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PHYS 234

Quantum Physics I

Dr. Robert Hill • Spring 2014 • University of Waterloo

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Abstract

These notes are intended as a resource for myself; past, present, or future students of this course, and anyone interested in the material. The goal is to provide an end-to-end resource that covers all material discussed in the course displayed in an organized manner. If you spot any errors or would like to contribute, please contact me directly.

Robert Hill is a low temperature experimentalist, but this course will be mostly theoretical.

Albert Einstein once said, "Quantum mechanics is certainly imposing. But an inner voice tells me that it is not yet the realthing. The theory says a lot, but does not really bring us any closer to the secret of the 'old one'. I, at any rate, am convinced that He does not throw dice."

Richard Feynman said "I think I can safely say that nobody understand quantum mechanics."

So we're in for miserable experience with this course then? Well not really, there are some good reasons to study Quantum Physics:

- It's Extremely interesting!
  - Physically
  - Mathematically
  - Philosophically
- It is the science behind future technology!
- Waterloo is Quantum Valley!

1 The Photoelectric and Compton Effects

1.1 Historical Background

In classical physics we always observed things as behaving like waves or as particles. For example, there is

- Particle-like behaviour of radiation
- Wave-like behaviour of matter
- Wave-particle duality that combines the two

Let's explore the two sides of the coin. First, **what is a particle?** Some words that describe it are *p*oint, localised, mass, solid and similarly **what is a wave?** It can be described with words like *interference, oscillation, delocalised*, and *medium*. One such thing that we have had trouble with describing is **light**. Is it a wave or a particle?

1.2 Einstein's Theory of Photoelectric Effect

Radiant energy (light) is quantized into concentrated bundles (photons)

$$E = hf$$

His Photoelectric Equation (1905) states that

$$K_{\text{max}} = hf - \omega_0$$

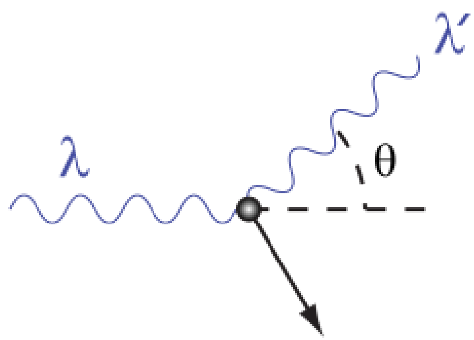

Figure 1.1: The Compton Effect. The scattered light has a different frequency; the frequency depends on the direction. A bigger deflection causes a bigger change in frequency.

1.3 Compton Effect

In the photoelectric effect, we treated light as being composed of individual light particles, called photons, that carry some energy. It then makes sense to think that the photons also have momentum.

Electromagnetic radiation is scattered by a target object. In classical theory, the charges in the target object will respond to the incoming wave and start to oscillate. All oscillating charges emit radiation at the frequency of oscillation, and this newly generated set of waves can also be detected at an angle Îž with respect to the incoming wave. This classical model explains why the sky is blue and all that jazz. The scattering process itself, though, does not change the frequency of incoming and outgoing radiation.

However, an experimental problem occurred. In experiments with X-ray radiation on a graphene target, one observes that two separate frequencies at an angle θ result in different intensities. This effect is independent of the material, though intensities may vary.

Definition 1.1 (Compton Shift)**.**

$$\Delta\lambda = \lambda_c(1 - \cos\theta)$$

Definition 1.2 (Compton wavelength)**.**

$$\lambda_c = \frac{h}{m_0 c}$$

2 De Broglie Wavelength and the Davisson-Germer Experiement

We have shown that wave phenomena can exhibit particle features. We can rewrite the momentum instead as $p = \frac{h}{\lambda}$ using a simple wave relationship. There is nothing in this reformed equation that has to do with light. This led to the following postulate.

2.1 The De Broglie Postulate (1924)

De Broglie's hypothesis was based on the grand symmetry of nature; if radiation has wave-particle duality, then so should matter.

Definition 2.1 (de Broglie Relation)**.**

$$\lambda = \frac{h}{p}$$

2.2 The Davisson-Germer Experiment

We must first understand the Bragg Grating; it is an optical filter that reflects particular wavelengths and transmits all others. Note that reflection, however, is common to both waves and particles.

2.3 Final Words

The observation of both phenomena in one and the same experiment leads us also to the concept of delocalization, which goes beyond the simple concept of "being extended", because single quantum objects seem to be able to simultaneously explore regions in space-time that cannot be explored by a single object in any classical way.

3 Linear Algebra Review

We're going to begin by reviewing some mathematics that will be needed in the course. This is a physics course so we're going to be a little loosey-goosey.

3.1 Vector Spaces

Definition 3.1 (Vector Space)**.** A **vector space** consists of a set of vectors : $(|\alpha\rangle, |\beta\rangle, |\gamma\rangle, \ldots)$ which is closed under vector addition and scalar multiplication.

Vector Addition produces another vector, that is

$$|\alpha\rangle + |\beta\rangle = |\gamma\rangle$$

it is also commutative

$$|\alpha\rangle + |\beta\rangle = |\beta\rangle + |\alpha\rangle$$

and associative

$$|\alpha\rangle + (|\beta\rangle + |\gamma\rangle) = (|\alpha\rangle + |\beta\rangle) + |\gamma\rangle$$

The null vector exists such that $|\alpha\rangle + |0\rangle = |\alpha\rangle$, and of course there is the inverse vector such that $|\alpha\rangle + |-\alpha\rangle = |0\rangle$.

Scalar Multiplication: The product of a scalar with a vector is another vector $(a|\alpha\rangle = |\gamma\rangle)$. Note that scalar multiplication is distributive with respect to vector addition

$$a(|\alpha\rangle + |\beta\rangle) = a|\alpha\rangle + a|\beta\rangle$$

Scalar multiplication is distributive with respect to scalar addition too

$$(a + b)|\alpha\rangle = a|\alpha\rangle + b|\alpha\rangle$$

and it is associative with respect to the product of scalars.

$$a(b|\alpha\rangle) = (ab)|\alpha\rangle$$

then multiplication by zero and by ± 1 has

$$0|\alpha\rangle = |0\rangle, \quad 1|\alpha\rangle = |\alpha\rangle, \quad -1|\alpha\rangle = -|\alpha\rangle = |-\alpha\rangle$$

Linear Combinations of Vectors: To generate a linear combination of vectors

$$|\lambda\rangle = a|\alpha\rangle + b|\beta\rangle + c|\gamma\rangle$$

(I) Any vector is linearly independent of a set of vectors if it cannot be written as a linear combination of them.

(II) A set of vectors is linearly independent if each is linearly independent of the rest.

(III) A colelction of vectors is said to **span** the space if every vector can be written as a linear combination of them.

(IV) A set of linearly independent vectors that span a space is called a **basis**.

(V) The number of vectors in the basis is called the **dimension** of the space.

Co-ordinate Representation: With respect to a given basis, $|e_1\rangle, |e_2\rangle, |e_3\rangle, \ldots, |e_n\rangle$, any given vector $|\alpha\rangle = a_1|e_1\rangle + a_2|e_2\rangle + \ldots + a_n|e_n\rangle$ is uniquely defined by the ordered n-tuple of its components.

$$|\alpha\rangle \iff \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

(the co-ordinate representation of $|\alpha\rangle$ with respect to the basis given by each $|e_1\rangle$.)
Coordinates depend on the chosen basis. In basis 1 $|\alpha\rangle = a_x|x\rangle + a_y|y\rangle \iff \begin{pmatrix} a_x \\ a_y \end{pmatrix}$ and then in basis 2 we see $|\alpha\rangle = a_{x'}|x'\rangle + a_{y'}|y'\rangle \iff \begin{pmatrix} a_{x'} \\ a_{y'} \end{pmatrix}$.

Addition of vectors by adding corresponding components (when in the same basis) works as you might expect, too:

$$|\alpha\rangle + |\beta\rangle \iff (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n$$

Also, scalar multiplication works by multiplying the scalar in each component

$$c|\alpha\rangle \iff (ca_1, ca_2, ca_3, \ldots, ca_n)$$

so of course,

$$|0\rangle = (0, 0, 0, \ldots, 0), \qquad |-\alpha\rangle = (-a_1, -a_2, \ldots, -a_n)$$

Inner Product: For every vector, $|\alpha\rangle$, in a vector space there exists a dual vector $\langle\alpha|$ in a corresponding dual vector space. Importantly, the dual vector to $c|\alpha\rangle$ is $C^*\langle\alpha|$ where $*$ denotes complex conjugation. So the inner product of $|\alpha\rangle$ and $|\beta\rangle$ is $\langle\alpha|\beta\rangle$ which is a scalar (complex number), hence $\langle\alpha|\beta\rangle$ is sometimes called **scalar product**.

(I) $\langle\beta|\alpha\rangle = \langle\alpha|\beta\rangle^*$

(II) $\langle\alpha|\alpha\rangle \geq 0$ (real and positive), so $\langle\alpha|\alpha\rangle = 0$ if $|\alpha\rangle = |0\rangle$.

(III) The norm of a vector $||\alpha|| = \sqrt{\langle\alpha|\alpha\rangle}$ generalized "length" of a vector.

(IV) Normalized $||\alpha|| = 1$.

(V) Orthogonal if $\langle\alpha|\beta\rangle = 0$, then $|\alpha\rangle$ is orthogonal to $|\beta\rangle$.

(VI) Orthogonal set $\langle a_i|a_j\rangle = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

Consider the orthonormal basis $|e_1\rangle, |e_2\rangle, \ldots, |e_n\rangle$, and

$$|\alpha\rangle = a_1|e_1\rangle + a_2|e_2\rangle + \ldots + a_n|e_n\rangle$$

$$|\beta\rangle = b_1|e_1\rangle + b_2|e_2\rangle + \ldots + b_n|e_n\rangle$$

where we have the column vectors

$$|\alpha\rangle = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \qquad |\beta\rangle = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

then with dual vectors

$$|\alpha\rangle = a_1^*|e_1\rangle + a_2^*|e_2\rangle + \ldots + a_n^*|e_n\rangle$$

$$|\beta\rangle = b_1^*|e_1\rangle + b_2^*|e_2\rangle + \ldots + b_n^*|e_n\rangle$$

so that we have row vectors

$$|\alpha\rangle = (a_1^*, a_2^*, \ldots, a_n^*) \qquad |\beta\rangle = (b_1^*, b_2^*, \ldots, b_n^*)$$

Now we can see that these results interact in a kind of cool way, check this out:

$$\langle\alpha|\beta\rangle = (a_1^*, a_2^*, \ldots, a_n^*) \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = a_1^*b_1 + a_2^*b_2 + \ldots + a_n^*b_n$$

which is a complex number. The components of the linear expansion are inner products too:

$$|\alpha\rangle = a_1|e_1\rangle + a_2|e_2\rangle + \ldots + a_n|e_n\rangle$$

Consider also that

$$\langle e_1|\alpha\rangle = \langle e_1|\left(a_1|e_1\rangle + a_2|e_2\rangle + \ldots + a_n|e_n\rangle\right)\rangle = a_1\langle e_1|e_1\rangle + a_2\langle e_1|e_2\rangle + \ldots + a_n\langle e_1|e_n\rangle = a_1$$

3.2 Matrices

Matrices represent linear transformations that take a vector in a vector space and map it to another vector.

$$|\alpha\rangle \longrightarrow |\alpha'\rangle = \hat{T}|\alpha\rangle$$

The transformation must be linear

$$\hat{T}(a|\alpha\rangle + b|\beta\rangle) = a\hat{T}|\alpha\rangle + b\hat{T}|\beta\rangle$$

Consider $\hat{T}$ acting on n basic vectors, $|e_i\rangle$

$$\hat{T}|e_1\rangle = T_{11}|e_1\rangle + T_{21}|e_2\rangle + \ldots + T_{n1}|e_n\rangle$$

That is, $|e_1\rangle$ is mapped to a new vector written as a linear combination of basis vectors, likewise

$$\begin{aligned}\hat{T}|e_2\rangle &= T_{12}|e_1\rangle + T_{22}|e_2\rangle + \ldots + T_{n2}|e_n\rangle \\ &\vdots \\ \hat{T}|e_n\rangle &= T_{1n}|e_1\rangle + T_{2n}|e_2\rangle + \ldots + T_{nn}|e_n\rangle\end{aligned}$$

which can be compactly expressed

$$\hat{T}|e_j\rangle = \sum_{i=1}^{n} T_{ij}|e_i\rangle \qquad (j = 1, 2, \ldots, n)$$

If $|\alpha\rangle$ is an arbitrary vector, expressed in terms of basis $|e_i\rangle$'s

$$|\alpha\rangle = a_1|e_1\rangle + a_2|e_2\rangle + \ldots + a_n|e_n\rangle = \sum_{j=1}^{n} a_j|e_j\rangle$$

(and recall $a_i = \langle e_1|\alpha\rangle$). Then the effect of $\hat{T}$ on $|\alpha\rangle$ is

$$\hat{T}|\alpha\rangle = \sum_{j=1}^{n} a_j\hat{T}|e_j\rangle = \sum_{j=1}^{n}\sum_{i=1}^{n} a_j T_{ij}|e_i\rangle = \sum_{i=1}^{n}\left(\sum_{j=1}^{n} T_{ij}a_j\right)|e_i\rangle$$

Hence $\hat{T}$ takes a vector $|\alpha\rangle$, with components $a_1, a_2, \ldots, a_n$ and maps to a new vector α' with components $a_i' = \sum_{j=1}^{n} T_{ij}a_j$. So $\hat{T}$ is characterized by n^2 elements, T_{ij}, which depend on the chosen basis. Express $\hat{T}$ as a matrix.

$$\begin{pmatrix} T_{11} & T_{12} & \ldots & T_{1n} \\ T_{21} & T_{22} & \ldots & T_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ T_{n1} & T_{n2} & \ldots & T_{nn} \end{pmatrix}$$

where T_{ij} is a matrix element, the row is i and column j. Now if we want to express $\hat{T}$ with respect to a particular set of basis vectors, the i-th element will define the values of matrix elements T_{ij}

$$\hat{T}|e_j\rangle = \sum_{i=1}^{n} |e_i\rangle \qquad (j = 1, 2, \ldots, n)$$

multiply on left by basis vector $|e_k\rangle$,

$$\begin{aligned}\langle e_k|\hat{T}|e_j\rangle &= \langle e_k|\sum_{i=1} T_{ij}|e_i\rangle \\ &= \langle e_k|\left(T_{ij}|e_1\rangle + T_{2j}|e_2\rangle + \ldots + T_{nj}|e_n\rangle\right) \\ &= \left(T_{ij} = \langle e_k|e_1\rangle + T_{2j}\langle e_k|e_2\rangle + \ldots + T_{kj}\langle e_k|e_k\rangle + \ldots + T_{nj}\langle e_k|e_n\rangle\right)\end{aligned}$$

where all terms except $\langle e_k|e_k\rangle$ go to 0. Now apply the orthonormal property of basis vectors $|e_i\rangle$ and thus

$$\langle e_k|\hat{T}|e_j\rangle = T_{kj} \quad \text{matrix element}$$

Once the basis is chosen, the i-th element will define the vector in coordinate representation and the linear transformation in matrix form.

Some matrix terminology,

Definition 3.2 (transpose)**.** The interchange of rows and columns of the matrix. Transpose of column is row and vice cersa. The transpose of a square matrix is to reflect elements in main diagonal.

Definition 3.3 (symmetric)**.** A matrix is equal to its transpose (square matrices only).

Definition 3.4 (conjugate)**.** The complex conjugate of every element.

Definition 3.5 (adjoint)**.** The conjugate transpose of a matrix. Inidicated by a dagger symbol $\hat{T}^\dagger$. A square matrix is **Hermitian** if matrix and adjoint are equal $\hat{T} = \hat{T}^\dagger$. Vector space and dual are related by adjoint.

$$|\alpha\rangle = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = a \implies \text{DUAL} = a^\dagger = (a_1^*, a_2^*, \dots, a_n^*)$$

The inner product $\langle\alpha|\beta\rangle = a^\dagger b$.

Definition 3.6 (product)**.** Multiplication may not be commutative : $\hat{T}\hat{S} \neq \hat{S}\hat{T}$. The difference between orders is commutator

$$[\hat{S}, \hat{T}] = \hat{S}\hat{T} - \hat{T}\hat{S} \quad \text{(will be zero if } \hat{T} \text{ and } \hat{S} \text{ commute)}$$

Definition 3.7 (eigenvalues, eigenvectors)**.** Every linear transformation has special vectors that transform into scalar multiples of themselves.

$$\hat{T}|\alpha\rangle = \lambda|\alpha\rangle$$

where $|\alpha\rangle$ is the eigenvector, and λ is the eigenvalue.

Example 3.1. Find the eigenvalues and normalized eigenvectors of $\begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}$.

$$\begin{vmatrix} 5-\lambda & -2 \\ -2 & 2-\lambda \end{vmatrix} = 0 \qquad \text{(Characteristic Equation)}$$

This resolves to solving

$$\begin{aligned} (5-\lambda)(2-\lambda) - (-2)(-2) &= 0 \\ (\lambda - 1)(\lambda - 6) &= 0 \end{aligned}$$

Therefore $\lambda = 1$ or $\lambda = 6$ are eigenvalues. Eigenvectors

- $\lambda_1 = 1 \implies |\lambda_1\rangle = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$

$$\begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = 1\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \implies 5x_1 - 2y_1 = x_1 \text{ and } -2x_1 + 2y_1 = y_1$$

Rearranging these equations gives us that $2x_1 - y_1 = 0$. This means that any vector on the line $2x_1 - y_1 = 0$ is an eigenvector. So one possible vector is $|\lambda_1\rangle = \begin{pmatrix}1\\2\end{pmatrix}$. Now we can normalize by introducing a normalization constant,

$$|\lambda_1\rangle = a\begin{pmatrix}1\\2\end{pmatrix}$$

then

$$\sqrt{\langle\lambda_1|\lambda_1\rangle} = 1 \implies \left(a(1,2)\begin{pmatrix}1\\2\end{pmatrix}\right)^{\frac{1}{2}} = a\sqrt{5} = 1 \implies a = \frac{1}{\sqrt{5}} \implies |\lambda_1\rangle = \frac{1}{\sqrt{5}} = \begin{pmatrix}1\\2\end{pmatrix}$$

4 Introduction to the formalism and structure of Quantum Mechanics

We're going to cover a few topics including Angular Momentum and Spin, the Stern-Gerlach Experiment, Quantum State Vectors, Computing Probabilities, and Operators and Measure.

4.1 Angular Momentum and Spin

Angular momentum and magnetic dipole moment (orbital). Consider an electron in a circular orbit, it has radius r, tangential velocity $\vec{v}$, and current going in the opposite direction of travel I, as well as dipole moment $\vec{\mu_L}$ and angular momentum $\vec{L}$. Now, to calculate the magnitude of the dipole moment,

$$\begin{aligned}
|\vec{\mu_L}| &= IA \quad \text{(product of current and area)} \\
&= \frac{e}{T}\pi r^2 \quad (T = \text{period of electron}) \\
&= \frac{e}{\left(\frac{2\pi r}{v}\right)}\pi r^2 \\
&= \frac{e}{2}vr \\
&= \frac{e}{2m_e}m_e vr \\
&= \frac{e}{2m_e}|\vec{L}|
\end{aligned}$$

The direction of $\vec{\mu_L}$ (follow the usual right hand rule)

$$\vec{\mu_L} = -\frac{e}{2m_e}\vec{L}$$

Next, we want to talk a little bit about **spin**. Spin is the intrinsic angular momentum $\vec{S}$ which leads to an intrinsic dipole moment $\vec{\mu_S}$. This intrinsic property is a fundamental nature of particle and cannot be taken away (c.f., mass or charge). In analogy with orbital angular momentum,

$$\vec{\mu_S} = g\frac{q}{2m}\vec{S}$$

where g is the **gyromagnetic ratio**, q is the **charge**, and m is the **mass** of the particle.

For an electron, $g \approx 2$, $q = -e$, $m = m_e$, which means

$$\vec{\mu_S} = -\frac{e}{m_e}\vec{S}$$

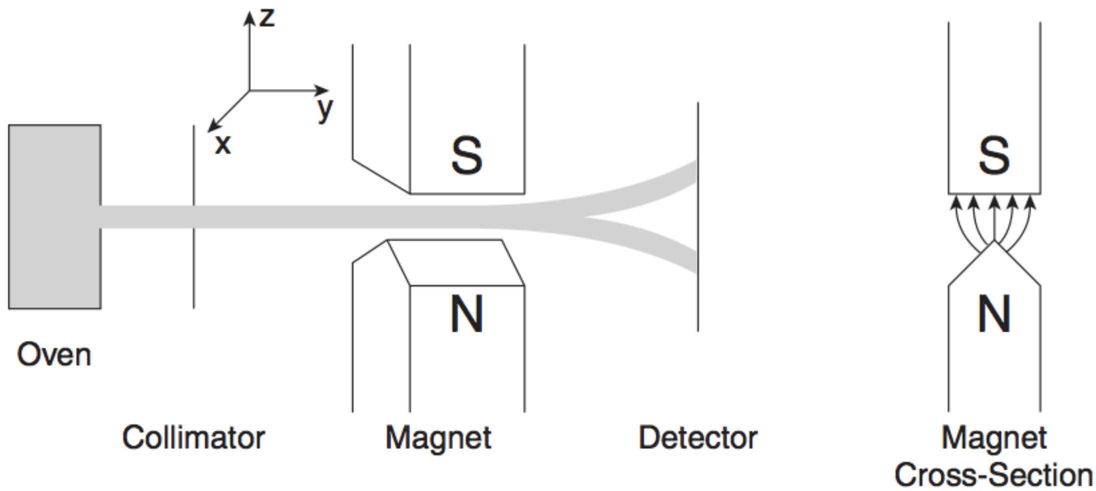

Figure 4.1: Stern-Gerlach experiment to measure the spin component of neutral particles along the z-axis. The magnet cross section at right shows the inhomogeneous field used in the experiment.

4.2 Stern Gerlach Experiment

This experiment was designed to measure the magnetic dipole moment of a particle (atom). A beam of atoms is passed through a magnetic field gradient and observations are made as to what happens to the trajectory.

So what are the physics in this experiment?
Potential energy of the magnetic dipole moment $\vec{\mu}$, in external field $\vec{B}$

$$E_{magn} = -\vec{\mu} \cdot \vec{B}$$

The force is negative of the gradient of the potential energy

$$\vec{F} = -\vec{\nabla}(-\vec{\mu} \cdot \vec{B})$$

In the Stern Gerlach experiment, the field gradient is in the z-direction, so only $\frac{dB_z}{dz} \neq 0$, so

$$\vec{F} = \mu_z \frac{dB_z}{dz} \hat{z} \qquad (\hat{z} \text{ is unit vector in } z\text{-direction})$$

Atoms experience a force in the z-direction proportionsl to the z-component of magnetic dipole moment μ_z because we designed an experiment where only $\frac{dB_z}{dz} \neq 0$.

What is the classical expectation for silver atoms? ($47e^-$, 47 photons, 60/62 neutrons). Note that

$$-\mu_L \text{ or } \mu_S \propto \frac{1}{m}, \qquad \text{so only consider electrons } (m_p \approx 2000 m_e)$$

and there is only one non-closed (tell electron that contributes to angular momentum), it is in an s-shell ($\vec{L} = 0$), leaving only instrinsic angular momentum. For silver atoms,

$$\vec{\mu} = -g \frac{e}{2m_e} \vec{S} \quad (\text{with } g \approx 2)$$

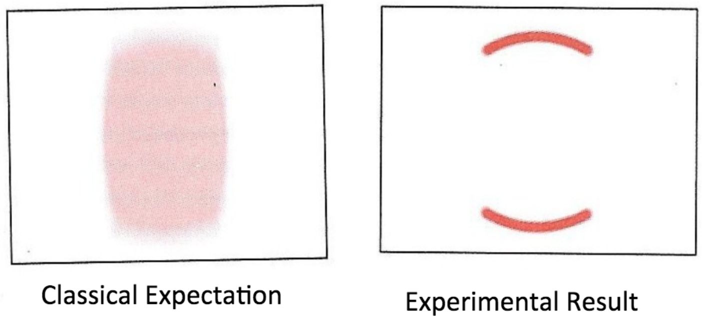

Figure 4.2: Space quantization as it appears in the experimental results of the Stern Gerlach experiment.

For a random gas of atoms, $\vec{\mu}$ is in all directions, so μ_z will have all possible values. So the force will range,

$$-\mu\frac{dB_z}{dz} \leq |\vec{F}| \leq +\mu\frac{dB_z}{dz}$$

which implies a circular beam spread in the z-direction.

It turns out that experimental results reveal that the beam is split into two. This is known as **space quantization**. This indicates that S_z has two possible values,

$$S_z = \pm\frac{\hbar}{2} \qquad \left(\hbar = \frac{h}{2\pi}\right)$$

Splitting is associated with the field gradient $\frac{dB_z}{dz}$ since it can change direction and splitting tracks the direction of field gradient. The weird thing here is that **there is no bias to atom deflection**. There is a 50% deflection up rate and 50% deflection down rate. An individual atom is deflected in a probabilistic way. So there is no way of determining precisely what happens to an individual atom.
No what we'd like to do is strip down the experiment to the essentisls and introduce language for additional study. First, there are **two possible outcomes**

$$S_z = +\frac{\hbar}{2} \text{ "spin up"} \qquad S_z = -\frac{\hbar}{2} \text{ "spin down"}$$

Definition 4.1 (observable)**.** The Quantum Mechanics term for the quantity being measured (S_z in this case.)

Definition 4.2 (analyser)**.** Stern Gerlach device is some form of an analyser $(x, y, z, \theta, \hat{n})$

Definition 4.3 (Postulate 1)**.** We label the input state with a left **ket**, ($|\Phi\rangle$), and label the output states with $|+\rangle$ for spin up and $|-\rangle$ for spin down.

5 Tutorials

5.1 Tutorial 1

1. Given two vectors

$$\vec{a} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 6 \\ 4 \\ 1 \end{pmatrix}$$

(a) Compute the inner product $\vec{a}^T\vec{b}$

$$1 \cdot 6 + 2 \cdot 4 + 4 \cdot 1 = 12$$

(b) Compute the outer product $\vec{a}\vec{b}^T$

$$6 \cdot 1 + 4 \cdot 2 + 1 \cdot 4 = 12$$

2. Given two matrices

$$A = \begin{pmatrix} 1 & 9 \\ 4 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 2 \\ 5 & 6 \end{pmatrix}$$

(a) Compute the product AB

$$\begin{pmatrix} 1 \cdot 0 + 9 \cdot 5 & 1 \cdot 2 + 9 \cdot 6 \\ 4 \cdot 0 + 3 \cdot 5 & 4 \cdot 2 + 3 \cdot 6 \end{pmatrix} = \begin{pmatrix} 45 & 56 \\ 15 & 20 \end{pmatrix}$$

(b) Do these matrices commute?
Let's check

$$\begin{pmatrix} 0 \cdot 1 + 2 \cdot 4 & 0 \cdot 9 + 2 \cdot 3 \\ 5 \cdot 1 + 6 \cdot 4 & 5 \cdot 9 + 6 \cdot 3 \end{pmatrix} = \begin{pmatrix} 8 & 6 \\ 29 & 63 \end{pmatrix}$$

No.

3. Given the matrix

$$C = \begin{pmatrix} 7 & 2 \\ 2 & 4 \end{pmatrix}$$

(a) Compute the eigenvalues of C The characteristic polynomial is

$$C(\lambda) = \det(C - \lambda I) = \det \begin{pmatrix} 7-\lambda & 2 \\ 2 & 4-\lambda \end{pmatrix} = (7-\lambda)(4-\lambda) - 4 = 24 - 11\lambda + \lambda^2 \implies \lambda = 8, 3$$

(b) Compute the eigenvectors of C

$$C - 8I = \begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix} \equiv \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix} \implies \vec{v} = t\begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad s, t \in \mathbb{R}$$

$$C - 3I = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \equiv \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix} \implies \vec{v} = t\begin{pmatrix} -1 \\ 2 \end{pmatrix} \qquad s, t \in \mathbb{R}$$

(c) Are the eigenvectors orthogonal?

Yes.

(d) Normalize the eigenvectors.

$$\begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}, \begin{pmatrix} \frac{-1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}$$

4. Given the complex number

$$z = 3 + 3i$$

(a) Compute the complex conjugate of z (denoted $\bar{z}$)

$$\bar{z} = 3 - 3i$$

(b) Compute the norm of z (denoted $|z|$)

$$|z| = \sqrt{z \cdot \bar{z}} = \sqrt{18}$$

(c) Express z as $z = re^{i\theta}$.

$$z = (\sqrt{18})e^{i\arctan\left(\frac{3}{3}\right)} = \sqrt{18}e^{\frac{\pi i}{2}}$$

(d) Express z as $z = r(cos\theta + i\sin\theta)$

$$z = \sqrt{18}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

5.2 Tutorial 2

Tutorial 2 concerns coin flipping, the SPINS program, and connections to Quantum Theory. We played games.