CS6046: Multi-Armed Bandits
Worksheet - 4

(1) Consider the linear bandit with two arms $a^1 = [\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]^\top$ and $a^2 = [-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]^\top$ (note that both a^1 and a^2 are column vectors). For the case of fixed design, consider the sequence $a_1 = a_2 = a_3 = a^2$ and $a_4 = a_5 = a^1$ (i.e., second arm is played 3 times, and first arm is played 2 times). Let $V_t = \sum_{s=1}^{t} a_s a_s^\top$. Sketch the unit ball $B = \{x \in \mathbb{R}^2 : ||x||^2_{V_t} \leq 1\}$.

(2) Let $\Omega = \{1, 2, 3, 4, 5, 6\}$, $\mathcal{F} = 2^\Omega$, and $\mathcal{P}(A) = \frac{|A|}{6}$. Let $X(\omega) = \omega$, $Y_1(\omega) = +1, \omega = 1, 2, 3$ and $Y_1(\omega) = -1, \omega = 3, 4, 5$, $Y_2(\omega) = +1, \omega = 1, 2, Y_2(\omega) = 0, \omega = 3, Y_2(\omega) = -1, \omega = 4, 5, 6$, and $Y_3(\omega) = +1, \omega = 1, 2, 3, 4, Y_3(\omega) = -1, \omega = 5, 6$. Write down $\mathcal{F}_1 = \sigma(Y_1), \mathcal{F}_2 = \sigma(Y_2), \mathcal{F}_3 = \sigma(Y_3), \mathcal{F}_4 = \sigma(Y_1, Y_2), \mathcal{F}_5 = \sigma(Y_2, Y_3), \mathcal{F}_6 = \sigma(Y_1, Y_3)$ and $\mathcal{F}_7 = \sigma(Y_1, Y_2, Y_3)$. Find $\mathbb{E}[X|\mathcal{F}_i]$ for all $i = 1, 2, \ldots, 7$.

(3) Continuing from the above question, take $\mathcal{F}_1 = \sigma(Y_1)$, $\mathcal{F}_2 = \sigma(Y_1, Y_2)$ and $\mathcal{F}_3 = \sigma(Y_3)$. Construct random variables $X_1 \neq X_2 \neq X_3$ such that $\mathbb{E}[X_2|\mathcal{F}_1] = X_1$, $\mathbb{E}[X_3|\mathcal{F}_2] = X_2$.

(4) Similar to martingales (defined in the class), we have sub-martingales for which $\mathbb{E}[X_t|\mathcal{F}_{t-1}] \geq X_{t-1}$ and super-martingales for which $\mathbb{E}[X_t|\mathcal{F}_{t-1}] \leq X_{t-1}$. Give example of X_1, X_2, X_3 such that it is a (i) sub-martingale and (ii) super-martingale.

(5) Let τ be a random variable taking values in the set $\{1, 2, \ldots, \} \cup \{\infty\}$ and X_1, X_2, X_3 be a martingale sequence.

(i) Give example for τ such that it is not a stopping time

(ii) What is X_τ?

(iii) Construct the example such that $\mathbb{E}[X_\tau] \neq \mathbb{E}[X_1]$