Donsker's Theorem: A brief review

Abhilash Singh Avik Biswas Sanket Agrawal

Department of Mathematics & Statistics
Indian Institute of Technology Kanpur

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Outline

1. Introduction
2. Donsker's Theorem
   - Heuristic approach
   - Donsker's result
   - Generalization
3. Simulation

Empirical Process

Let $Z_1, \ldots Z_n \sim F$ be a random sample taking values in $\mathcal{X}$. Define the *Empirical measure* as,

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For any measurable function g, define

$$F_n g := \int g dF_n, \forall n; \quad Fg := \int g dF$$

Consider $\mathcal{G} = \{g \text{ defined on } \mathcal{X}, g \text{ measurable}\}$. Then, the collection of random elements $\{X_n g : g \in \mathcal{G}\}$ is called an *empirical process indexed by* $\mathcal{G}$.

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$$\hat{F}_n(t) := F_n I_{(-\infty,t]} = \frac{1}{n}\sum_{i=1}^{n} I_{(-\infty,t]}(Z_i); \quad t \in \mathbb{R}$$

$$X_n(t) := X_n I_{(-\infty,t]} = \sqrt{n}(\hat{F}_n(t) - F(t)); \quad t \in \mathbb{R}$$

Motivations

- Glivenko-Cantelli Theorem
- By Central Limit Theorem (CLT),

$$X_n(t) \xrightarrow{d} N\left(0, F(t)(1 - F(t))\right), \quad \text{for each } t \in \mathbb{R}.$$

What happens for collective t?

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Donsker's Theorem
(*uniform CLT, functional CLT, Invariance principle, and what not...*)

Doob's Heuristic approach

Let $F = U[0,1]$. For any $k \in \mathbb{N}$ and $0 \leq t_1 < t_2 < \cdots < t_k \leq 1$,

$$(X_n(t_1), \ldots, X_n(t_k)) \stackrel{CLT}{\to} N_k(0, \Sigma),$$

where $E(i,j) = \min\{t_i, t_j\} - t_i \cdot t_j$. Following is from Doob (1949).

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> We shall assume, until a contradiction frustrates our devotion to heuristic reasoning, that *in calculating asymptotic $x_n(t)$ process distributions when $n \rightarrow \infty$ we may simply replace the $x_n(t)$ processes by the $x(t)$ process*. It is clear that this cannot be done in all possible situations, but let the reader who has never used this sort of reasoning exhibit the first counter example.

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Hence,

$$X_n(t) \stackrel{??}{\to} X(t)$$

such that finite dimensional distribution of $X(t)$ for any k is $N_k(0, \Sigma)$.

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- What does $\xrightarrow{d}$ mean here?
- Generalizations?

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- $\mathrm{Cov}(B(s), B(t)) = \min\{s, t\} - st$.
- The function $t \mapsto B(t)$ is (almost surely) continuous on $[0, 1]$.

The space D

Definition (D)

The space $D([0,1], \|\cdot\|_\infty)$ is the space of all cádlaǵ functions on $[0,1]$ endowed with uniform metric.

cádlaǵ = right continuous and left limit exists at each point.

When D is endowed with Skorokhod metric, it is called the Skorokhod space.

Convergence

Say that, G_n converges weakly to G as elements of $D([0,1], \|\cdot\|)$ iff,

$$E(f(G_n)) \to E(f(G)) \text{ for all } f \in \mathcal{C}_b(D[0,1]).$$

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Remedies?

1. Projection $\sigma-$field.
2. Use a different metric (Skorokhod metric).
3. More general notion of weak convergence.

Generalization

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(Donsker's Class). A class $\mathcal{G}$ of measurable functions $g : \mathcal{X} \to \mathbb{R}$ is *Donsker* if the empirical process $\{X_n g : g \in \mathcal{G}\}$ indexed by $\mathcal{G}$ converges in distribution in the space $l^\infty(\mathcal{G})$ to a tight random element.

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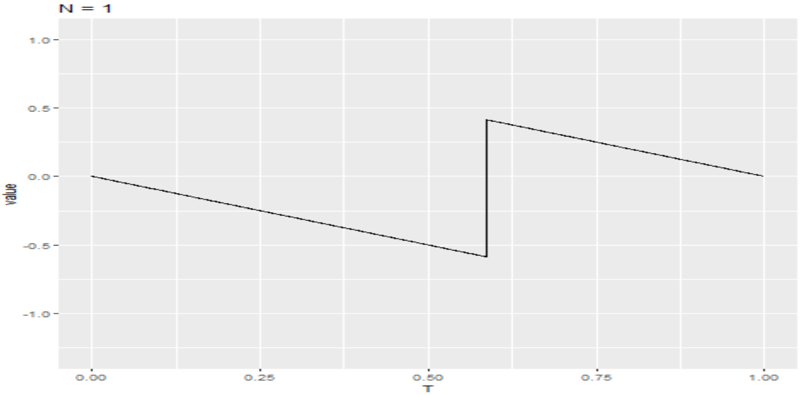
Sufficient conditions: *Covering numbers* and *Bracketing numbers*

Donsker's Theorem at work!



*Codes and GIF are available at: https://github.com/babasanku/DonskerTheorem

References

- Sen, B. A Gentle Introduction to Empirical Process Theory and Applications. Lecture Notes. Columbia University.
- Doob, J. L. (1949). Heuristic approach to the Kolmogorov-Smirnov theorems. *Ann. Math. Statistics*, 20:393–403.
- Donsker, M. D. (1952). Justification and extension of Doob's heuristic approach to the Komogorov-Smirnov theorems. *Ann. Math. Statistics*, 23:277–281.
- Pollard, D. (1984). *Convergence of stochastic processes* (Springer Series in Statistics). Springer-Verlag, New York.
- Vaart, A. (1998). *Asymptotic statistics* (Cambridge Series in Statistical and Probabilistic Mathematics). Cambridge University Press, Cambridge.

Questions????