Numerical Methods

Reference Books:

1) R. L. Burden and J. D. Faires, Numerical Analysis, Brookes/Cole, 2011

2) C. F. Gerald and P. O. Wheatley, Applied Numerical Analysis, 7th edition Pearson Education, 2003.

Types of errors in Numerical Procedures:

Truncation Error:

* These kind of errors caused by method itself.

Eg: Approximate e^x by the cubic polynomial.

$$P_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

We have observed that approximating e^x with cubic gives an inaccurate answer. This error is due to truncating the infinite series.

Round-off Error:

Error due to the computer imperfection is the round-off error.

Eg:

$x_0 = 1$

$x_1 = 1.1$

$x_2 = 1.1234567\ldots$

$\simeq 1.12345\bar{6}$

Absolute Error: (A.E)

$AE = |\text{True value} - \text{Approximate value}|$

Remark:

1) A given size of error is usually more serious when the

magnitude of the true value is small.

Eg!

1) 1036.52 ± 0.01

is accurate to five digits

2) $0.005 \pm \boxed{0.01}$

A.E is quite large

Relative Error: (RE)

$$RE = \frac{A.E}{|\text{True value}|}, \quad \text{True value} \neq 0$$

* Suppose the true value is 0 then R.E is undefined

Significant Digits:

Def. The number p^* is said to be approximate value of p to t significant digits if t is the largest non-negative integer for which

$$\frac{|p - p^*|}{|p|} \leq 5 \times 10^{-t}$$

Ex 1:

p 0.1 $t = 4$

$\max |p - p^*|$ 0.00005 significant digits accuracy

Eg 2:

$p = 10000$

$\max |p - p^*| = 5$

$\frac{5}{10000} \le \cancel{5} \times 10^{-t}$

$t = 4$

Eg 3:

a) $p = 0.3 \times 10^1$, $p^* = 0.31 \times 10^1$

$A.E = 0.1$

$R.E = 0.333\bar{3} \times 10^{-1}$

b) $p = 0.3 \times 10^4$, $p^* = 0.31 \times 10^4$

$A.E = 0.1 \times 10^3$

$RE = 0.333\bar{3} \times 10^{-1}$

* In measure of accuracy, the A.E can be misleading and the R.E more meaningful.

Nonlinear Equations:

Equations with one variable:

We are looking to find a root (or) solution of an equation of the form

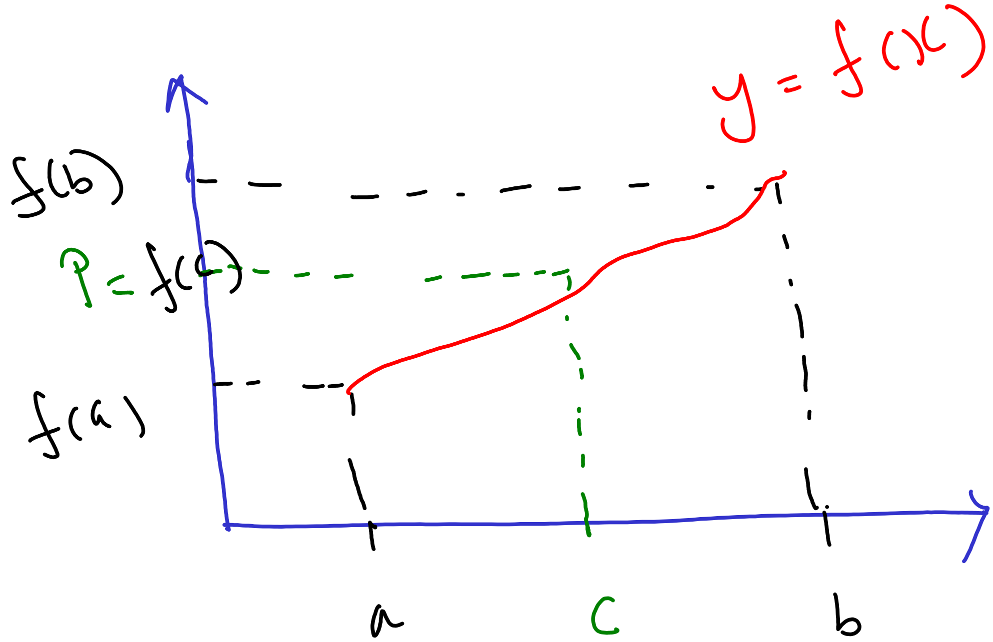
$$f(x) = 0$$

for a given $f(x)$.

Remark: A root of $f(x)$ is also called a zero of the function f.

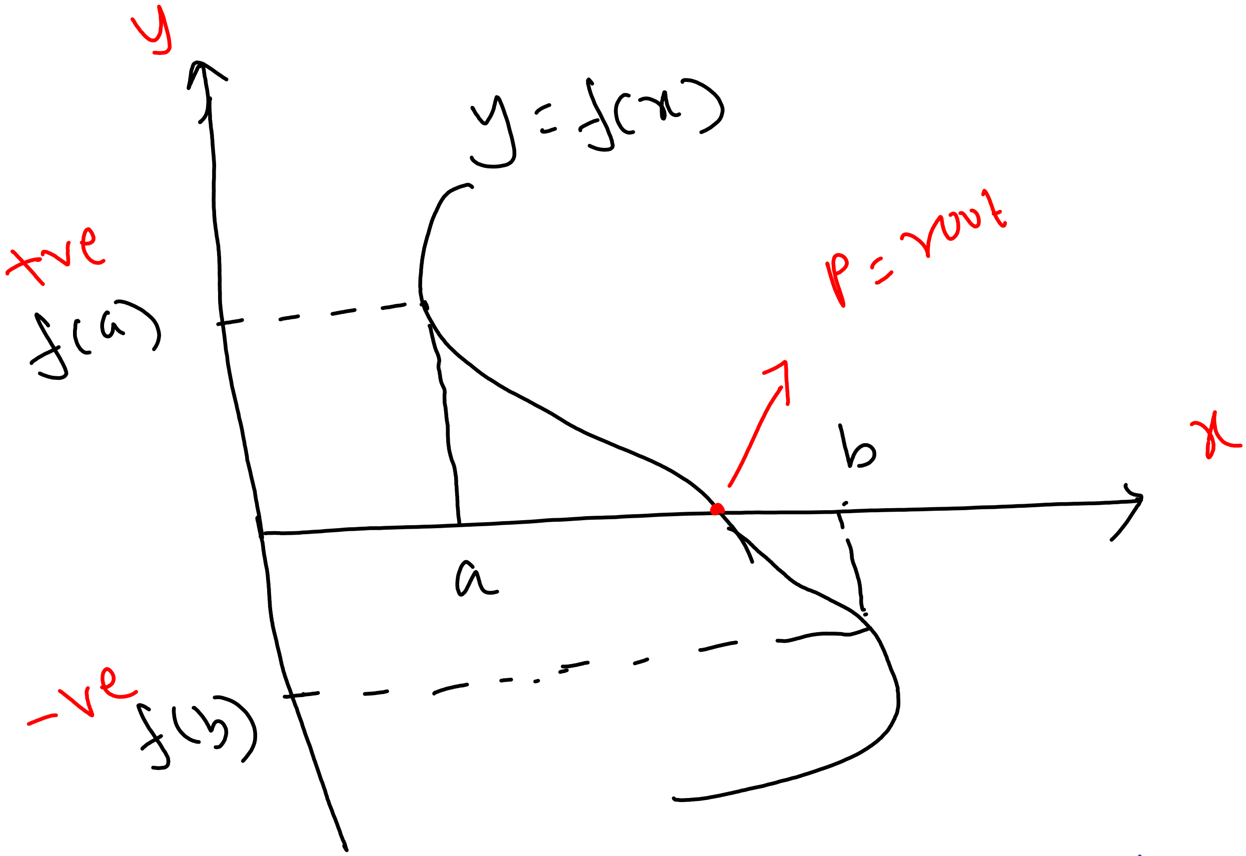
Intermediate Value Theorem: (IVT)

Thm: If f is a continuous fn. on a closed interval $[a,b]$ and if 'p' is any value between $f(a)$ and $f(b)$ then $p = f(c)$ for some c in $[a,b]$



Remark:

If "f" is a continuous fn. then any interval on which "f" changes sign contains a zero function

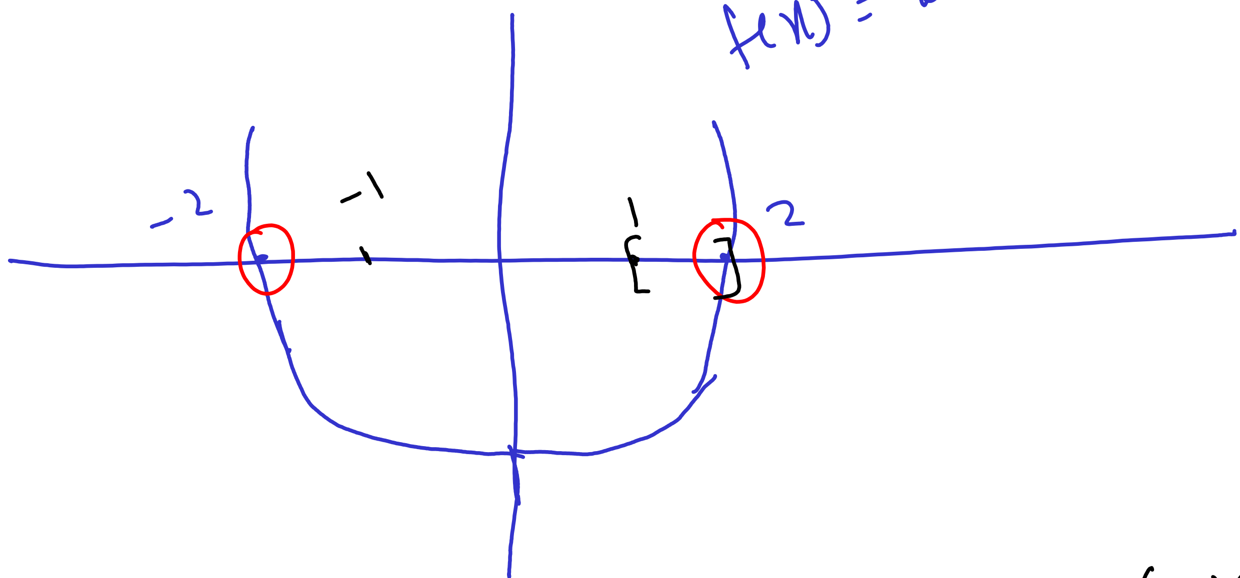


Remark: Suppose a continuous function f on $[a,b]$ is given with $f(a)$ and $f(b)$ of opposite sign. Then by <u>Intermediate Value Theorem</u> there exists a point $c \in [a,b]$ for which $f(c) = 0$

Eg: $f(x) = x^2 - 4$ Does $f(x)$ has solution? Then find the interval of soln. using IVT.

Yes, $x = 2$ is a solution

$f(x) = x^2 - 4$



$f(0) = -4$ (-ve)

$f(1) = -3$ (-ve)

$f(2) = 0$ (+ve)

Root lies b/w $[1,2]$

$f(0) = -4$ (-ve)

$f(-1) = -3$ (-ve)

$f(-2) = 0$ (-ve)

Root lies b/w $[-1,-2]$

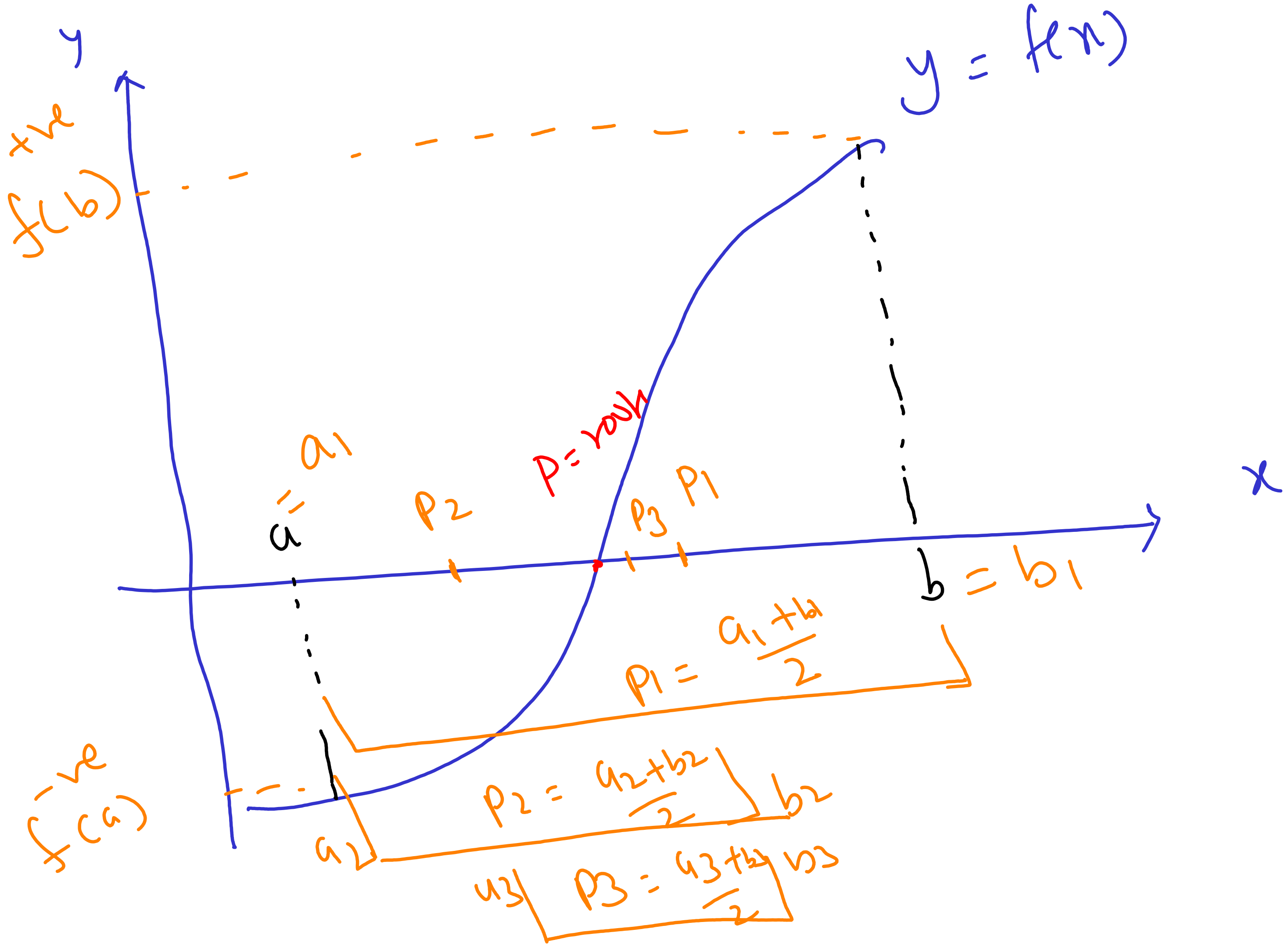
(1) <u>Bisection Method:</u>

<u>Assumptions:</u>

(1) f is a continuous function on $[a,b]$ with $f(a)$ and $f(b)$ of opposite in sign.

(2) This method works when there is more than one root in [a, b].

(3) The Bisection method calls for a repeated halving of subintervals of [a, b].



Algorithm:

(i) Set $a_1 = a$ & $b_1 = b$

and $p_1 \in [a, b]$ be a mid point

$$p_1 = a_1 + \frac{(b_1 - a_1)}{2}$$

$$= \frac{a_1 + b_1}{2}$$

Qn: $[a_1, p_1]$ (or) $[p_1, b_1]$?

(ii) If $f(p_1) = 0$ then $p = p_1$

and we are done

(iii) If $f(p_1) \neq 0$ then

$f(p_1)$ has the same sign as either

$f(a_1)$ (or) $f(b_1)$

(1) If $f(p_1)$ and $f(a_1)$ have the same sign then

$$p \in [p_1, b_1]$$

set $a_2 = p_1$, $b_2 = b_1$

(or)

(2) If $f(p_1)$ and $f(a_1)$ have the opposite sign then

$$p \in [a_1, p_1]$$

set $a_2 = a_1$, $b_2 = p_1$

Then repeat the procedure to the new intervals $[a_2, b_2]$, $[a_3, b_3]$ etc.,

Stopping Criteria:

We can select a tolerance $\epsilon > 0$ and generate a sequence $\{p_n\}_{n=1}^{N}$

until one of the following conditions

$$|p_N - p_{N-1}| < \epsilon \quad (1)$$

$$\frac{|p_N - p_{N-1}|}{|p_N|} < \epsilon \quad (2)$$

$p_N \neq 0$

$$|f(p_N)| < \epsilon \quad (3)$$

Eg 1: Solve $x^3 + 4x^2 - 10 = 0$ has a root in $[1, 2]$ atleast 10^{-4} accuracy.

Sol:

$f(1) = -5$ (−ve)

$f(2) = 14$ (+ve)

$\therefore$ roots lie b/w $[1, 2]$

Relative Error Test:

$$\frac{|p_N - p_{N-1}|}{|p_N|} < 10^{-4}$$

p_1

$p_2 = p_{N-1}$

$p_3 = p_N$

$\vdots$

Bisection method.

$$f(x) = x^3 + 4x^2 - 10$$

$$p_1 = \frac{a_1 + b_1}{2} = \frac{1+2}{2} = 1.5$$

$$f(p_1) = +ve$$

$\therefore$ roots lies b/w $[1, 1.5]$

$$p_2 = \frac{1 + 1.5}{2} = 1.25$$

$$f(p_2) = -ve$$

$\therefore$ roots lies b/w $[1.25, 1.5]$

$$P_3 = \frac{1.25 + 1.5}{2} = 1.375$$

$f(P_3) = +ve$

$\therefore$ roots lies b/w $[1.25, 1.375]$

Repeat the Same proceddure

$P_{13} = 1.365112$

It	a_n	b_n	P_n	$f(a_n)$	$f(b_n)$	Relative Error
1	1	2	1.5	-ve	+ve	0.3333
2	1	1.5	1.25	"	"	0.2
3	1.25	1.5	1.375	"	"	0.09091
4	1.25	1.375	1.3125	"	"	0.04762
⋮						
12	1.364746	1.365234	1.36499	"	"	0.00039
13	1.364990	1.365234	1.365112	"	"	0.00009

Thm: (Error Bound)

Suppose that f is continuous function on [a, b] and f(a) . f(b) < 0. The bisection method generates a sequence $\{p_n\}_{n=1}^{\infty}$ approximately a zero of f and satisfies

$$|p_n - p| \le \frac{b-a}{2^n}, \quad n \geq 1$$

Eg: Determine the number of iterations necessary to solve

$$f(x) = x^3 + 4x^2 - 10 = 0$$

with Level of Significance (LoS) 10^{-4} using $a = 1$ and $b = 2$ in bisection method.

Sol:

$$|p_n - p| \le \frac{2-1}{2^n} < 10^{-4}$$

$$\Rightarrow \log_{10} 2^{-n} < \log_{10} 10^{-4}$$

$$\Rightarrow -n \log_{10} 2 < -4$$

$$\Rightarrow n > \frac{4}{\log_{10} 2} \approx 13.28$$

We need 13 iterations to ensure an approximate solution accurate within 10^{-4} in bisection method.

Note:

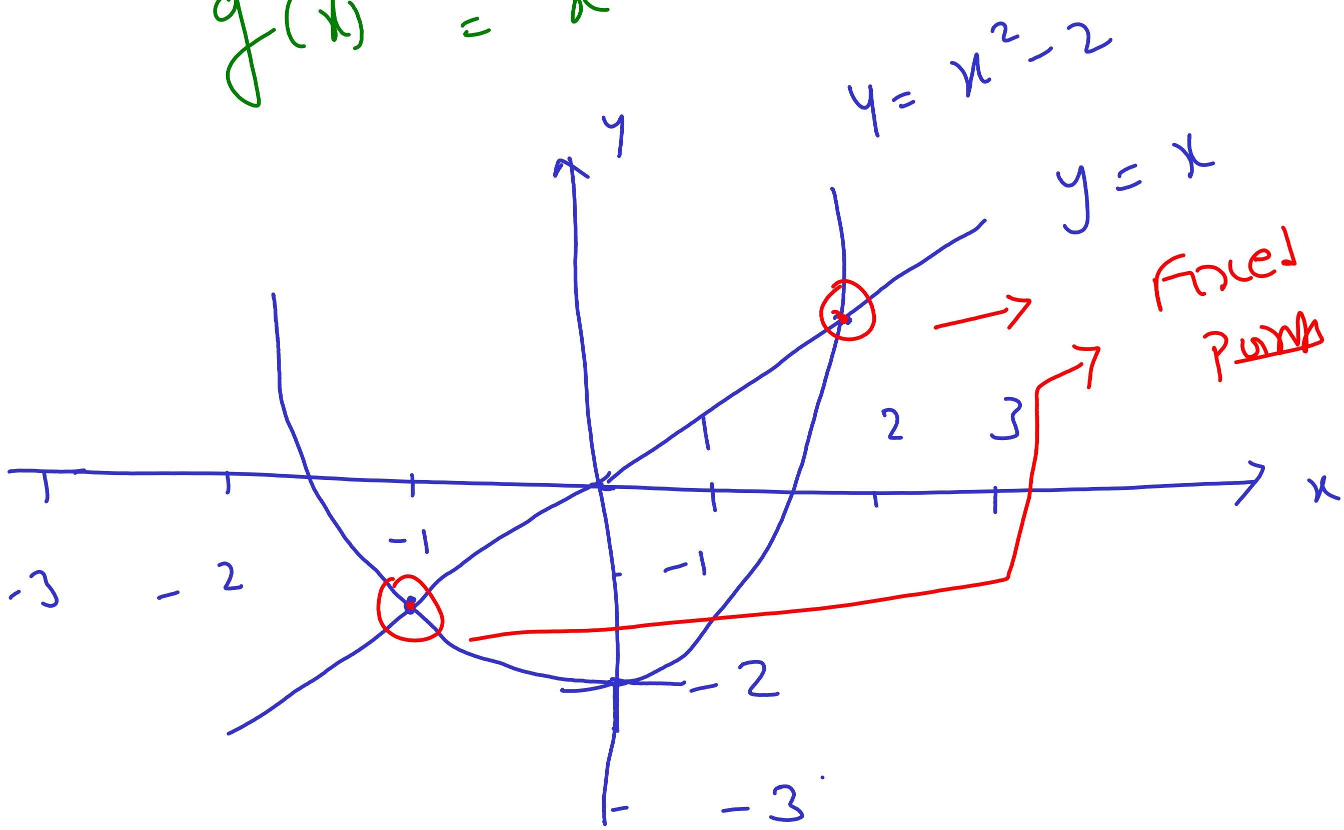
* Bisection method is relatively slow to converge for solution. It means that "N" quite large before $|p - p_n|$ is suff. small.
* This method always converges to solution

(2) Fixed Point Iteration method:

Def: The number 'p' is a fixed point for a given function $g(x)$ if $g(p) = p$

Eg: Determine any fixed points of

$$g(x) = x^2 - 2$$



$$g(p) = p \Rightarrow p^2 - 2 = p$$

$$\Rightarrow p^2 - p - 2 = 0 \qquad p = -1, 2$$

Remark:

Given a root-finding problem $f(x)=0$ we can define function $g(x)$ with fixed point at p in a number of ways.

For eg: $$g(x) = x - f(x)$$

Thm: If g is a continuous function on $[a,b]$ and $g(x) \in [a,b]$ $\forall$ $x \in [a,b]$ then $g(x)$ has at least one fixed point in $[a,b]$.

Global Max $g(x) \in [a,b]$

Global Min $g(x)$

Thm:

Suppose $g'(x)$ exists on (a,b) and a positive constant $K<1$ exists with $\boxed{|g'(x)| \le K}$ $\forall\ x \in (a,b)$.

Then $\exists$ exactly one fixed point in $[a,b]$

Eg: Show that $g(x) = \frac{x^2-1}{3}$ has a unique fixed point on $[-1,1]$.

Sol:

* $g(x) = \frac{x^2-1}{3}$ is continuous fn. on $[-1,1]$.

* $g'(x) = \frac{2x}{3}$ exists on $(-1,1)$

$* \quad g'(x) = 0$

$\Rightarrow \quad \frac{2x}{3} = 0 \quad \Rightarrow \quad x = 0$ is an extremum point.

$g''(x)\Big|_{x=0} = 2/3 > 0$

$\therefore \quad x = 0$ is minimum and $g(0) = -1/3$

Boundary point: $g(-1) = g(1) = 0$

$\therefore$ Maximum occurs at $\underline{-1, 1}$

Here max and min values of $g(x)$ $\in [-1, 1]$. $\therefore$ by Thm 1 $\exists$ at least one fixed point.

Here $|g'(x)| = \left|\frac{2x}{3}\right| \leq \frac{2}{3} < 1$

$\forall \; x \in (-1, 1)$

$\therefore$ By Thm 2, $\exists$ a unique fixed point on $[-1, 1]$

$$\frac{x^2-1}{3} = x \Rightarrow x^2 - 3x - 1 = 0$$

$$x = \frac{3-\sqrt{13}}{2}$$

Thm: (Fixed point method)

Let g be a cts fn. such that $g(x) \in [a,b]$ $\forall$ x in (a,b). Suppose that $g'(x)$ exists in (a,b) and $\exists$ a constant $0 < k < 1$ with $|g'(x)| \leq k$ $\forall$ x in (a,b). Then for any number p_0 in $[a,b]$ the sequence defined by

$$\boxed{p_n = g(p_{n-1})} \quad n \geq 1$$

converges to the unique fixed point p in $[a, b]$.

Note: Initial guess p_0

$$p_1 = g(p_0)$$
$$p_2 = g(p_1)$$
$$p_3 = g(p_2)$$
$$\vdots$$
$$p_n = g(p_{n-1})$$

Thm: (Error Bound)

If g satisfies the hypothesis of the previous theorem then bound for the error involved using p_n to approximate p are given by

$$|p_n - p| \le k^n \max(p_0 - a,\, b - p_0)$$

& $$|p_n - p| \le \frac{k^n}{1-k} |p_1 - p_0| \quad \forall\, n \ge 1$$

where $|g'(x)| \le k$

Ex: Solve $x^3 + 4x^2 - 10 = 0$ using fixed point method in $[1,2]$.

Sol: $g(1) = -ve$

$g(2) = +ve$ $\therefore$ roots lies b/w $[1,2]$.

$x = g_1(x) = x - x^3 - 4x^2 + 10$

~~$g_2(x) = x$~~

$\Rightarrow 4x^2 = 10 - x^3$

$g_2(x) = x = \frac{1}{2}\sqrt{10 - x^3}$

$p_0 = 1.5$

$p_1 = g_1(p_0)$

$= -0.875$

$p_2 = g_1(p_1)$

$= 6.732$

$p_3 = g_1(p_2)$

$= -469.7$

$p_4 = g_1(p_3)$

$= 1.03 \times 108$

not converging

why?

$p_0 = 1.5$

$p_1 = g_2(p_0)$

$= 1.28695$

$p_2 = g_2(p_1)$

$= 1.4025408$

$p_3 = 1.345458$

$p_4 = 1.375170$

$\vdots$

$p_{30} = 1.36523$

Converging

why?

$$g(x) = x - x^3 - 4x^2 + 10$$

$$g'(x) = 1 - 3x^2 - 8x$$

$|g'(x)|$?

max/min of $g'(x)$ in $[1, 2]$

$-6x - 8 = 0 \Rightarrow x = -\frac{4}{3}$

$g'(x)\big|_{x=-4/3} = 6.333$

$g'(x)\big|_{x=1} = -10$

$g'(x)\big|_{x=2} = -27$

$\therefore \quad |g'(x)| = |6.333| > 1$

$\forall x \text{ in } (1, 2)$

$\therefore$ there is no unique fixed point.

Hence $g_1(x)$ will not given conversing solution.

$$g_2(x) = \frac{1}{2}(10-x^3)^{1/2}$$

$$g_2'(x) = -\frac{3}{4} x^2 (10-x^2)^{-1/2} < 0$$

$\forall x \in [1, 2]$

$\therefore$ $g_2(x)$ is strictly decreasing function

But $|g_2'(2)| \simeq 2.12$

$\therefore$ choosing $p_0 = 1.5$ shows that it sufficient to converse b/w $[1, 1.5]$

and $|g_2'(x)| < 1$ on $x \in [1, 1.5]$

Remark: If you choose

$$g_3(x) = x - \frac{x^3+4x^2-10}{3x^2+8x}$$

then fixed point method converges within 3 or 4 iterations

Newton's method:

Suppose f is a continuous fn. and twice differentiable function.

Let $p_0 \in [a, b]$ be an approximation to p such that

$f'(p_0) \neq 0$ and $|p - p_0|$ is small

Taylor polynomial for $f(x)$ and p_0

$$f(p) = f(p_0) + (p - p_0) f'(p_0) + \frac{(p - p_0)^2}{2!} f''(\xi(p))$$

Error $O(p - p_0)$

where $\xi(p)$ lies b/w p and p_0. If p is a root of $f(x)$ then $f(p) = 0$

By Taylor formula

$$f(p_0) + (p-p_0) f'(p_0) + O(p-p_0) = 0$$

Assume $O(p-p_0)$ is sufficiently small.

It means $(p-p_0)^2, (p-p_0)^3, \ldots$ are very small close to zero. Then

$$f(p_0) + (p-p_0) f'(p_0) = 0$$

p_0 initial guess

$$p - p_0 = -\frac{f(p_0)}{f'(p_0)}$$

$$p \simeq p_0 - \frac{f(p_0)}{f'(p_0)}$$

Now generate $\{p_n\}_{n=0}^{\infty}$ by

$$\boxed{p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}} \quad \forall\ n \geq 1$$

$f'(p_{n-1}) \neq 0$

Thm: (Newton method)

Suppose f is continuous and twice diff. fns on (a,b) is such that $f(P)=0$ & $f'(P)\neq 0$. Then $\exists$ $\delta>0$ such that a sequence $\{p_n\}_{n=1}^{\infty}$ converging to p for any initial approximation $p_0 \in [p-\delta, p+\delta]$

$$\boxed{p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}}$$

$\forall$ $n \geq 1$ and $f'(p_{n-1}) \neq 0$.

Eg 1: Solve $3x = \cos x + 1$ with 5 significant digits of accuracy.

Fixed point Method:

$3x = \cos x + 1$

$\Rightarrow g(x) = x = \frac{\cos x + 1}{3}$

$g'(x) = \frac{-\sin x}{3}$

$|g'(x)| \le 1/3 < 1$

$f(0) = -ve$

$f(1) = +ve$

$\therefore$ roots lie b/w $[0,1]$

Initial guess

$p_0 = 0.6$

$p_1 = g(p_0)$

$= 0.60844$

$p_2 = g(p_2)$

$= 0.60684$

$p_3 = 0.60715$

$p_4 = 0.60709$

$p_5 = 0.60710$

$p_6 = 0.60710$

Newton's method:

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$$

$$p_1 = p_0 - \frac{f(p_0)}{f'(p_0)}$$

$$f(x) = 3x - \cos x - 1$$

$$f'(x) = 3 + \sin x$$

$$p_n = p_{n-1} - \left(\frac{3p_{n-1} - \cos p_{n-1} - 1}{3 + \sin p_{n-1}} \right)$$

Initial guess $p_0 = 0.6$

$$p_1 = p_0 - \left(\frac{3p_0 - \cos p_0 - 1}{3 + \sin p_0} \right)$$

$$= 0.60710$$

$$p_2 = 0.60710$$

Remark:

* Newton's method converge quickly than other methods.

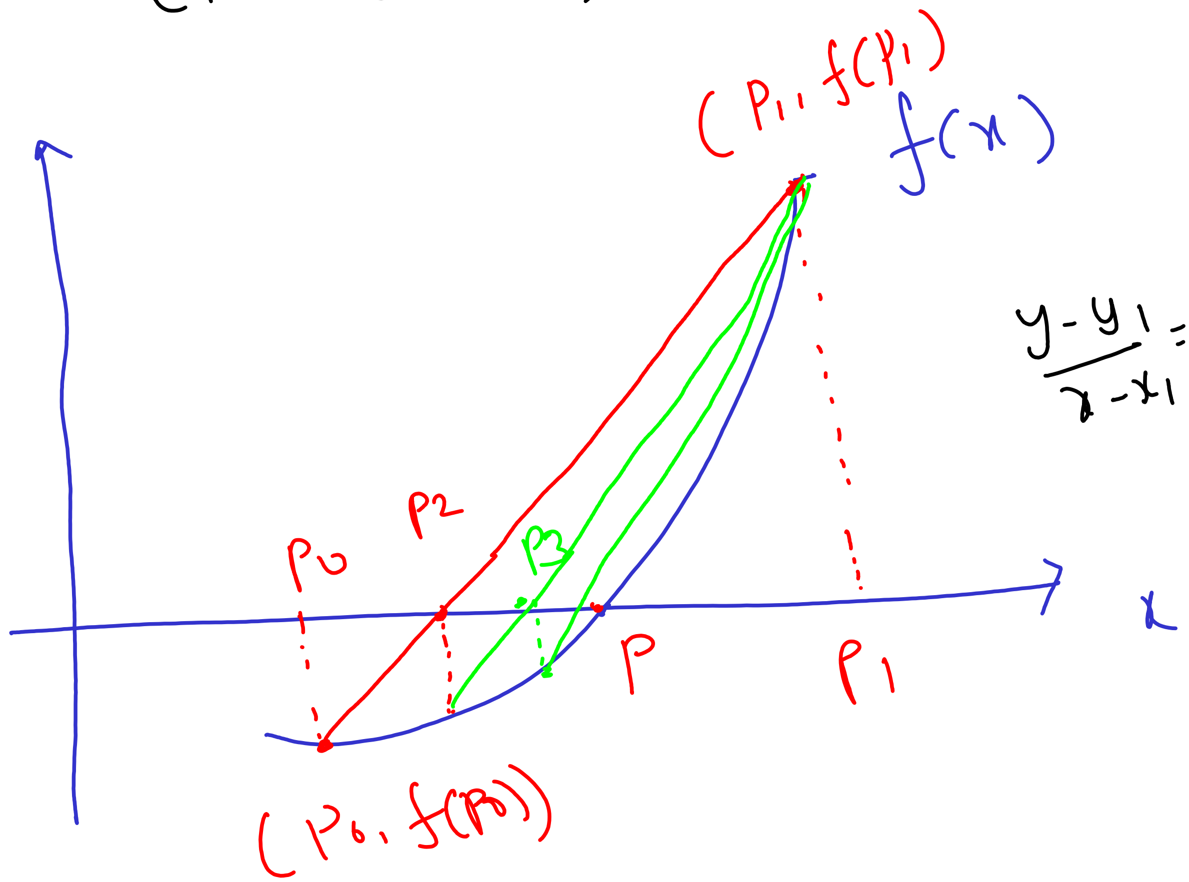
* It is an extremly powerful method to solve nonlinear equations.

* Major weakness is need to know the value of the derivative of f at each approximation.

The Second method:

* Starting with two initial approximations p_0 and p_1, the approximate p_2 is x-intercept of line joing $(p_0, f(p_0))$ and $(p_1, f(p_1))$

* Similarly p_3 is approximation x-intercept of the line joining $(p_1, f(p_1))$ and $(p_2, f(p_2))$ and so on.



$$\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$$

Geometrical Interpretation

* We derive Secant method from Newton's formula in the following way.

Second method Derivation:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

Suppose x_0 is very close to x_1 then

$$f'(x_1) = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$\boxed{f'(x) \simeq \frac{f(x_1) - f(x_0)}{x_1 - x_0}}$$

Newton's method formula:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= x_1 - \frac{f(x_1)}{\frac{f(x_1) - f(x_0)}{x_1 - x_0}}$$

$$\therefore x_2 = x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)}$$

Suppose $\{P_n\}$ sequence of approximations

$$\boxed{P_n = P_{n-1} - \frac{f(P_{n-1})(P_{n-1} - P_{n-2})}{f(P_{n-1}) - f(P_{n-2})}}$$

Eg: Solve $3x = \cos x + 1$ using the secant method.

Sol: $f(x) = 3x - \cos x - 1$

$$p_n = p_{n-1} - \frac{(3p_{n-1} - \cos p_{n-1} - 1)(p_{n-1} - p_{n-2})}{(3p_{n-1} - \cos p_{n-1} - 1) - (3p_{n-2} - \cos p_{n-2} - 1)}$$

$$p_2 = p_1 - \frac{(3p_1 - \cos p_1 - 1)(p_1 - p_0)}{(3p_1 - \cos p_1 - 1) - (3p_0 - \cos p_0 - 1)}$$

$f(0) = -ve$; $f(1) = +ve$

$\therefore$ roots lies b/w $[0, 1]$

$p_0 = 0$; $p_1 = 1$

$p_2 = 0.57808$

$p_3 = 0.60595$

$p_4 = 0.60710$

$p_5 = 0.60710$

Different initial guess

$p_0 = 0.5$

$p_1 = 0.7$

$p_2 = 0.60555$

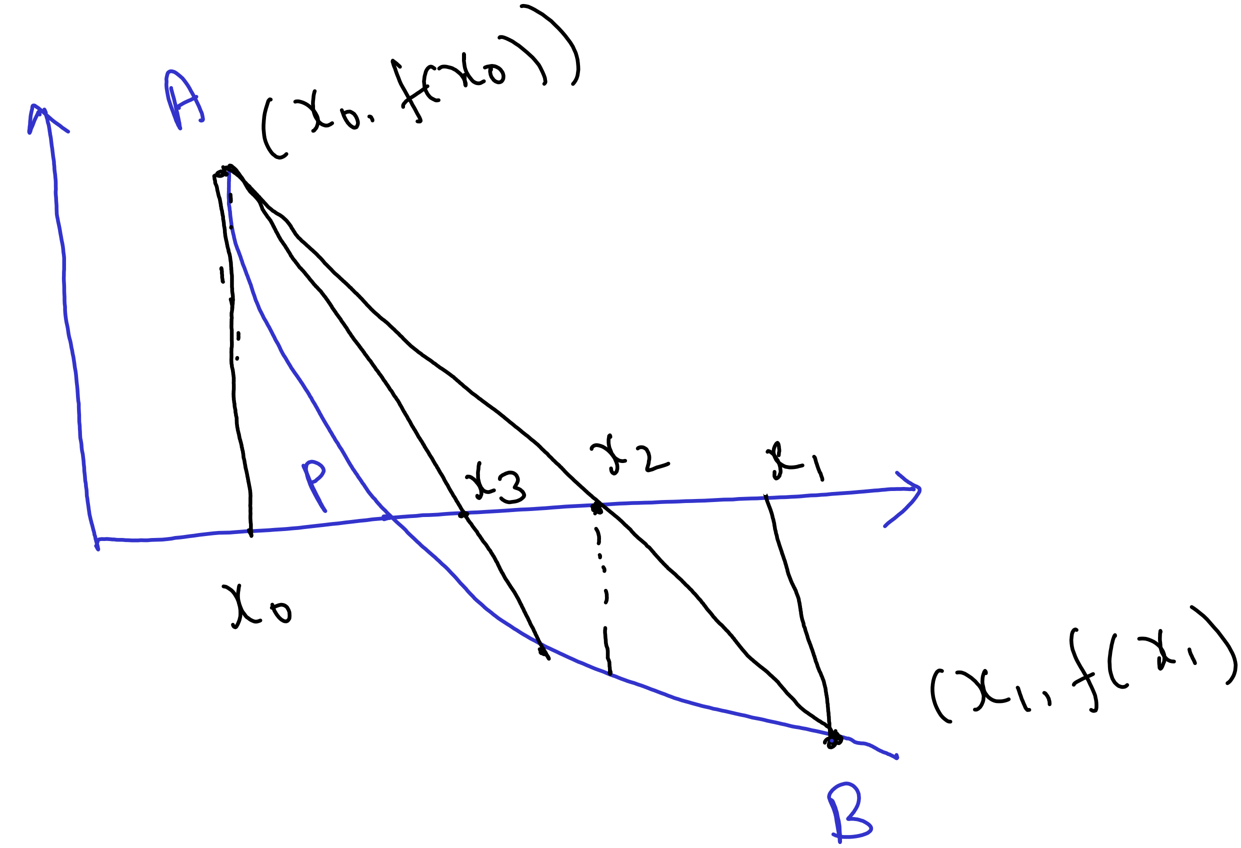
$p_3 = 0.60708$

Regula Falsi Method:

(Method of False Position)

* This method generates approximation as the secant method but it includes a test to assure that the roots is always bracketed b/w successive iteration.

Eqn. of Chrod joining the points $A[x_0, f(x_0)]$, $B[x_1, f(x_1)]$ is



$$\frac{y - f(x_0)}{x - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

At root P, $y = f(x) \Rightarrow f(P) = 0$

$$\Rightarrow \quad -f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

$$\Rightarrow \quad x = x_0 - \frac{f(x_0)(x_1 - x_0)}{f(x_1) - f(x_0)}$$

$$\Rightarrow \boxed{x_2 = x_0 - f(x_0)\,\frac{(x_1 - x_0)}{f(x_1) - f(x_0)}}$$

Eg: Solve by method of false position $x^3 - 2x - 5 = 0$ with 4 signi. digits

Sol:

$$f(x) = x^3 - 2x - 5$$

$f(0) = -ve$; $f(1) = -ve$; $f(2) = -ve$

$f(3) = +ve$

$\therefore$ roots lies b/w $[2, 3]$

$x_0 = 2, \quad x_1 = 3$

$$x_2 = x_0 - f(x_0)\,\frac{x_1 - x_0}{f(x_1) - f(x_0)}$$

$$x_2 = 2 - \frac{(-1)(2-3)}{(16+1)}$$

$$x_2 = 2.0588$$

$$f(x_2) = f(2.0588) = -0.39076$$
(-ve)

$\therefore$ roots lies b/w 2.0588 and 3

$x_0 = 2.0588$ and $x_1 = 3$

$$x_3 = x_0 - \frac{f(x_0)(x_1 - x_0)}{f(x_1) - f(x_0)}$$

$$\boxed{x_3 = 2.08126}$$

$$f(x_3) = -0.14724 \text{ (-ve)}$$

$\therefore$ root lies b/w 2.08126 and 3

$$\boxed{x_4 = 2.08964} \text{ etc}$$

Order of Convergence:

Def: Suppose $\{p_n\}_{n=0}^{\infty}$ is a sequence that converges to p with $p_n \neq p$ for all n. If positive constants λ and α exists with

$$\lim_{n \to \infty} \frac{|p_{n+1} - p|}{|p_n - p|^{\alpha}} = \lambda$$

then $\{p_n\}_{n=0}^{\infty}$ converges to p of order α with asymptotic error λ.

Remark:

* High order of convergence means converge more rapidly than sequence with lower order.

* If $\alpha = 1$ the sequence is linearly convergent.

* If $\alpha = 2$ the sequence is quadratically convergent.

Ex: Suppose that $\{P_n\}_{n=0}^{\infty}$ is linearly convergent to zero ($P = 0$) with

$$\lim_{n \to \infty} \frac{|P_{n+1}|}{|P_n|} = 0.5$$

and $\{q_n\}_{n=0}^{\infty}$ is quadratically convergent to zero with some $\lambda = 0.5$

$$\lim_{n \to \infty} \frac{|q_{n+1}|}{|q_n|^2} = 0.5$$

For linear convergence,

$$|p_n - 0| = |p_n| \simeq 0.5\,|p_{n-1}|$$

$$\simeq (0.5)(0.5)\,|p_{n-2}|$$

$$\simeq (0.5)^2\,|p_{n-2}|$$

$$\vdots$$

$$\simeq (0.5)^n\,|p_0|$$

For quadratic convergence

$$|q_n - 0| = |q_n| \simeq (0.5)\,|q_{n-1}|^2$$

$$\simeq (0.5)\left\{(0.5)\,|e_{n-2}|^2\right\}^2$$

$$\simeq (0.5)^3\,|e_{n-2}|^4$$

$$\simeq (0.5)^7\,|e_{n-3}|^8$$

$$\vdots$$

$$\simeq (0.5)^{2^n-1}\,|e_0|^{2^n}$$

n	linear $(0.5)^n$	quadratic $(0.5)^{2^n-1}$
1	5×10^{-1}	5×10^{-1}
2	2.5×10^{-1}	1.25×10^{-1}
3	1.25×10^{-1}	7.8125×10^{-3}
$\vdots$ 7	7.8125×10^{-3}	5.8775×10^{-39}

Thm:

Let $g \in C[a,b]$ be such that $g(x)$ in $[a,b]$ $\forall x \in (a,b)$. Suppose that $g'(x)$ is continuous on (a,b) and $K < 1$ exists with $|g'(x)| \le K$ $\forall x \in (a,b)$. If $g'(p) \neq 0$ then for any number $p_0 \neq p$ in $[a,b]$

$$p_n = g(p_{n-1}) \text{ for } n \geq 1$$

converges linearly to the unique fixed point p in $[a,b]$

Thm 2:

Let p be a solution of $x = g(x)$

Suppose that $g'(p) = 0$ and g'' is

continuous with $|g''(x)| < M$ on

an open interval I containing p

Then $p_n = g(p_{n-1})$ for $n \geq 1$

converges at least quadratically to p.

Def:

A solution p of $f(x) = 0$ is

a zero of multiplicity m of f

if for $x \neq p$ we can write

$$f(x) = (x-p)^m q(x)$$

where $\lim_{x \to p} q(x) \neq 0$

Thm:

(1) $f \in C^1(a,b)$ has a simple zero at p in (a,b) if and only if $f(p) = 0$ but $f'(p) \neq 0$.

cts and diff. 1 time

(2) $f \in C^m(a,b)$ has a zero of multiplicity m at (a,b) if and only if

m time diff.

$$0 = f(p) = f'(p) = f''(p) = \cdots = f^{(m-1)}(p)$$

but $f^{m}(p) \neq 0$

Eg:

$$f(x) = e^{x} - x - 1$$

$$f'(x) = e^{x} - 1$$

$$f''(x) = e^{x}$$ and so on

$f(0) = 0$; $f'(0) = 0$; $f''(0) = 1$

$\therefore$ f has a zero of multiplicity 2

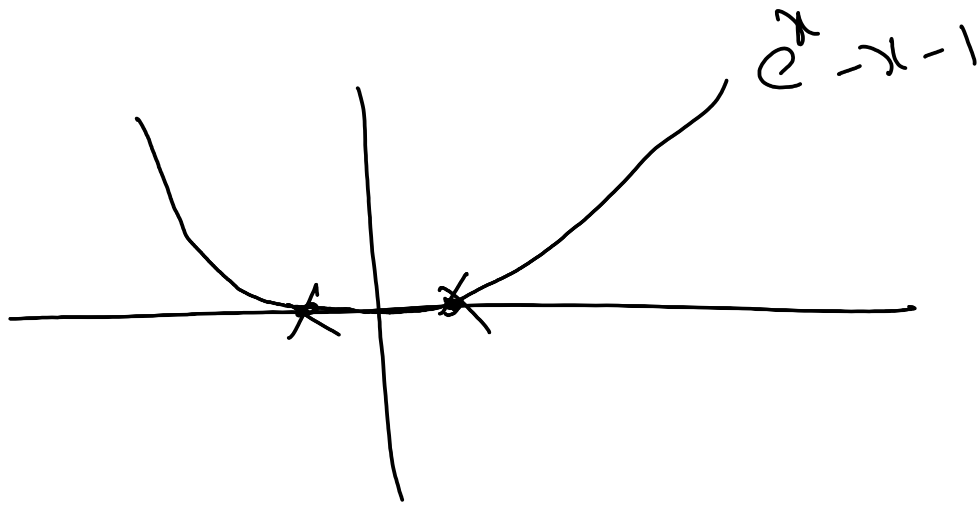
at $x = 0$.

Using Newton's method

$p_0 = 0.5$

$p_1 \simeq 0.58158$ | $p_1 = -0.23421$

$p_2 \simeq 0.31506$


$e^x - x - 1$