Interpolation & polynomial Approximation:

Weierstrass Approximation Thm:

Suppose that f is defined and continuous on $[a,b]$. For each $\epsilon > 0$ there exists a polynomial $p(x)$ with

$$| f(x) - p(x) | < \epsilon \quad \forall x \in [a,b]$$

Eg: Taylor Series

$$f(x+h) = f(x) + \frac{h}{1!} f'(x) + \frac{h^2}{2!} f''(x) + \cdots$$

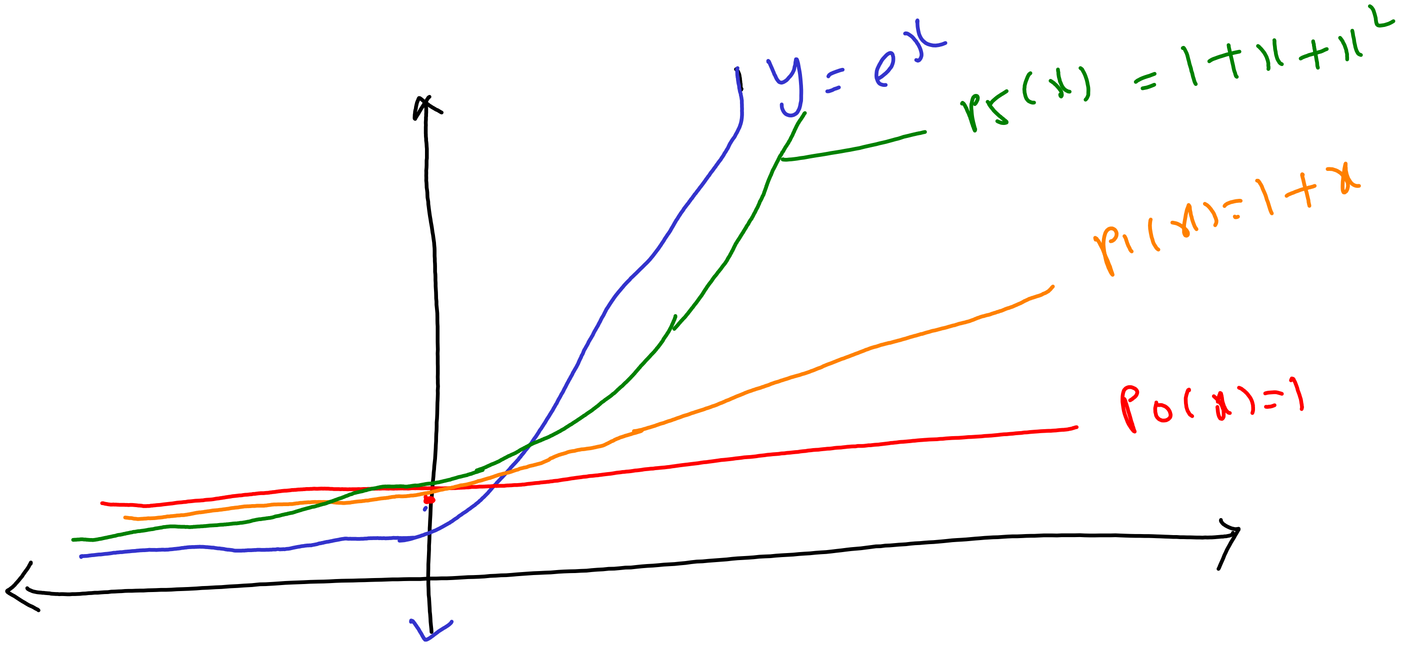
$$f(x) = e^x$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$p_0(x) = 1$ poly. of degree 0

$p_1(x) = 1 + x$ " 1

$p_2(x) = 1 + x + x^2$ " 2



Lagarange Interpolating Polynomial:

Problem of determining a polynomial of degree that passes through the distinct pts (x_0, y_0) and (x_1, y_1) is the same as approximating a function by means of first-degree polynomial interpolating.

Using this polynomial for approximation within the interval given by the end points is called polynomial interpolation.

Define

$$L_0(x) = \frac{x - x_1}{x_0 - x_1} \quad \text{and} \quad L_1(x) = \frac{x - x_0}{x_1 - x_0}$$

You have to construct in a such a way that

$$L_0(x_0) = 1 \; ; \; L_1(x_0) = 0$$

$$L_0(x_1) = 0 \; ; \; L_1(x_1) = 1$$

Thus linear Lagrange interpolating poly. through (x_0, y_0) and (x_1, y_1) is given as

$$P(x) = L_0(x) f(x_0) + L_1(x) f(x_1)$$

$$\boxed{P(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1)}$$

1) Determine the linear Lagrange inter-polating poly. that passes through $(2,4), (5,1)$

$x_0 = 2 \quad y_0 = f(x_0) = 4$

$x_1 = 5 \quad y_1 = f(x_1) = 1$

$$L_0(x) = \frac{x - x_1}{x_0 - x_1} = \frac{x-5}{-3}$$

$$L_1(x) = \frac{x - x_0}{x_1 - x_0} = \frac{x-2}{3}$$

$$\therefore P(x) = -\left(\frac{x-5}{3}\right)(4) + \left(\frac{x-2}{3}\right)(1)$$

$$= -x + 6$$

Generalization of Lagrange Interpolation:

To generalize the concept of linear interpolation, consider the construction of poly. of degree n passes through

$(x_0, f(x_0))$ $(x_1, f(x_1))$ .... $(x_n, f(x_n))$

First construct a function $L_{n,k}(x)$

satisfies $k = 0, 1, \ldots n$

$L_{n,k}(x_i) = 0$ when $i \neq k$

$L_{n,k}(x_k) = 1$

To satisfy $L_{n,k}(x_i) = 0 \quad i \neq k$ numerator should be

$$(x - x_0)(x - x_1) \dots (x - x_{k-1}) \cancel{(x - x_k)}$$
$$(x - x_{k+1}) \dots (x - x_n)$$

Therefore

$$L_{n,k}(x) = \frac{(x - x_0)(x - x_1) \dots (x - x_{k-1})(x - x_{k+1}) \dots (x - x_n)}{(x_k - x_0)(x_k - x_1) \dots (x_k - x_{k-1})(x_k - x_{k+1}) \dots (x_k - x_n)}$$

$L_{n,k}(x_k) = 1$;

$L_{n,k}(x_{k-1}) = 0$; $L_{n,k}(x_0) = 0, \dots$

$$\boxed{L_{n,k}(x_k) = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{(x - x_i)}{x_k - x_i}}$$

$\therefore$ required polynomial is

$$P(x) = f(x_0) L_{n,0}(x) + f(x_1) L_{n,1}(x) + \dots + f(x_n) L_{n,n}(x)$$

$$\boxed{P(x) = \sum_{k=0}^{n} f(x_k) L_{n,k}(x)}$$

1) Find the second Lagrange poly for $f(x) = \frac{1}{x}$ using $x_0 = 2$, $x_1 = 2.75$, $x_2 = 4$

Sol:

$$L_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} = \frac{2}{3}(x - 2.75)(x - 4)$$

$$L_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} = \frac{-16}{15}(x - 2)(x - 4)$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{2}{5}(x-2)(x-2.75)$$

$f(x) = 1/x$

$$\Rightarrow f(x_0) = \frac{1}{2}; \quad f(x_1) = \frac{1}{2.75}; \quad f(x_2) = \frac{1}{4}$$

$$\therefore \quad P(x) = \sum_{k=0}^{2} f(x_k) L_k(x_k)$$

$$= \left(\frac{2}{3}\right)\left(\frac{1}{2}\right)(x-2.75)(x-4)$$

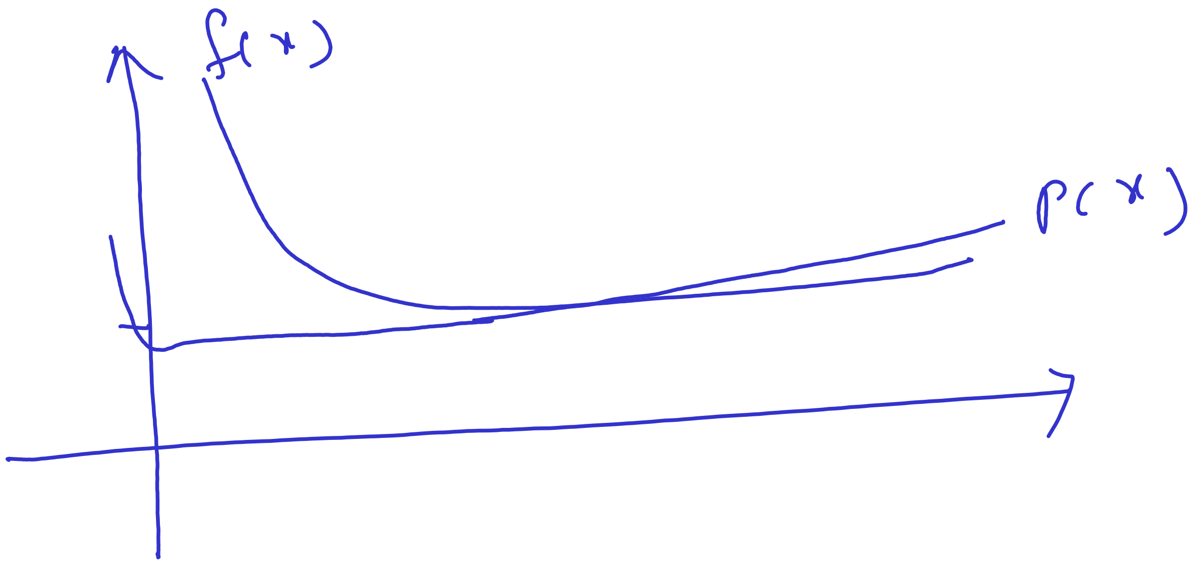
$$-\frac{64}{165}(x-2)(x-4)$$

$$+\frac{1}{10}(x-2)(x-2.75)$$

$$= \frac{x^2}{22} - \frac{35}{88}x + \frac{49}{44}$$

At $x = 3$ $\quad f(3) = 0.33333$

$$P(3) = 0.32955$$



Thm:

Suppose $x_0, x_1, \dots x_n$ are distinct numbers in $[a,b]$ and $f \in C^{n+1}[a,b]$. Then for each $x \in [a,b]$ a number $\xi(x)$ b/w $x_0, x_1, \dots x_n$ and hence

$$f(x) = P(x) + \underbrace{\frac{f^{(n+1)}(\xi(x))}{n+1}(x-x_0)(x-x_1)\dots(x-x_n)}_{\text{Error}}$$

where $P(x)$ is the interpolating polynomial.

Eg 1:

In previous problem we found the second Lagrange polynomial for $f(x) = \frac{1}{x}$ on $[2,4]$ using $x_0 = 2$, $x_1 = 2.75$ and $x_2 = 4$. Determine the max. error from that polynomial to approximate $f(x)$ for x in $[2,4]$.

Sol:

$$f(x) = \frac{1}{x}$$

$$f'(x) = \frac{-1}{x^2}; \quad f''(x) = \frac{2}{x^3}; \quad f'''(x) = \frac{-6}{x^4}$$

Error:

$$\frac{f'''(\xi(x))}{3!}(x-x_0)(x-x_1)(x-x_2)$$

$$= \frac{-6}{3!}(\xi(x))^{-4}(x-2)(x-2.75)(x-4)$$

$\xi(x)$ in $[2,4]$

Max. value of $(\xi(x))^{-4}$ in $(2,4)$ is

$$\max_{x\in(2,4)} (\xi(x))^{-4} = 2^{-4} = \frac{1}{16}$$

Next determine the max value of

let $g(x) = -(x-2)(x-2.75)(x-4)$

$g'(x) = 0 \Rightarrow$ following critical pts

$x = \frac{7}{3}$ $\quad \left|g(7/3)\right| = \left|\frac{25}{108}\right|$ max.

$x = \frac{7}{2}$ $\quad \left|g(7/2)\right| = \left|-\frac{9}{16}\right| = \frac{9}{16}$

$\therefore$ the max. error is

$$Er = \frac{f'''(\xi(x))}{3!}(x-x_0)(x-x_1)(x-x_2)$$

$$\le \frac{\cancel{6}}{\cancel{3!}} \frac{1}{16} \times \frac{9}{16} \simeq 0.035$$

2) Suppose a table is to be prepared for $f(x) = e^x$ for x in $[0,1]$. Assume that number of decimal places to be given

per entry is $d \geq 8$ and that difference b/w adjacent x values, the step size is h. What step size "h" will ensure that linear interpolation gives an absolute error at most 10^{-6} $\forall x \in [0,1]$?

Sol:

Let $x_0, x_1, \ldots$ be the numbers at which f is evaluated. $x \in [0,1]$. $\therefore$ Error is

$$\text{Error} = \frac{f^{(2)}(\xi(x))}{2!}(x - x_j)(x - x_{j+1})$$

$h = x_{j+1} - x_j$

Assumed $x_j \leq x \leq x_{j+1}$

$\therefore$ step size is h, $x_j = jh$

$x_{j+1} = (j+1)h$

$$\therefore \text{Error} \leq \frac{f^{(2)}(\xi(x))}{2!}\left|(x - jh)(x - (j+1)h)\right|$$

$$f(x) = e^x \qquad f''(x) = e^x$$

$$\max_{x \in (0,1)} f''(\xi(x)) = e$$

$$\therefore \text{Error} \le \frac{e}{2} \max_{x_j \le x \le x_{j+1}} |(x - jh)(x - (j+1)h)|$$

$$g(x) = (x - jh)(x - (j+1)h)$$

$$g'(x) = 2\left(x - jh - \frac{h}{2}\right) \quad (\text{verify})$$

$$\text{critical point} \quad x = jh + \frac{h}{2}$$

$$g\left(jh + \frac{h}{2}\right) = \frac{h^2}{4} \quad (\text{verify})$$

$$\therefore \text{Error} \le \frac{e}{2} \times \frac{h^2}{4} = \frac{eh^2}{8}$$

$$\text{Given Error} \le 10^{-6}$$

$$\Rightarrow \frac{eh^2}{8} \le 10^{-6} \Rightarrow \boxed{h \simeq 0.0017}$$