Initial Value Problems for ODE

We are interested to find the approximate solution $y(t)$ to a problem of the form

$$\frac{dy}{dt} = f(t,y) \quad , \quad a \le t \le b$$

$$y(a) = \alpha$$

Basic Results:

(i) A function $f(t,y)$ is said to satisfy a Lipschitz condition in the variable y on a set $D \subset \mathbb{R}^2$ if a constant $L > 0$ exists with

$$|f(t,y_1) - f(t,y_2)| \le L|y_1 - y_2|$$

whenever (t,y_1) and (t,y_2) in D.

* L is called a Lipschitz constant

Eg: $f(t,y) = t|y|$

$D : \{(t,y) \mid 1 \le t \le 2, \; -3 \le y \le 4\}$

$$|f(t,y_1) - f(t,y_2)|$$

$$= |t|y_1| - t|y_2||$$

$$= |t|\,|y_1 - y_2|$$

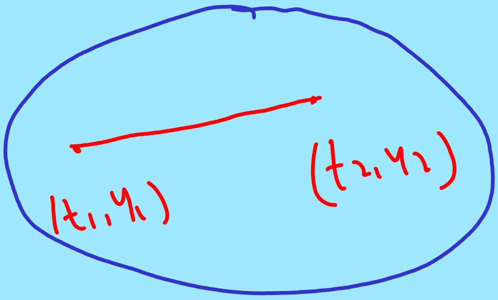
$$\le 2|y_1 - y_2|$$

$\therefore$ f satisfies a Lipschitz condition on D.

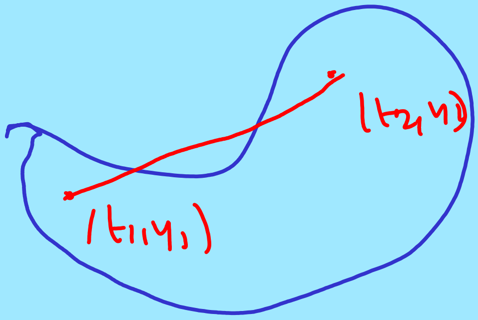
(2) A set $D \subset \mathbb{R}^2$ is said to be *convex* if whenever (t_1, y_1) & (t_2, y_2) in D then

$$((1-\lambda)t_1 + \lambda t_2, (1-\lambda)y_1 + \lambda y_2)$$

also belongs to D and λ in $[0,1]$



Convex



Not convex

Thm:

Suppose $f(t,y)$ is defined on a convex set $D \subset \mathbb{R}^2$. If a constant $L > 0$ exists with

$$\left| \frac{\partial f}{\partial y}(t,y) \right| \leq L \quad \forall\ (t,y) \text{ in } D$$

then f satisfies a Lipschitz condition on D.

Eg:

$$f(t,y) = y - t^2 + 1, \quad 0 \leq t \leq 2$$

$$\left| \frac{\partial f}{\partial y} \right| = |1| = 1$$

$\therefore$ f is Lipschitz

Thm:

Suppose that $D = \{ (t, y) \mid a \le t \le b, -\infty < y < \infty \}$

and that $f(t, y)$ is continuous on D.

If f is satisfying a Lipschitz condition on D then the IVP

$$y'(t) = f(t, y) \qquad a \le t \le b$$

$$y(a) = \alpha$$

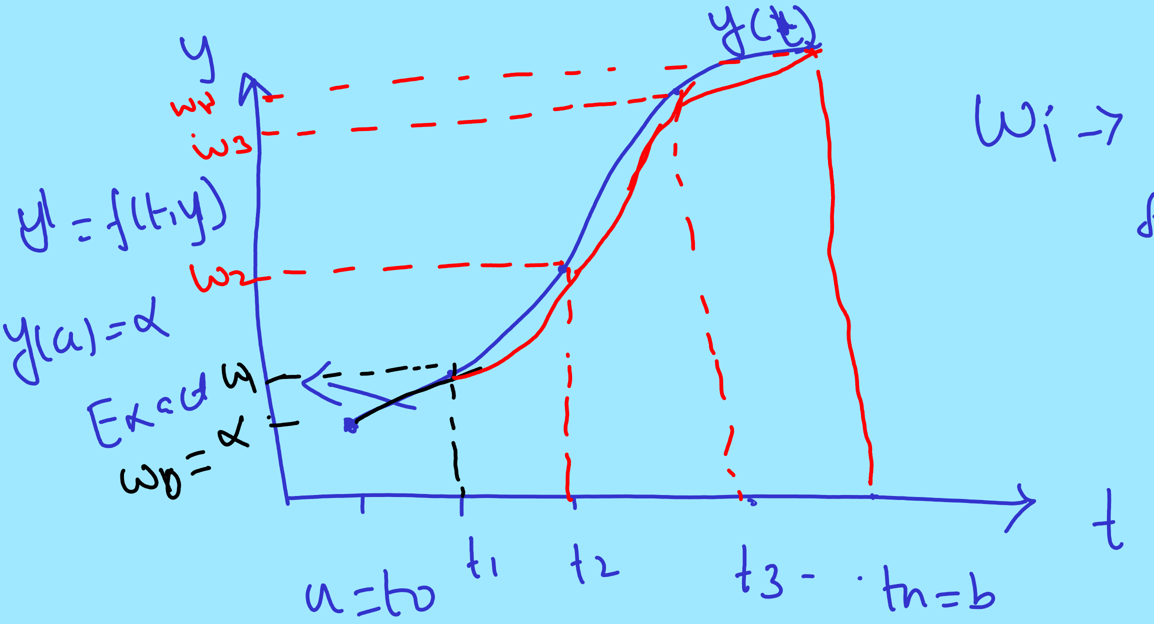
has a unique solution $y(t)$, for $a \le t \le b$.

Euler's method:

The main objective of Euler's method is to find app. solution to the IVP

$$\frac{dy}{dt} = f(t, y) \qquad a \le t \le$$

$$y(a) = \alpha$$



$w_i \to$ App. soln

- One-step Euler method
- Series of steps Euler method

* Approximations to y will be generated at various values called <u>mesh points</u> in $[a, b]$

* Choose a positive integer N and selecting mesh points

$$t_i = a + ih, \quad i = 0, 1, 2, \ldots N$$

$$\Rightarrow \quad h = \frac{b - a}{N} = \underbrace{t_{i+1} - t_i}_{\text{step size}}$$

From Taylor's Series

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \dots$$

$$y(t_{i+1}) = y(t_i) + h\, y'(t_i) + \frac{h^2}{2!} y''(\xi_i)$$

$$\qquad\qquad\qquad\quad \| \qquad\qquad$$

where $t_i < \xi_i < t_{i+1}$ $\qquad f(t_i, y_i)$

$$y(t_{i+1}) = y(t_i) + h f(t_i, y(t_i)) + \frac{h^2}{2} y''(\xi_i)$$

Euler's method constructs,

$$w_i \approx y(t_i), \quad i = 1, 2 \dots N$$

Thus Euler's method is

$$\boxed{\begin{aligned} w_0 &= \alpha \\ w_{i+1} &= w_i + h f(t_i, w_i) \end{aligned}}$$

$$i = 0, 1, 2, \dots N-1$$

1) $y' = y - t^2 + 1, \quad 0 \le t \le 2$

Given

$y(0) = 0.5, \quad h = 0.5$

By Euler method

$w_0 = y(0) = 0.5$

$y(0.5) = w_1 = w_0 + h f(t_0, w_0)$

$= w_0 + (0.5)(w_0 - 0^2 + 1)$

$t_i = a + ih$

$= 1.25$

$t_1 = a + h$

$= 0.5$

$y(1) = w_2 = w_1 + h f(t_1, w_1)$

$= w_1 + (0.5) f(0.5, 1.25)$

$= 1.25 + (0.5)(1.25 - 0.5^2 + 1)$

$= 2.25$

$y(1.5) = w_3 = w_2 + h f(t_2, w_2)$

$= 2.25 + (0.5) f(1, 2.25)$

$= 3.375$

$$y(2) = w_4 = w_3 + h f(t_3, w_3)$$

$$= 3.375 + (0.5) f(1.5, 3.375)$$

$$= 4.4375$$

2) Solve $y' = y - t^2 + 1$, $0 \le t \le 2$

$y(0) = 0.5$, $\underline{\underline{N = 10}}$

$$h = \frac{b-a}{N} = \frac{2}{10} = 0.2$$

$$w_{i+1} = w_i + h f(t_i, w_i)$$

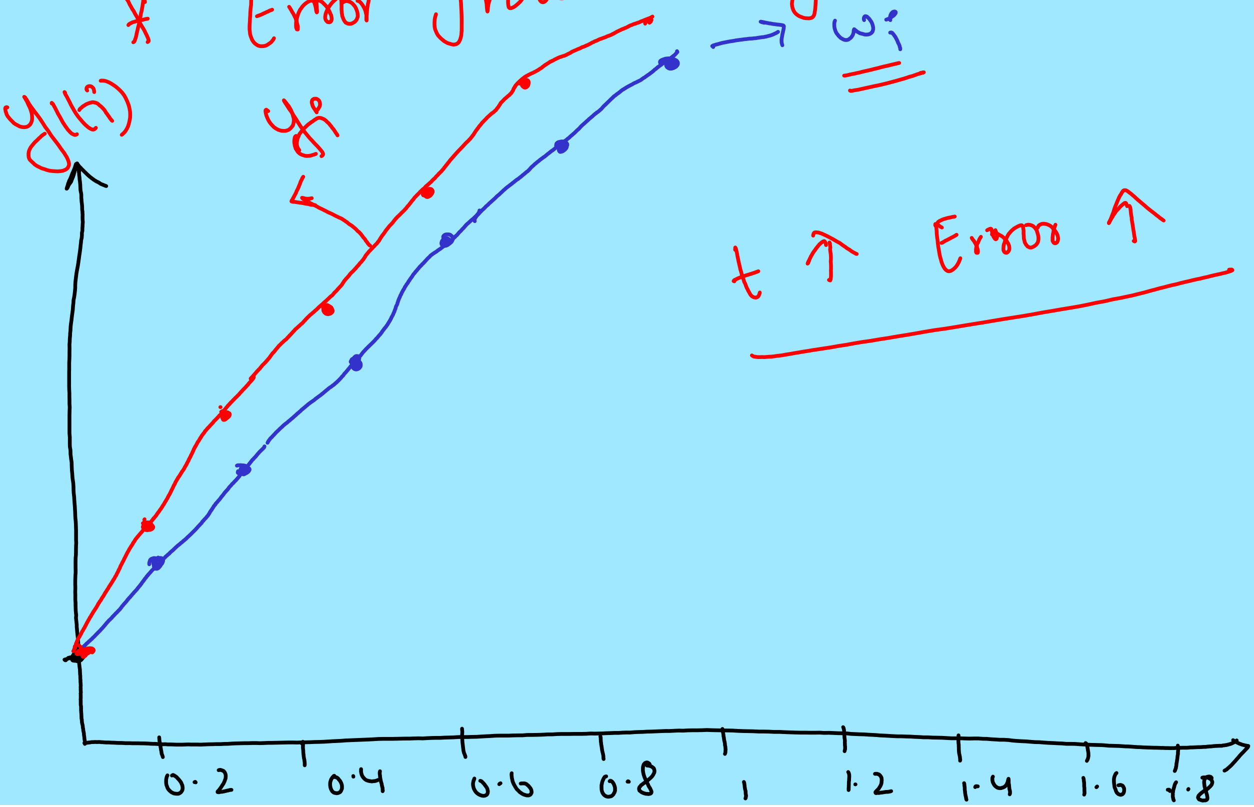
Compare with exact solution

$$y(t) = (t+1)^2 - 0.5 e^t$$

t_i	w_i	y_i	$\lvert y_i - w_i \rvert$
0	0.5	0.5	0
0.2	0.8	0.8292986	0.0292986
0.4	1.152	1.214087	0.0620877

0.6	1.5504	1.6489406	0.0985406
0.8	1.98848	2.1272295	0.1387495
1	2.458176	2.6408591	0.1826831
1.2	2.9498112	3.1799415	0.2301303
1.4	3.4517734	3.732400	0.2806266
1.6	3.9501281	4.2834838	0.3333557
1.8	4.4281538	4.8151763	0.3870225
2	4.8657845	5.3054720	$\underline{\underline{0.4396874}}$

* Error grows slowly when t increase



t ↑ Error ↑

Error Bound

Thm:

Suppose f is continuous and satisfies a Lipschitz condition with L on

$$D = \{ (t, y),\ a \le t \le b,\ -\infty < y < \infty \}$$

and a constant M exists with

$$\| y''(t) \| \le M \quad \forall\ t \in [a, b]$$

where $y(t)$ is a solution of

$$y' = f(t, y), \quad a \le t \le b.$$

$$y(a) = \alpha$$

Let w_i, $i = 1, 2, \ldots N$ be the app. solutions generated by Euler's method for some N,

$$\boxed{|y(t_i) - w_i| \le \frac{hM}{2L}\left[e^{L(t_i-a)} - 1\right]}$$

1) Find a bounds for the approximation error and compare with actual errors of Euler's method with $h = 0.2$ for the following problem

$$y' = y - t^2 + 1, \quad 0 \le t \le 2$$

$$y(0) = 0.5$$

Soln: Exact soln:

$$y(t) = (t+1)^2 - 0.5e^t, \quad 0 \le t \le 2$$

$$y'(t) = 2(t+1) - 0.5e^t$$

$$y''(t) = 2 - 0.5e^t$$

$\therefore \ |y''(t)|| \leq 0.5e^2 - 2$

$g(t) = 2 - 0.5e^t$

$t \in [0, 2] \quad g'(t) = -0.5e^t = 0$

$e^t = 0 \Rightarrow t = -\infty$

$\boxed{M = 0.5e^2 - 2}$

$t = 0 \Rightarrow |g(t)| = 1.5$

$f(t, y) = y - t^2 + 1$

$t = 2 \Rightarrow |g(t)| = |-1.69|$

$0.5e^2 - 2$

$\left|\frac{\partial f}{\partial y}\right| = |1| = L$

$\therefore \ \boxed{L = 1}$ Given $\boxed{h = 0.2}$, $\underline{\underline{a = 0}}$

$\therefore$ Error bound

$$|y_i - w_i| \leq \frac{(0.2)(0.5e^2 - 2)}{2(1)} \left(e^{(1)(t_i - 0)} - 1\right)$$

$$\Rightarrow |y_i - w_i| \leq (0.1)(0.5e^2 - 2)(e^{t_i} - 1)$$

At $t = 0.2$

$$|y(0.2) - w_1| \le (0.1)(0.5e^2 - 2)(e^{0.2} - 1)$$

$$= 0.03752$$ Error bound

Computed error $|y(0.2) - w_1| \le \underline{0.29298}$

At $t = 0.4$

$$|y(0.4) - w_2| \le (0.1)(0.5e^2 - 2)(e^{0.4} - 1)$$

$$= 0.08334$$

previous problem

Computed error

$$|y(0.4) - w_2| = 0.\underline{06208}$$

High-order Taylor methods

Consider IVP

$$y' = f(t, y), \quad a \le t \le b$$

$$y(a) = \alpha$$

has $(n+1)$ continuous derivatives.

By Taylor polynomial about t_i,

$$y(t_{i+1}) = y(t_i) + h y'(t_i) + \frac{h^2}{2} y''(t_i) + \cdots + \frac{h^{n+1}}{(n+1)!} y^{(n+1)}(\xi_i)$$

for $\xi_i \in (t_i, t_{i+1})$. $\quad$ — (1)

Given $y' = f(t, y(t))$

By successive differentiation of $y(t)$, we get

$$y''(t) = f'(t, y(t))$$

$$y'''(t) = f''(t, y(t))$$

$$\vdots$$

$$y^{(k)}(t) = f^{(k-1)}(t, y(t))$$

②

Sub. ② in ①, we get

$$y(t_{i+1}) = y(t_i) + h f(t_i, y(t_i))$$

$$+ \frac{h^2}{2} f'(t_i, y(t_i))$$

$$+ \cdots + \frac{h^n}{n!} f^{(n-1)}(t_i, y(t_i))$$

$$+ \frac{h^{n+1}}{(n+1)!} f^{(n)}(\xi_i, y(\xi_i))$$

Now, we construct Taylor method of order n

$$w_0 = \alpha$$

$$w_{i+1} = w_i + h f(t_i, w_i) + \frac{h^2}{2} f'(t_i, w_i) + \frac{h^3}{6} f''(t_i, w_i) + \dots + \frac{h^n}{n!} f^{(n-1)}(t_i, w_i)$$

Note: Euler's method is Taylor's method of order one

Eg 1: Apply Taylor's method of order (a) Two (b) Four with $N = 10$ to the IVP $y' = y - t^2 + 1$, $0 \le t \le 2$

$y(0) = 0.5$

(a) Taylor method of order two

$$f(t,y) = y - t^2 + 1$$

$$f'(t,y) = y' - 2t$$
$$= y - t^2 + 1 - 2t$$

$$w_0 = \alpha$$

$$w_{i+1} = w_i + h(w_i - t_i^2 + 1) + \frac{h^2}{2}(w_i - t_i^2 + 1 - 2t_i)$$

$h = \frac{b-a}{N} = 0.2$

$$y(0) = w_0 = 0.5$$

$$w_1 = w_0 + (0.2)(w_0 - (0)^2 + 1) + \left(\frac{0.2^2}{2}\right)(w_0 - 0^2 + 1 - 0)$$

$$y(0.2) \approx w_1 = 0.83$$

$$w_2 = w_1 + (0.2)(w_1 - (0.2)^2 + 1) + \frac{(0.2)^2}{2}(w_1 - (0.2)^2 + 1 - 2(0.2))$$

$y(0.4) \simeq w_2 = 1.215800$

t_i	Taylor order 2	Error $\lvert y_i - w_i \rvert$
0	0.5	0
0.2	0.83	0.000701
0.4	1.215800	0.001712
0.6	1.652076	0.003135
0.8	2.132333	0.005103
1	2.648646	0.007787
1.2	3.191348	0.011407
1.4	3.748645	0.016245
1.6	4.306146	0.022663
1.8	4.846299	0.031122
2	5.347684	0.042212

(b) Taylor method of order 4

$$f(t,y) = y - t^2 + 1$$

$$f'(t,y) = y - t^2 + 1 - 2t$$

$$f''(t,y) = y' - 2t - 2$$

$$= y - t^2 + 1 - 2t - 2$$

$$= y - t^2 - 1 - 2t$$

$$f'''(t,y) = y' - 2t - 2$$

$$= y - t^2 + 1 - 2t - 2$$

$$= y - t^2 - 2t - 1$$

$$w_0 = 0.5$$

$$w_{i+1} = w_i + h(w_i - t_i^2 + 1)$$

$$+ \frac{h^2}{2}(w_i - t_i^2 + 1 - 2t_i)$$

$$+\frac{h^3}{6}\left(w_i - t_i^2 - 2t_i - 1 \right)$$

$h = 0.2$

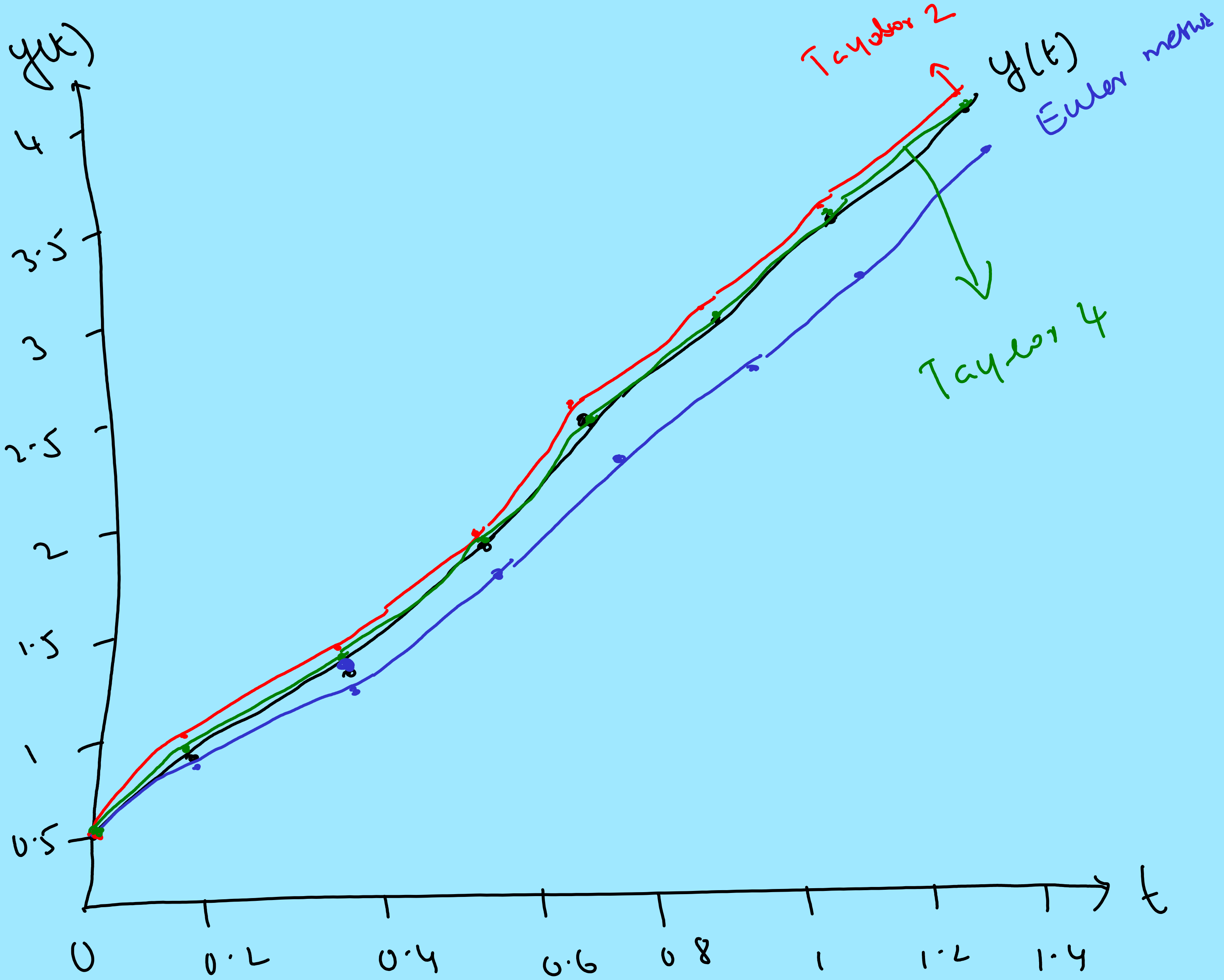
$$+\frac{h^4}{24}\left(w_i - t_i^2 - 2t_i - 1 \right)$$

$$y(0.2) \simeq w_1 = 0.8293$$

$$y(0.4) \simeq w_2 = 1.214091$$

t_i	Taylor Order 4	Error
0	0.5	0
0.2	0.8293	0.000001
0.4	1.214091	0.000003
0.6	1.648947	0.000006
0.8	2.127240	0.000010
1	2.640874	0.000015

1.2	3.179964	0.000023
1.4	3.732432	0.000032
1.6	4.283529	0.000045
1.8	4.815238	0.000062
2	5.305555	0.000083



Runge-kutta method

* Disadvantage of Taylor method is require to compute and evaluate the derivatives of $f(t,y)$.

* This is time-consuming procedure

* Runge-kutta (RK) methods have the high-order local truncation errors of the Taylor method but no need to compute the derivatives of $f(t,y)$

R-K method of order Two:

* We need to determine a_1, α_1 and β_1 with property that $a_1 f(t+\alpha_1, y+\beta_1)$

approximates

$$T^{(2)}(t,y) = f(t,y) + \frac{h}{2} f'(t,y)$$

$\leftarrow$ (1)

with error $O(h^2)$.

We know that (By chain rule)

$$f'(t,y) = \frac{\partial f}{\partial t}(t,y) + \frac{\partial f}{\partial y}(t,y)\,y'(t)$$

$\leftarrow$ (2)

We are looking to solve

$$y'(t) = f(t,y) \quad \leftarrow (3)$$

$$y(a) = \alpha$$

Sub (2), (3) in (1),

$$T^{(2)}(t,y) = f(t,y) + \frac{h}{2}\frac{\partial f}{\partial t}(t,y) + \frac{h}{2}\frac{\partial f}{\partial y}\cdot f(t,y)$$

$\leftarrow$ (4)

Expand $a_1 f(t+\alpha_1, y+\beta_1)$ in Taylor polynomial of degree one,

$$a_1 f(t+\alpha_1, y+\beta_1)$$

$$= a_1 f(t,y) + a_1\alpha_1 \frac{\partial f}{\partial t}(t,y)$$

$$+ a_1\beta_1 \frac{\partial f}{\partial y}(t,y) + a_1 R(t+\alpha_1, y+\beta_1)$$

L (5)

where

$$R(t+\alpha_1, y+\beta_1) = \frac{\alpha_1^2}{2} \frac{\partial^2 f}{\partial t^2}(\xi,\mu) + \alpha_1\beta_1 \frac{\partial^2 f}{\partial t \partial y}(\xi,\mu)$$

$$+ \beta_1^2 \frac{\partial^2 f}{\partial y^2}(\xi,\mu)$$

$\xi \in (t, t+\alpha_1)$

$\mu \in (y, y+\beta_1)$

Comparing (4), (5), we get

$f(t,y)$: $\quad a_1 = 1$

$\frac{\partial f}{\partial t}$: $\quad a_1 \alpha_1 = \frac{h}{2} \Rightarrow \alpha_1 = h/2$

$\frac{\partial f}{\partial y}$: $\quad a_1 \beta_1 = h/_2 f(t,y) \Rightarrow \beta_1 = \frac{h}{2} f(t,y)$

$$\therefore \boxed{T^2(t,y) = f\left(t + \frac{h}{2}, y + \frac{h}{2} f(t,y)\right) - R_1\left(t + \frac{h}{2}, y + \frac{h}{2} f(t,y)\right)}$$

$R_1\left(t + h/_2, y + \frac{h}{2} f(t,y)\right)$ Bounded

$$= \frac{h^2}{8} \underbrace{\frac{\partial^2 f}{\partial t^2}(\xi,\mu)} + \frac{h^2}{4} f(t,y) \underbrace{\frac{\partial^2 f}{\partial t \partial y}(\xi,\mu)}$$

$$+ \frac{h^2}{8} f(t,y) \underbrace{\frac{\partial^2 f}{\partial y^2}}(\xi,\mu)$$

$\simeq O(h^2)$

* Order of error in new method is the same as Taylor method.

* This new method is a specific RK method also know as <u>mid-point method</u>.

From Taylor method of order 2

$$w_0 = \alpha$$

$$w_{i+1} = w_i + h f(t_i, w_i) + \frac{h^2}{2} f'(t_i, w_i)$$

$$= w_i + h \Big[\underbrace{f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i)} \Big]$$

$$= w_i + h\, T^{(2)}(t_i, w_i)$$

$$w_{i+1} = w_i + h f\left(t_i + \frac{h}{2}, w_i + \frac{h}{2} f(t_i, w_i)\right)$$

Mid-point method:

$$w_0 = \alpha$$

$$w_{i+1} = w_i + h f\left(t_i + \frac{h}{2}, w_i + \frac{h}{2} f(t_i, w_i)\right)$$

$$i = 0, 1, \dots N-1$$

Modified - Euler method:

$$T^3(t, y) = f(t, y) + \frac{h}{2} f'(t, y) + \frac{h^2}{6} f''(t, y)$$

is approximating

$$a_1 f(t, y) + a_2 f(t + \alpha_2, y + \delta_2 f(t, y))$$

Modified - Euler method chooses

$$a_1 = a_2 = \frac{1}{2}$$

$$\alpha_2 = \delta_2 = h$$

Modified - Euler method

$$w_0 = \alpha$$

$$w_{i+1} = w_i + \frac{h}{2}\Big[f(t_i, w_i)$$

$$+ f(\overset{t_i + h}{t_{i+1}}, w_i + h f(t_i, w_i)) \Big]$$

$$i = 0, 1, \ldots N-1$$

Eg: Use the midpoint and the modified Euler method with $N = 10$, $h = 0.2$, $t_i = 0.2i$ and $w_0 = 0.5$ to approximate the solution of

$$y' = y - t^2 + 1 \quad 0 \leq t \leq 2.$$

Sol:

$$f(t, y) = y - t^2 + 1$$

Mid point method

$$w_{i+1} = w_i + h f\left(t_i + \frac{h}{2}, w_i + \frac{h}{2} f(t_i, w_i)\right)$$

$$= w_i + h\left[w_i + \frac{h}{2} f(t_i, w_i) - \left(t_i + \frac{h}{2}\right)^2 + 1\right]$$

$$= w_i + h\left[w_i + \frac{h}{2}\left(w_i - t_i^2 + 1\right) - t_i^2 - \frac{h^2}{4} - t_i h + 1\right]$$

$$= w_i + (0.2)\left[w_i + \frac{0.2}{2}\left(w_i - (0.04)i^2 + 1\right) - 0.04i - \frac{(0.2)^2}{4} - (0.2)i(0.2) + 1\right]$$

$$\boxed{w_{i+1} = 1.22\, w_i - 0.0088 i^2 - 0.008 i + 0.218}$$

Modified-Euler method

$$w_{i+1} = w_i + \frac{h}{2}\left[f(t_i, w_i) + f(t_{i+1}, w_i + h f(t_i, w_i)) \right]$$

$$= w_i + \frac{h}{2}\left\{ \left[w_i - t_i^2 + 1 \right] + \left\{ w_i + h f(t_i, w_i) - t_{i+1}^2 + 1 \right\} \right\}$$

$$= w_i + \frac{h}{2}\left\{ w_i - t_i^2 + 1 + w_i - t_{i+1}^2 + 1 + h(w_i - t_i^2 + 1) \right\}$$

$$\boxed{w_{i+1} = 1.22 w_i - 0.0088 i^2 - 0.008 i + 0.216}$$

$i = 0, 1, 2 \ldots N-1$

$i = 0, 1, 2, \ldots 9$

$i = 0$

mid : $w_1 = 1.22\,(w_0) + 0.218$

$= 1.22\,(0.5) + 0.218$

$= 0.828$

Euler $w_1 = 1.22\,(0.5) + 0.216$

$= 0.826$

$\underline{\underline{i = 1}}$ (0.2)

mid : $w_2 = 1.22\,w_1 - 0.0088(0.2)^2$

$- 0.008\,(0.2) + 0.218$

$= 1.21136$

Euler $w_2 = 1.22\,w_1 - 0.0088\,(0.2)^2$

$- 0.008\,(0.2) + 0.216$

$= 1.20692$

t_i	$y(t_i)$	mid w_i	Er	Euler w_i
0	0.5	0.5	0	0.5
0.2	0.82929	0.828	0.0012	0.826
0.4	1.21408	1.21136	0.0027	1.20692
0.6	1.64894	1.6446892	0.00428	1.63724
0.8	2.127229	2.1212842	0.00594	2.11023
1	2.640859	2.633168	0.007692	2.61768
1.2	3.1799415	3.1704634	0.009478	3.14957
1.4	3.7324000	3.7211654	0.01123	3.69368
1.6	4.2834838	4.2706218	0.0128	4.23809
1.8	4.8157763	4.8009886	0.01421	4.75568
2	5.3054720	5.2903695	0.01510	5.2334

Higher-Order RK Method:

Runge-Kutta Method of Order Four

$$w_0 = \alpha$$

$$k_1 = h f(t_i, w_i)$$

$$k_2 = h f\left(t_i + \frac{h}{2}, w_i + \frac{k_1}{2}\right)$$

$$k_3 = h f\left(t_i + \frac{h}{2}, w_i + \frac{k_2}{2}\right)$$

$$k_4 = h f(t_{i+1}, w_i + k_3)$$

$$w_{i+1} = w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

for each $i = 0, 1, 2 \ldots N-1$

local truncation error $O(h^4)$

1) Use RK method of order 4 with $h = 0.2$, $N = 10$ and $t_i = 0.2i$ to obtain the app. soln. of

$$y' = y - t^2 + 1, \quad 0 \le t \le 2$$

$$y(0) = 0.5$$

Sol:

$$w_0 = 0.5$$

$$y(0.2) \simeq w_1$$

$$K_1 = 0.2 f(t_0, w_0)$$

$$= 0.2 f(0, 0.5)$$

$$= 0.3$$

$$K_2 = 0.2 f\left(0 + \frac{0.2}{2}, 0.5 + \frac{0.3}{2}\right)$$

$$= 0.2\, f(0.1,\ 0.65)$$

$$= 0.328$$

$$k_3 = 0.2\, f\left(0.1,\ 0.5 + \frac{0.328}{2}\right)$$

$$= 0.2\, f(0.1,\ 0.664)$$

$$= 0.3308$$

$$k_4 = 0.2\, f(0.2,\ 0.5 + 0.3308)$$

$$= 0.2\, f(0.2,\ 0.8308)$$

$$= 0.35816$$

$$w_1 = w_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 0.5 + \frac{1}{6}(0.3 + 2(0.3308) + 2(0.3308) + 0.35816)$$

$= 0.8292933$

t_i	Exact	RK-4	Error
0	0.5	0.5	0
0.2	0.8292986	0.8292933	0.0000053
0.4	1.2140877	1.2140762	0.0000114
0.6	1.6489406	1.6489220	0.0000186
0.8	2.1272295	2.1272027	0.0000269
1	2.6408591	2.6408227	0.0000364
1.2	3.1799415	3.1798942	0.0000474
1.4	3.7324	3.7323401	0.0000599
1.6	4.2834858	4.2834095	0.0000763
1.8	4.8151763	4.8150857	0.0000906
2	5.3054720	5.3053630	0.0001089

Step	2	3	4	$5 \le n \le 7$	$8 \le n \le 9$	$10 \le n$
Error	$O(h^2)$	$O(h^3)$	$O(h^4)$	$O(h^{n-1})$	$O(h^{n-2})$	$O(h^{n-3})$

Multistep Methods:

* Methods using the approximation at more than one mesh point to determine the approximation at the next point are called multistep methods

Def:

An m-step multistep method for solving IVP

$$y' = f(t, y), \quad a \le t \le b$$

$$y(a) = \alpha$$

has following difference equation to find w_{i+1} at t_{i+1}

$$w_{i+1} = a_{m-1} w_i + a_{m-2} w_{i-1} + \dots + a_0 w_{i+1-m} + h\left[b_m f(t_{i+1}, w_{i+1}) + b_{m-1} f(t_i, w_i) + \dots + b_0 f(t_{i+1-m}, w_{i+1-m}) \right]$$

$i = m-1, m, \dots N-1$

$$h = \frac{b-a}{N}$$

$a_0, \dots a_{m-1}, b_0, \dots b_{m-1}$ are constants

$\underline{w_0 = \alpha}, \; w_1 = \alpha_1, \; w_2 = \alpha_2, \dots w_{m-1} = \alpha_{m-1}$

are specified

Remark:

* $b_m = 0$ then above method is called **explicit**

* $b_m \neq 0$ then above method is called **implicit**

Explicit Multistep method:

Adams-Bashforth method

Two-Step method:

$$w_0 = \alpha, \quad w_1 = \alpha_1$$

$$w_{i+1} = w_i + \frac{h}{2}\left[3f(t_i, w_i) - f(t_{i-1}, w_{i-1})\right]$$

$i = 1, 2, \ldots N-1$

$$LTE = O(h^2)$$

Three-Step method:

$$w_0 = \alpha, \quad w_1 = \alpha_1, \quad w_2 = \alpha_2$$

$$w_{i+1} = w_i + \frac{h}{12}\left(23 f(t_i, w_i)\right.$$

$$\left. - 16 f(t_{i-1}, w_{i-1}) + 5 f(t_{i-2}, w_{i-2})\right]$$

$i = 2, 3, \ldots N-1$

$$LTE = O(h^3)$$

Four - Step Method:

$w_0 = \alpha$, $w_1 = \alpha_1$, $w_2 = \alpha_2$, $w_3 = \alpha_3$

$$w_{i+1} = w_i + \frac{h}{24}\left(55 f(t_i, w_i)\right.$$

$$- 59 f(t_{i-1}, w_{i-1}) + 37 f(t_{i-2}, w_{i-2})$$

$$\left. - 9 f(t_{i-3}, w_{i-3})\right]$$

$i = 3, 4, \ldots N-1$, $LTE = O(h^4)$

Eg: Consider IVP

$$y' = y - t^2 + 1, \quad 0 \le t \le 2$$

$$y(0) = 0.5$$

Use the exact soln. $y(t) = (t+1)^2 - 0.5e^t$ as starting values and $h = 0.2$ to find the app. solution using explicit Adams-Bashforth four-step method.

Sol:

$$w_{i+1} = w_i + \frac{h}{24}\left(55 f(t_i, w_i)\right.$$

$$- 59 f(t_{i-1}, w_{i-1}) + 37 f(t_{i-2}, w_{i-2})$$

$$\left. - 9 f(t_{i-3}, w_{i-3})\right]$$

$h = 0.2, \quad t_i = 0.2i$

$$= w_i + \frac{h}{24}\Big[55(w_i - t_i^2 + 1)$$

$$-59(w_{i-1} - t_{i-1}^2 + 1)$$

$$+37(w_{i-2} - t_{i-2}^2 + 1)$$

$$-9(w_{i-3} - t_{i-3}^2 + 1)\Big]$$

$$= w_i + \frac{\overset{0.2}{h}}{24}\Big[55w_i - \overset{55}{(0.2)^2 i^2} + 55$$

$$-59w_{i-1} + 59(0.2)^2(i-1)^2 - 59$$

$$+37w_{i-2} - 37(0.2)^2(i-2)^2 + 37$$

$$-9w_{i-3} + 9(0.2)^2(i-3)^2 - 9\Big]$$

$$= \frac{1}{24}\Big[35\underline{w_i} - 11.8\underline{w_{i-1}} + 7.4\underline{w_{i-2}}$$

$$-1.8 w_{i-3} - 0.192 i^2 - 0.192 i + 4.736)!$$

ti	Exact		Bashforth	Err
0	0.5	w_{i-3}		
0.2	0.829286	w_{i-2}		
0.4	1.2140877	w_{i-1}		
0.6	1.6489406	w_i		
0.8	2.1272295		2.1273124	0.0000828
1	2.6408891		2.6410810	0.0002219
1.2	3.1799415		3.1803480	0.0004065
1.4	3.7324		3.7330601	0.0006601
1.6	4.2834838		4.2844931	0.0010093

Implicit Method

* Backward Euler Method:

$$\frac{dy}{dt} = f(t, y), \quad a \le t \le b$$

$$y(a) = \alpha$$

$$w_{i+1} = w_i + h f(t_{i+1}, w_{i+1})$$

Adams-Moulton Implicit methods

Two-Step method

$$w_0 = \alpha, \quad w_1 = \alpha_1$$

$$w_{i+1} = w_i + \frac{h}{12}\left[5 f(t_{i+1}, w_{i+1}) + 8 f(t_i, w_i) - f(t_{i-1}, w_{i-1})\right]$$

$$i = 1, 2, \dots N-1$$

$$LTE = O(h^3)$$

Three-step method

$$w_0 = \alpha, \quad w_1 = \alpha_1, \quad w_2 = \alpha_2$$

$$w_{i+1} = w_i + \frac{h}{24}\Big[9 f(t_{i+1}, w_{i+1}) + 19 f(t_i, w_i) - 5 f(t_{i-1}, w_{i-1}) + f(t_{i-2}, w_{i-2})\Big]$$

$$i = 2, 3, \dots N-1$$

$$LTE = O(h^4)$$

Four-step method:

$$w_0 = \alpha, \quad w_1 = \alpha_1, \quad w_2 = \alpha_2, \quad w_3 = \alpha_3$$

$$w_{i+1} = w_i + \frac{h}{720}\Big[251 f(t_{i+1}, w_{i+1})$$

$+ 646 f(t_i, w_i) - 264 f(t_{i-1}, w_{i-1}) + 106 f(t_{i-2}, w_{i-2})$
$- 19 f(t_{i-3}, w_{i-3})]$

$i = 3, 4, \ldots N-1$

$LTE = O(h^5)$

1) Consider the IVP

$$y' = y - t^2 + 1, \quad 0 \le t \le 2$$

$$y(0) = 0.5$$

use $h = 0.2$ to find the app. solution by the Adams - Moulton three-step method.

Sol: $w_0 = \alpha_0$, $w_1 = \alpha_1$, $w_2 = \alpha_2$

$$w_{i+1} = w_i + \frac{h}{24}\left[9 f(t_{i+1}, w_{i+1}) + 19 f(t_i, w_i) - 5 f(t_{i-1}, w_{i-1}) + f(t_{i-2}, w_{i-2})\right]$$

$$w_{i+1} = w_i + \frac{0.2}{24}\Big[9(w_{i+1} - t_{i+1}^2 + 1) + 19(w_i - t_i^2 + 1) - 5(w_{i-1} - t_{i-1}^2 + 1) + (w_{i-2} - t_{i-2}^2 + 1)\Big]$$

$t_i = 0.2i$, $h = 0.2$

Cos w_{i+1} (or) $e^{w_{i+1}}$

$$w_{i+1,c} = \frac{1}{24}\Big[1.8\, w_{i+1,p} + 27.8\, w_i - w_{i-1} + 0.2\, w_{i-2} - 0.192\, i^2 - 0.192\, i + 4.736\Big]$$

Predictor - Corrector method

* Combination of an explicit method to predict and an implicit to improve the prediction is called predictor - corrector method.

* Here we use Adams-Bashforth as predictor method.

$\therefore$ from Taylor method of order 2

$$\boxed{w_{i+1,p} = y(0.6)_p = 1.652076}$$

$w_0 = y(0) = 0.5$

$w_1 = y(0.2) = 0.829286$

$w_2 = y(0.4) = 1.214087$

$$w_{i+1,c} = y(0.6)_c$$

$$= \frac{1}{24}\left[1.8 \times 1.652076 + 27.8 \times 1.214087 - 0.829286 + 0.2 \times 0.5 - 0.192 \times 2^2 - 0.192 \times 2 + 4.736\right]$$

$$= 1.649169$$