System of Equations:

Direct method

Gauss Elimination Method:

Consider $n \times (n+1)$ matrix can be used to represent the linear system

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \qquad [A, b] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} & b_n \end{array}\right]$$

$\therefore$ given linear system can be written in the form

$$Ax = b$$

where $x = [x_1, x_2 \dots x_n]^T$ and array $[A|b]$ is called an augmented matrix.

Solve:

$$x_1 + x_2 + 3x_4 = 4$$

$$2x_1 + x_2 - x_3 + x_4 = 1$$

$$3x_1 - x_2 - x_3 + 2x_4 = -3$$

$$-x_1 + 2x_2 + 3x_3 - x_4 = 4$$

Augmented matrix $[A|B]$ is

$$\simeq \left[\begin{array}{cccc:c} 1 & 1 & 0 & 3 & 4 \\ 2 & 1 & -1 & 1 & 1 \\ 3 & -1 & -1 & 2 & -3 \\ -1 & 2 & 3 & -1 & 4 \end{array}\right]$$

$$\simeq \left[\begin{array}{cccc|c} 1 & 1 & 0 & 3 & 4 \\ 0 & -1 & -1 & -5 & -7 \\ 0 & -4 & -1 & -7 & -15 \\ 0 & 3 & 3 & 2 & 8 \end{array}\right] \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 + R_1 \end{array}$$

$$\simeq \left[\begin{array}{cccc|c} 1 & 1 & 0 & 3 & 4 \\ 0 & -1 & -1 & -5 & -7 \\ 0 & 0 & 3 & 13 & 13 \\ 0 & 0 & 0 & -13 & -13 \end{array}\right] \begin{array}{l} R_3 \to R_3 - 4R_2 \\ R_4 \to R_4 + 3R_2 \end{array}$$

From $E4$

$$\Rightarrow \quad -13x_4 = -13$$

$$\boxed{x_4 = 1}$$

From $E3$

$$\Rightarrow \quad 3x_3 + 13x_4 = 13$$

$$\Rightarrow \quad \boxed{x_3 = 0} \quad (\because x_4 = 1)$$

from $E2$ $\quad -x_2 - x_3 - 5x_4 = -7$

$$\Rightarrow \quad \boxed{x_2 = 2}$$

From E_1

$$x_1 + x_2 + 3x_4 = 4$$

$$\boxed{x_1 = -1}$$

Generalization of Gauss Elimination Method:

Rewrite the augmented matrix as

$$\tilde{A} = [A, b] = \begin{bmatrix} a_{11} & a_{12} \dots & a_{1n} & : & a_{1,n+1} \\ a_{21} & a_{22} \dots & a_{2n} & : & a_{2,n+1} \\ \vdots & & & & \\ a_{n1} & a_{n2} \dots & a_{nn} & : & a_{n,n+1} \end{bmatrix}$$

where we assumed that $b_i = a_{i,n+1}$
$i = 1$ to n.

Suppose $a_{11} \neq 0$, we perform

(1) $\left(E_j - \left(\frac{a_{j1}}{a_{11}}\right) E_1 \right) \to E_j$

for each $j = 2, 3, \dots n$. To eliminate the co-eff. of x_1 in each of these rows.

(2) Though entries in rows 2, 3, ... are expected to change however we use the same notation a_{ij} for convenient

(3) Then perform the operation

$$\left(E_j - \frac{a_{ji}}{a_{ii}} E_i \right) \to E_j$$

$$j = i+1, i+2, \dots n$$

provided $a_{ii} \neq 0$

Therefore the resulting matrix is of the form

$$\tilde{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & ; & a_{1,n+1} \\ 0 & a_{22} & \dots & a_{2n} & ; & a_{2,n+1} \\ \vdots & & & & & \vdots \\ 0 & 0 & \dots & 0 \ a_{nn} & ; & a_{n,n+1} \end{bmatrix}$$

Thus the new linear system is given in the following form

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = a_{1,n+1}$$

$$a_{22}x_2 + \cdots + a_{2n}x_n = a_{2,n+1}$$

$$\ddots$$

$$a_{nn}x_n = a_{n,n+1}$$

can be solved by backward substitution.

Gauss Jordan Method:

To find inverse of a matrix:

1) $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix}$

Augment matrix A with identity matrix

$$[A, I] = \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$R_2 \to R_2 - 3R_1 \qquad R_3 \to R_3 - R_1 \qquad \simeq \begin{bmatrix} 1 & -1 & 2 & : & 1 & 0 & 0 \\ 0 & 3 & -5 & : & -3 & 1 & 0 \\ 0 & 1 & 0 & : & -1 & 0 & 1 \end{bmatrix}$$

$$R_3 \to 3R_3 - R_2 \qquad \simeq \begin{bmatrix} 1 & -1 & 2 & : & 1 & 0 & 0 \\ 0 & 3 & -5 & : & -3 & 1 & 0 \\ 0 & 0 & 5 & : & 0 & -1 & 3 \end{bmatrix}$$

$$R_1 \to 5R_1 - 2R_3 \qquad R_2 \to R_2 + R_3 \qquad \simeq \begin{bmatrix} 5 & -5 & 0 & : & 5 & 2 & -6 \\ 0 & 3 & 0 & : & -3 & 0 & 3 \\ 0 & 0 & 5 & : & 0 & -1 & 3 \end{bmatrix}$$

$$R_1 \to 3R_1 + 5R_2 \qquad \simeq \begin{bmatrix} 15 & 0 & 0 & : & 0 & 6 & -3 \\ 0 & 3 & 0 & : & -3 & 0 & 3 \\ 0 & 0 & 5 & : & 0 & -1 & 3 \end{bmatrix}$$

$R_1 \to R_1/15$
$R_2 \to \frac{R_2}{3}$
$R_3 \to \frac{R_3}{5}$

$$\simeq \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 0 & 2/5 & -1/5 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & \frac{-1}{5} & \frac{3}{5} \end{array}\right]$$

$$\therefore \quad A^{-1} = \begin{bmatrix} 0 & 2/5 & -1/5 \\ -1 & 0 & 1 \\ 0 & -1/5 & 3/5 \end{bmatrix}$$

Iterative Methods:

Gauss Jaccobi Method:

Diagonally Dominant:

We say that an $n \times n$ matrix A is diagonally dominant if and only if

for each $i = 1, 2, \ldots n$

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \qquad i = 1, 2, \ldots n$$

Eg $6x_1 - 2x_2 + x_3 = 11$

$x_1 + 2x_2 - 5x_3 = -1$

$-2x_1 + 7x_2 + 2x_3 = 5$

Check the above system is diagonally dominant.

Sol:

No. It can be re written as dia. dominant

$6x_1 - 2x_2 + x_3 = 11$ $\quad 6 > 2+1$

$-2x_1 + 7x_2 + 2x_3 = 5$ $\quad 7 > 2+2$

$x_1 + 2x_2 - 5x_3 = -1$ $\quad 5 > 1+2$

Jacobi method: (Gauss Jacobi method)

The Jacobi iteration method is used to solve the equation $Ax = b$. To obtain x_i we use following algorithm provided $a_{ii} \neq 0$.

$$\boxed{x_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} -\frac{a_{ij}}{a_{ii}} x_j + \frac{b_i}{a_{ii}}}$$

$$i = 1, 2, \ldots n$$

$$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 = b_1$$

$$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 = b_2$$

$$a_{31} x_1 + a_{32} x_2 + a_{33} x_3 = b_3$$

(1) Convert the system as diagonally dominant

(2) $$x_1 = \frac{b_1 - a_{12} x_2 - a_{13} x_3}{a_{11}}$$

For each $K \geq 1$, generate the components $x_i^{(K)}$ of $x^{(K)}$ from $x_i^{(K-1)}$ by

$$x_i^{(K)} = \frac{1}{a_{ii}} \left[\sum_{\substack{j=1 \\ i \neq j}}^{n} -a_{ij} x_j^{(K-1)} + b_i \right]$$

Remark:

Convert $Ax = b$ into an equivalent system

$$x = Tx + C$$ for some matrix T and vector c

Guess $x^{(0)}$ initial vector. Then sequence of approximate soln vector is given by

$$x^{(k)} = T x^{(k-1)} + C$$

for each $k = 1, 2 \ldots n$. This is similar like of the fixed point method

Solve:

$$10x_1 - x_2 + 2x_3 = 6$$

$$-x_1 + 11x_2 - x_3 + 3x_4 = 25$$

$$2x_1 - x_2 + 10x_3 - x_4 = -11$$

$$3x_2 - x_3 + 8x_4 = 15$$

Use Jacobi method to find approximate solutions with initial guess $x^{(0)} = (0,0,0,0)$

until $\frac{\|x^k - x^{k-1}\|}{\|x^k\|} < 10^{-3}$.

At least 3 digits accuracy

Sol:

(1) Problem is diagonally dominant

(2) $$x_1 = \frac{x_2 - 2x_3 + 6}{10}$$

$$x_2 = \frac{x_1 + x_3 - 3x_4 + 25}{11}$$

$$x_3 = \frac{-2x_1 + x_2 + x_4 - 11}{10}$$

$$x_4 = \frac{-3x_2 + x_3 + 15}{8}$$

(3) $$x^{(0)} = \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ (0, & 0, & 0, & 0) \end{array}$$

$$x_1^{(1)} = \frac{6}{10} = 0.6$$

$$x_2^{(1)} = \frac{25}{11} = 2.2727$$

$$x_3^{(1)} = \frac{-11}{10} = -1.1$$

$$x_4^{(1)} = \frac{15}{8} = 1.875$$

$$x_1^{(2)} = \frac{2.2727 - 2(-1.1) + 6}{10}$$

$$= 1.04727$$

$$x_2^{(2)} = \frac{0.6 + (-1.1) - 3(1.875) + 25}{11}$$

$$= 1.715909$$

$$x_3^{(2)} = -0.80523$$

$$x_4^{(2)} = 0.8852$$

it	x_1	x_2	x_3	x_4
0	0	0	0	0
1	0.6	2.2727	-1.1	1.875
2	1.04727	1.715909	-0.80523	0.8852
3	0.9326	2.053	-1.0493	1.1309
4	1.0152	1.9537	-0.9681	0.9739
5	0.9890	2.0114	-1.0103	1.0214
6	1.0032	1.9922	-0.9945	0.9924
7	0.9982	2.0023	-1.0020	1.0036
8	1.0006	1.9987	-0.9990	0.9985
9	0.9987	2.0004	-1.0004	1.0006
10	1.0001	1.9998	-0.9998	0.9998

$$\text{Tollerance} = \frac{\|x^{(10)} - x^{(9)}\|_\infty}{\|x^{(10)}\|_\infty}$$

$$= \frac{8 \times 10^{-4}}{1.9998} < 10^{-3}$$

$$\|x^{(10)}\|_\infty = \max\{\|x_1\|, \|x_2\|, \|x_3\|, \|x_4\|\}$$

Remark:

Jacobi method can be written as

$$x^{(k)} = T x^{(k-1)} + C$$

by splitting A into its diagonal and off diagonal form,

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix} - \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ a_{21} & 0 & & 0 \\ \vdots & & \ddots & \vdots \\ a_{n1} & \cdots & \cdots & 0 \end{bmatrix}$$

$$
- \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & 0 & \cdots & a_{2n} \\ \vdots & & \ddots \; 0 & a_{n-1\,n} \\ 0 & \cdots\; 0 & \cdots & 0 \end{bmatrix}
$$

$$= D - L - U$$

$$\therefore \quad Ax = b$$

$$\Rightarrow (D-L-U)x = b$$

$$\Rightarrow Dx = (L+U)x + b$$

Suppose D^{-1} exists $a_{ii} \neq 0$ then

$$x = D^{-1}(L+U)x + D^{-1}b$$

For $k = 1, 2, \ldots n$,

$$x^{(k)} = D^{-1}(L+U)x^{(k-1)} + D^{-1}b$$

$$\Rightarrow x^{(k)} = T x^{(k-1)} + C \quad \text{when}$$

$T = D^{-1}(L+U)$ and $C = D^{-1}b$

Gauss Seidel method:

The components of $x^{(k-1)}$ are used to compute all the components of $x_i^{(k)}$ of $x^{(k)}$. But $i>1$ the components $x_1^{(k)}, \ldots, x_{i-1}^{(k)}$ of $x^{(k)}$ have already been computed and are expected to be better approximation to the actual solution.

$$x_i^{(k)} = \frac{1}{a_{ii}}\left[-\sum_{j=1}^{i-1} a_{ij}\, x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij}\, x_j^{(k-1)} + b_i\right]$$

$$i = 1, 2, \ldots n$$

This modification in Gauss-Jacobi method is called as Gauss-Seidel method.

Eg: Same problem as in previous method

(1) Diagonally dominant

(2) $x_1 = 6 + x_2 - 2x_3 / 10$

$x_2 = 25 + x_1 + x_3 - 3x_4 / 11$

$x_3 = -11 - 2x_1 + x_2 + x_4 / 10$

$x_4 = 15 - 3x_2 + x_3 / 8$

(3) $x^{(0)} = (0, 0, 0, 0)$

(4) $x_1^{(1)} = 6 + 0 - 0 / 10 = 0.6$

$x_2^{(1)} = 25 + 0.6 + 0 - 0 / 11 = 2.3272$

$$x_3^{(1)} = \frac{-11 - 2(0.6) + 2.3272 + 0}{10}$$

$= -0.98728$

$$x_4^{(1)} = \frac{15 - 3(2.3272) + (-0.98728)}{8}$$

$= 0.87889$

$$x_1^{(k)} = 6 + x_2^{(k-1)} - 2x_3^{(k-1)} \;|\; 10$$

$$x_2^{(k)} = 25 + x_1^{(k)} + x_3^{(k-1)} - 3x_4^{(k-1)} \;|\; 11$$

$$x_3^{(k)} = -11 - 2x_1^{(k)} + x_2^{(k)} + x_4^{(k-1)} \;|\; 10$$

$$x_4^{(k)} = 15 - 3x_2^{(k)} + x_3^{(k)} \;|\; 8$$

it	x_1	x_2	x_3	x_4
0	0	0	0	0
1	0.6	2.3272	-0.9873	0.87885
2	1.030	2.037	-1.014	0.9844
3	1.0065	2.0036	-1.0025	0.9985
4	1.0009	2.0003	-1.0003	0.9999
5	1.0001	2.0000	-1.0000	1.0000

$$\text{Tol} = \frac{\| x5 - x4 \|_\infty}{\| x5 \|_\infty} = \frac{0.0008}{2} = 0.0004 < 10^{-3}$$

Note:

Jacobi method required twice as many as iterations for the same accuracy than Seidel method. It means Seidel method converges quickly than Jacobi method.

Iterative Method to find Eigenvalues:

Power Method:

* A standard iterative procedure for computing approximate values of the eigen-values of an $n \times n$ matrix is called the Power method.

* This method applies to any $n \times n$ matrix that has a dominant eigen values.

* It means $|\lambda|$ is greater than the absolute values of the other eigen values.

* Power method helps to find the dominant eigen values.

Thm:

Suppose A is $n \times n$ a real symmetric matrix. Let $x \neq 0$ be any real vector with n components. Let

$$y = Ax, \quad m_0 = x^T x, \quad m_1 = x^T y$$

$$m_2 = y^T y$$

Then $q = \frac{m_1}{m_0}$ is an approximation for an eigenvalue λ of A.

Remark:

If $\overset{\text{app.}}{q} = \overset{\text{exact}}{\lambda} - \epsilon$ so that ϵ is the error of q then

$$|\epsilon| \le \sqrt{\frac{m_2}{m_0} - q^2}$$

Power Method:

(1) Start from any initial vector $x_0 \neq 0$ with n components

(2) $x_1 = A x_0$

$x_2 = A x_1$

$\vdots$

$x_n = A x_{n-1}$

Remark: $Ax = (a_{ij}]_{n \times n} [x_i]_{n \times 1}$

$= [\lambda_i]_{n \times 1}$

$$= \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix}$$

$$= \lambda_1 \begin{bmatrix} 1 \\ \lambda_2/\lambda_1 \\ \vdots \\ \lambda_n/\lambda_1 \end{bmatrix}$$

if $\lambda_1 > \lambda_2 > \lambda_3 > \dots > \lambda_n$

Note We have to scale with the largest (in magnitude only) value of the vector.

1) Use power method to find the dominant eigen value of $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

Sol: Initial guess $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \neq 0$

$$(5-\lambda)(2-\lambda) - 4 = 0$$

$$\lambda^2 - 7\lambda + 6 = 0$$

$-7 \mid 6$

-1

-6

$$x_1 = A x_0 = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 1 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 0.2 \end{bmatrix}$$

$$x_2 = A x_1 = \begin{bmatrix} 5.8 \\ 1.4 \end{bmatrix}$$

$$= 5.8 \begin{bmatrix} 1 \\ 0.2423 \end{bmatrix}$$

$$x_3 = A x_2$$

$$= 5.9652 \begin{bmatrix} 1 \\ 0.2485 \end{bmatrix}$$

$$x_4 = A x_3$$

$$= 5.994 \begin{bmatrix} 1 \\ 0.24974 \end{bmatrix}$$

$$x_5 = A x_4$$

$$= 5.99896 \begin{bmatrix} 1 \\ 0.24995 \end{bmatrix}$$

Dominant eigenvalue

Corresponding EV

2)

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

(1)

$$x_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \neq 0$$

$$x_1 = Ax_0$$

$$= \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix}$$

$$x_2 = Ax_1$$

$$= \begin{bmatrix} 2.5 \\ -2 \\ 0.5 \end{bmatrix} = 2.5 \begin{bmatrix} 1 \\ -0.8 \\ 0.2 \end{bmatrix}$$

$$x_3 = 2.8 \begin{bmatrix} 1 \\ -1 \\ 0.43 \end{bmatrix}$$

$$x_4 = 3.43 \begin{bmatrix} 0.87 \\ -1 \\ 0.54 \end{bmatrix}$$

$$x_5 = 3.41 \begin{bmatrix} 0.8 \\ -1 \\ 0.61 \end{bmatrix}$$

$$x_6 = 3.41 \begin{bmatrix} 0.74 \\ -1 \\ 0.64 \end{bmatrix}$$

Restrictions:

* It gives only dominant eigenvalue
* If A has more than one dominant eigen value then power method may not converges.