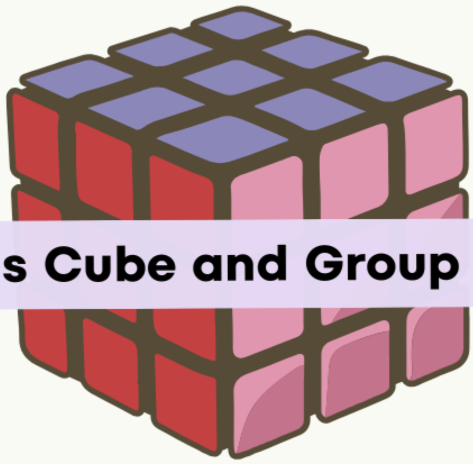
ALGEBRA

PROJECT REPORT



Rubik's Cube and Group Theory

by Sanket Motagi
SE23UCAM020

by

Sanket Motagi

SE23UCAM020

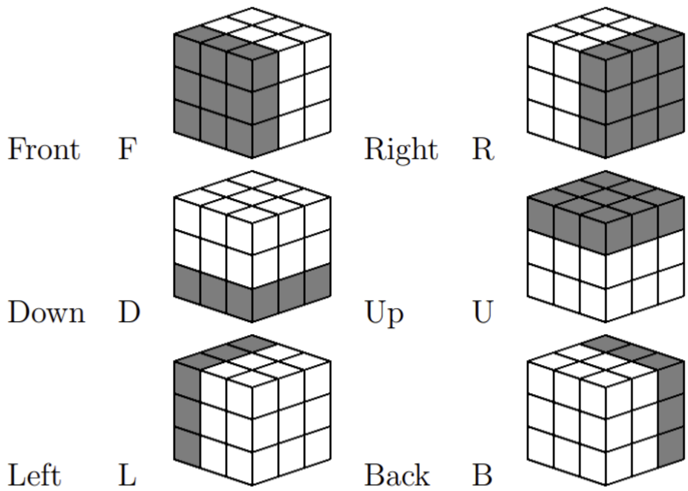
Contents

§1. Introduction

Almost everyone has tried to solve a Rubik's cube. The first attempt often ends in vain with only a jumbled mess of colored **cubies** (referred to one small cube in the bigger Rubik's cube) in no coherent order. Solving the cube becomes almost trivial once a certain core set of algorithms, called **macros**, are learned. Using basic group theory, the reason these solutions are not incredibly difficult to find will become clear.

§2. Notation

Throughout the report, the following notations will be used to refer to the sides of the cube.



The same notation will be used to refer to face rotations. For example, F means to rotate the front face 90 degrees clockwise. A counterclockwise rotation is denoted by lowercase letters (f) or by adding a ' (F'). A 180 degree turn is denoted by adding a superscript 2 (F^2), or just the move followed by a 2 (F2).

To refer to an individual cubie or a face of a cubie, we use one letter for the center cubies, two letters for the edge cubies, and three letters for the corner cubies, which give the faces of the cube that the cubie is part of. The first of the three letters gives the side of the cubie we are referring to. For example, in the picture below, the red square is at FUR, yellow at RUF, blue at URF, and green at ULB:

§3. Groups

Definition 3.1 (Groups).
A *group* G consists of a set of objects and a binary operator, $*$, on those objects satisfying the following four conditions.

The conditions are:

- The operation $*$ is closed, so for any group elements h and g in G, $h * g$ is also in G

- The operation $*$ is associative, so for any elements f, g and h,

$$(f * g) * h = f * (g * h)$$

- There is an identity element $e \varepsilon G$ such that

$$e * g = g * e = g$$

- Every element in G has an inverse g^{-1} relative to the operation $*$ such that

$$g * g^{-1} = g^{-1} * g = e$$

Theorems about Groups

1. The identity element, e, is unique
2. If $a * b = e$, then $a = b^{-1}$
3. If $a * x = b * x$, then $a = b$
4. The inverse of (ab) is $b^{-1}a^{-1}$
5. $(a^{-1})^{-1} = e$

§4. Cube Moves as Group Elements

We can conveniently represent cube permutations as group elements. We will call the group of permutations R, for Rubik (not a symbol for real numbers).

§4.1. The Binary Operator for the Rubik Group

The binary operator, $*$, will be a concatenation of subsequences of cube moves, or rotations of a face of the cube. We will almost always omit the $*$ symbol, and interpret fg as $f * g$. This operation is clearly closed, since any face rotation still leaves us with a permutation of the cube, which is in R. Rotations are also associative: it does not matter how we group them, as long as the order in which operations are performed is conserved. The identity element e corresponds to not changing the cube at all.

Inverses

The inverse of a group element g is usually written as g^{-1}. Let F be the cube move that rotates the front face clockwise. Then f, the inverse of F, moves the front face counterclockwise. Suppose there is a sequence of moves, say FR, then its inverse if rf: to invert the operations they must be done in reverse order. So the inverse of an element essentially "undoes" it.

§4.2. Permutations

The different move sequences of cube elements can be viewed as permutations, or rearrangements, of the cubies. Before getting into cube moves' permutations, it is easier to discuss these permutations first using numbers. An example of a permutation written in ***canonical cycle notation*** is:

$$(1)(234)$$

This means that 1 stays in place, and elements 2, 3 and 4 are cycled. For example, 2 goes to 3, 3 goes to 4, and 4 goes to 2. Or in other words, 2 is mapped to 3, 3 is mapped to 4 and 4 is mapped to 2. (234) $\rightarrow$ (423).

Note: If P (a permutation) consists of multiple cycles of varying length, then the **order** of that permutation is the least common multiple (LCM) of the lengths of the cycles, since that number of cycle steps will return both chains to their starting states. Below are a few examples:

$$(123)(231) = (132) \to \text{order } 3$$

$$(23)(456)(345) = (243)(56) \to \text{order } 6$$

$$(12) \to \text{order } 2$$

§4.3. Parity

Permutations can also be described in terms of their parity. Any length n cycle of a permutation can be expressed as the product of 2-cycles[1].

Definition 4.3.1 (Parity).
The **Parity** of a length n cycle is given by the number of 2-cycles it is composed of.

§5. Subgroups

Given a group R, if $S \subseteq R$ is any subset of the group, then the subgroup H "generated" by S is the smallest group of R that contains all the elements of S. For instance, {F} generates a group that is a subgroup of R consisting all possible different cube permutations you can get to by rotating the front face, $\{F, F^2, F^3, F^4\}$. The group generated by {F, B, U, L, R, D} is the whole group R. Below are some examples of some generators of subgroups of R:

- Any single face rotation, e.g., {F}
- Any two face opposite face rotations, e.g., {LR}
- The two moves {RF}

§5.1. Order

We define the **order** of an element g as the number m, such that $g^m = e$, the identity. The order of an element is also the size of the subgroup it generates. So we can use the notion of order to describe cube move sequences in terms of how many times you have to repeat a particular move before returning to the identity.

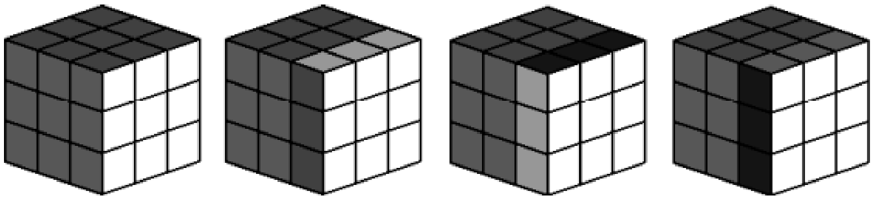
For example, the move F generates a subgroup of order 4, since rotating a face 4 times returns to the original state.

§5.2. Cosets

If G is a group and H is a subgroup of G, then for an element g of G:
- $gH = \{gh : h \varepsilon H\}$ is a left coset of H in G.
- $Hg = \{gh : h \varepsilon H\}$ is a right coset of H in G.

So for instance if H is the subgroup of R generated by F, then one right coset is shown below:

[1]for proof, see Davis, Tom. Permutation Groups and Rubiks Cube. May 6 2000



Lemma 5.2.1.
If H is a finite subgroup of a group G and H contains n elements, then any right cost of H contains n elements.

Lemma 5.2.2.
Two right cosets of a subgroup H in a group G are either identical or disjoint.

§5.3. Lagrange's Theorem

Theorem 5.3.1 (Lagrange's Theorem)**.**
The size of any group $H \subseteq G$ must be a divisor of the size of G. So $m|H| = |G|$ for some $m \geq 1 \ \varepsilon \ N^{+}$.

Proof: Click here

§6. Solving the Cube

§6.1. The "Bottom Up" Method

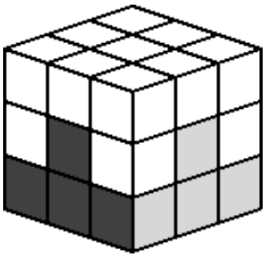
This is one of the most intuitive, but probably one of the slowest, ways to solve the cube. It averages about 100 moves per solution.

1st layer

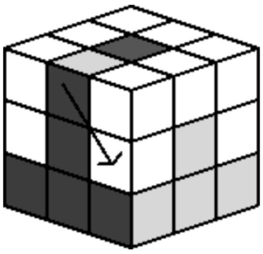
This first layer must be done by inspection. There is usually no set algorithm to follow. It is helpful to focus on getting a cross first with the edge pieces correctly in place, and then solving the corners one by one.

2nd layer

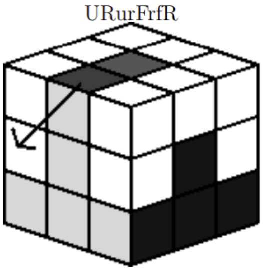
Now rotate the bottom (solved) layer so that its edges on the other faces are paired with the correct center pieces. Your cube should look as follows:



For this layer we only have to solve the four middle layer edge pieces. If an edge piece is in the top layer, use the following macros:

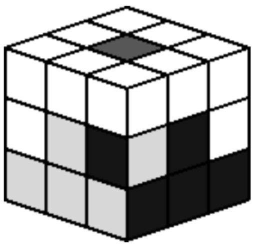


URurFrfR



ulULfLFl

If an edge pieces is not in the top layer, but is not oriented correctly, use the following to put the piece in the top layer and then proceed as above:



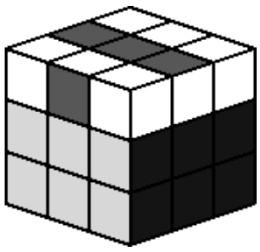
URurFrfR

The second layer should now be solved.

3rd layer

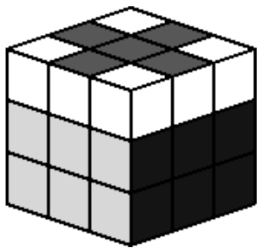
We will do this layer in 4 steps:

1. Flip the edges to form a cross on the top: To flip a top layer edge correctly, use this macro:

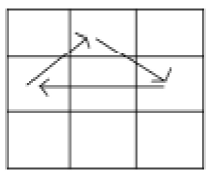


FRUrufU

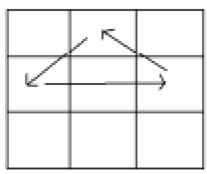
Repeat until all the edge pieces form a cross on the top:



2. Position the top layer edges correctly: Now position the top layer so that one of the edges is solved. If all the edges are solved, move on to the next step. If not, use the following algorithms to permute the edges correctly:



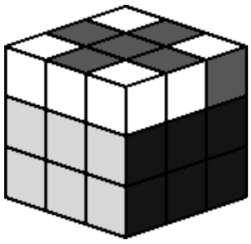
$\mathrm{RU^2ruRur}$



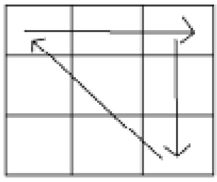
$\mathrm{RUrURU^2r}$

If none of these work, apply one of them until you get to a position here one of these will work, then proceed.

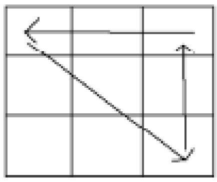
3. Flip the top layer corners: For each corner that does not have the correct color on the top layer, position it at UBR and perform RDrd repeatedly until it is oriented with the correct color on top. Then, without rotating the cube, position the next unsolved corner at UBR and repeat the process. The bottom two layers will appear to be a mess, but they will be correct once all the four corners are facing the correct direction.



4. Position the top layer corners correctly: Now the top layer should have all the same color faces, but the corners might not be oriented correctly. Position one corner correctly, and then determine whether the others are solved, need to be rotated clockwise, or need to rotated counterclockwise, and then apply the following (let x = rD^2R):



xU^{2}xuxux



xUxUxU^{2}x

Congratulations! You solved a cube!

§6.2. Other methods

The above solution is by no means the only one. Some other popular methods include:

- **CFOP Method:** : Cross, First two layers, Orient last layer, Permute last layer. Invented in the 1980s by Jessica Fridrich.
- **Petrus Method:** solve a 2 by 2 by 2 block first, expand this to 2 by 2 by 3, fix the improperly oriented edges on the outside layer, and then solve the rest.

§7. References

90% of the report references one MIT paper on the same topic. https://web.mit.edu/sp.268/www/rubik.pdf