Newton's Divided Difference Formula:

* Divided difference methods are used to successively generate the polynomials themselves.

Zero divided difference of f:

$$f[x_i] = f(x_i)$$

First divided difference of f:

Notation

$$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}$$

Second divided difference of f:

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}$$

Thus k^{th} difference formula of f

$$f[x_i, x_{i+1}, x_{i+2}, \ldots x_{i+k-1}, x_{i+k}]$$

$$= \frac{f[x_{i+1}, x_{i+2}, \ldots x_{i+k}] - f[x_i, x_{i+1} \ldots x_{i+k-1}]}{x_{i+k} - x_i}$$

Interpolating polynomial

$$P_n(x) = f(x_0) + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2] + \cdots + (x - x_0)(x - x_1) \ldots (x - x_{n-1}) f[x_0, x_1, \ldots x_n]$$

$$\boxed{P_n(x) = f(x_0) + \sum_{k=1}^{n} f[x_0, x_1 \ldots x_k] (x - x_0)(x - x_1) \ldots (x - x_{k-1})}$$

This is called newton's divided diff. formula

Table:

x	$f(x)$	1st	2nd
x_0	$f[x_0]$	$f[x_0, x_1] = \dfrac{f[x_1] - f[x_0]}{x_1 - x_0}$	$f[x_0, x_1, x_2] = \dfrac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$
x_1	$f[x_1]$	$f[x_1, x_2] = \dfrac{f[x_2] - f[x_1]}{x_2 - x_1}$	$f[x_1, x_2, x_3] = \dfrac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1}$
x_2	$f[x_2]$	$f[x_2, x_3] = \dfrac{f[x_3] - f[x_2]}{x_3 - x_2}$	
x_3	$f[x_3]$		

3rd diff. formula

$$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}] = \frac{f[x_{i+1}, x_{i+2}, x_{i+3}] - f[x_i, x_{i+1}, x_{i+2}]}{x_{i+3} - x_i}$$

Eg: Construct divided diff table

x_i	$f(x_i)$	Diff 1	Diff 2	Diff 3	Diff 4
1	0.7651977				
		-0.4837057			
1.3	0.620086		-0.1087339		
		-0.5489460		0.0658784	
1.6	0.4554022		-0.0494433		0.0018251
		-0.5786120		0.0680685	
1.9	0.2818186		0.0118183		
		-0.575210			
2.2	0.1103623				

1st diff:

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0}$$

$$= \frac{0.620086 - 0.7651977}{0.3}$$

$$= -0.4837057$$

$$f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1} = -0.548946$$

2nd diff:

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

$$= \frac{-0.5489460 + 0.4837057}{1.6 - 1.0}$$

$$= -0.1087339$$

3rd diff.

$$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0}$$

$$= \frac{-0.0494433 + 0.1087339}{1.9 - 1.0}$$

$$= 0.0658784$$

4th diff:

$$f[x_0, x_1, x_2, x_3, x_4] = \frac{f[x_1, x_2, x_3, x_4] - f[x_0, x_1, x_2, x_3]}{x_4 - x_0}$$

$$= \frac{0.0680685 - 0.0658784}{2.2 - 1}$$

$$= 0.0018251$$

Therefore

$$p(x) = f(x) + \sum_{k=1}^{3} f[x_0, x_1, \ldots x_k] (x - x_0)(x - x_1) \ldots (x - x_k)$$

$$= 0.7651977 - 0.4837057 (x - 1.0)$$
$$- 0.1087339 (x - 1.0)(x - 1.3)$$
$$+ 0.0658784 (x - 1.0)(x - 1.3)(x - 1.6)$$
$$+ 0.0018251 (x - 1.0)(x - 1.3)(x - 1.6)(x - 1.9)$$

is a polynomial of degree 4.

Note: We can use Newton's divided difference formula where nodes are arranged in either equal (or) unequal spacing.

Newton's forward Difference formula:

(for equally spacing nodes)

Let $h = x_{i+1} - x_i$, $i = 0, 1, 2, 3, \ldots n-1$

$x = x_0 + sh \Rightarrow s = \dfrac{x - x_0}{h}$

$\Rightarrow x - x_i = (s - i)h$

$f[x_0, x_1] = \dfrac{f[x_1] - f[x_0]}{x_1 - x_0} \quad \Delta f(x_0)$

h

$= \dfrac{\Delta f(x_0)}{h}$ where $\Delta f(x_0) = f[x_1] - f[x_0]$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

$x_2 - x_0$
$= x_2 - x_1 + x_1 - x_0$
$= h + h$
$= 2h$

$$= \frac{\frac{1}{h}\Delta f(x_1) - \frac{1}{h}\Delta f(x_0)}{2h}$$

$$= \frac{1}{2h^2}\Delta^2 f(x_0)$$

Similarly

$$f[x_0, x_1, x_2, x_3] = \frac{1}{6h^3}\Delta^3 f(x_0)$$

$$= \frac{1}{3!\,h^3}\Delta^3 f(x_0)$$

$$\Rightarrow f[x_0, x_1, \dots x_k] = \frac{1}{k!\,h^k}\Delta^k f(x_0)$$

Therefore,

$$P_n(x) = f(x_0) + (x - x_0) f[x_0, x_1]$$
$$+ (x - x_0)(x - x_1) f[x_0, x_1, x_2]$$
$$+ \dots + (x - x_0)\dots(x - x_{k-1}) f(x_0, \dots x_k]$$

$$= f(x_0) + (sh) \frac{1}{h} \Delta f(x_0)$$
$$+ (sh)(s-1)h \frac{1}{2!h^2} \Delta^2 f(x_0)$$
$$+ \dots + sh(s-1)h(s-2)h \dots \frac{1}{k!h^k} \Delta^k f(x_0)$$

$$= f(x_0) + s\Delta f(x_0) + \frac{s(s-1)}{2!} \Delta^2 f(x_0)$$
$$+ \dots \frac{s(s-1)(s-2)\dots(s-(k-1))}{k!} \Delta^k f(x_0)$$

$$\boxed{P_n(x) = f(x_0) + \sum_{k=1}^{n} h^k \binom{s}{h} \frac{\Delta^k f(x_0)}{h^k}}$$

This is called Newton's forward difference formula

Backward Difference Formula:

If the interpolating nodes are reordered from <u>last</u> to <u>first</u> as $x_n, x_{n-1}, x_{n-2}, \ldots x_0$ we get

$$P_n(x) = f[x_n] + f[x_n, x_{n-1}](x - x_n)$$
$$+ f[x_n, x_{n-1}, x_{n-2}](x - x_n)(x - x_{n-1})$$
$$+ \cdots + f[x_n, x_{n-1}, x_{n-2}, \ldots x_0]$$
$$(x - x_n)(x - x_{n-1}) \cdots (x - x_1)$$

Let $x = x_n + sh$

$$x - x_i = (s + n - i)h$$

$$P_n(x) = f[x_n] + sh\, f[x_n, x_{n-1}]$$
$$+ s(s+1)h^2 f[x_n, x_{n-1}, x_{n-2}]$$
$$+ \cdots + s(s+1)\cdots(s+n-1)h^n$$
$$f[x_n, x_{n-1} \ldots x_0]$$

If nodes are equally spaced

$$f[x_n, x_{n-1}] = \frac{1}{h} \nabla f(x_n)$$

$$f[x_n, x_{n-1}, x_{n-2}] = \frac{1}{2h^2} \nabla^2 f(x_n)$$

$\vdots$

$$f[x_n, x_{n-1}, \ldots x_{n-k}] = \frac{1}{k!\,h^k} \nabla^k f(x_n)$$

$$\binom{-s}{k} = \frac{-s(-s-1)(-s-2)\ldots(-s-k+1)}{k!}$$

$$= (-1)^k \, s(s+1)(s+2)\ldots(s+k-1)$$

$$\therefore \boxed{p_n(x) = f(x_n) + \sum_{k=1}^{n} (-1)^k h^k \binom{-s}{k} \frac{\nabla^k f(x_n)}{h^k}}$$

This is called Newton's Backward diff. formula.

Eg 1) Consider the previous problem table. Find $f(x)$ at $x = 1.1$ & 2.0

x	$f(x)$	$\Delta f(x_0)$	$\Delta^2 f(x_0)$	$\Delta^3 f(x_0)$	Δ^4	Δ^5
1.0	0.7651977					
		-0.483057				
1.3	0.6200860		-0.1087339			
		-0.5489460		0.0658784		
1.6	0.4554022		-0.0494433		0.0018251	
		-0.5786120		0.0680685		
1.9	0.2818166		0.0118183			
		-0.5715210				
2.2	0.1103623					

a) Here $x = 1.1$ is very close to x_0 and x_1. Here (case) we apply Forward diff. formula.

Here $h = 0.3$

$$s = \frac{x - x_0}{h} = \frac{1.1 - 1}{0.3} = \frac{1}{3}$$

$$P_4(x) = f(x_0) + sh \frac{\Delta f(x_0)}{h} + s(s-1)h^2 \frac{\Delta^2 f(x_0)}{h^2} + \cdots$$

→ table

table

$$= 0.7651977 + \left(\frac{1}{3}\right)(0.3)(-0.4837057)$$

$$+ \left(\frac{1}{3}\right)\left(\frac{-2}{3}\right)(0.3)^2(-0.1087339)$$

$$+ \left(\frac{1}{3}\right)\left(\frac{-2}{3}\right)\left(\frac{-5}{3}\right)(0.3)^3(0.0658784)$$

$$+ \left(\frac{1}{3}\right)\left(\frac{-2}{3}\right)\left(\frac{-5}{3}\right)\left(\frac{-8}{3}\right)(0.3)^4(0.0018251)$$

$$p_4(1.1) = 0.719646$$

b) $x = 2.0$ is very close to x_n and x_{n-1}, therefore we have to apply backward diff. formula.

$$p_n(x) = f(x_n) + sh\frac{\nabla f(x_n)}{h} + s(s+1)h^2\frac{\nabla^2 f(x_n)}{2h^2} + \dots$$

$h = 0.3$ $\qquad s = \dfrac{x - x_n}{h} = \dfrac{2 - 2.2}{0.3} = \dfrac{-2}{3}$

$$P(2.0) = 0.1103623 - \left(\frac{2}{3}\right)(0.3)(-0.57152)$$
$$-\left(\frac{2}{3}\right)\left(-\frac{2}{3}+1\right)(0.3)^2\,(0.0118183)$$
$$-\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)\left(\frac{4}{3}\right)(0.3)^3\,(0.0680685)$$
$$-\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)\left(\frac{4}{3}\right)\left(\frac{7}{3}\right)(0.3)^4\,(0.0018251)$$

$$= 0.2238754$$

Central Difference :

* To approximate $f(x)$ when x lies near the center of the table

* For central diff. we choose x_0 near the point x to be approximated

x	$f(x)$	1st	2nd	3rd
x_{-2}	$f(x_{-2})$			
		$f(x_{-2}, x_{-1})$		
x_{-1}	$f(x_{-1})$		$f(x_{-2}, x_{-1}, x_0)$	
x_0	$f(x_0)$	$f(x_{-1}, x_0)$		$f(x_{-2}, x_{-1}, x_0, x_1)$
			$f(x_{-1}, x_0, x_1)$	
x_1	$f(x_1)$	$f(x_0, x_1)$		$f(x_{-1}, x_0, x_1, x_2)$
		$f(x_1, x_2)$	$f(x_0, x_1, x_2)$	
x_2	$f(x_2)$			

$$p_n(x) = P_{2m+1}(x)$$

$$= f(x_0) + \frac{sh}{2}\left(f[x_{-1}, x_0] + f[x_0, x_1]\right)$$

$$+ s^2h^2 f[x_{-1}, x_0, x_1]$$

$$+ \frac{s(s^2-1)h^3}{2}\left(f[x_{-2}, x_{-1}, x_0, x_1] + f[x_{-1}, x_0, x_1, x_2]\right)$$

$$+ \dots + s^2(s^2-1)(s^2-4)\cdots(s^2-(m-1)^2)h^{2m} f[x_{-m}, x_{-m+1}, \dots x_m]$$

$$+ \frac{s(s^2-1)\ldots\ldots(s^2-m^2)\,h^{2m+1}}{2} \left(f[x_{-m-1}, \ldots x_m] + f[x_{-m}, \ldots x_{m+1}]\right)$$

If $n = 2m+1$ is odd

If $n = 2m$ is even then we use the same above formula but we delete the last line.

This formula is also called as

Stirling's formula

Eg: Consider the same problem and find $f(1.5)$ with $x_0 = 1.6$

$h = 0.3$

$x_0 = 1.6$

$$s = \frac{x - x_0}{h} = \frac{-1}{3}$$

1	0.7651977				
		−0.4837057			
1.3	0.6200860		−0.1087339		
		−0.5489460		0.0658784	
1.6	0.4554022		−0.0494433		0.0018251
		−0.5786120		0.0680685	
1.9	0.2818186		0.0118183		
		−0.5715210			
2.2	0.1103623				

x_0

$$f(1.5) = P(1.5)$$

$$= 0.4554022 + \left(\frac{-1}{3}\right)\left(\frac{0.3}{2}\right)\left(-0.5489460 - 0.5786120\right)$$

$$+ \left(-\frac{1}{3}\right)^2 (0.3)^2 (-0.0494433)$$

$$+ \left(\frac{1}{2}\right)\left(-\frac{1}{3}\right)\left(\left(-\frac{1}{3}\right)^2 - 1\right)(0.3)^3 \left(0.0658784 + 0.0680685\right)$$

$$+ \left(-\frac{1}{3}\right)^2\left(\left(-\frac{1}{3}\right)^2 - 1\right)(0.3)^4 (0.0018251)$$

$$= 0.511820$$