Numerical Differentiation:

* We are introducing procedures for approximating derivatives using the polynomial that approximate the function.

Taylor Series:

$$f(x+h) = f(x) + \frac{f'(x)}{1!}h + \frac{f''(x)}{2!}h^2 + \cdots$$

$$f(x+h) \simeq f(x) + f'(x)\frac{h}{1!} + f''(\xi(x))\frac{h^2}{2!}$$

$$\Rightarrow f(x+h) - f(x) \simeq f'(x)h + f''(\xi(x))\frac{h^2}{2!}$$

$$\Rightarrow f'(x) \simeq \frac{f(x+h) - f(x)}{h} - \frac{h}{2} f''(\xi(x))$$

Suppose x is x_0 then we have

$$\boxed{f'(x_0) = \frac{f(x_0+h)-f(x)}{h} - \frac{h}{2} f''(\xi(x))}$$

For small value of h the diff. quotient $\frac{f(x_0+h)-f(x)}{h}$ can be used to approximate $f'(x_0)$ with an error $\frac{M|h|}{2}$ where M is a bound on $\|f''(\xi(x))\|$ for $\xi(x)$ b/w x_0 and x_0+h.

Remark:

$$f'(x_0) = \frac{f(x_0+h)-f(x_0)}{h}$$

* If $h>0$ then above formula is called as Forward-diff. formula.
* If $h<0$ then above formula is called as Backward-diff. formula.

Eg 1: Use forward diff. formula to approximate $f'(x)$ of $f(x) = \ln x$ at $x_0 = 1.8$ using $h = 0.05$ and 0.01. further determine the bounds for the approximation.

Sol:

$$f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h}$$

$$f'(1.8) = \frac{f(1.8+h) - f(1.8)}{h}$$

when $h = 0.05$

$$f'(1.8) = \frac{\ln 1.85 - \ln 1.8}{0.05} = 0.547979$$

when $h = 0.01$

$$f'(1.8) = 0.5540180$$

Error Bound:

$$M \frac{|h|}{2} \quad \text{when}$$

$$M = \max |f''(\xi(x))|$$

$h = 0.01$

$$f(x) = \ln x$$

$$f'(x) = \frac{1}{x} \; ; \quad f''(x) = \frac{-1}{x^2}$$

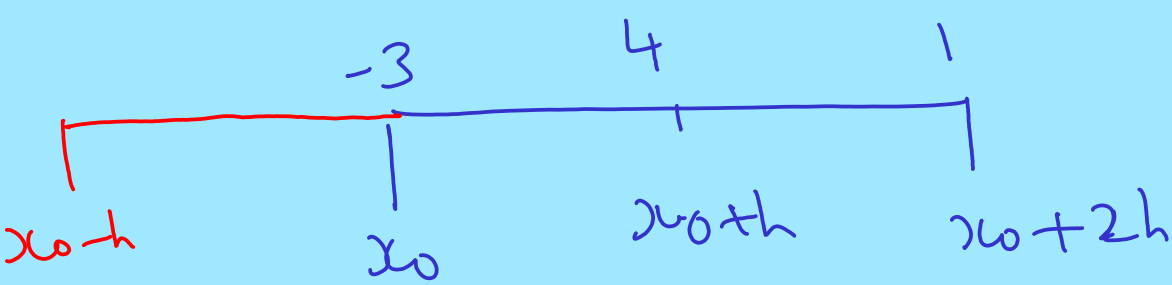
$$f''(\xi(x)) = \frac{-1}{\xi(x)^2} \quad \text{where} \quad 1.8 < \xi(x) < 1.81$$

$$M = \left| f''(\xi(x)) \right| = \left| \frac{1}{\xi(x)^2} \right| = \frac{1}{1.8^2}$$

$$\therefore \quad \text{Error} = \frac{M|h|}{2}$$

$$\leq \frac{0.01}{2(1.8)^2} = 0.0015432$$

$h = 0.05$ Ex:



Three-Point endpoint formula:

$$f'(x_0) = \frac{1}{2h}\left[-3f(x_0) + 4f(x_0+h) - f(x_0+2h)\right] + \frac{h^2}{3} f^{(3)}(\xi(x))$$

$x_0 < \xi(x) \leq x_0+h$

L(1)

Three-Point midpoint formula:

$$f'(x_0) = \frac{1}{2h}\left[f(x_0+h) - \underline{\underline{f(x_0-h)}}\right] - \frac{h^2}{6} f^{(3)}(\xi(x))$$

L(2)

$x_0-h < \xi(x) < x_0+h$

Remark:

* Though errors in (1) and (2) are $O(h^2)$ the error in (2) is approximately half of the error in (1).

* f needs be to be evaluated at only points in ② but three evaluations are needed in ①.

Five-point midpoint formula:

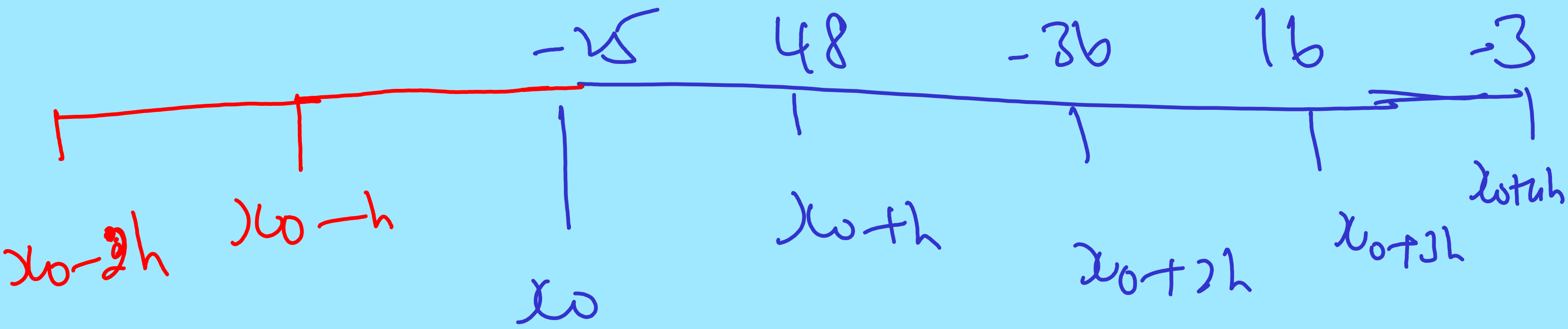
$$f'(x_0) = \frac{1}{12h}\Big[f(x_0-2h) - 8 f(x_0-h) + 8 f(x_0+h) - f(x_0+2h)\Big] + \frac{h^4}{30} f^{(5)}(\xi(x))$$

$$x_0-2h < \xi(x) < x_0+2h$$

Five-point Endpoint formula:

$$f'(x_0) = \frac{1}{12h}\Big[-25 f(x_0) + 48 f(x_0+h) - 36 f(x_0+2h) + 16 f(x_0+3h) - 3 f(x_0+4h)\Big] + \frac{h^4}{5} f^{(5)}(\xi(x))$$

$x_0 < \xi(x) < x_0 + h$



Eg1. Values for $f(x) = xe^x$ are given in the table. Use all the possible three and five point formulas to app. $f'(2.0)$

x	$f(x)$
1.8	10.889365
1.9	12.703199
2.0	14.778112
2.1	17.148957
2.2	19.855030

Sol:

$h = 0.1$

Three-Pt. End point formula:

$$f'(x_0) = \frac{1}{2h}\left[-3f(x_0) + 4f(x_0+h) - f(x_0+2h)\right]$$

$$f'(2.0) = \frac{1}{2\times(0.1)}\left[-3f(2) + 4f(2.1) - f(2.2)\right]$$

$$= 22.0323$$

$h = -0.1$

$$f'(2.0) = \frac{1}{2\times(0.1)}\left(-3f(2) + 4f(1.9) - f(1.8)\right]$$

$$= 22.0545$$

Three-point mid point formula:

$$f'(x_0) = \frac{1}{2h}\left[f(x_0+h) - f(x_0-h)\right]$$

$\underline{\underline{h = 0.1}}$

$$f'(2.0) = \frac{1}{2\times(0.1)}\left(f(2.1) - f(1.9)\right)$$

$$= 22.22879$$

$\underline{\underline{h = 0.2}}$

$$f'(2.0) = \frac{1}{2\times 0.2}\left(f(2.2) - f(1.8)\right)$$

$$= 22.414163$$

Five point Midpoint formula:

$$f'(x_0) = \frac{1}{12h}\left(f(x_0-2h) - 8f(x_0-h) + 8f(x_0+h) - f(x_0+2h)\right]$$

$\underline{\underline{h = 0.1}}$

$$f'(2.0) = \frac{1}{12\times 0.1}\left(f(1.8) - 8f(1.9) + 8f(2.1) - f(2.2)\right]$$

$$= 22.16999$$

Given fn. $f(x) = xe^x$

$f'(x) = xe^x + e^x$

$f'(2.0) = 22.167168$

Errors:

		Error
Three pt:	$h=0.1 \Rightarrow$	0.13485
	$h=-0.1 \Rightarrow$	0.113
Three mid:	$h=0.1 \Rightarrow$	-0.0616
	$h=0.2 \Rightarrow$	-0.247
Five pt:	$h=0.1 \Rightarrow$	0.0001693

Second Derivative mid point formula:

$$f''(x_0) = \frac{1}{h^2}\left[f(x_0-h) - 2f(x_0) + f(x_0+h)\right] - \frac{h^2}{12} f^{(4)}(\xi(x))$$

$x_0 - h < \xi(x) < x_0 + h$

1) Use previous table to find

$f''(2.0)$

$h = 0.1$

$$f''(2.0) = \frac{1}{0.1^2}\left[f(1.9) - 2f(2) + f(2.1)\right]$$

$$= 29.5932$$

$h = 0.2$

$$f''(2.0) = \frac{1}{0.2^2}\left[f(1.8) - 2f(2) + f(2.2)\right]$$

$$= 29.70427$$

Exact value:

$$f''(x) = (x+2)e^{x}$$

$$f''(2) = 29.556224$$

		Error
$h=0.1$	$\Rightarrow$	-0.037
$h=0.2$	$\Rightarrow$	-0.148