $(2, 0)$.

1. The critical points:

$$f_x = y - 1 = 0 \implies y = 1$$
$$f_y = x - 1 = 0 \implies x = 1$$

So the only critical point is $(1, 1)$, and it is inside the triangle. Also,

$$f(1, 1) = 2$$

2. The boundary points are the points on the lines $x = 0$, $y = 0$, and $y = -2x + 4$. So,

   i. $x = 0, 0 \leq y \leq 4$ then $f(0, y) = -y + 3$ has maximum 3 and minimum -1.

   ii. $y = 0, 0 \leq x \leq 2$ then $f(x, 0) = -x + 3$ has maximum 3 and minimum 1.

   iii. $y = -2x + 4, 0 \leq x \leq 2$ then $f(x, -2x + 4) = -2x^2 + 5x - 1$. From first year calculus, find the maximum and minimum of $g(x) = -2x^2 + 5x - 1$ on $0 \leq x \leq 2$. Then, $\underbrace{g(0) = -1, g(2) = 1}_{\text{endpoints}}$, and $g\left(\frac{5}{4}\right) = \frac{17}{8}$ (critical point).

3. Compare all values and find that the maximum of f is 3 and the minimum of f is -1.

Example 11.2. Suppose we have $f(x, y) = xy$. Find the maximum and minimum of f on $S = \{(x, y) \mid x^2 + y^2 \leq 1\}$ (points within a unit circle).

- The critical points are found first, so

$$f_x = y = 0 \implies y = 0$$
$$f_y = x = 0 \implies x = 0$$

so the only critical point is $(0, 0)$ which is in S, and $f(0, 0) = 0$.

- Look at the boundary of the function $(x^2 + y^2 = 1)$, one way is to set $y^2 = 1 - x^2$, so $y = \pm\sqrt{1 - x^2}$. The other way is to parameterize the circle $(0 \leq t \leq 2\pi)$,

$$x(t) = \cos t$$
$$y(t) = \sin t$$

(then $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$). The value of f on the curve is

$$g(t) = f(x(t), y(t)) = f(\cos t, \sin t) = \cos t \cdot \sin t$$

So we need to find the maximum and minimum of g on $[0, 2\pi]$. The critical points of g:

$$\begin{aligned} g'(t) &= (\cos t)(\cos t) + (-\sin t)(\sin t) \\ &= \cos^2 t - \sin^2 t \\ &= \cos 2t \qquad \text{(Identity)} = 0 \end{aligned}$$

which resolves to $2t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$ (since $0 \leq 2t \leq 44\pi$) so $t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$. Thus the points for respective t values are,

$$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \quad \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \quad \frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, \quad \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}$$

Now we plug in these t values,

$$\begin{aligned} g(0) &= 0 \\ g(2\pi) &= 0 \\ g\left(\frac{\pi}{4}\right) &= \frac{1}{2} \\ g\left(\frac{3\pi}{4}\right) &= \frac{-1}{2} \\ g\left(\frac{5\pi}{4}\right) &= \frac{1}{2} \\ g\left(\frac{7\pi}{4}\right) &= \frac{-1}{2} \end{aligned}$$

Clearly $\frac{1}{2}$ is the absolute maximum of f, and $\frac{-1}{2}$ is the absolute minimum of f.

Question. What if it's difficult to parameterize the boundary?

Idea. Think of the boundary curve as the level curve of a fucntion $g(x, y)$:

$$g(x, y) = k$$

Set $\nabla f = \lambda \nabla g$ (where λ is a constant), so they're parallel.

Idea of Proof: Parameterize the boundary by $(x(t), y(t))$; define $u(t) = f(x(t), y(t))$, then set $u'(t) = 0$, then $\nabla f \bullet (x'(t), y'(t)) = 0$. Where $(x'(t), y'(t))$ is a tangent vector to the curve, by chain rule. So ∇f is perpendicular to the boundary curve, so ∇f is parallel to ∇g. (see page 115-116 for a rigorous proof).

11.1 Method of Lagrange Multipliers

To find the maximum and minimum of $f(x, y)$ subject to contraint $g(x, y) = k$, evaluate f at all points (a, b) which satisfy:

- $\nabla f(a, b) = \lambda \nabla g(a, b)$ and $g(a, b) = k$.
- $\nabla g(a, b) = (0, 0)$ and $g(a, b) = k$.
- (a, b) is an endpoint of the curve $g(x, y) = k$.

Then, the maximum of f is the largest value of f is obtained from points found in $(i) - (iii)$ and the minimum of f is the smallest of the points found in steps 1 - 3.

Comments:

- Assumed $g \in C^1$.

- Condition (i) is 3 equations, 3 unknowns (a, b, λ):

$$\begin{aligned} f_x(a,b) &= \lambda g_x(a,b) \\ f_y(a,b) &= \lambda g_y(a,b) \\ g(a,b) &= k \end{aligned}$$

  λ is called a **Lagrange Multiplier**.

- If $g(x,y) = k$ is an unbounded curve, one must consider $\lim_{||(x,y)|| \to \infty}$ for (x,y) satisfying $g(x,y) = k$.
- Generalizes to $f(x_1, x_2, \ldots, x_n)$ easily.

Example 11.3. Find the abolute maximum point of $f(x,y) = x^2 + y$ on the region $S = \{(x,y) \mid x^2 + y^2 \leq 4\}$.

1. Find critical points of f inside S.

$$\begin{aligned} f_x = 2x = 0 &\implies x = 0 \\ f_y = 1 = 0 &\implies \text{false} \end{aligned}$$

   There are no critical points.

2. Find the maximum of f on the boundary $x^2 + y^2 \leq 4$. Let $g(x,y) = x^2 + y^2$ and let $k = 4$. Use Lagrange Multipliers with the constraint $g(x,y) = x^2 + y^2 = 4$.

   i. Set $\nabla f = \lambda \nabla g$, $g = 4$.

$$2x = \lambda \cdot 2x \tag{1}$$
$$1 = \lambda \cdot 2y \tag{2}$$
$$x^2 + y^2 = 4 \tag{3}$$

   Equation (1) implies $\lambda = 1$ or $x = 0$.
   If $\lambda = 1$, then Equation (2) implies $y = \frac{1}{2}$, and if $y = \frac{1}{2}$ then $x = \pm\frac{\sqrt{15}}{2}$. So this case gets

$$\left(\frac{\sqrt{15}}{2}, \frac{1}{2}\right)$$

   otherwise if $x = 0$, then by Equation (3), $y = \pm 2$, so another point is

$$(0, \pm 2)$$

   ii. Set $\nabla g = \vec{0}$, $g = 4$, then

$$\begin{aligned} 2x &= 0 \\ 2y &= 0 \\ x^2 + y^2 &= 4 \end{aligned}$$

   this system has no solution.

   iii. This equation has no endpoints (it's a circle).

3. Compare values:

$$f\left(\frac{\sqrt{15}}{2},\frac{1}{2}\right)=\left(\frac{\sqrt{15}}{2}\right)^2+\frac{1}{2}=\frac{17}{4}$$
$$f(0,2)=0^2+2=2$$
$$f(0,-2)=0^2-2=-2$$

Therefore the absolute maximum value of f is $\frac{17}{4}$ and occurs at $\left(\frac{\sqrt{15}}{2},\frac{1}{2}\right)$.

Example 11.4. Let's say we have a Nuclear Reactor, and because we're doing applied math it is a cylinder with radius $r>0$ and height $h>0$, and has equation

$$\frac{4}{r^2}+\frac{\pi^2}{h^2}=k, \qquad \text{a constant}$$

Find the minimum volume of such a reactor.

We want to minimize the volume, so first

$$f(r,h)=\pi r^2 h$$

is the equation for volume of a cylinder, and it is subject to the contraint

$$\underbrace{\frac{4}{r^2}+\frac{\pi^2}{h^2}}_{g(r,h)}=k$$

Using Lagrange Multipliers,

i. Set $\nabla f=\lambda\nabla g$, and $g=k$, and $\nabla f=(f_r,f_h)=(2\pi rh,\pi r^2)$, and $\nabla g=\left(\frac{-8}{r^3},\frac{-2\pi^2}{h^3}\right)$ so we get three equations,

$$2\pi rh=\lambda\left(\frac{-8}{r^3}\right) \tag{1}$$
$$\pi r^2=\lambda\left(\frac{-2\pi^2}{h^3}\right) \tag{2}$$
$$\frac{4}{r^2}+\frac{\pi^2}{h^2}=k \tag{3}$$

The first equation implies

$$\lambda=\frac{-r^3\pi rh}{4}=\frac{-\pi}{4}r^4h$$

and the second implies

$$\lambda=\frac{-r^2h^3}{2\pi}$$

So,

$$\frac{-\pi}{4}r^4h=\frac{-r^2h^3}{2\pi}$$
$$r^2=\frac{2h^2}{\pi^2}$$

Pluging this r^2 into Equation 3 we get

$$\frac{4}{\left(\frac{2h^2}{\pi^2}\right)} + \frac{\pi^2}{h^2} = k \implies h = \sqrt{\frac{3\pi^2}{k}} \implies r = \sqrt{\frac{6}{k}}$$

So one point is

$$\left(\sqrt{\frac{6}{k}}, \sqrt{\frac{3\pi^2}{k}}\right)$$

ii. Set $\nabla g = 0$, $g = k$,

$$\begin{aligned} \frac{-8}{r^3} &= 0 \\ \frac{-2\pi^2}{h^3} &= 0 \\ \frac{4}{r^2} + \frac{\pi^2}{h^2} &= k \end{aligned}$$

No points from this step (no solution).

iii. What about endpoints of $\frac{4}{r^2} = \frac{\pi^2}{h^2} = k$? This curve is infinite, solving for h describes the contraint curve

$$h = \frac{\pi r}{\sqrt{kr^2 - 4}}$$

which has an asymptote at $r = \frac{2}{\sqrt{k}}$. So we should take a limit to get the "endpoint",

$$\begin{aligned} \lim_{r\to\infty} f(r,h) &= \lim_{r\to\infty} f\left(r, \frac{\pi r}{\sqrt{kr^2-4}}\right) \\ &= \lim_{r\to\infty} \pi r^2 \left(\frac{\pi r}{\sqrt{kr^2-4}}\right) \\ &= \lim_{r\to\infty} \frac{\pi^2 r^2}{\sqrt{kr^2-4}} \\ &= \lim_{r\to\infty} \frac{3\pi^2 r^2}{\frac{1}{2}(2kr)(kr^2-4)^{-\frac{1}{2}}} && \text{(L'Hopital's Rule)} \\ &= \infty \end{aligned}$$

Similarly, $\lim_{h\to\infty} f(r,h) = \infty$. So the minimum value is

$$\begin{aligned} f\left(\sqrt{\frac{6}{k}}, \sqrt{\frac{3\pi^2}{k}}\right) &= \pi \frac{6}{k}\sqrt{\frac{3\pi^2}{k}} \\ &= \frac{6\sqrt{3}\pi^2}{k^{\frac{3}{2}}} \end{aligned}$$

Example 11.5. Find the point(s) on the surface $z = xy + 5$ that are closest to the origin.

We want to minimize distance from (x, y, z) to the origin:

$$\sqrt{x^2 + y^2 + z^2}$$

Better yet, minimize the squared distance, then take a square root at the end. So, minimize

$$f(x, y, z) = x^2 + y^2 + z^2$$

subject to contraint $\underbrace{z - xy}_{g(x,y,z)} = 5$.

We use the Lagrange Multipliers Method:

(i) Set $\nabla f = \lambda \nabla g$ and $g = 5$.

$$\begin{aligned} 2x &= \lambda(-y) && (1) \\ 2y &= \lambda(-x) && (2) \\ 2z &= \lambda \cdot 1 && (3) \\ z - xy &= 5 && (4) \end{aligned}$$

Then

$$\frac{(1)}{(2)} \implies \frac{2x}{2y} = \frac{-\lambda y}{-\lambda x} \text{if } x \neq 0,\ y \neq 0$$

$$\begin{aligned} \frac{x}{y} &= \frac{y}{x} \\ x^2 &= y^2 \\ y &= \pm x \\ \implies \lambda &= \frac{2x}{-y} && \text{(from (1))} \\ \implies z &= \frac{\lambda}{2} && \text{(from (2))} \end{aligned}$$

Putting this result into (4) gets

$$\begin{aligned} \frac{-2}{2} - x(x) = -1 - x^2 = 5 (\text{if } \lambda = -2) \\ \frac{2}{2} - x(-x) = 1 + x^2 = 5 (\text{if } \lambda = -2) \end{aligned}$$

The first case is impossible, and the second implies that $x^2 = 4$. So if $\lambda = 2 \implies y = -x$, and $z = \frac{\lambda}{2} = 1$. So our set is,

$$(2, -2), (-2, 2)$$

What if $x = 0$ or $y = 0$? Then $z = 5$ from (4) and $\lambda = 10$ from (3) and thus $y = 0$ from (1) or (2). We get $(0, 0, 5)$.

(ii) Set $\nabla g = 0$ and $g = 5$. Then

$$\begin{aligned} -y &= 0 \\ -x &= 0 \\ 1 &= 0 \\ z - xy &= 5 \end{aligned}$$

Clearly no good, so there are no points from this approach.

(iii) The boundary curve of the surface is $z - xy = 5$. Since the surface is unbounded, we check

$$\lim_{||(x,y,z)||\to\infty} f(x,y,z) = \lim_{||(x,y,z)||\to\infty} ||(x,y,z)||^2 = \infty$$

which is clearly not a minimum. Then we check the points,

$$\begin{aligned} f(2,-2,1) &= 4+4+1 = 9 \\ f(-2,2,1) &= 4+4+1 = 9 \\ f(0,0,5) &= 0+0+5^2 = 25 \end{aligned}$$

So the minimum squared disrance is 9 thus the minimum disrance is 3 for the points $(2,-2,1)$ and $(-2,2,1)$.

12 Coordinate Systems

Note. For this section, checking the course notes is more useful, since there's lots of sketches.

Example 12.1. Convert to polar coordinates.

(a) $x^2 + y^2 = 1$. So $r^2 = 1$, and so $r = 1$ (take $r > 0$). Which is a circle.

(b) $x^2 + y^2 = \sqrt{x^2 + y^2} - x$. First,

$$\begin{aligned} r^2 &= r - r\cos\theta \\ r &= 1 - \cos\theta \\ \text{OR} \quad r &= 0 \end{aligned}$$

The $r = 0$ case is just a point on the origin, and otherwise the shape can be sketched as a "Cardioid" aka a heart.

Example 12.2. Find the area enclosed by Cardiois in the last example.

"Cut up" $0 \le \theta \le 2\pi$ into increments of size $\Delta\theta$. Then the area is $\frac{1}{2}r^2\Delta\theta = \frac{1}{2}f(\theta)^2\Delta\theta$. We sum over all the θ calues and let $\Delta\theta \to 0$. Then

$$\begin{aligned} A &= \int_0^{2\pi} \frac{1}{2}(1-\cos\theta)^2 d\theta \\ &= \frac{1}{2}\int (1 - \cancelto{0}{2\cos\theta} + \cos^2\theta)d\theta \\ &= \frac{1}{2}(2\pi) + \frac{1}{2}\int_0^{2\pi} \frac{1}{2} + \frac{\cancelto{0}{\cos(2\theta)}}{2} d\theta \\ &= \frac{3\pi}{2} \end{aligned}$$

12.1 Cylindrical Coordinates

If we place the pole at the origin and the polar axis along the positive x-axis in polar coordinates and place the axis of symmetry along the z-axis we then can relate a point P in cylindrical and Cartesian coordinates by

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}$$

and

$$\begin{aligned} r &= \sqrt{x^2+y^2} \\ \tan\theta &= \frac{y}{x} \\ z &= z \end{aligned}$$

Example 12.3. Consider the following

(a) $r = 1$ represents a cylinder of radius 1.

(b) $z = r$. Then,

$$z = \sqrt{x^2+y^2}$$

which is a cone.

Example 12.4. Convert into cylindrical coordinates,

(a) $z = xy$. Then

$$z = (r\cos\theta)(r\sin\theta) = r^2\cos\theta\sin\theta = \frac{1}{2}r^2\sin 2\theta$$

(b) $\underbrace{x^2+y^2}_{r^2} + z^2 = 1$ which implies $z^2 = 1 - r^2$ so $z = \pm\sqrt{1-r^2}$. Th e best way to write it is probably $r^2 + z^2 = 1$.

12.2 Spherical Coordinates

Let ρ be the distance from a point to the origin, and like before the angle from the x-axis to the line projected by line from the origin is θ, and the similar angle from the z-axis is ϕ. That is, a point P can be represented as

(ρ, ϕ, θ). Then,

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$

where $\rho \geq 0$ and $0 \leq \theta \leq 2\pi$, and $0 \leq \phi \leq \pi$.

Example 12.5. Discuss the points $(\rho, \phi, \theta) = \left(1, \frac{\pi}{2} . \frac{\pi}{4}\right)$ and $\left(2, \pi, \frac{3\pi}{4}\right)$ and convert to Cartesian.

(A) $(x, y, z) = \left(1 \sin\frac{\pi}{2} \cos\frac{\pi}{4}, 1 \sin\frac{\pi}{2} \sin\frac{\pi}{4}, 1 \cos\frac{\pi}{2}\right) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$

(B) $(x, y, z) = (0, 0, 2(-1)) = (0, 0, -2)$.

Example 12.6. Some spherical coordinates:

(a) $\rho = 1$ is sphere of radius 1.

(b) $\phi = \frac{\pi}{3}$ is a cone with every line having angle $\frac{\pi}{3}$ with the origin vertical axis (z-axis).

(c) Right $y > 0$ half of the yz-plane

12.3 Mappings

Definition 12.1 (mapping)**.** Consider $(u, v) = F(x, y) = (F_1(x, y), F_2(x, y))$. This is essentially like taking a point (x, y) in a Domain D on the $x - y$ plane and mapping it to some point (u, v) in some other subset R on the $u - v$ plane. This is called a "mapping" or "transformation".

Example 12.7. Find the image of $S = \{(x, y) \mid |x| \leq 1, |y| \leq 1\}$ under the mapping

$$(u, v) = F(x, y) = (x + y, y)$$

The region S is just a square, where x and y are between -1 and 1. We can seperate this into four lines, for each side of the square, so

$$\begin{aligned} y &= 1 & \qquad -1 &\leq x \leq 1 \\ x &= 1 & -1 &\leq y \leq 1 \\ y &= -1 & -1 &\leq x \leq 1 \\ x &= -1 & -1 &\leq y \leq 1 \end{aligned}$$

The mapping is

$$u = x + y$$
$$v = y$$

Looking at the sides of the square:

(1) $x = 1, -1 \leq y \leq 1$ implies

$$\begin{aligned} u &= 1 + y & & \\ v &= y & \qquad -1 &\leq v \leq 1 \end{aligned}$$

(2) $y = 1, -1 \leq x \leq 1$ implies

$$u = x + 1 \qquad 0 \leq u \leq 2$$
$$v = 1$$

You can keep doing this for the next two lines, and the resultant mapping on the $u - v$ plane is a parallelagram. Additionally, all inside points get mapped to the inside of the new shape.

Does the inside always necessarily get mapped to the inside of the new figure? The easiest way is to check a point inside S, say $(x, y) = (0, 0)$ which implies $(u, v) = F(0, 0) = (0, 0)$. Yes, inside gets mapped to inside.

Example 12.8. Consider the mapping

$$(x, y) = F(r, \theta) = (r \cos \theta, r \sin \theta)$$

Find the image of $S = \{(r, \theta) \mid 0 \leq r \leq 1, 0 \leq \theta \leq \pi\}$. Again we clearly have a square region for S.

- Side 1: $\theta = \pi$ and $0 \leq r \leq 1$, which implies

  $$x = r \cos \theta = -r$$
  $$y = r \sin \theta = 0$$

  which implies $y = 0$, $-1 \leq x \leq 0$.

- Side 2:

  $$x = 1 \cos \theta$$
  $$y = 1 \sin \theta \qquad 0 \leq \theta \leq \pi$$

  which implies $x^2 + y^2 = c^2 + s^2 = 1$. Which lies on a circle, and since $0 \leq \theta \leq \pi$, we have $\sin \theta \geq 0$ and $y \geq 0$, which is a semicircle.

- Side 3: $\theta = 0$, $0 \leq r \leq 1$ and

  $$x = r \cos 0 = r$$
  $$y = r \sin 0 = 0$$

  so $y = 0$ with $0 \leq x \leq 1$ which closes off the semicircle.

- Side 4: $r = 0$, $0 \leq \theta \leq \pi$

  $$x = 0 \cos \theta = 0$$
  $$y = 0 \sin \theta = 0$$

  so $(x, y) = 0$ which is just one point, the origin.

Does the inside map to the inside? Try $(r, \theta) = \left(\frac{1}{2}, \frac{\pi}{2}\right)$ which implies $(x, y) = \left(\frac{1}{2} \cos \frac{\pi}{2}, \frac{1}{2} \sin \frac{\pi}{2}\right) = \left(0, \frac{1}{2}\right)$ which is inside the new shape. We're assuming that if one point maps from the inside to the inside, then they all do.

12.4 Linear Approximation of a Mapping

Say we have some point (a, b) on the xy-plane, then the mapping F will map this point to some other point (c, d) in the uv-plane. Then the question is, will the point $(a + \Delta x, b + \Delta y)$ map to some other point $(c + \Delta u, d + \Delta v)$.

Question. How does a small change in (x, y) affect (u, v)?

Answer. By increment form of the linear approximation, then thinking of $F(x, y) = (f(x, y), g(x, y))$ has $u = f(x, y)$ and $v = g(x, y)$ then

$$\Delta u \approx f_x(a, b)\Delta x + f_y(a, b)\Delta y$$

and

$$\Delta v \approx g_x(a, b)\Delta x + g_y(a, b)\Delta y$$

In matrix form:

$$\begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix} \approx \underbrace{\begin{pmatrix} f_x(a,b) & f_y(a,b) \\ g_x(a,b) & g_y(a,b) \end{pmatrix}}_{DF(a,b)} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} \qquad DF(a, b) = \text{"derivative matrix"}$$

$$\begin{aligned} \Delta\vec{u} &\approx DF(\vec{a})\Delta\vec{x} \\ F(\vec{a} + \Delta\vec{x}) - F(\vec{a}) &\approx DF(\vec{a})\Delta\vec{x} \\ F(\vec{a} + \Delta\vec{x}) &\approx F(\vec{a}) + DF(\vec{a})\Delta\vec{x} \end{aligned}$$

Example 12.9. Consider $F(x, y) = (x^2 + 2y, xe^y)$, estimate the image of the point $(2.01, -0.02)$.

Use $(a, b) = (2, 0)$. Then,

$$\begin{aligned} DF(x, y) &= \begin{pmatrix} 2x & 2 \\ e^y & xe^y \end{pmatrix} \\ \implies DF(2, 0) &= \begin{pmatrix} 4 & 2 \\ 1 & 2 \end{pmatrix} \end{aligned}$$

So,

$$\begin{aligned} F(2, 01, -0.02) &= F(2 + 0.01, 0 - 0.02) \\ &\approx F(2, 0) + DF(2, 0)\begin{pmatrix} 0.01 \\ -0.02 \end{pmatrix} \\ &= \begin{pmatrix} 4 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 0.01 \\ -0.02 \end{pmatrix} \\ &= \begin{pmatrix} 4 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ -0.03 \end{pmatrix} \\ &= \begin{pmatrix} 4 \\ 1.97 \end{pmatrix} \end{aligned}$$

so the image is approximately $(4, 1.97)$.

12.5 Composite Mappings

Suppose you have some point (x, y) in the xy-plane and a mapping G that maps it to some point (u, v) into the uv-plane, and then some other mapping F that maps this new point to the pq-plane then we can define a mapping that does both steps in one called $F \circ G$.

First apply the map $(u, v) = G(x, y)$ and then apply $(p, q) = F(u, v)$ to get the composite map $F \circ G(x, y)$ or $(p, q) = F(G(x, y))$. Then,

$$\begin{aligned} F(G(x,y)) &= F(u(x,y), v(x,y)) \\ &= (p(u(x,y), v(x,y)), q(u(x,y), v(x,y))) \end{aligned}$$

Question. How does the derivative matrix of $F \circ G$ relate to the derivative matricies of F and G?

Theorem 12.1 (chain rule in matrix form)**.** If G has continuous partial derivatives at $\vec{x} = (x, y)$ and F has continuous partials at $\vec{u} = G(\vec{x})$ then $F \circ G$ has continuous partials at $\vec{x}$ and

$$D(F \circ G) = DF(\vec{u}) \cdot DG(\vec{x})$$

Proof. Note that $\frac{\partial q}{\partial x} = \frac{\partial}{\partial x}\left(q(u(x,y), v(x,y))\right)$, then we show

$$\begin{pmatrix} \frac{\partial p}{\partial x} & \frac{\partial p}{\partial y} \\ \frac{\partial q}{\partial x} & \frac{\partial p}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial p}{\partial u} & \frac{\partial p}{\partial v} \\ \frac{\partial q}{\partial u} & \frac{\partial q}{\partial v} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

Then by the usual chain rule

$$\begin{aligned} \frac{\partial p}{\partial x} &= \frac{\partial}{\partial x}\left(p(u(x,y), v(x,y))\right) \\ &= \frac{\partial p}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial p}{\partial v}\frac{\partial v}{\partial x} \end{aligned}$$

which proves the first entry. The rest are similar. □

Example 12.10. Find $D(F \circ G)$ where $F(u, v) = (u^2 + 2v, ue^v)$ and $G(x, y) = \left(\underbrace{y\cos x}_{u}, \underbrace{\frac{\ln y}{x}}_{v}\right)$.

By Chain rule,

$$\begin{aligned} D(F \circ G) &= DF(u,v)DG(x,y) \\ &= \begin{pmatrix} 2u & 2 \\ e^v & ue^v \end{pmatrix} \begin{pmatrix} -y\sin x & \cos x \\ \frac{-\ln y}{x^2} & \frac{1}{xy} \end{pmatrix} \\ &= \begin{pmatrix} -2uy\sin x - \frac{2\ln y}{x^2} & 2u\cos x + \frac{2}{xy} \\ -ye^v\sin x - \frac{ue^v\ln y}{x^2} & e^v\cos x + \frac{ue^v}{xy} \end{pmatrix} \\ &= \cdots \text{(sub in } u = y\cos x,\ v = \tfrac{\ln y}{x}) \end{aligned}$$

Example 12.11. Find $D(F \circ G)$ if $F(u, v) = (u\cos v, u\sin v)$ and $G(x, y) = \left(\sqrt{x^2 + y^2}, \arctan\left(\frac{y}{x}\right)\right)$ where $x > 0$ and $y > 0$.

By Chain Rule,

$$\begin{aligned}
D(F\circ G) &= DF(u,v)DG(x,y)\\
&= \begin{pmatrix}\cos v & -u\sin v\\ \sin v & u\cos v\end{pmatrix}\begin{pmatrix}\frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}}\\ \frac{\left(\frac{-y}{x^2}\right)}{1+\left(\frac{y}{x}\right)^2} & \frac{\left(\frac{1}{x}\right)}{1+\left(\frac{y}{x}\right)^2}\end{pmatrix}\\
&= \begin{pmatrix}\frac{x\cos v}{\sqrt{x^2+y^2}}+\frac{yu\sin v}{x^2+y^2} & \frac{y\cos v}{\sqrt{x^2+y^2}}-\frac{xu\sin v}{x^2+y^2}\\ \frac{x\sin v}{\sqrt{x^2+y^2}}+-\frac{yu\sin v}{x^2+y^2} & \frac{y\sin v}{\sqrt{x^2+y^2}}+\frac{xu\cos v}{x^2+y^2}\end{pmatrix}
\end{aligned}$$

Note. $u=\sqrt{x^2+y^2}$ so $\frac{u}{x^2+y^2}=\frac{1}{\sqrt{x^2+y^2}}$, and $u=\arctan\left(\frac{y}{x}\right)\implies \tan v=\frac{y}{x}$ then by drawing the triangle to figure out other relations we see $\cos v=\frac{x}{\sqrt{x^2+y^2}}$, $\sin v=\frac{y}{\sqrt{x^2+y^2}}$. So plug in and simplify,

$$D(F\circ G)=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

This mapping is essentially just converting from polar coordinates to cartesian, and then back.
Comment: If we first subtitute

$$\begin{aligned}
(F\circ G)(x,y) &= F(G(x,y))\\
&= F(\sqrt{x^2+y^2},\arctan\left(\frac{y}{x}\right))\\
&= \left(\sqrt{x^2+y^2}\cos\left(\arctan\left(\frac{y}{x}\right)\right),\sqrt{x^2+y^2}\sin\left(\arctan\left(\frac{y}{x}\right)\right)\right)\\
&= \left(\sqrt{x^2+y^2}\frac{x}{\sqrt{x^2+y^2}},\sqrt{x^2+y^2}\frac{y}{\sqrt{x^2+y^2}}\right)\\
&= (x,y)
\end{aligned}$$

So

$$\begin{aligned}
D(F\circ G) &= \begin{pmatrix}\frac{\partial}{\partial x}(x) & \frac{\partial}{\partial y}(x)\\ \frac{\partial}{\partial x}(y) & \frac{\partial}{\partial y}(y)\end{pmatrix}\\
&= \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}
\end{aligned}$$

13 Inverse Mappings

Definition 13.1 (one-to-one)**.** A mapping from D_{xy} to D_{uv} is **one-to-one** on $D_{xy}\subseteq\mathbb{R}^2$ if and only if

$$F(a,b)=F(c,d)\implies (a,b)=(c,d)$$

for all (a,b) and $(c,d)\in D_{xy}$.

Theorem 13.1. If F is one-to-one on D_{xy} then F has an **inverse mapping** F^{-1} such that

$$(x,y)=F^{-1}(u,v)\iff (u,v)=F(x,y).$$

Then we have $(F^{-1} \circ F)(x, y) = (x, y)\ \forall (x, y) \in D_{xy}$.

Question. How do we determine if a given mapping is invertible?
We look at $DF(x, y)$.

Theorem 13.2. Let F be one-to-one on D_{xy} with image D_{uv} and inverse map F^{-1}. If F has continuous partial derivatives at $(x, y) \in D_{xy}$ and F^{-1} has continuous partial derivatives at $(u, v) = F(x, y)$ then

$$DF^{-1}(u, v))DF(x, y) = I$$

Proof. By earlier property,

$$\begin{aligned} (F^{-1} \circ F)(x, y) &= (x, y) \\ \implies D(F^{-1} \circ F)(x, y) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \end{aligned}$$

By chain rule in matrix form,

$$\begin{aligned} D(F^{-1} \circ F)(x, y) &= I \\ DF^{-1}(x, y) &= I \end{aligned}$$

$\square$

- So the derivative matrix is invertible if the mapping F is invertible. Is the converse true? Does the derivative matrix being invertible imply that the mapping F is invertible?
- How do you check if DF is invertible? Show that the determinant is not zero.

Definition 13.2 (Jacobian)**.** The **Jacobian** of a mapping

$$(u, v) = F(x, y) = (f(x, y), g(x, y))$$

is denoted by

$$\frac{\partial(u, v)}{\partial(x, y)}$$

where

$$\frac{\partial(u, v)}{\partial(x, y)} = \det(DF(x, y)) = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{vmatrix}$$

Note. If Theorem 13.2 holds, then $\det(DF^{-1}(u, v)DF(x, y)) = \det(I) = 1$, then

$$\frac{\partial(x, y)}{\partial(u, v)} \cdot \frac{\partial(u, v)}{\partial(x, y)} = 1$$

So,

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{1}{\frac{\partial(x,y)}{\partial(u,v)}} \qquad \text{(inverse property)}$$

Example 13.1. Find the Jacobian of $G(x,y) = (\sqrt{x^2+y^2}, \arctan\left(\frac{y}{x}\right)$

$$\begin{aligned}
\frac{\partial(u,v)}{\partial(x,y)} &= \det\begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{-y}{x^2}\right) & \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right) \end{pmatrix} \\
&= \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{pmatrix} \\
&= \frac{x^2+y^2}{(x^2+y^2)^{\frac{3}{2}}} \\
&= \frac{1}{\sqrt{x^2+y^2}}
\end{aligned}$$

Example 13.2. Same for $(x,y) = F(u,v) = (u\cos v, u\sin v)$.

$$\begin{aligned}
\frac{\partial(x,y)}{\partial(u,v)} &= \det\begin{pmatrix} \cos v & -u\sin v \\ \sin v & u\cos v \end{pmatrix} \\
&= u
\end{aligned}$$

Notice that

$$\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\sqrt{x^2+y^2}} \cdot u = \frac{1}{u} \cdot u, \quad u \neq 0$$

Are the mappings inverses of each other? If $u = 0$, by inverse property not satisfied. By contrapositive of "F inverstible implies $\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)} = 1$" we get F is not invertible at all points $(0,v)$ for any v.

Note $(x,y) = F(0,v) = (0,0)$. Polar coordinates have a problem here! Are there any other (u,v) "problem" points? If we plug in the point $(u, 2n\pi)$, then $F(u, 2n\pi + v_0) = F(u, v_0)$ so F is not one-to-one so it is not invertible.

The moral of the story is that we must restrict to neighbourhoods.

Theorem 13.3 (Inverse Mapping Theorem)**.** Let $(u,v) = F(x,y)$ where $u(x,y)$ and $v(x,y)$ have continuous partial derivatives at (a,b). If $\frac{\partial(u,v)}{\partial(x,y)} \neq 0$ at (a,b) then there exists a neighbourhood of (a,b) in which F has an inverse mapping F^{-1} which has continuous partial derivatives.

13.1 Inventing Mappings

Example 13.3. Invent a mapping that maps the ellipse $x^2+4xy+5y^2 = 4$ onto the unit circle. Shwo that it is invertible and find the inverse.

Then, $x^2+4xy+5y^2 = 4 \implies (x+2y)^2+y^2 = 4 \implies \left(\frac{x+2y}{2}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$. Then let $u = \frac{x+2y}{2}$ and $v = \frac{y}{2}$ to get

$u^2 + v^2 = 1$. Now we show that this mapping is invertible; The Jacobian is

$$\begin{aligned}
\frac{\partial(u,v)}{\partial(x,y)} &= \det\begin{pmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{pmatrix} \\
&= \frac{1}{4} - 0 \\
&= \frac{1}{4} \\
&\neq 0
\end{aligned}$$

So there exists an inverse mapping on a neighbourhood of each point. To find the inverse of the entire mapping, we solve

$$\begin{aligned}
v &= \frac{x+2y}{2} \\
v &= \frac{y}{2}
\end{aligned}$$

for x and y. Well the second equation implies $v = 2v$ and then from the second get $x = 2u - 4v$. So F^{-1} is given by $(x, y) = F^{-1}(u, v) = (2u - 4v, 2v)$.

Note that for this example we didn't even need to calculate the Jacobian because just finding the inverse was enough. For more complicated u and v where we couldn't solve for x and y then we would rely more on the Jacobian.

Example 13.4. Map a parallelogram with corners at $(-1, 1)$, $(3, 1)$, $(-3, -1)$ and $(1, -1)$ to a square.

We have equations for all four sides, namely

$$\begin{aligned}
y &= x - 2 \\
y &= -1 \\
y &= x + 2 \\
y &= 1
\end{aligned}$$

Let $v = y$, so $y = \pm 1$ implies $v = \pm 1$, this covers the two top and bottom straight lines already. Now for the other non-straight lines,

$$y = x - 2 \implies y - x = -2 \implies \frac{y-x}{2} = -1$$

and

$$y = x + 2 \implies y - x = 2 \implies \frac{y-x}{2} = 1$$

then let $u = \frac{y-x}{2}$ to get lines $u = -1$ and $u = 1$. Next we check the point $(0, 0)$ on the parallelogram, and it maps to the inside of the square also at $(0, 0)$. So our conclusion is that

$$F(x, y) = (u, v) = \left(\frac{y-x}{2}, y\right)$$

maps the parallelogram to the square.

Comment: The Jacobian is

$$\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{pmatrix} = -\frac{1}{2}$$

so by Inverse Mapping Theorem there exists an inverse mapping around a neighbourhood of each point. Also, the area of the square is $2 \times 2 = 4$ and the area of the parallelogram is $2 \times 4 = 8$. Notice that they differ by a factor of a half.

$$\frac{1}{2} = \left|\frac{-1}{2}\right| = \left|\frac{\partial(u,v)}{\partial(x,y)}\right|$$

is the area scale factor.

Suppose we have some point $P = (x, y)$ that is the bottom left most corner of a rectangle with topleft corner R, and bottomright corner Q, then the topright corner is $(x + \Delta x, y + \Delta y)$ thus area $\Delta x\Delta y$.A mapping F converts these all to $P' = (u, v)$, R', Q'. Then the area of this nrw shape is approximately the area of the paralleogram for $\Delta x\Delta y$. Then the rectangle is formed by vertical line PR, and horizontal line PQ which maps to the parallelogram with $P'R'$ and $P'Q'$. Then,

$$PQ = \begin{pmatrix} \Delta x \\ 0 \end{pmatrix} \implies P'Q' \approx \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}\begin{pmatrix} \Delta x \\ 0 \end{pmatrix} = \begin{pmatrix} u_x\Delta x \\ v_x\Delta x \end{pmatrix}$$

$$PR = \begin{pmatrix} 0 \\ \Delta y \end{pmatrix} \implies P'R' \approx \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}\begin{pmatrix} 0 \\ \Delta y \end{pmatrix} = \begin{pmatrix} u_y\Delta y \\ v_y\Delta y \end{pmatrix}$$

Then the area of our paralellogram is

$$\begin{aligned} \left|\det\begin{pmatrix} u_x\Delta x & u_y\Delta y \\ v_x\Delta x & v_y\Delta y \end{pmatrix}\right| &= \left|\det\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}\right|\Delta x\Delta y \\ &= \left|\frac{\partial(u,v)}{\partial(x,y)}\right|\Delta x\Delta y && \text{by definition} \end{aligned}$$

So the area of the image of the rectangle is approximately

$$\left|\frac{\partial(u,v)}{\partial(x,y)}\right|\Delta x\Delta y$$

and this is approximation becomes more accurate as $\Delta x, \Delta y \to 0$. The Jacobian gives an "area scale factor".

Example 13.5. Approximate the area of a small rectangle with $\Delta x\Delta y$ located at $(x, y) = (0, 1)$ under the map $(u, v) = F(x, y) = (2x^2 + 2xy, y^2e^x)$.

The Jacobian is

$$\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{pmatrix} 4x+2y & 2x \\ y^2e^x & 2ye^x \end{pmatrix} = \underbrace{\det\begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}}_{\text{at } (0,1)} = 4$$

So the area of the image is $|4|\Delta x\Delta y = 4\Delta x\Delta y$.

Generalization. $(u_1, u_2, \ldots, u_n) = F(x_1, x_2, \ldots, x_n)$. Then the derivative matrix is

$$DF(x_1, \ldots, x_n) = \begin{pmatrix} D_1F_1 & D_2F_1 & \cdots & D_nF_1 \\ D_1F_2 & D_2F_2 & \cdots & D_nF_2 \\ \vdots & \vdots & \ddots & \vdots \\ D_1F_n & D_2F_n & \cdots & D_nF_n \end{pmatrix}$$

The Jacobian is

$$\frac{\partial(u_1, u_2, \ldots, u_n)}{\partial(x_1, x_2, \ldots, x_n)} = \det(DF)$$

If $n = 3$ then the absolute value of the Jacobian is the volume scale factor.

Example 13.6. $(u, v, w) = F(x, y, z) = (xy, xz, yz)$. Then,

$$\begin{aligned} \frac{\partial(u, v, w)}{\partial(x, y, z)} &= \det \begin{pmatrix} y & x & 0 \\ z & 0 & x \\ 0 & z & y \end{pmatrix} \\ &= -xzy - yxz \\ &= -2xyz \end{aligned}$$

For example, at point $(1, 1, 1)$ a rectangular prism (box) with volume $\Delta x \Delta y \Delta z$ would get mapped to a parallelpiped with volume $|-2(1)(1)(1)|\Delta x \Delta y \Delta z = 2\Delta x \Delta y \Delta z$.

14 Double Integrals

Recall from single variable functions,

$$\int_a^b f(x)dx = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i)\Delta x_i$$

Then for $f(x, y)$ we consider some function $z = f(x, y)$ and we integrate over some region $D_{xy} \subseteq \mathbb{R}^2$ (think of a flat surface at $z = 0$). Imagine that we consider this domain as a grid being composed of some number of cells with width Δx and heigh Δy. Then, suppose we turn each cell into a column of height $z = f(x, y)$. Then,

Definition 14.1 (double integral)**.**

$$\iint\limits_{D_{xy}} f(x, y)dxdy = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \sum_{i=1}^{n} f(x_1, y_i)\Delta x_i \Delta y_i$$

We can interpret $\iint_{D_{xy}} f(x, y)dA$ is the volume under a surface if $f > 0$. Additionally, $\iint_{D_{xy}} 1dA$ is the area of D_{xy}. Another example,

$$\iint\limits_{D_{xy}} \rho(x, y)dA$$

where ρ represents area density (kg/m^2), then this double integral represents the totla mass of the plate. One more, suppose we want to calculate the average value of $f(x, y)$ over D_{xy},

$$f_{avg} = \frac{\iint_{D_{xy}} f(x, y)dA}{\iint_{D_{xy}} 1dA}$$

which is analogous to

$$\frac{\int_a^b f(x)dx}{b - a} = \frac{\int_a^b f(x)dx}{\int_a^b 1dx}$$

14.1 Evaluating $\iint$

Key Idea. Consider the cross-sections. Suppose we have some region D_{xy}, and we want to integrate over this area, starting from x_l up to x_u, where the curve is closed for some lower function $y_l(x)$ and upper function $y_u(x)$. Then

$$\iint\limits_{D_{xy}} f(x,y)dA = \int_{x_l}^{x_u} \left(\underbrace{\int_{y_l(x)}^{y_u(x)} f(x,y)dy}_{\text{function of } x} \right) dx$$

called an "iterated integral".

Another way. Suppose we had another surface with two distinct curves that can be thought of with respect to y like $x = x_u(x)$ on the right and $x = x_l(x)$ on the left. Then describe D_{xy} by $y_l \leq y \leq y_u$ and $x_l(y) \leq x \leq x_u(y)$, then we can write

$$\iint\limits_{D_{xy}} f(x,y)dA = \int_{y_l}^{y_u} \left(\underbrace{\int_{x_l(y)}^{x_u(y)} f(x,y)dy}_{\text{function of } y} \right) dy$$

Example 14.1. Evaluate $\iint_D x^2ydA$ where $D = \{(x,y) \mid 0 \leq x \leq 2, 0 \leq y \leq 1\}$.

Solution. One way:

$$\begin{aligned}\iint\limits_D x^2ydA &= \int_0^2 \left(\int_0^1 (x^2y)dy \right) dx \\ &= \int_0^2 \left(x^2 \frac{y^2}{2} \Big|_{y=0}^{y=1} \right) dx \\ &= \int_0^2 \left(\frac{1}{2}x^2 - 0 \right) dx \\ &= \frac{4}{3}\end{aligned}$$

Another way:

$$\begin{aligned}\iint\limits_D x^2ydA &= \int_0^1 \left(\int_0^2 x^2ydx \right) dy \\ &= \frac{4}{3}\end{aligned}$$

Note. Since y is a constant with respect to x, and we have constant bounds, we could write

$$\begin{aligned}\int_0^1 \left(\int_0^2 x^2ydx \right) dy &= \int_0^1 \left(y \int_0^2 x^2dx \right) dy \\ &= \int_0^2 x^2dx \int_0^1 ydy\end{aligned}$$

Example 14.2. Let D be bounded by $y = \sqrt{x}$ and $y = x^3$. Evaluate

$$\iint_D x^2y^2 dA = I$$

Solution. First visualize this region as the the "cuban cigar" closed from $(0,0)$ to $(1,1)$ (the two intersecting points of the functions). We can describe D as having $0 \leq x \leq 1$ and being careful about y, since it depends on the function values. The lower curve is $y = x^3$ and the upper curve is $y = \sqrt{x}$. Then, $x^3 \leq \sqrt{x}$. Now we can write it as an iterated integral,

$$\begin{aligned} I &= \int_0^1 \left(\int_{x^3}^{\sqrt{x}} x^2y^2 dy \right) dx \\ &= \text{(exercise)} \end{aligned}$$

Another way: Think of the region instead as horizontal strips, where $x = y^{\frac{1}{3}}$ on the right and $x = y^2$ on the left, then $y^2 \leq x \leq y^{\frac{1}{3}}$ and $0 \leq y \leq 1$. So,

$$\begin{aligned} I &= \int_0^1 \left(\int_{y^2}^{y^{\frac{1}{3}}} (x^2y) dx \right) dy \\ &= \int_0^1 \left(\frac{x^3}{3} y^2 \Big|_{x=y^2}^{x=y^{\frac{1}{3}}} \right) dy \\ &= \int_0^1 \left(\frac{y}{3} y^2 - \frac{y^6}{3y^2} \right) dy \\ &= \frac{5}{108} \end{aligned}$$

Example 14.3. Let D be bounded by $y = x^2$ and $y = x + 2$. Find $\iint_D (x + 2y) dA$.

First make a sketch of both curves in the xy-plane, and figure out exactly which curve we're talking about. Then find the intersection points

$$\begin{aligned} x^2 &= x + 2 \\ x^2 - x - 2 &= 0 \\ x &= 2, -1 \end{aligned}$$

So our region is between -1 and 2 and is bounded above by $y = x + 2$ and below by $y = x^2$. So,

$$\begin{aligned} \iint_D (x + 2y) dA &= \int_{x=-1}^{x=2} \left(\int_{y=x^2}^{y=x+2} (x + 2y) dy \right) dx \\ &= \int_{-1}^2 \left((xy + y^2) \Big|_{y=x^2}^{y=x+2} \right) dx \\ &= \int_{-1}^2 (x(x + 2) + (x + 2)^2 - (x^3 + x^4)) dx \\ &= \frac{333}{20} \end{aligned}$$

Comment. Going the other way (integrating horizontally, not vertically): first we find the y values of those two boundary values for x (-1 and 2) and we get $(-1,1)$ and $(2,4)$. To get an idea of the problem, sketch the curve and notice that for y values above 1 we would be taking horizontal bars that on the left touch $y=x+2$ whereas on the right they touch $y=x^2$. This is a bit of a problem, so we need to subdivide the two regions into D_1 (the lower one) and D_2 (the upper one).

$$\begin{aligned}\iint\limits_D (x+2y)dA &= \iint\limits_{D_1}(x+2y)dA + \iint\limits_{D_2}(x+2y)dA \\ &= \int_{y=0}^{1}\left(\int_{-\sqrt{y}}^{\sqrt{y}}(x+2y)dx\right)dy + \int_{y=1}^{4}\left(\int_{y-2}^{\sqrt{y}}(x+2y)dx\right)dy \\ &= \frac{333}{20}\end{aligned}$$

- So the shape of the region (partially) determines the order of integration
- Any other considerations?

Example 14.4. Evaluate $I=\iint_D e^{y^2}dA$ where D bounded by lines $x=0$, $y=1$ and $y=x$. (Again make a sketch, this region is a triangle).

One way:

$$I=\int_0^1\left(\underbrace{\int_x^1 e^{y^2}dy}_{\text{impossible ☹}}\right)dx$$

Try the other order:

$$\begin{aligned}I &= \int_0^1\left(\int_0^y e^{y^2}dx\right)dy \\ &= \int_0^1\left(\left[e^{y^2}\right]\Big|_{x=0}^{x=y}\right)dy \\ &= \int_0^1\left(e^{y^2}y-0\right)dy \\ &= \frac{1}{2}e^{y^2}\Big|_0^1 \\ &= \frac{1}{2}(e-1)\end{aligned}$$

- So the integrand (the function in the $\iint$) is also a consideration.

Properties of $\iint$

Sum.

$$\iint\limits_D (f(x,y)+g(x,y))\,dA = \iint\limits_D f(x,y)dA + \iint\limits_D g(x,y)$$

Scalar Multiple.

$$\iint\limits_D kf(x,y)dA = k\iint\limits_D f(x,y)dA$$

Inequality. If $f(x,y) \leq g(x,y)$ for all $x, y \in D$ then

$$\iint\limits_D f(x,y)dA \leq \iint\limits_D g(x,y)dA$$

Triangle Inequality. (absolute value inequality)

$$\left|\iint\limits_D f(x,y)dA\right| \leq \iint\limits_D |f(x,y)|\, dA$$

Decomposition Property. If $D_1 \subseteq D$ and $D_2 \subseteq D$ and $D_1 \cap D_2 = \emptyset$ then

$$\iint\limits_D f dA = \iint\limits_{D_1} f dA + \iint\limits_{D_2} f dA$$

Theorem 14.1 (Change of Variable Theorem)**.**

$$\iint\limits_{D_{xy}} G(x,y)dxdy = \iint\limits_{D_{uv}} G(f(u,v), g(u,v)) \underbrace{\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv}_{dA}$$

where

- D_{xy} and D_{uv} are closed, bounded sets with piecewise smooth boundary curve
- $(x,y) = F(u,v) = (f(u,v), g(u,v))$ is an invertible mapping of D_{uv} onto D_{xy} and $f \in C'$, $g \in C'$, $\frac{\partial(x,y)}{\partial(u,v)} \neq 0$ on D_{uv}.
- $G(x,y)$ is continuous on D_{xy}.

Example 14.5. Find $\iint_D \frac{1}{x^2+y^2+1} dA$ on $D = \{(x,y) \mid 4 \leq x^2 + y^2 \leq 16\}$

This is a little complicated in Cartesian coordinates. The idea is to use polar coordinates to map to rectangle in the $r\theta$-plane. The mapping is

$$(x,y) = (r\cos\theta, r\sin\theta)$$

and the Jacobian is

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix}\cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta\end{pmatrix} = r\cos^2\theta - (-r\sin^2\theta) = r$$

By Change of Variable Theorem,

$$\begin{aligned}
\iint_{D_{xy}} \frac{1}{x^2+y^2+1} dxdy &= \iint_{D_{r\theta}} \frac{1}{(r\cos\theta)^2+(r\sin\theta)^2+1} |r|\, drd\theta \\
&= \int_0^{2\pi} \left(\int_2^4 \frac{r}{r^2+1} dr \right) d\theta \\
&= \int_0^{2\pi} \frac{1}{2} \ln(r^2+1) \Big|_2^4 d\theta \\
&= \int_0^{2\pi} \left(\frac{1}{2} (\ln 17 - \ln 5) \right) d\theta \\
&= \pi \ln \left(\frac{17}{5} \right)
\end{aligned}$$

Example 14.6. $\iint_D y dA$ where $D = \{(x,y) \mid x^2+y^2 \le 2y\}$.

First, $x^2+y^2 \le 2y \implies x^2+(y-1)^2 \le 1$. A clever way is to use modified polar coordinates:

$$(x,y) = (r\cos\theta, r\sin\theta + 1)$$

$D_{r\theta}$ is described by $0 \le r \le 1$ and $0 \le \theta \le 2\pi$ so by Change of Variable Theorem, (note that $\frac{\partial(x,y)}{\partial(u,v)} = r$ still; shift of a constant doesn't affect the derivative matrix)

$$\begin{aligned}
\iint_D y dxdy &= \iint_{D_{r\theta}} (r\sin\theta+1) r drd\theta \\
&= \int_0^{2\pi} \int_0^1 (r^2\sin\theta + r) drd\theta \\
&= \int_0^{2\pi} \left(\frac{1}{3} r^3 \sin\theta + \frac{1}{2} r^2 \right) \Big|_0^1 d\theta \\
&= \int_0^{2\pi} \left(\frac{1}{3}\sin\theta + \frac{1}{2} \right) d\theta \\
&= \pi
\end{aligned}$$

Comments.

- If we used "regular" polar ($x = r\cos\theta, y = r\sin\theta$) then describe D by $0 \le \theta \le \pi$ and $0 \le r \le 2\sin\theta$. Since $x^2+y^2 \le 2y$, $r \le 2\sin\theta$ and we get

$$\int_0^{2\pi} \left(\int_0^{2\sin\theta} (\underbrace{r\sin\theta}_{y}) r dr \right) d\theta = \ldots$$

- The mapping is not invertible at the origin ($r = 0$) - we ignore that since the value of an integral is not affected by only one point

Example 14.7. Find $\iint_{D_{xy}} xy dxdy$ where D_{xy} is defined by the region enclosed by $y = x$, $y = x - 1$, $y = -x$ and $y = -x + 2$

We can describe the lines differently,

$$x - y = 0 \qquad y = x - 1 \qquad x + y = 2 \qquad x + y = 0$$

The idea is to let $(u, v) = (x + y, x - y)$ and use Change of Variable Theorem, and the answer is $\frac{1}{4}$ (exercise)

Example 14.8. D_{xy} in the first quadrant is bounded by $xy = 1$, $xy = 2$, $x^2 - y^2 = 1$, $x^2 - y^2 = 2$. Evaluate $\iint_{D_{xy}} (x^2 + y^2) dA$.

Let $u = x^2 - y^2$ and $v = xy$, then the region in the uv-plane is

$$\begin{cases} 1 \leq u \leq 2 \\ 1 \leq v \leq 2 \end{cases}$$

By the Change of Variable Theorem,

$$\iint\limits_{D_{xy}} (x^2 + y^2) dA = \iint\limits_{D_{uv}} ((x(u, v))^2 + (y(u, v))^2) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dudv$$

Note.

$$\frac{\partial(u, v)}{\partial(x, y)} = \det \begin{pmatrix} 2x & -2y \\ y & x \end{pmatrix} = 2(x^2 + y^2) \neq 0 \text{ on } D_{xy} \text{ so } \exists \text{ an inverse mapping}$$

By inverse propery,

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{2(x^2 + y^2)}$$

So,

$$\begin{aligned} \iint\limits_{D_{xy}} (x^2 + y^2) dA &= \iint\limits_{D_{uv}} \cancel{(x^2 + y^2)} \left| \frac{1}{2\cancel{(x^2 + y^2)}} \right| dudv \\ &= \frac{1}{2} \int_1^2 \int_1^2 dudv \\ &= \frac{1}{2} \end{aligned}$$

Example 14.9 (A famous integral)**.** Evaluate

$$I = \int_0^\infty e^{-x^2} dx$$

Multiply by I:

$$\begin{aligned} I^2 &= \int_0^\infty e^{-x^2} dx \cdot \int_0^\infty e^{-y^2} dy \\ &= \int_0^\infty e^{-x^2} \left(\int_0^\infty e^{-y^2} dy \right) dx \\ &= \int_0^\infty \left(\int_0^\infty e^{-(x^2+y^2)} dy \right) dx \end{aligned}$$

Use polar coordinates:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad , \qquad \frac{\partial(x,y)}{\partial(u,v)} = r$$

So,

$$\begin{aligned} I^2 &= \int_0^{\frac{\pi}{2}} \int_0^{\infty} e^{-r^2} \cdot |r| dr d\theta \\ &= \int_0^{\infty} re^{-r^2} dr \int_0^{\frac{\pi}{2}} \\ &= -\frac{1}{2} e^{-r^2}\Big|_0^{\infty} \cdot \theta\Big|_0^{\frac{\pi}{2}} \\ &= -\frac{1}{2}(0-1)\frac{\pi}{2} \\ &= \frac{\pi}{4} \end{aligned}$$

So

$$\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

Comments:

- Technically an improper integral:

$$\int_0^{\infty} = \lim_{R\to\infty} \int_0^R$$

- The "error function"

$$Erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

15 Triple Intesdfgghjklbbnmwertuqazwsxedcrfvtgbyhnujmik,olp;/grals

Consider $f(x,y,z)$ and $D \subset \mathbb{R}^3$, slice D into rectangular boxes, with volume $\Delta V_i = \Delta x_i \Delta y_i \Delta z_i$, and define

Definition 15.1 (triple integral)**.**

$$\iiint\limits_D f(x,y,z) dv = \lim_{\substack{\Delta x_i \to 0 \\ \Delta y_i \to 0 \\ \Delta z_i \to 0}} \sum_{i=1}^{n} f(x_1, y_i, z_i)\Delta V_i$$

Interpretations.

$$\iiint\limits_d 1 dV = \text{ volume of } D$$

$$\iint\limits_D \underbrace{\rho(x,y,z)}_{\text{mass density}} dV = \text{ the mass of } D$$

$$\text{Average value of } f \text{ over } D = \frac{\iiint_D f(x,y,z)dV}{\iint_D 1 dv}$$

15.1 Evaluating $\iiint$

Suppose we have some shape in $\mathbb{R}^3$ with some upper surface $z_u(x,y)$ and lower surface $z_l(x,y)$ over some region D_{xy}, then

$$\iiint_D f(x,y,z)dV = \iint_{D_{xy}} \left(\int_{z_l(x,y)}^{z_u(x,y)} f(x,y,z)dz \right) dxdy$$

D is described by

$$\begin{cases} (x,y) \in D_{xy} \\ z_l(x,y) \leq z \leq z_u(x,y) \end{cases}$$

Example 15.1. Find the mass of a tetrahedron formed by the planes $x=0$, $y=0$, and $z=0$, and $\frac{x}{a}+\frac{y}{x}+\frac{z}{c}=1$, where the density is $\rho(x,y,z)=c-z$ $(a>0, b>0, c>0 \ \text{ const})$

Mass $= \iiint_D \rho(x,y,z)dV$ where D is drawn in the book on page 15. D is described by $0 \leq z \leq c\left(1-\frac{x}{a}-\frac{y}{b}\right)$ and $(x,y) \in D_{xy}$ has

$$\begin{cases} 0 \leq x \leq a \\ 0 \leq y \leq \frac{-b}{a}x+b \end{cases}$$

So the mass is

$$\begin{aligned} m &= \iiint_D (c-z)dV \\ &= \iint_{D_{xy}} \left(\int_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} (c-z)dz \right) dxdy \\ &= \frac{1}{8}abc^2 \end{aligned}$$

Example 15.2. Evalute $\iiint_D \frac{z}{4-x}dV$ where D bounded by cylinder $x^2+z^2=4$ and planes $x+y=2$ and $2x+y=6$, $z=0$, and $z=0$ in the first octant.

See textbook.

Example 15.3. Find the volume of the region bounded by $z=x^2+y^2$, $z=y+2$.

The volume is $\iiint_D 1dV$ where D is drawn in class but here I'll just describe it; it's a cone.

If we only consider the zy-plane then it's just a parabola cut by $z=y+2$, and since $z=x^2+y^2$,

$$x = \pm\sqrt{z-y^2}$$

so

$$-\sqrt{z-y^2} \leq x \leq \sqrt{z-y^2}$$

So the volume is

$$\begin{aligned}
\iiint\limits_{D} 1dV &= \iint\limits_{D_{yz}} \left(\int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} 1dx\right) dydz \\
&= \int_{-1}^{2} \left(\int_{y^2}^{y+2} \left(\int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} 1dx\right) dz\right) dy \\
&= \int_{-1}^{2} \left(\int_{y^2}^{y+2} \left(2\sqrt{z-y^2}\right) dz\right) dy \\
&= \int_{-1}^{2} \left(\frac{4}{3}(z-y^2)^{\frac{3}{2}}\Big|_{y^2}^{y+2}\right) dy \\
&= \frac{4}{3}\int_{-1}^{2} \left((y+2-y^2)^{\frac{3}{2}}\right) dy
\end{aligned}$$

Note that

$$2-(y^2-y) = 2-\left(\left(y-\frac{1}{2}\right)^2-\frac{1}{4}\right) = \frac{9}{4}-\left(y-\frac{1}{2}\right)^2 = \left(\frac{3}{2}\right)^2-\left(y-\frac{1}{2}\right)^2$$

also

$$u = a\sin\theta \implies a^2-u^2 = a^2\cos^2\theta$$

so let $y-\frac{1}{2} = \frac{3}{2}\sin\theta$ so at $y=-1, \theta = \frac{-\pi}{2}$ and $y = 2 \implies \theta = \frac{\pi}{2}$, also $dy = \frac{3}{2}\cos\theta$.

$$\begin{aligned}
&= \frac{4}{3}\int_{-1}^{2} \left((y+2-y^2)^{\frac{3}{2}}\right) dy \\
&= \frac{4}{3}\int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \left(\left(\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2\sin^2\theta\right)^{\frac{3}{2}} \left(\frac{3}{2}\cos\theta d\theta\right) \\
&= \frac{4}{2}\int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \left(\frac{3}{2}\right)^2 (\cos\theta)^3\frac{3}{2}\cos\theta d\theta \\
&= \frac{27}{4}\int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} (\cos^4\theta d\theta) \\
&= \frac{27}{4}\int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \left(\frac{1}{2}+\frac{1}{2}\cos 2\theta\right)^2 d\theta \\
&= \frac{81\pi}{32}
\end{aligned}$$

Theorem 15.1 (Change of Variable's Theorem)**.** Let $(x,y,z) = F(u,v,w) = (f(u,v,w), g(u,v,w), h(u,v,w))$ and an invertible mapping of D_{uvw} into D_{xyz} with $f,g,h \in C^1$ and $\frac{\partial(x,y,z)}{\partial(u,v,w)} \neq 0$ on D_{uvw}. If $G(x,y,z)$ is continuous on D_{xyz} then

$$\iiint\limits_{D_{xyz}} G(x,y,z)dV = \iiint\limits_{D_{uvw}} G(f(u,v,w), g(u,v,w), h(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw$$

Example 15.4. Find the volume in the first octant bounded by the hyperbolic cylinders $xy=1$, $xy=4$, $xz=1$, $xz=9$, $yz=4$, $yz=9$.

Consider the mapping

$$(u, v, w) = F(x, y, z)$$

given by

$$u = xy, \qquad v = xz \qquad w = yz$$

then D_{xyz} mapped to D_{uvw} described by

$$1 \leq u \leq 4 \qquad 1 \leq v \leq 9 \qquad 4 \leq w \leq 9$$

The Jacobian:

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \det \begin{pmatrix} y & x & 0 \\ z & 0 & x \\ 0 & z & y \end{pmatrix} = -2xyz$$

so

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{1}{-2xyz} = \frac{-1}{2\sqrt{u}\sqrt{v}\sqrt{w}}$$

By Change of Variable's Theorem,

$$\begin{aligned} V &= \iiint_{D_{xyz}} 1 dV \\ &= \iiint_{D_{uvw}} 1 \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| dudvdw \\ &= \int_1^4 \int_1^9 \int_4^9 \left(\frac{1}{2\sqrt{u}\sqrt{v}\sqrt{w}} \right) dudvdw \\ &= 8 \end{aligned}$$

Example 15.5. Show tht the volume of a sphere of radius R is $\frac{4}{3}\pi R^3$. Consider the sphere centered at the orgiin $D_{xyz} = \{(x,y,z) \mid x^2 + y^2 + z^2 \leq R^2\}$.

Converting to spherical coordinates:

$$D_{\rho\theta\phi} = \{(\rho, \theta, \phi) \mid 0 \leq \rho \leq R, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi\}$$

Then

$$\begin{aligned} V &= \iint_{D_{xyz}} 1 dV \\ &= \iint_{D_{\rho\theta\phi}} 1 \left| \frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} \right| dV \\ &= \int_{\rho=0}^{R} \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \rho^2 \sin\phi d\phi d\theta d\rho \\ &= \frac{4}{3}\pi R^3 \end{aligned}$$

Example 15.6. Find the volume of an ice cream cone. (region bounded below by $z = \sqrt{x^2 = y^2}$ and above by $x^2 + y^2 + z^2 = z$).

From the intersection of surfaces, $z = \sqrt{x^2+y^2} \implies z^2 = x^2+y^2$. Plug into the sphere equation and we get $2z^2 = z$ which implies $z = 0$ or $2z = 1 \implies z = \frac{1}{2}$. First note that,

$$\begin{aligned}
x^2+y^2+z^2 &= z \\
x^2+y^2+\left(z^2 - z + \frac{1}{4}\right) &= \frac{1}{4} \\
x^2+y^2+\left(z-\frac{1}{2}\right)^2 &= \left(\frac{1}{2}\right)^2
\end{aligned}$$

Try spherical coordinates:

$$\begin{aligned}
z &= \sqrt{x^2+y^2} \\
\rho\cos\phi &= \sqrt{(\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2} \\
&= \sqrt{\rho^2\sin^2\phi} \\
&= \rho\sin\phi \\
1 &= \tan\phi \implies \phi = \frac{\pi}{4}
\end{aligned}$$

Then

$$\begin{aligned}
x^2+y^2+z^2 &= z \\
\rho^2 &= \rho\cos\phi \\
\rho &= \cos\phi
\end{aligned}$$

Also,

$$D_{\rho\theta\phi} = \{(\rho,\theta,\phi) \mid 0 \le \rho \le \cos\phi, 0 \le \theta \le 2\pi, 0 \le \phi \le \frac{\pi}{4}\}$$