NUMERICAL ANALYSIS OF A LINEAR BLACK-SCHOLES MODEL

MA202 PROJECT REPORT

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- <u>**Problem Statement:**</u>

“Options are the contracts that give the buyer the right to buy or sell a certain amount of an underlying asset at a predetermined price at or **before the contract expires”** we can say An option is a contract that allows an investor to buy or sell something.

In this option, the buyer purchased an underlying instrument at a **premium**, strike price, or exercise price over a set period (before the expiry date). In the call option, the holder has the right to purchase a stock, whereas, in the put option, the holder has the right to sell a stock. We can think of the call option as a deposit for a future purchase.

So to predict the value of the option based on the given dataset, we are performing the numerical analysis of a linear Black-Scholes model equation.

Previously, there was no efficient mathematical model for determining the value of an option. As a result, the Black Scholes model serves as an analytical framework for options trading. The Black Scholes equation is a well-known partial differential equation in financial mathematics.

In this project, we will look at solving the Black Scholes model with European call and put options.

So in this project, we will also discuss the weighted average method with various weights used for numerical approximations. The modified Linear Black-Scholes equation is solved to find the option's value. We will use the delta-defining sequence of the generalized Dirac-delta function and the Mellin transformation to obtain an integral formula.

Abstract:

In our project, we have used the Black-Scholes Option Model (BSOPM) to develop the comparative analytical approach and numerical technique for analyzing the prices of call and put options.
We build our model so that it predicts the prices of a stock market in a frontier market, also known as the closing price. The Black-Scholes equation

has also been used lately to determine the stock varying prices by valuing the equity option. We modified the Black Scholes equation into Schrodinger's equation since it's easy to solve it numerically; then, by the method of explicit finite difference and applying proper boundary conditions, we were able to solve the Black Scholes method. The graph obtained through the output of code helped obtain the trend for the company's stock price and helped predict its price in the mean duration of time.

Introduction:

Stock price prediction of a specific company in a frontier market using the Black-Scholes formula has not been studied previously. Frontier markets do not follow the method of an emerging market, where options are traded; asset prices such as stock prices can be predicted using BSOPM. This is because BSOPM has parameters such as strike price, spot price, and expiry/date that have not been introduced in the frontier market.

In recent years, applications of machine learning and data mining techniques have led to the development of various business intelligence systems. It is because the algorithms can be trained and developed using ML to predict future stock prices based on the historical prices of the stocks. Black-Scholes option pricing method is based on the Brownian motion, so many different approaches have been made to derive the Black-Scholes PDE in the light of quantum mechanics.

We have developed a process for predicting stock prices in frontier markets using the Black-Scholes option pricing model by modifying the parameters of Black Scholes equation.

We found that the market crash is directly related to volatility. As the volatility increases continuously at a high rate, month by month, and the increase in the stock price of the companies in a frontier market, we can say that the market is likely to crash in a very short period. This is because the volatility can not remain high for a longer duration as people change their way of trading stocks because of the overpricing.

Therefore, the prices of stocks in the market must switch, i.e., there has to be regular increments or decrements.

Call and Put options:

Before going into the call and put options we need to have basic knowledge of some terms like

Strike Price: “The price at which buyers and sellers decide to buy or sell the underlying asset at the end of a specified period is referred to as the strike price or the Specified price”.

Spot Price: The current price of the underlying asset in the stock market is referred to as the spot price.

Expiry or Maturity date: “It is a date at which the actual options trading starts. And the specified time during which the sale can be made”.

Option Premium: The option premium is the non-refundable amount paid by the option buyer to the option seller upfront (also known as the option writer) or we can say it is the amount paid by the buyer while signing for the owning the right of the stock.

Premium: It is the value that needs to be paid when signing the to own the right of the stock. Or we can say a price needs to be paid if you purchase a call option.

Exercise price: “The exercise price is the price at which the underlying asset must be for the put option contract to be worth something.”

There are two types of styles present in the options market - the American and European styles. The fundamental difference between both styles is that European Style Options can only be executed at expiration.
In contrast, American Style Options can be executed at any time before or after maturity.

This project primarily focuses on the European Options.

- **Call option:**

A call option is called a "call" option because it allows the stock owner to "call the stock away" from the seller. It will enable the owner to set the maximum purchase price of the stock.
"Basically, a call option is a contract that allows the owner or the buyer the right, but not the obligation, to purchase a specified amount of an asset at a specified price which is premium, and within a specified time frame, which is maturity." For instance, if the stock is unprofitable, he can simply let the option expire worthless. Call options can be purchased for speculative purposes or sold for income or tax management.

Types of call options:

- **Long call option: "**A long call option is that in which the buyer has all the rights, but not the obligation, to purchase a stock in the future at a strike price." The advantage of a long call option is that we can plan to

purchase a stock at a lower price in advance. It is also for future price, as its name implies.

- **Short call option:** It is the opposite of the long call option in that the seller sells the stock in the future at a fixed price, unlike the long call option. "Short call options are generally used for the call options where the option seller already owns the underlying asset.” If the trade does not go their way, the call allows them to limit their losses.

- **<u>Put option:</u>**

  A put option is called a "put" because it provides the stock owner to "take the stock away" from the buyer. It will enable the owner to set the stock's minimum selling price. “A put gives the holder the right, but not the obligation, to sell the underlying stock at a predetermined price and within a predetermined time frame.”The value of a put option increases as the underlying stock price falls; the value of a put option decreases as the underlying stock price rises.

- **<u>Put vs. Calls:</u>**

  “A call option gives the holder a right to buy a stock at a certain price whereas in put option, the holder has a right to sell the stock at a certain price.”
  When an investor buys a call option, they expect the value of the underlying asset to go up or rise, and when the investor purchases a put option they expect the value to decline in price.

  The best thing about owning an option is a fixed amount to lose and a theoretically unlimited profit potential.

<u>The formula used for Call and put values:-</u>

According to the Black Scholes method, there are six parameters involved with it:-

- **S- Underlying Price**
- **σ- Volatility**
- **r- risk-free rate**
- **q- compounded dividend yield**

- **t- Time for expiration**
- **K- Strike price**

$$C = S\, e^{-qt}\, N(d_1) - K\, e^{-rt}\, N(d_2)$$

$$P = K\, e^{-rt}\, N(-d_2) - S\, e^{-qt}\, N(-d_1)$$

The formulas for d_1 and d_2 are:

$$d_1 = \frac{\ln(\frac{S}{K}) + t(r - q + \frac{\sigma^2}{2})}{\sigma\sqrt{t}}$$

$$d_2 = d_1 - \sigma\sqrt{t}$$

where $N(x)$ is the standard normal cumulative distribution function:

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{-t^2}{2}} dt$$

Black-Scholes Option Value		
Input Data		
Stock Price now (S)		100
Strike Price(K)		100
Number of periods to Exercise in years (t)		5
Compounded Risk-Free Interest Rate (r)		3.66%
Standard Deviation (annualized σ)		62.00%
Output Data		
Present Value of Exercise Price (PV(EX))		83.2768
σ*t^.5		1.3864
d1		0.8252
d2		-0.5612
Delta N(d1) Normal Cumulative Density Function		0.7954
Bank Loan N(d2)*PV(EX)		23.9285
Value of Call		**55.6080**
Value of Put		**38.8849**

Image:-[Black Scholes model using excel sheet]

Assumptions:

- No taxes
- No transaction costs
- Shares are infinitely divisible.
- Constant risk-free (RF) rate for borrowing/lending
- The value of the standard deviation of log returns(**σ**) remains stable.

- General short selling of stocks
- No dividends
- Continuous trading
- The stock prices evolve with a specific process through time.
- We assume European call in our methods.

Physical Model modification of BSOPM from Quantum Physics:

The Black Scholes' Merton equation was created by three outstanding economists, Fischer Black, Myron Scholes, and Robert Merton. Many methods can be employed to derive the Black Scholes PDE; in this example, we used the quantum physics method. On a non-dividend-paying underlying stock, this equation is used to price European call and put options.

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

Here, $V = V(S, t)$ is the pay-off function,

$S = S(t)$ the stock price with $S = S(t) \geq 0$,

t = time with $t \in (0; T)$ for T is the time of maturity,

r = Risk-free interest rate,

σ = Volatility Condition,

The option pricing method introduced by Black, Scholes, and Merton is composed of a random differential equation or precisely a stochastic differential equation, whose price depends on the stock price that evolved with changing time. However, the above Black and Scholes PDE could be written as a diffusion equation:

$$\delta_t \pi = -\frac{1}{2}(\sigma s)^2 \delta_{ss}\pi - rs\delta_s \pi + r\pi$$

Standard Brownian Motion process is the root idea behind the definition of Black-Scholes equation, which is given as follows:

$$ds(t) = \mu s(t)dt + \sigma s(t)dW(t)$$

In the equation above:

W = wiener process or standard Brownian motion process(a continuous stochastic process)

Furthermore, the random term in the diffusion equation (stochastic equation) must be delta-correlated because the market changes information instantly concerning the future market, this is the efficient market hypothesis.

This implies that theoretical prices are driven by White noise (A white noise process is the process of uncorrelated random variables, having zero mean value and a finite variance). White noise combines two processes, named white shot noise and wiener process.

Samples of white noises:

These are the sequence of uncorrelated random variables with zero mean value and finite variance and are independent, having identical probability distributions.

Note: If the mean value of the normal distribution is zero for each sample, then the signal is a Gaussian white noise (a zero-mean, stable, and ergodic random process described by these two essential properties:

- Any two GWN values are statistically independent, regardless of their temporal proximity.

- This characteristic directly impacts the autocorrelation function of a GWN).

But in our case, the mean is not absolute zero; it is slightly greater than zero, so there are clear chances that white noise is not applicable in BSOPM.

- Additionally, the stock prices in the Black-Scholes-Merton equation are sequential in time because samples dependent on time are also sequential.
- Because of this, the parameter time ‘t’ is changed sequentially.
- Hence, the time of expiration depends on the stock price of that particular day and varies with the stock’s price.
- The implied volatility also goes over time.
- The characteristics of the options differ only by the strike price.

ITO’s lemma: Ito's lemma is an identity used in Ito's calculus to find the differential of a stochastic process's time-dependent function. It shows how the value of an option changes as a function of the stock price.

Equation of ITO’s lemma:

$$dV = \frac{dV}{dS}dS + \frac{dV}{dt}dt + \frac{1}{2}\frac{d^2V}{dS^2}dS^2 + \frac{1}{2}\frac{d^2V}{dt^2}dt^2 + \frac{d^2V}{dtdS}dtdS$$

Delta Hedge Portfolio consists of two terms: a long position in call and a short position in stocks.

$$\Pi = V - \Delta S$$

Many attempts have been made to change the original BlackScholes PDE by considering the impact of transaction prices, price slippage, and market illiquidity on option costs separately.

The application of numerical methods for PDEs, particularly finite difference approaches, will be emphasized. We'll use this link to present the main theoretical features of the

Black-Scholes PDE's solutions, which are equivalent to the classical heat equation. The solution to the heat equation will then be utilized to solve the equation of the Linear Black-Scholes model.

The function $V = V(S, t)$ is representing the price of an option as a function of time and the cost of an asset.

<u>Conversion of Black Scholes differential equation into linear diffusion equation:</u>

We start with the equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

That has some initial terminal conditions,

$V(S,T) = max(S - K, 0)$

$V(0,t) = 0,\ and\ V(S,t) \sim S\ as\ S \sim \infty,\ 0 \leq t \leq T$

The value of option V(S,t) is defined from S$\sim (0,\infty)\ and\ t \sim (0,T)$

Step 1:

The equation can be rewritten as the following the equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2\left(S\frac{\partial}{\partial S}\right)^2 V + \left(r - \frac{1}{2}\sigma^2\right)S\frac{\partial V}{\partial S} - rV = 0$$

Substituting S=e^y & t=T-τ

The transformed derivatives after substituting are:

$$S\frac{\partial}{\partial S} \to \frac{\partial}{\partial y}, \qquad \frac{\partial}{\partial t} \to -\frac{\partial}{\partial \tau}$$

Step 2:

Substitute u=$e^{r\tau}V$ we get the following equation, in this the zeroth-order derivative term gets eliminated.

$$\frac{\partial u}{\partial \tau} - \frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial y^2} - \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial u}{\partial y} = 0.$$

Step 3:

After we get the above equation then, substituting,

$$x = y + (r - \sigma^2/2)\tau$$

Helps in the elimination of the first-order term and gives the following heat equation:

$$\frac{\partial u}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2}$$

Using Wick's rotation, through the transformation of real-term into imaginary term;

$\tau = - it$ and substituting it back into the above-obtained Heat's equation we get:

$$i\frac{\partial V}{\partial t} = - \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial x^2}$$

If we compare the above equation with Schrodinger's equation of a free particle, we find that h/2m=1; m = $\frac{1}{\sigma^2}$. Since the left side of the above equation has imaginary number i and the Wick-rotated time, 'τ', which is also imaginary. The function V(x,τ) changes with imaginary time, representing an option that gets traded in the future. If we were supposed to compare it with the heat equation, then the function V(x,τ) gets spread throughout the space is an option. The time it takes to calculate the stock price, which is made up of the function V(x,τ) at a right-hand location of x, is unknown. As a result, we just take the time to count the number of trading days on which the stock was traded.

Numerical Solution:

The Schrodinger equation is a partial differential equation that can be solved using various methods, but we have involved the Explicit Finite Difference technique.

Explicit Finite Difference:

One of the common ways to discretize a partial differential equation is the explicit finite difference method. It comes directly from the truncation of the Taylor series method and applying the boundary conditions since it is a Partial Differential equation.

We denote the time by upper case with parentheses and space by the lower case in the way that

$$\Psi_{r+1}^{(t)} = \Psi(r+h, t)$$

Note: h: Grid spacing

r: It is an integer that serves as an index for a general point on the grid.

We will deploy the Taylor series approximation in the process of this method:

$$\Psi_{r+1}^{(t)} = \Psi_r^{(t)} + h\Psi'^{(t)}_r + \frac{h^2}{2!}\Psi''^{(t)}_r + O(h^3)$$

$$\Psi_{r-1}^{(t)} = \Psi_r^{(t)} - h\Psi'^{(t)}_r + \frac{h^2}{2!}\Psi'^{(t)}_r + O(h^3)$$

From some approximation and manipulation, we get:

$$\Psi'^{(t)}_r = \frac{\Psi_{r+1}^{(t)} - \Psi_r^{(t)}}{h} + O(h) \;\; or \;\; \Psi'^{(t)}_r = \frac{\Psi_r^{(t)} - \Psi_{r-1}^{(t)}}{h} + O(h)$$

We can further bring some changes in the above formulas involving the forward step and backward step by subtracting the above two equations for the first derivative:

$$\Psi'^{(t)}_r \approx \frac{\Psi_{r+1}^{(t)} - \Psi_{r-1}^{(t)}}{2h}$$

Similarly for the second-order derivative we would add the two equations and using them to find second derivative:

$$\Psi''^{(t)}_{r} = \frac{-\Psi^{(t)}_{r+1} + 2\Psi^{(t)}_{r} - \Psi^{(t)}_{r-1}}{h^2} \text{ .}$$

Applying the appropriate boundary conditions for call and put options, we could solve the above PDE.

Boundary conditions for call options involve:

$$V(0,t) = 0 \; for \; 0 \leq t \leq T,$$

$$V(S,t) = S - Ke^{-r(T-t)} \; as \; S \to \infty$$

$$V(S,T) = max(S-K,0) \; for \; 0 \leq S < \infty$$

Boundary conditions for put options involve:

$$V(0,t) = Ke^{-r(T-t)} \; for \; 0 \leq t \leq T,$$

$$V(S,t) = 0 \; as \; S \to \infty$$

$$V(S,T) = max(K-S,0) \; for \; 0 \leq S < \infty$$

Using the above approximation and time evolution, we get the Crank-Nicolson method which is an easy method to teach. It has errors of order two, which reduce the overall error hence an efficient way to be used. However, there are pretty disadvantages to using this method too.

Results and Discussion:

The Black Scholes equation helped us determine the stock price for any frontier market, including its buying and selling prices. This method can be beneficial for calculating the values of volatility, strike price, and time of expiration. The above graph obtained through numerical analysis depicts the trend of the company's stock price. Through the historical values of the company's stock price, we can predict its value for later months, in which the value of volatility and strike price remains constant. Since trading occurs continuously throughout the day, causing the stock (close) price to change, Black-Schole’s prediction predicts a price comparable to the day's opening price. In contrast, the original closure price differs from Black-Schole's prediction.

Future Scope:

Stock market and trading culture is a trending topic worldwide. Many people want to learn, invest, and profit from the immense ocean of stocks. But it is not that easy; a lot of research and practice should be involved while investing in a stock. To make the objective clear and easy, we tried to develop a solution that could predict the relative value of the stock from the previous trends. But as we did the basic model with few assumptions so we can’t only rely on that. It can be just used to get the idea of the trend, and by following a little bit with news and trade trends, we can do the right thing.

As our model is fundamental and we can’t rely on that, by using machine learning, we can continue to work on improving our solution, which can be done by recording the past trends, recent data, and the frequency by which solution changes and answers. Machine learning and the stock market and trading world have a lot of scope in the future. Mostly, everything will be worked on in the future (like newly introduced NFTs). We can use a machine learning approach to reduce errors and get a more reliable solution.

The frontier market doesn’t use the black Scholes equation as the strike price, and volatility has not been introduced yet. So, in the future, we can use this equation in the stock market better.

Algorithm:

1. Setting the upper and the lower limits of stock price.
2. Converting the upper and lower limits into distance to get the length of the rod.
3. Initializing ‘u’ and ‘x’. ‘x’ has been initialized from x[1] to x[N] and leaving out x[0] and x[N+1] that will be updated later with help of boundary conditions.
4. Using the error function difference method to update the value of ‘u’ a large number of times.
5. Appending and prepending the boundary conditions that we left earlier to ‘u’.
6. Converting back the values into the financial variables from heat variables.
7. Plotting the graph of ‘u’ against ‘x’.
8. Link to the code :-

Citations:

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