Division Algebras, Triality, and Exceptional Magic

A. Garrett Lisi

Pacific Science Institute, Makawao, HI, USA

E-mail: Gar@Li.si

Abstract: We describe in explicit detail the rich relationship between division algebras, split composition algebras, triality, Clifford algebras, spinors, triality Lie algebras, generalized reflections, exceptional magic square Lie algebras, and Exceptional Unification in particle physics.

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1 Introduction

A growing community of researchers has become interested in the application of division algebras and the corresponding split composition algebras to a structural description of the Standard Model of particle physics.[1–18] The use of triality in this context, relating three generations of fermions, has gathered increasing interest.[19] The existence of the quaternion group, Q_8, within the CPT Group generated by charge, parity, and time conjugation symmetries, and its extension to the $CPTt$ Group—acting on three generations of fermions related by triality—strongly indicates division algebras are intricately woven into the fabric of reality. Despite this indication of its usefullness, explicit mathematical descriptions of exactly how triality can be used in model building are sparse. It is the purpose of this work to remedy this deficit—to provide a detailed description of the mathematical scaffolding relating division algebras, triality, Clifford algebras, and Lie algebras related to the Standard Model and gravity. This paper largely follows and complements Baez's excellent paper, "The Octonions",[20], but in more painful detail, and with an eye to physics.

We begin by introducing division algebras: the complex numbers, $\mathbb{C}$, quaternions, $\mathbb{H}$, and octonions, $\mathbb{O}$, and the related split-signature composition algebras, $\mathbb{C}'$, $\mathbb{H}'$, and $\mathbb{O}'$, and use them to construct Clifford algebras. The Clifford bivectors generate *Spin* groups, which act on spinors with negative and positive-chiral parts. A structural isomorphism (or "confusion") then exists, between 2, 4, or 8-dimensional vectors, v, negative-chiral spinors, ψ, positive-chiral spinors, χ, and sets of three division algebra elements. (We casually use "division algebra", $\mathbb{D}$, to also encompass the corresponding split-signature composition algebras, and sometimes not the reals.)

A real, cyclic, trilinear triality function is defined by the division algebra product (or vice versa), and is invariant under the triality group of symmetries on its arguments. These symmetries relate to generalized reflections, producing duality automorphisms related to twistor incidence relations,[21] as well as triality automorphisms, which transform and cycle the arguments. Each division algebra has a Lie algebra, its triality algebra, corresponding to this triality group. Each triality algebra is a subalgebra of the *triality Lie algebra* formed by the joining of a triality algebra with the three elements corresponding to a vector, negative-chiral spinor, and positive-chiral spinor. Using these elements, these triality Lie algebras can be expressed heuristically as $su(3, \mathbb{D})$.[22] Triality inner automorphisms act within these Lie algebras, and can be displayed graphically in their root systems.

Two division algebras can also be combined to construct a compound division algebra representation of a Clifford algebra. The corresponding two triality Lie algebras combine to give Lie algebras in the exceptional magic square.[23] These *magic square Lie algebras*, and their triality automorphisms, are explicitly formulated. Detailed examples are provided for Lie algebras e_6, e_7, and e_8, each of which is relevant to Exceptional Unification models in physics. The algebra of the $SO(10)$ Grand Unified Theory embeds in e_6. Three generations of Dixon algebra fermions, related to $\mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$, embed in complex e_7. And three generations of fermions, related by triality, embed in real forms of e_8, along with Standard Model gauge, Higgs, and (Euclidean) gravitational fields. The paper ends with a brief description of division algebra automorphisms, which brings the remaining exceptional Lie algebra, g_2.

2 Division Algebra Representation of Clifford Algebras

A n-dimensional division algebra, $\mathbb{D}$, or split-signature composition algebra, $\mathbb{D}'$, is spanned by its basis elements, e_a, which have a conjugation,

$$\tilde{e}_0 = e_{\tilde{0}} = e_0 = 1 \qquad \tilde{e}_1 = e_{\tilde{1}} = -e_1 \qquad \dots \qquad \tilde{e}_{n-1} = e_{\widetilde{n-1}} = -e_{n-1}$$

and a multiplication table, $e_a e_b = M_{ab}{}^c e_c$, allowing the definition of its metric,

$$(e_a, e_b) = \tfrac{1}{2}(\tilde{e}_a e_b + \tilde{e}_b e_a) = n_{ab}$$

with $n_{ab} = \delta_{ab}$ for the usual division algebras, and n_{ab} having split signature, $\{+,-\}$, for the split-algebras. Under conjugation, division algebra multiplication satisfies

$$\widetilde{(e_a e_b)} = e_{\tilde{b}} e_{\tilde{a}} \qquad M_{ab}{}^{\tilde{c}} = M_{\tilde{b}\tilde{a}}{}^{c}$$

Standard multiplication tables, $M_{ab}{}^c e_c$, for the division algebras and their split-algebras are:

$$\mathbb{C} : \begin{bmatrix} e_0 & e_1 \\ e_1 & -e_0 \end{bmatrix} \qquad \mathbb{C}' : \begin{bmatrix} e_0 & e_1 \\ e_1 & e_0 \end{bmatrix}$$

$$\mathbb{H} : \begin{bmatrix} e_0 & e_1 & e_2 & e_3 \\ e_1 & -e_0 & e_3 & -e_2 \\ e_2 & -e_3 & -e_0 & e_1 \\ e_3 & e_2 & -e_1 & -e_0 \end{bmatrix} \qquad \mathbb{H}' : \begin{bmatrix} e_0 & e_1 & e_2 & e_3 \\ e_1 & e_0 & e_3 & e_2 \\ e_2 & -e_3 & -e_0 & e_1 \\ e_3 & -e_2 & -e_1 & e_0 \end{bmatrix}$$

$$\mathbb{O} : \begin{bmatrix} e_0 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\ e_1 & -e_0 & e_4 & e_7 & -e_2 & e_6 & -e_5 & -e_3 \\ e_2 & -e_4 & -e_0 & e_5 & e_1 & -e_3 & e_7 & -e_6 \\ e_3 & -e_7 & -e_5 & -e_0 & e_6 & e_2 & -e_4 & e_1 \\ e_4 & e_2 & -e_1 & -e_6 & -e_0 & e_7 & e_3 & -e_5 \\ e_5 & -e_6 & e_3 & -e_2 & -e_7 & -e_0 & e_1 & e_4 \\ e_6 & e_5 & -e_7 & e_4 & -e_3 & -e_1 & -e_0 & -e_2 \\ e_7 & e_3 & e_6 & -e_1 & e_5 & -e_4 & -e_2 & -e_0 \end{bmatrix} \qquad \mathbb{O}' : \begin{bmatrix} e_0 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\ e_1 & -e_0 & e_3 & -e_2 & -e_5 & e_4 & -e_7 & e_6 \\ e_2 & -e_3 & -e_0 & e_1 & -e_6 & e_7 & e_4 & -e_5 \\ e_3 & e_2 & -e_1 & -e_0 & -e_7 & -e_6 & e_5 & e_4 \\ e_4 & e_5 & e_6 & e_7 & e_0 & e_1 & e_2 & e_3 \\ e_5 & -e_4 & -e_7 & e_6 & -e_1 & e_0 & e_3 & -e_2 \\ e_6 & e_7 & -e_4 & -e_5 & -e_2 & -e_3 & e_0 & e_1 \\ e_7 & -e_6 & e_5 & -e_4 & -e_3 & e_2 & -e_1 & e_0 \end{bmatrix} \tag{2.1}$$

Division algebra multiplication allows the construction of chiral Clifford basis elements of $Cl(n)$, $Cl(0,n)$, or $Cl(\frac{n}{2},\frac{n}{2})$, which act on chiral *division algebra spinors*,

$$\gamma_c = \begin{bmatrix} 0 & \pm\tilde{e}_c \\ e_c & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & \pm(\bar{\Gamma}_c)^a{}_b \\ (\Gamma_c)^b{}_a & 0 \end{bmatrix} = \begin{bmatrix} 0 & \pm M_{\tilde{c}\tilde{b}}{}^a \\ M_{ca}{}^{\tilde{b}} & 0 \end{bmatrix} \qquad \Psi = \begin{bmatrix} \psi \\ \tilde{\chi} \end{bmatrix} = \begin{bmatrix} \psi^a e_a \\ \chi^b \tilde{e}_b \end{bmatrix} \sim \begin{bmatrix} \psi^a Q^-_a \\ \chi^b Q^+_b \end{bmatrix} \tag{2.2}$$

with $(\bar{\Gamma}_c) = n_{cc}(\Gamma_c)^T$ and multiplication understood to be to the right by division algebra elements (accounting for non-associativity of octonions), or represented equivalently as $2n \times 2n$ real matrices built from the multiplication table coefficients,

$$(\Gamma_c)^b{}_a = M_{ca}{}^{\tilde{b}} \qquad (\bar{\Gamma}_c)^a{}_b = M_{\tilde{c}\tilde{b}}{}^a = M_{bc}{}^{\tilde{a}} = M_c{}^a{}_{\tilde{b}} = (\Gamma_c)_b{}^a$$

These representative Clifford basis vector elements satisfy the fundamental Clifford identity,

$$\gamma_a \cdot \gamma_b = \tfrac{1}{2}\left(\gamma_a\gamma_b + \gamma_b\gamma_a\right) = \pm\tfrac{1}{2}\begin{bmatrix} \tilde{e}_a e_b + \tilde{e}_b e_a & 0 \\ 0 & e_a\tilde{e}_b + e_b\tilde{e}_a \end{bmatrix} = \pm n_{ab} = \eta_{ab} \tag{2.3}$$

with the "$\pm$" signature usually chosen to be "$-$" for our purposes. Division algebra multiplication coefficients, and the corresponding Clifford matrix elements, satisfy a cyclic identity,

$$\bar{\Gamma}_{abc} = \bar{\Gamma}_{bca} = \bar{\Gamma}_{cab} = \Gamma_{acb} = \Gamma_{cba} = \Gamma_{bac} = M_{ab\tilde{c}} = M_{bc\tilde{a}} = M_{ca\tilde{b}} = M_{\tilde{a}\tilde{c}b} = M_{\tilde{c}\tilde{b}a} = M_{\tilde{b}\tilde{a}c} \tag{2.4}$$

in which n_{ab} is used to lower indices. The Clifford pseudoscalar for each division algebra and split-algebra is $\gamma = \gamma_0 ... \gamma_{n-1} = \pm\left[\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right]$.

The $\frac{1}{2}n(n-1)$ representative Clifford bivector basis elements, for $c < d$, are:

$$\gamma_{cd} = \begin{bmatrix} \pm\tilde{e}_c e_d & \\ & \pm e_c\tilde{e}_d \end{bmatrix} \sim \begin{bmatrix} \pm(\bar{\Gamma}_c)^a{}_b(\Gamma_d)^b{}_e & \\ & \pm(\Gamma_c)^b{}_a(\bar{\Gamma}_d)^a{}_f \end{bmatrix} = \begin{bmatrix} \pm M_{\tilde{c}\tilde{b}}{}^a M_{de}{}^{\tilde{b}} & \\ & \pm M_{ca}{}^{\tilde{b}} M_{\tilde{d}\tilde{f}}{}^a \end{bmatrix}$$

with it understood that, for example, e_d, multiplies to the right before $\tilde{e}_c$ multiplies the result. Since $e_c\tilde{e}_d = -e_d\tilde{e}_c$ for $c \neq d$, we also have the reverse-indexed bivectors, $\gamma_{dc} = -\gamma_{cd}$, and will sometimes account for these with a $1/2$ in sums, such as for $B = \frac{1}{2}B^{cd}\gamma_{cd}$. These bi-product division algebra operators are the chiral basis elements of the corresponding spin Lie algebra, which act on division algebra spinors. Spinors are the fundamental representation space of spin groups, which have spin Lie algebras spanned by Clifford algebra bivectors that are represented by matrices that act on the spinors. For the complex numbers and quaternions, multiplication is associative, so these bi-product basis elements are themselves purely imaginary complex numbers or quaternions, $\tilde{e}_c e_d \in \mathbb{B} = \mathrm{Im}(\mathbb{D})$, and the corresponding spin Lie algebras, $\mathbb{B}_2 = so(2) = u(1)$ and $\mathbb{B}_4 = so(4) = su(2) + su(2)$, are 1 and 6-dimensional. For the octonions, multiplication is not associative, and these octonionic bi-products span 28-dimensional $\mathbb{B}_8 = so(8) \neq \mathrm{Im}(\mathbb{O})$.

From the division algebra representation of Clifford algebras we have a natural *confusion* between sets of three n-dimensional division algebra elements and corresponding sets of Clifford algebra vectors, negative real chiral spinors, and positive real chiral spinors,

$$\begin{array}{lcl} v = v^c e_c & \sim & v = v^c\gamma_c \\ \psi = \psi^a e_a & \sim & \psi = \psi^a Q^-_a \\ \tilde{\chi} = \chi^b \tilde{e}_b & \sim & \chi = \chi^b Q^+_b \\ \tilde{\chi} = v\,\psi & \sim & \chi = v\,\psi \end{array}$$

It is the chiral division algebra representative matrices of Clifford algebras, $\Gamma_c{}^b{}_a = M_{ca}{}^{\tilde{b}}$, that allow this direct identification between a set of vector, negative, and positive chiral spinors, (v, ψ, χ), and a triplet of division algebra elements, and their multiplication,

$$\chi^b e_{\tilde{b}} = \tilde{\chi} = v\,\psi = v^c\psi^a M_{ca}{}^{\tilde{b}} e_{\tilde{b}} \qquad \sim \qquad \chi^b = v^c\psi^a(\Gamma_c)^b{}_a$$

This confusion of vectors and spinors with division algebra elements, and the division algebra construction of Clifford algebras, leads to the explicit construction and understanding of the structure of many Lie algebras and their automorphisms.

3 Generalized Reflections and Triality

Division algebras (and their split versions) have a cubic form—a real, cyclic, trilinear *triality function*, $T(v,\psi,\chi)$, of three elements, or, equivalently, of vectors and chiral spinors,

$$\chi^b v^c \psi^a \Gamma_{cba} = \chi^T v\,\psi = T(v,\psi,\chi) = (\tilde{\chi}, v\,\psi) = \tfrac{1}{2}\left(\chi(v\psi) + \widetilde{\chi(v\psi)}\right) = \chi^b v^c \psi^a M_{ca\tilde{b}}$$

The triality function is cyclic, $T(v,\psi,\chi) = T(\psi,\chi,v)$, by virtue of the cyclic nature of division algebra multiplication, (2.4). Although one usually considers the triality function as built from the division algebra product, it is possible, alternatively, to use the existence of a triality function, as a cyclic cubic form on a vector space, to define the division algebra product. The triality function is invariant under the *triality group*, $\mathrm{Tri}(\mathbb{D})$, with elements $r \in \mathrm{Tri}(\mathbb{D})$ satisfying:

$$r\ :\ (v,\psi,\chi)\ \mapsto\ (v',\psi',\chi') \qquad \ni \qquad T(v',\psi',\chi') = T(v,\psi,\chi)$$

The triality group acts linearly on (v,ψ,χ) as its representation space.

Consider reflections, R^u_v, through a unit-length Clifford vector or division algebra element,

$$-\,u\cdot u = u^a u^b n_{ab} = s_u = \tilde{u}u = \pm 1$$

in which the signature, s_u, is space-like, $+1$, for division algebra elements or $Cl(0,n)$ Clifford vectors, or can be time-like, -1, for some split-composition algebra elements or the corresponding time-like $Cl(\frac{n}{2},\frac{n}{2})$ Clifford vectors. Note that we have chosen the "$-$" sign in (2.2, 2.3), to later match Lie algebra elements. Reflections can then be expressed as

$$\begin{array}{llll}
v' = R^u_v v = -uvu^- & v' = R^u_v v = -s_u u\tilde{v}u & v' = R^u_m \chi = \sqrt{s_u}\,\tilde{\chi}\tilde{u} & v' = R^u_p \psi = \sqrt{s_u}\,\tilde{u}\tilde{\psi} \\
\Psi' = R^u_v \Psi = \sqrt{s_u}\,u\gamma\Psi & \psi' = R^u_v \chi = \sqrt{s_u}\,\tilde{u}\tilde{\chi} & \psi' = R^u_m \psi = -s_u u\tilde{\psi}u & \psi' = R^u_p v = \sqrt{s_u}\,\tilde{v}\tilde{u} \\
 & \chi' = R^u_v \psi = \sqrt{s_u}\,\tilde{\psi}\tilde{u} & \chi' = R^u_m v = \sqrt{s_u}\,\tilde{u}\tilde{v} & \chi' = R^u_p \chi = -s_u u\tilde{\chi}u
\end{array}$$

and, since triality is cyclic, we also have *generalized reflections*, R^u_m and R^u_p, acting as reflections through negative and positive chiral spinors. These generalized reflections, through a space-like or time-like unit element, u, can be equivalently expressed as operations on Clifford basis vector and spinor elements,

$$\begin{array}{ccc}
R^u_v & R^u_m & R^u_p \\
\gamma'_c = (\delta^a_c - 2s_u u^a u_c)\gamma_a & \gamma'_c = \sqrt{s_u}\,u^a (\Gamma_a)^b{}_c Q^+_b & \gamma'_c = \sqrt{s_u}\,u^b (\bar{\Gamma}_b)^a{}_c Q^-_a \\
Q^{-\prime}_a = \sqrt{s_u}\,u^c (\bar{\Gamma}_c)^b{}_a Q^+_b & Q^{-\prime}_a = (\delta^b_a - 2s_u u^b u_a) Q^-_b & Q^{-\prime}_a = \sqrt{s_u}\,u^c (\Gamma_c)^b{}_a \gamma_b \\
Q^{+\prime}_b = \sqrt{s_u}\,u^c (\Gamma_c)^a{}_b Q^-_a & Q^{+\prime}_b = \sqrt{s_u}\,u^c (\bar{\Gamma}_c)^a{}_b \gamma_a & Q^{+\prime}_b = (\delta^a_b - 2s_u u^a u_b) Q^+_a
\end{array} \tag{3.1}$$

The triality function is anti-invariant under these generalized reflections, such as

$$T(v',\psi',\chi') = T(R^u_v v, R^u_v \chi, R^u_v \psi) = T(-s_u u\tilde{v}u, \sqrt{s_u}\,\tilde{u}\tilde{\chi}, \sqrt{s_u}\,\tilde{\psi}\tilde{u}) = (u\psi, (-u\tilde{v}u)\tilde{u}\tilde{\chi}) = -T(v,\chi,\psi)$$

Generalized reflections through non-unit-length u give *duality functions*, such as $v' = R^\chi_p \psi = \widetilde{\psi\chi}$. These duality functions come from dualizing the triality function: if we demand that $T(v,\psi,\chi) = 1$, then we can obtain an expression for the vector, negative spinor, or positive spinor from the two others,

$$v = \frac{1}{|\psi\chi|^2}\widetilde{\psi\chi} \qquad\qquad \psi = \frac{1}{|\chi v|^2}\widetilde{\chi v} \qquad\qquad \chi = \frac{1}{|v\psi|^2}\widetilde{v\psi}$$

These "incidence relations" are at the heart of the twistor program.[21]

Combining two generalized reflections of the same type gives a generalized rotation—an element of the triality group. Combining two generalized reflections of different types gives an element of the triality group that isn't a rotation. Combining four generalized reflections through two unit-length elements, u and w, gives a *triality automorphism*,

$$t^{uw} = R^w_p R^u_v R^{\tilde{u}}_m R^{\tilde{u}}_p$$

an element of the triality group that takes vectors to negative spinors, negative spinors to positive spinors, and positive spinors to vectors,

$$t^{uw} \;:\; (v,\psi,\chi) \;\mapsto\; (v',\psi',\chi') = \big(\sqrt{s_u s_w}\,\tilde{w}(u\psi), \sqrt{s_u s_w}(\chi u)\tilde{w}, s_u s_w w(\tilde{u}v\tilde{u})w\big)$$

with $T(v',\psi',\chi') = T(v,\psi,\chi)$. Via Clifford algebra confusion, pairs of unit elements, u and w, produce general triality automorphisms of sets of three division algebra elements or of the corresponding Clifford vector and spinors,

$$\begin{aligned}
v' &= \sqrt{s_u s_w}\,\tilde{w}(u\psi) = \sqrt{s_u s_w}\, w^d u^c \psi^a M_{\tilde{d}\tilde{b}}{}^f M_{ca}{}^{\tilde{b}} e_f \\
&\qquad\qquad \sim \quad v' = \sqrt{s_u s_w}\, w^d u^c \psi^a (\bar{\Gamma}_d)^f{}_b (\Gamma_c)^b{}_a Q^-_f = \sqrt{s_u s_w}\, w u \psi \\
\psi' &= \sqrt{s_u s_w}(\chi u)\tilde{w} = \sqrt{s_u s_w}\,\chi^b u^c w^d M_{bc}{}^{\tilde{a}} M_{\tilde{a}\tilde{d}}{}^f e_f \\
&\qquad\qquad \sim \quad \psi' = \sqrt{s_u s_w}\,\chi^b u^c w^d (\Gamma_d)^f{}_a (\bar{\Gamma}_c)^a{}_b Q^+_f = (\chi^T u w)^T = \sqrt{s_u s_w}\, w^T u^T \chi \\
\chi' &= s_u s_w w(\tilde{u}v\tilde{u})w = s_u s_w w^a u^b v^c u^d w^e M_{af}{}^g M_{\tilde{b}c}{}^a M_{a\tilde{d}}{}^f M_{ge}{}^h e_h \\
&\qquad\qquad \sim \quad \chi' = v^b(\delta^a_b - 2 s_w w^a w_b)(\delta^c_a - 2 s_u u^c u_a)\gamma_c = s_u s_w w u v u w
\end{aligned} \tag{3.2}$$

Choosing $u = 1$ ($\sim\; u = \gamma_0$) and $w = 1$ ($\sim\; w = \gamma_0$) gives the *canonical triality automorphism*,

$$t \;:\; (v,\psi,\chi) \;\mapsto (v',\psi',\chi') = (\psi,\chi,v)$$

consistent with the invariance of triality under cyclic permutation of its arguments.

The *triality algebra* of a division algebra, $\mathrm{tri}(\mathbb{D})$, is the Lie algebra of its triality group, and its elements, $R \in \mathrm{tri}(\mathbb{D})$, satisfy:

$$R \;:\; (v,\psi,\chi) \;\mapsto\; (v',\psi',\chi') \qquad\qquad T(v',\psi',\chi') = 0$$

For the complex numbers, quaternions, and octonions, and their split versions, the triality algebras are:

$$\begin{aligned}
\mathrm{tri}(\mathbb{C}) &= u(1) + u(1) & \qquad\qquad \mathrm{tri}(\mathbb{C}') &= gl(1) + gl(1) \\
\mathrm{tri}(\mathbb{H}) &= su(2) + su(2) + su(2) & \mathrm{tri}(\mathbb{H}') &= sl(2) + sl(2) + sl(2) \\
\mathrm{tri}(\mathbb{O}) &= so(8) & \mathrm{tri}(\mathbb{O}') &= so(4,4)
\end{aligned}$$

4 Triality Lie Algebras

The best way to understand how triality algebras act on the corresponding triplets, (v, ψ, χ), is via their embedding in the corresponding *triality Lie algebras*,

$$\begin{aligned}
su(3) &= \; \mathrm{tri}(\mathbb{C}) + \mathbb{C} + \mathbb{C} + \mathbb{C} &&= u(1) + u(1) + (1+\bar{1})_v + (1+\bar{1})_m + (1+\bar{1})_p \\
sp(3) &= \; \mathrm{tri}(\mathbb{H}) + \mathbb{H} + \mathbb{H} + \mathbb{H} &&= su(2) + su(2) + su(2) + (2,2,1)_v + (2,1,2)_m + (1,2,2)_p \\
f_{4(-52)} &= \; \mathrm{tri}(\mathbb{O}) + \mathbb{O} + \mathbb{O} + \mathbb{O} &&= so(8) + 8_v + 8_{s-} + 8_{s+}
\end{aligned}$$

$$\begin{aligned}
sl(3) &= \mathrm{tri}(\mathbb{C}') + \mathbb{C}' + \mathbb{C}' + \mathbb{C}' = gl(1) + gl(1) + (1+\bar{1})_v + (1+\bar{1})_m + (1+\bar{1})_p \\
sp(6,\mathbb{R}) &= \mathrm{tri}(\mathbb{H}') + \mathbb{H}' + \mathbb{H}' + \mathbb{H}' = sl(2) + sl(2) + sl(2) + (2,2,1)_v + (2,1,2)_m + (1,2,2)_p \\
f_{4(4)} &= \mathrm{tri}(\mathbb{O}') + \mathbb{O}' + \mathbb{O}' + \mathbb{O}' = so(4,4) + 8_v + 8_{s-} + 8_{s+}
\end{aligned}$$

Generalized reflections, and triality automorphisms, are real automorphisms of triality Lie algebras. The structures of these Lie algebras fully elucidate these symmetries, and are worth examining in each case.

4.1 $su(3)$

The eight Lie algebra basis generators for the special unitary group, $SU(3)$, may be represented by 3×3 traceless, anti-Hermitian matrices of complex numbers, related to the Gell-Mann matrices,

$$\begin{aligned}
A &= \begin{bmatrix} i\,B^1 + \frac{i}{\sqrt{3}}B^2 & -v^0 + i\,v^1 & \psi^0 + i\,\psi^1 \\ v^0 + i\,v^1 & -i\,B^1 + \frac{i}{\sqrt{3}}B^2 & -\chi^0 + i\,\chi^1 \\ -\psi^0 + i\,\psi^1 & \chi^0 + i\,\chi^1 & -\frac{2i}{\sqrt{3}}B^2 \end{bmatrix} = \begin{bmatrix} V - M & -v^* & \psi \\ v & P - V & -\chi^* \\ -\psi^* & \chi & M - P \end{bmatrix} \\
&= B^1 T_1 + B^2 T_2 + v^a \gamma_a + \psi^a Q_a^- + \chi^a Q_a^+ \\
&= V\,H_v + M\,H_m + P\,H_p + (v\,E_v^- - v^* E_v^+) + (\psi\,E_m^- - \psi^* E_m^+) + (\chi\,E_p^- - \chi^* E_p^+) \\
&\in su(3)
\end{aligned} \tag{4.1}$$

with $\{v, \psi, \chi\}$ complex numbers, $v = v^0 + i\,v^1 = v^0 e_0 + v^1 e_1$, and $\{V, M, P\}$ pure imaginary numbers. Note that since $su(3)$ elements are traceless, V, M, and P correspond to only two degrees of freedom, B^1 and B^2—the same $su(3)$ element is specified if V, M, and P are all shifted by a constant. These V, M, and P generators are motivated by the existence of three overlapping $su(2)$ subalgebras, spanned by $\{H_v, \gamma_a\}$, $\{H_m, Q_a^-\}$, and $\{H_p, Q_a^+\}$. As a triality Lie algebra, $su(3)$ relates to the complex division algebra representation of $Cl(0,2)$, with basis vectors $\gamma_0 = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ and $\gamma_1 = \left[\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right]$. This $\mathbb{C}$ division algebra representation of $Cl(0,2)$ is not the representation from the complex multiplication table, (2.1), which is instead, from $(\Gamma_c)^b{}_a = M_{ca}{}^{\tilde{b}}$,

$$M_{00}{}^{\tilde{0}} = 1 \;\; M_{01}{}^{\tilde{1}} = M_{10}{}^{\tilde{1}} = -1 \;\; M_{11}{}^{\tilde{0}} = -1 \qquad \Gamma_0 = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \;\; \Gamma_1 = \begin{bmatrix} & -1 \\ -1 & \end{bmatrix} \;\; \gamma_c = \begin{bmatrix} 0 & -\bar{\Gamma}_c \\ \Gamma_c & 0 \end{bmatrix}$$

In general, a triality Lie algebra can either be described directly as $\sim su(3,\mathbb{D})$, or equivalently by constructing the corresponding Clifford algebra and its bivector and vector matrix representatives (identified with the upper-left 2×2 block in $su(3,\mathbb{D})$) which act on negative and positive chiral spinors, then closing the algebra via the Lie brackets between spinors. The representation of the Clifford algebra may be by division algebra elements, by their matrix representatives, or from the equivalent division algebra multiplication table coefficients.

We can compute the $su(3)$ Lie brackets directly from the commutator of its representative matrices, (4.1),

$$\left[\left(B^1_1, B^2_1, v_1, \psi_1, \chi_1\right), \left(B^1_2, B^2_2, v_2, \psi_2, \chi_2\right)\right] = \left(B^1_3, B^2_3, v_3, \psi_3, \chi_3\right)$$

$$\begin{aligned}
2i\, B^1_3 &= -2(v_1^* v_2 - v_2^* v_1) - (\psi_1 \psi_2^* - \psi_2 \psi_1^*) - (\chi_1 \chi_2^* - \chi_2 \chi_1^*) \\
\tfrac{2i}{\sqrt{3}} B^2_3 &= (\chi_1 \chi_2^* - \chi_2 \chi_1^*) + (\psi_1^* \psi_2 - \psi_2^* \psi_1) \\
v_3 &= -2i(B^1_1 v_2 - B^1_2 v_1) + (\chi_1^* \psi_2^* - \chi_2^* \psi_1^*) \\
\psi_3 &= i(B^1_1 + \sqrt{3} B^2_1)\psi_2 - i(B^1_2 + \sqrt{3} B^2_2)\psi_1 + (v_1^* \chi_2^* - v_2^* \chi_1^*) \\
\chi_3 &= i(B^1_1 - \sqrt{3} B^2_1)\chi_2 - i(B^1_2 - \sqrt{3} B^2_2)\chi_1 + (\psi_1^* v_2^* - \psi_2^* v_1^*)
\end{aligned}$$

Alternatively, using our basis elements, $\{T_1, T_2, \gamma_a, Q^-_a, Q^+_a\}$, the non-vanishing $su(3)$ brackets between them are, explicitly,

$$\begin{array}{lll}
[T_1, \gamma_0] = -2\,\gamma_1 & \left[T_1, Q^-_0\right] = +Q^-_1 & \left[T_1, Q^+_0\right] = +Q^+_1 \\
[T_1, \gamma_1] = +2\,\gamma_0 & \left[T_1, Q^-_1\right] = -Q^-_0 & \left[T_1, Q^+_1\right] = -Q^+_0 \\
 & \left[T_2, Q^-_0\right] = +\sqrt{3}\,Q^-_1 & \left[T_2, Q^+_0\right] = -\sqrt{3}\,Q^+_1 \\
 & \left[T_2, Q^-_1\right] = -\sqrt{3}\,Q^-_0 & \left[T_2, Q^+_1\right] = +\sqrt{3}\,Q^+_0 \\
[\gamma_0, \gamma_1] = -2\,T_1 & \left[Q^-_0, Q^-_1\right] = T_1 + \sqrt{3}\,T_2 & \left[Q^+_0, Q^+_1\right] = T_1 - \sqrt{3}\,T_2 \\
\left[\gamma_a, Q^-_b\right] = -M_{\tilde{a}\tilde{b}}{}^c\, Q^+_c & \left[\gamma_a, Q^+_b\right] = M_{\tilde{a}\tilde{b}}{}^c\, Q^-_c & \left[Q^-_a, Q^+_b\right] = -M_{\tilde{a}\tilde{b}}{}^c\, \gamma_c
\end{array} \tag{4.2}$$

and their anti-symmeterized partners.

With orthogonal Cartan subalgebra basis generators, $\{T_1, T_2\}$, or non-orthogonal Cartan basis generators, $\{H_v, H_m, H_p\}$, the root vectors and their Lie brackets are:

$$\begin{array}{lll}
E^+_v = \tfrac{1}{2}(-\gamma_0 - i\,\gamma_1) & [T_1, E^\pm_\alpha] = \pm i\, g^1_\alpha\, E^\pm_\alpha & [E^\pm_v, E^\pm_m] = \mp E^\mp_p \\
E^-_v = \tfrac{1}{2}(+\gamma_0 - i\,\gamma_1) & [T_2, E^\pm_\alpha] = \pm i\, g^2_\alpha\, E^\pm_\alpha & \left[E^\pm_m, E^\pm_p\right] = \mp E^\mp_v \\
E^+_m = \tfrac{1}{2}(-Q^-_0 - i\,Q^-_1) & & \left[E^\pm_p, E^\pm_v\right] = \mp E^\mp_m \\
E^-_m = \tfrac{1}{2}(+Q^-_0 - i\,Q^-_1) & [H_v, E^\pm_\alpha] = \pm i\, v_\alpha\, E^\pm_\alpha & [E^+_v, E^-_v] = -i\,T_1 = -i\,H_v \\
E^+_p = \tfrac{1}{2}(-Q^+_0 - i\,Q^+_1) & [H_m, E^\pm_\alpha] = \pm i\, m_\alpha\, E^\pm_\alpha & [E^+_m, E^-_m] = -i\,(-\tfrac{1}{2}T_1 - \tfrac{\sqrt{3}}{2}T_2) = -i\,H_m \\
E^-_p = \tfrac{1}{2}(+Q^+_0 - i\,Q^+_1) & [H_p, E^\pm_\alpha] = \pm i\, p_\alpha\, E^\pm_\alpha & \left[E^+_p, E^-_p\right] = -i\,(-\tfrac{1}{2}T_1 + \tfrac{\sqrt{3}}{2}T_2) = -i\,H_p
\end{array}$$

with the $\{g^1_\alpha, g^2_\alpha, v_\alpha, m_\alpha, p_\alpha\}$ roots in Table 1. This structure of $su(3)$ is consistent with its triality decomposition, in which each of the three triples, $\{H_{v/m/p}, E^+_{v/m/p}, E^-_{v/m/p}\} \sim \{H, E^+, E^-\}$, corresponds to a different $su(2)$, related to each other by triality, with disjoint root vectors but overlapping Cartan generators. The relevant triality function is,

$$T(v, \psi, \chi) = v^c \psi^a \chi^b M_{ab\tilde{c}} = v^0 \psi^0 \chi^0 - v^0 \psi^1 \chi^1 - v^1 \psi^1 \chi^0 - v^1 \psi^0 \chi^1$$

and the canonical inner triality automorphism of $su(3)$ is:

$$t : A \mapsto A' = g_t \, A \, g_t^- \qquad\qquad g_t = \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix} \in SU(3)$$

in which g_t is an element of the 3×3 representation of the $SU(3)$ Lie group, and t transforms the generators, root vectors, and Cartan subalgebra elements as:

$$t \quad : \quad \gamma_a \mapsto Q_a^- \mapsto Q_a^+ \mapsto \gamma_a \qquad E_v^\pm \mapsto E_m^\pm \mapsto E_p^\pm \mapsto E_v^\pm \qquad H_v \mapsto H_p \mapsto H_m \mapsto H_v$$

This triality automorphism corresponds to a rotation on root space coordinates, (g_α^1, g_α^2), by t^-, and a transformation of the Cartan subalgebra basis elements, $\{T_1, T_2\}$, by the *triality matrix*, t,

$$\begin{bmatrix} g_\alpha^1{}' \\ g_\alpha^2{}' \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} g_\alpha^1 \\ g_\alpha^2 \end{bmatrix} \qquad \begin{bmatrix} T_1{}' \\ T_2{}' \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \qquad t = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

Within the triality algebra of $su(3)$, which is also its Cartan subalgebra, the two basis generators, $\{T_1, T_2\}$, are each rotated between three directions by the canonical triality automorphism. Specifically, $T_1^I = H_v = T_1$, $T_1^{II} = H_p = -\frac{1}{2}T_1 + \frac{\sqrt{3}}{2}T_2$, and $T_1^{III} = H_m = -\frac{1}{2}T_1 - \frac{\sqrt{3}}{2}T_2$.

$su(3)$		g^1	g^2	v	m	p
▲	v^+	$+2$	0	$+2$	-1	-1
▼	v^-	-2	0	-2	$+1$	$+1$
▲	m^+	-1	$-\sqrt{3}$	-1	$+2$	-1
▼	m^-	$+1$	$+\sqrt{3}$	$+1$	-2	$+1$
▲	p^+	-1	$+\sqrt{3}$	-1	-1	$+2$
▼	p^-	$+1$	$-\sqrt{3}$	$+1$	$+1$	-2

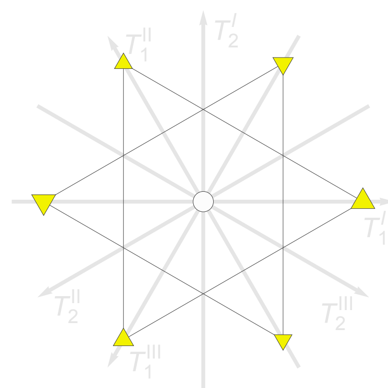


Table 1. Roots of $su(3)$ with respect to the orthogonal Cartan subalgebra basis generators, $\{T_1, T_2\}$, or non-orthogonal Cartan basis generators, $\{H_v, H_m, H_p\}$, and their automorphism under triality.

The split-complex numbers, a composition algebra, are represented by $\{e_0' = 1, e_1' = I\}$, with $M_{11}'{}^0 = I^2 = 1$. Repeating our Lie algebra construction, using real Gell-Mann matrices, we get the Lie algebra $sl(3)$, with non-vanishing brackets:

$$\begin{array}{lll}
[T_1', \gamma_0'] = -2\,\gamma_1' & [T_1', Q_0'^-] = +Q_1'^- & [T_1', Q_0'^+] = +Q_1'^+ \\
[T_1', \gamma_1'] = -2\,\gamma_0' & [T_1', Q_1'^-] = +Q_0'^- & [T_1', Q_1'^+] = +Q_0'^+ \\
 & [T_2', Q_0'^-] = +\sqrt{3}\,Q_1'^- & [T_2', Q_0'^+] = -\sqrt{3}\,Q_1'^+ \\
 & [T_2', Q_1'^-] = +\sqrt{3}\,Q_0'^- & [T_2', Q_1'^+] = -\sqrt{3}\,Q_0'^+ \\
[\gamma_0', \gamma_1'] = -2\,T'^1 & [Q_0'^-, Q_1'^-] = T'^1 + \sqrt{3}\,T'^2 & [Q_0'^+, Q_1'^+] = T'^1 - \sqrt{3}\,T'^2 \\
[\gamma_a', Q_b'^-] = -M_{\tilde{a}\tilde{b}}'{}^c\,Q_c'^+ & [\gamma_a', Q_b'^+] = M_{\tilde{a}\tilde{b}}'{}^c\,Q_c'^- & [Q_a'^-, Q_b'^+] = -M_{\tilde{a}\tilde{b}}'{}^c\,\gamma_c'
\end{array} \tag{4.3}$$

The triality automorphism structure for $sl(3)$ is the same as for $su(3)$.

4.2 $sp(3)$

The structure of the 21-dimensional symplectic Lie algebra, $sp(3)$—the triality Lie algebra of the quaternions—is similar to that of $su(3)$. Instead of 3×3 traceless matrices of complex numbers, elements of $sp(3)$ can be represented by matrices of quaternions,

$$A = \begin{bmatrix} M & -\tilde{v} & \psi \\ v & P & -\tilde{\chi} \\ -\tilde{\psi} & \chi & V \end{bmatrix} \in sp(3) = su(3, \mathbb{H}) \tag{4.4}$$

with $\{v, \psi, \chi\}$ quaternions and $\{M, P, V\}$ purely imaginary quaternions. The Lie brackets are thus:

$$[(M_1, P_1, V_1, v_1, \psi_1, \chi_1), (M_2, P_2, V_2, v_2, \psi_2, \chi_2)] = (M_3, P_3, V_3, v_3, \psi_3, \chi_3)$$

$$\begin{aligned}
M_3 &= M_1 M_2 - M_2 M_1 - (\tilde{v}_1 v_2 - \tilde{v}_2 v_1) - (\psi_1 \tilde{\psi}_2 - \psi_2 \tilde{\psi}_1) \\
P_3 &= P_1 P_2 - P_2 P_1 - (\tilde{\chi}_1 \chi_2 - \tilde{\chi}_2 \chi_1) - (v_1 \tilde{v}_2 - v_2 \tilde{v}_1) \\
V_3 &= V_1 V_2 - V_2 V_1 - (\tilde{\psi}_1 \psi_2 - \tilde{\psi}_2 \psi_1) - (\chi_1 \tilde{\chi}_2 - \chi_2 \tilde{\chi}_1) \\
v_3 &= (P_1 v_2 - P_2 v_1) + (v_1 M_2 - v_2 M_1) + (\tilde{\chi}_1 \tilde{\psi}_2 - \tilde{\chi}_2 \tilde{\psi}_1) \\
\psi_3 &= (M_1 \psi_2 - M_2 \psi_1) + (\psi_1 V_2 - \psi_2 V_1) + (\tilde{v}_1 \tilde{\chi}_2 - \tilde{v}_2 \tilde{\chi}_1) \\
\chi_3 &= (V_1 \chi_2 - V_2 \chi_1) + (\chi_1 P_2 - \chi_2 P_1) + (\tilde{\psi}_1 \tilde{v}_2 - \tilde{\psi}_2 \tilde{v}_1)
\end{aligned}$$

Each of the three diagonal matrix elements is a $su(2)$ subalgebra, which act on two out of the three off-diagonal elements. The Cartan subalgebra basis consists of one element from each diagonal element, $\{T_3^M, T_3^P, T_3^V\}$.

The $sp(3)$ Lie algebra, and its brackets, relate to the Clifford algebra matrix representation of $Cl(0,4)$ from the quaternion multiplication table, as in (2.2), in which case (4.4) gives a 12×12 real representation. Or, using the usual Pauli matrix representation of quaternions,

$$e_0 = \sigma_0 \qquad e_1 = -i\,\sigma_1 \qquad e_2 = -i\,\sigma_2 \qquad e_3 = -i\,\sigma_3$$

(4.4) results in a 6×6 complex representation. Using the quaternion mulitplication table, (2.1), the $Cl(0,4)$ chiral representative matrices (2.2) are $(\Gamma_c)^b{}_a = M_{ca}{}^{\tilde{b}}$ and $-(\bar{\Gamma}_c)^a{}_b = -M_{\tilde{c}\tilde{b}}{}^a$, so

$$v \sim v^c \Gamma_c = \begin{bmatrix} v^0 & v^1 & v^2 & v^3 \\ -v^1 & v^0 & -v^3 & v^2 \\ -v^2 & v^3 & v^0 & -v^1 \\ -v^3 & -v^2 & v^1 & v^0 \end{bmatrix} \qquad -\tilde{v} \sim -v^c \bar{\Gamma}_c = \begin{bmatrix} -v^0 & v^1 & v^2 & v^3 \\ -v^1 & -v^0 & -v^3 & v^2 \\ -v^2 & v^3 & -v^0 & -v^1 \\ -v^3 & -v^2 & v^1 & -v^0 \end{bmatrix}$$

The negative and positive $Cl(0,4)$ chiral bivector matrices separate into independent degrees of freedom, corresponding to $so(4) = su(2)_M + su(2)_P$,

$$M \sim B_M = -\tfrac{1}{2} B^{ab} \bar{\Gamma}_a \Gamma_b = B_M^A \, \Gamma_A \qquad P \sim B_P = -\tfrac{1}{2} B^{ab} \Gamma_a \bar{\Gamma}_b = B_P^A \, \Gamma_A$$

in which $B_{M/P}^A = \mp B^{0A} + \frac{1}{2} \epsilon^{BCA} B^{BC}$, and the capital indices, $\{A, B, C\}$, range over $\{1, 2, 3\}$.

The $sp(3)$ Lie algebra elements can be written in an orthoganal, Killing-normalized basis as

$$A = M^A T^M_A + P^A T^P_A + V^A T^V_A + v^a \gamma_a + \psi^a Q^-_a + \chi^a Q^+_a = \begin{bmatrix} M^A e_A & -v^a \frac{1}{\sqrt{2}} \tilde{e}_a & \psi^a \frac{1}{\sqrt{2}} e_a \\ v^a \frac{1}{\sqrt{2}} e_a & P^A e_A & -\chi^a \frac{1}{\sqrt{2}} \tilde{e}_a \\ -\psi^a \frac{1}{\sqrt{2}} \tilde{e}_a & \chi^a \frac{1}{\sqrt{2}} e_a & V^A e_A \end{bmatrix}$$

with the $sp(3)$ Lie brackets between these basis elements computed explicitly:

$$\begin{array}{rlrlrl}
[T^M_A, T^M_B] &= T^M_C (2M_{[AB]}{}^C) & [\gamma_a, Q^-_b] &= Q^+_c \frac{1}{\sqrt{2}}(-M_{\tilde{b}\tilde{a}}{}^c) & [T^M_A, \gamma_b] &= \gamma_c (-M_{bA}{}^c) \\
[T^P_A, T^P_B] &= T^P_C (2M_{[AB]}{}^C) & [\gamma_a, Q^+_b] &= Q^-_c \frac{1}{\sqrt{2}}(M_{\tilde{a}\tilde{b}}{}^c) & [T^M_A, Q^-_b] &= Q^-_c (M_{Ab}{}^c) \\
[T^V_A, T^V_B] &= T^V_C (2M_{[AB]}{}^C) & [Q^-_a, Q^+_b] &= \gamma_c \frac{1}{\sqrt{2}}(-M_{\tilde{b}\tilde{a}}{}^c) & [T^P_A, \gamma_b] &= \gamma_c (M_{Ab}{}^c) \\
[\gamma_a, \gamma_b] &= \multicolumn{3}{l}{T^M_C \frac{1}{2}(M_{\tilde{b}a}{}^C - M_{\tilde{a}b}{}^C) + T^P_C \frac{1}{2}(M_{b\tilde{a}}{}^C - M_{a\tilde{b}}{}^C)} & [T^P_A, Q^+_b] &= Q^+_c (-M_{bA}{}^c) \\
[Q^-_a, Q^-_b] &= \multicolumn{3}{l}{T^V_C \frac{1}{2}(M_{\tilde{b}a}{}^C - M_{\tilde{a}b}{}^C) + T^M_C \frac{1}{2}(M_{b\tilde{a}}{}^C - M_{a\tilde{b}}{}^C)} & [T^V_A, Q^-_b] &= Q^-_c (-M_{bA}{}^c) \\
[Q^+_a, Q^+_b] &= \multicolumn{3}{l}{T^P_C \frac{1}{2}(M_{\tilde{b}a}{}^C - M_{\tilde{a}b}{}^C) + T^V_C \frac{1}{2}(M_{b\tilde{a}}{}^C - M_{a\tilde{b}}{}^C)} & [T^V_A, Q^+_b] &= Q^+_c (M_{Ab}{}^c)
\end{array} \tag{4.5}$$

$sp(3)$		M	P	V
	$M^\pm$	± 2	0	0
	$P^\pm$	0	± 2	0
	$V^\pm$	0	0	± 2
	$\psi^\pm_e$	± 1	0	± 1
	$\psi^\pm_o$	± 1	0	∓ 1
	$\chi^\pm_e$	0	± 1	± 1
	$\chi^\pm_o$	0	∓ 1	± 1
	$v^\pm_e$	± 1	± 1	0
	$v^\pm_o$	∓ 1	± 1	0

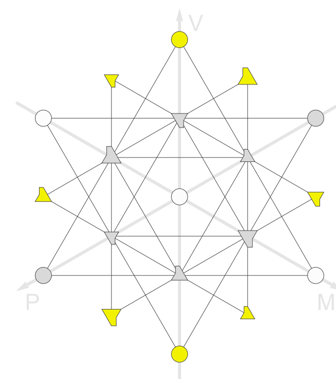


Table 2. Roots of $sp(3)$ with respect to the orthogonal Cartan subalgebra basis generators, $\{T^M_3, T^P_3, T^V_3\}$, and their automorphism under canonical triality.

Generalized quaternionic reflections give automorphisms of $sp(3)$. For example, the generalized reflection along a unit positive spinor, R^1_p, gives

$$R^1_p \quad : \quad (M, P, V, v, \psi, \chi) \quad \mapsto \quad (M', P', V', v', \psi', \chi') \;=\; (M, V, P, \tilde{\psi}, \tilde{v}, -\tilde{\chi}) \tag{4.6}$$

which is an inner automorphism,

$$R^1_p \;:\; A \;\mapsto\; A' = g_R\, A\, g_R^- \qquad\qquad g_R = \begin{bmatrix} -1 & & \\ & & 1 \\ & 1 & \end{bmatrix} \;\in\; SP(3) = SU(3, \mathbb{H})$$

This corresponds to a reflection of roots in root space coordinates, $(\alpha_M, \alpha_P, \alpha_V)$, by the R matrix, and a transformation of the Cartan subalgebra basis elements, $\{M, P, V\}$, by R,

$$\begin{bmatrix} \alpha'_M \\ \alpha'_P \\ \alpha'_V \end{bmatrix} = \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix} \begin{bmatrix} \alpha_M \\ \alpha_P \\ \alpha_V \end{bmatrix} = \begin{bmatrix} \alpha_M \\ \alpha_V \\ \alpha_P \end{bmatrix} \qquad \begin{bmatrix} M' \\ P' \\ V' \end{bmatrix} = \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix} \begin{bmatrix} M \\ P \\ V \end{bmatrix} = \begin{bmatrix} M \\ V \\ P \end{bmatrix} \qquad R = \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix}$$

The collection of all possible generalized quaternion reflections, including their compositions, is the automorphism group, $SP(3)$, of the $sp(3)$ Lie algebra. The canonical triality automorphism of $sp(3)$ is:

$$t \quad : \quad \gamma_a \mapsto Q_a^- \mapsto Q_a^+ \mapsto \gamma_a \qquad T_A^M \mapsto T_A^V \mapsto T_A^P \mapsto T_A^M$$

which, as a composition of reflections, $t = R_p^1 R_v^1 R_m^1 R_p^1$, is an inner automorphism,

$$t \;:\; A \;\mapsto\; A' = g_t \, A \, g_t^- \qquad g_t = \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix} \;\in\; SP(3) = SU(3, \mathbb{H})$$

This corresponds to a rotation of roots in root space coordinates, $(\alpha_M, \alpha_P, \alpha_V)$, by t^-, and a transformation of the Cartan subalgebra basis elements, $\{M, P, V\}$, by the triality matrix, t,

$$\begin{bmatrix} \alpha'_M \\ \alpha'_P \\ \alpha'_V \end{bmatrix} = \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix} \begin{bmatrix} \alpha_M \\ \alpha_P \\ \alpha_V \end{bmatrix} = \begin{bmatrix} \alpha_P \\ \alpha_V \\ \alpha_M \end{bmatrix} \qquad \begin{bmatrix} M' \\ P' \\ V' \end{bmatrix} = \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix} \begin{bmatrix} M \\ P \\ V \end{bmatrix} = \begin{bmatrix} V \\ M \\ P \end{bmatrix} \qquad t = \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}$$

Although one can describe these sorts of Lie algebra automorphisms as reflections or rotations of roots in root space, which correspond to reflections and rotations within the Cartan subalgebra and maps between root vectors, these root maps do not specify the signs of the maps between root vectors. To obtain a consistent set of signs for such maps, it is usually easiest to describe the automorphisms directly, as transformations of the Lie algebra generators, and then transform to the Cartan-Weyl basis to get a complete description of the maps between root vectors, including signs.

To describe the split real form, $sp(6, \mathbb{R})$, of the $sp(3)$ Lie algebra, we can use split-quaternions in (4.4) instead of quaternions, representing the basis elements as 2×2 real matrices,

$$e'_0 = \sigma_0 \qquad e'_1 = \sigma_1 \qquad e'_2 = -i\,\sigma_2 \qquad e'_3 = \sigma_3$$

Automorphisms of this split real Lie algebra are similar to those of compact $sp(3)$; however, an interesting difference can occur. For example, a reflection, R_v^3, around the $u = e'_3$ unit split-quaternion vector has $s_u = -1$, and the corresponding reflection element,

$$g_R = \begin{bmatrix} & -ie'_3 & \\ ie'_3 & & \\ & & -1 \end{bmatrix} \;\in\; SP(6, \mathbb{R}) = SU(3, \mathbb{H}')$$

produces a real $sp(6, \mathbb{R})$ inner automorphism,

$$M' = e'_3 P e'_3 \qquad P' = e'_3 M e'_3 \qquad V' = V \qquad v' = e'_3 \tilde{v} e'_3 \qquad \psi' = -ie'_3 \tilde{x} \qquad \chi' = -i\tilde{\psi} e'_3$$

which doesn't look like a real automorphism, but is. This will be further discussed in the next section.

4.3 f_4

The triality Lie algebra of the octonions, f_4, has a similar structure to that of $sp(3)$:

$$A = \tfrac{1}{2}B^{ab}\gamma_{ab} + v^a\gamma_a + \psi^a Q^-_a + \chi^a Q^+_a = \begin{bmatrix} B_M & -\tilde{v} & \psi \\ v & B_P & -\tilde{\chi} \\ -\tilde{\psi} & \chi & B_V \end{bmatrix} \;\tilde{\in}\; su(3,\mathbb{O}) \sim f_4 \tag{4.7}$$

but cannot be directly identified with $su(3,\mathbb{O})$ because of octonion non-associativity. To make the identification of elements of f_4 with matrices, (4.7), we need to define multiplication compositionally—with octonions multiplying to their right before they are then multiplied by octonions from their left. The f_4 representative matrix is then made of octonions and octonion bi-products. The matrix representation of $Cl(0,8)$ vectors, bivectors, and chiral spinors, from compositional octonionic multiplication, is

$$B = \tfrac{1}{2}B^{ab}\gamma_{ab} = \begin{bmatrix} -\tfrac{1}{2}B^{ab}\bar{\Gamma}_a\Gamma_b & \\ & -\tfrac{1}{2}B^{ab}\Gamma_a\bar{\Gamma}_b \end{bmatrix} \qquad v = v^c\gamma_c = \begin{bmatrix} & -v^c\bar{\Gamma}_c \\ v^c\Gamma_c & \end{bmatrix} \qquad \Psi = \begin{bmatrix} \psi^a Q^-_a \\ \chi^b Q^+_b \end{bmatrix}$$

using the octonion multiplication table, (2.1), and $(\Gamma_c)^b{}_a = M_{ca}{}^{\tilde{b}}$. The $su(3,\mathbb{O})$-inspired Lie brackets for the f_4 basis generators are:

$$\begin{array}{lll} [\gamma_{ab},\gamma_{cd}] = 2\left\{n_{ac}\gamma_{bd} - n_{ad}\gamma_{bc} - n_{bc}\gamma_{ad} + n_{bd}\gamma_{ac}\right\} & & \\ [\gamma_{ab},\gamma_c] = 2\left\{-n_{bc}\gamma_a + n_{ac}\gamma_b\right\} & [\gamma_c, Q^-_a] = Q^+_b(\Gamma_c)^b{}_a & [\gamma_a,\gamma_b] = 2\,\gamma_{ab} \\ [\gamma_{ab}, Q^-_c] = Q^-_d(-\bar{\Gamma}_a\Gamma_b)^d{}_c & \left[\gamma_c, Q^+_b\right] = Q^-_a(-\bar{\Gamma}_c)^a{}_b & \left[Q^-_a, Q^-_b\right] = \gamma_{cd}(\mp\bar{\Gamma}^c\Gamma^d)_{ab} \\ [\gamma_{ab}, Q^+_c] = Q^+_d(-\Gamma_a\bar{\Gamma}_b)^d{}_c & \left[Q^-_a, Q^+_b\right] = \gamma_c(\mp\bar{\Gamma}^c)_{ab} & \left[Q^+_a, Q^+_b\right] = \gamma_{cd}(\mp\Gamma^c\bar{\Gamma}^d)_{ab} \end{array} \tag{4.8}$$

in which $n_{ab} = \delta_{ab}$ is used to raise or lower indices, and we have 28 bivector generators, γ_{ab}, identified with $-\gamma_{ba}$ for $a < b$. The choice of $-$ signs above corresponds to the compact real form, $f_{4(-52)}$, while the choice of $+$ signs gives $f_{4(-20)}$. For the split real form, $f_{4(4)}$, we use the split-octonionic representation of $Cl(4,4)$, with the split metric, $n'_{ab} = \mathrm{diag}(++++----)$. The positive and negative spinor metrics come from multiplying the positive signature Clifford vectors, $\gamma_0\gamma_1\gamma_2\gamma_3$, giving $n'^-_{ab} = n'^+_{ab} = n'_{ab}$. Using this Clifford representation and these metrics, the $f_{4(4)}$ brackets are the same as for $f_{4(-52)}$, above, except for some signs,

$$\left[Q^-_a, Q^+_b\right] = \gamma_c(-\bar{\Gamma}^c)_{ab} \qquad \left[Q^-_a, Q^-_b\right] = \gamma_{cd}(+\bar{\Gamma}^c\Gamma^d)_{ab} \qquad \left[Q^+_a, Q^+_b\right] = \gamma_{cd}(+\Gamma^c\bar{\Gamma}^d)_{ab}$$

For a canonical triality automorphism to exist, he metrics of the vectors, negative spinors, and positive spinors in $f_{4(-52)}$ or $f_{4(4)}$ must be equal and match the Killing form.

The canonical triality automorphism between vectors and spinors,

$$t \quad : \quad \gamma_a \mapsto Q^-_a \mapsto Q^+_a \mapsto \gamma_a$$

is an automorphism of $f_{4(-52)}$, with corresponding automorphisms of its $so(8)$ subalgebra,

$$t \;:\; \gamma_{ab} \mapsto \gamma'_{ab} = \gamma_{cd}\, t^{cd}{}_{ab} = \tfrac{1}{2}\left[\gamma'_a, \gamma'_b\right] = \tfrac{1}{2}\left[Q^-_a, Q^-_b\right] = \gamma_{cd}\left(\mp\tfrac{1}{2}\bar{\Gamma}^c\Gamma^d\right)_{ab}$$

and

$$t^2 \ : \ \gamma_{ab} \mapsto \gamma''_{ab} = \gamma_{cd}\, t^{2\ cd}{}_{ab} = \tfrac{1}{2}\left[\gamma''_a, \gamma''_b\right] = \tfrac{1}{2}\left[Q^+_a, Q^+_b\right] = \gamma_{cd}\left(\mp\tfrac{1}{2}\Gamma^c\bar{\Gamma}^d\right)_{ab}$$

in which $a < b$ and the sum is over the 28 basis bivectors, $c < d$.

The description of f_4 can be made more explicitly octonionic by remembering that $\bar{\Gamma}_a\Gamma_b$ comes from multiplying to the right by e_b then multiplying to the right by $\tilde{e}_a$, so each $so(8)$ chiral bivector element corresponds to a sum of compositional octonion multiplications, $B_- \sim -\frac{1}{2}B^{ab}\tilde{e}_a e_b$. The canonical triality automorphism of $so(8)$ thus corresponds to a transformation of compositional multiplications of octonions,

$$t(\tilde{e}_a e_b) = t_{ab}{}^{cd}\,\tilde{e}_c e_d = e_a\tilde{e}_b$$

We can then describe the diagonal entries of the $\sim su(3,\mathbb{O})$ matrix explicitly as:

$$\begin{aligned}
B_M &= -\tfrac{1}{2}B^{ab}\tilde{e}_a e_b \\
B_P &= t(B_M) = -\tfrac{1}{2}B^{ab}t_{ab}{}^{cd}\,\tilde{e}_c e_d = -\tfrac{1}{2}B^{ab}e_a\tilde{e}_b \\
B_V &= t(B_P) = t^2(B_M) = -\tfrac{1}{2}B^{ab}t_{ab}{}^{cd}\,e_c\tilde{e}_d = -\tfrac{1}{2}B^{ab}t^{2\ cd}_{ab}\,\tilde{e}_c e_d
\end{aligned}$$

This allows us to write the Lie brackets between $f_{4(-52)}$ elements,

$$\left[(B_1, v_1, \psi_1, \chi_1), (B_2, v_2, \psi_2, \chi_2)\right] = (B_3, v_3, \psi_3, \chi_3)$$

using compositional octonionic multiplication and triality,

$$\begin{aligned}
B_3 &= B_1B_2 - B_2B_1 - (\tilde{v}_1v_2 - \tilde{v}_2v_1) - t^2(\tilde{\psi}_1\psi_2 - \tilde{\psi}_2\psi_1) - t(\tilde{\chi}_1\chi_2 - \tilde{\chi}_2\chi_1) \\
v_3 &= t^2(B_1)v_2 - t^2(B_2)v_1 + \tilde{\chi}_1\tilde{\psi}_2 - \tilde{\chi}_2\tilde{\psi}_1 \\
\psi_3 &= B_1\psi_2 - B_2\psi_1 + \tilde{v}_1\tilde{\chi}_2 - \tilde{v}_2\tilde{\chi}_1 \\
\chi_3 &= t(B_1)\chi_2 - t(B_2)\chi_1 + \tilde{\psi}_1\tilde{v}_2 - \tilde{\psi}_2\tilde{v}_1
\end{aligned} \tag{4.9}$$

in which the bivectors here are pairs of octonions multiplying to the right in order, such as $B_1\psi_2 = -\frac{1}{2}B_1^{ab}\psi_2^c\,\tilde{e}_a(e_be_c)$.

Generalized reflection symmetries of $f_{4(-52)}$, $f_{4(-20)}$, or $f_{4(4)}$ through a space-like or time-like unit vector, u, are real Lie algebra automorphisms, and their combinations comprise the corresponding F_4 Lie group. As F_4 group elements acting via their adjoint action on Lie algebra elements represented as elements of $su(3,\mathbb{O})$, (4.7), the three types of generalize reflections are

$$R^u_v \ : \ \begin{bmatrix} & \sqrt{s_u}\tilde{u} & \\ \sqrt{s_u}u & & \\ & & -1 \end{bmatrix} \qquad R^u_m \ : \ \begin{bmatrix} & & \sqrt{s_u}u \\ & -1 & \\ \sqrt{s_u}\tilde{u} & & \end{bmatrix} \qquad R^u_p \ : \ \begin{bmatrix} -1 & & \\ & & \sqrt{s_u}\tilde{u} \\ & \sqrt{s_u}u & \end{bmatrix}$$

Explicitly, in addition to the action of R^u_v, R^u_m, and R^u_p, on f_4 vector and chiral spinor basis generators, (3.1), these each extend to corresponding actions on the $so(8)$ or $so(4,4)$ subalgebra basis generators,

$$\begin{aligned}
R^u_v &\ : \ \gamma_{ab} \ \mapsto \ \gamma'_{ab} = \tfrac{1}{2}\left[R^u_v\gamma_a, R^u_v\gamma_b\right] = u\gamma_{ab}u = \tfrac{1}{2}(\delta^c_a - 2s_uu^cu_a)(\delta^d_b - 2s_uu^du_b)\gamma_{cd} \\
R^u_m &\ : \ \gamma_{ab} \ \mapsto \ \gamma'_{ab} = \tfrac{1}{2}\left[R^u_m\gamma_a, R^u_m\gamma_b\right] = \tfrac{1}{2}s_uu^e(\Gamma_e)^c{}_a u^f(\Gamma_f)^d{}_b(\mp\Gamma^g\bar{\Gamma}^h)_{cd}\gamma_{gh} \\
R^u_p &\ : \ \gamma_{ab} \ \mapsto \ \gamma'_{ab} = \tfrac{1}{2}\left[R^u_p\gamma_a, R^u_p\gamma_b\right] = \tfrac{1}{2}s_uu^e(\bar{\Gamma}_e)^c{}_a u^f(\bar{\Gamma}_f)^d{}_b(\mp\bar{\Gamma}^g\Gamma^h)_{cd}\gamma_{gh}
\end{aligned}$$

producing automorphisms of the f_4 Lie algebra via these maps of basis generators. For triality automorphisms, $t^{uw} : (\gamma_c, Q^-_a, Q^+_b, \gamma_{ab}) \mapsto (\gamma'_c, {Q^-_a}', {Q^+_b}', \gamma'_{ab})$, using unit vectors u and w, from (3.2), we have

$$\begin{aligned}
\gamma'_c &= \sqrt{s_u s_w}\, w^d (\bar\Gamma_d)^f{}_b u^a (\Gamma_a)^b{}_c Q^-_f \\
{Q^-_a}' &= \sqrt{s_u s_w}\, w^c (\Gamma_c)^f{}_d u^b (\bar\Gamma_b)^d{}_a Q^+_f \\
{Q^+_b}' &= (\delta^a_b - 2 s_w w^a w_b)(\delta^c_a - 2 s_u u^c u_a)\gamma_c \\
\gamma'_{ab} &= \tfrac{1}{2}\,[\gamma'_a, \gamma'_b] = \tfrac{1}{2} s_u s_w w^d (\bar\Gamma_d)^k{}_i u^c (\Gamma_c)^i{}_a w^f (\bar\Gamma_f)^m{}_j u^e (\Gamma_e)^j{}_b (\mp \bar\Gamma^g \Gamma^h)_{km} \gamma_{gh}
\end{aligned}$$

As an F_4 Lie group element acting via the adjoint, this typical generalized triality element is

$$t^{uw} = R^w_p R^u_v R^{\tilde u}_m R^{\tilde u}_p \quad : \quad \begin{bmatrix} & & 1 \\ \sqrt{s_u s_w}\, w\tilde u & & \\ & \sqrt{s_u s_w}\, \tilde w u & \end{bmatrix}$$

with it again understood that this compositional multiplication operation is ordered to act right-first—for example, $w\tilde u \psi = w(\tilde u \psi)$.

Note that real automorphisms of $f_4(-20)$ or the split real Lie algebra, $f_{4(4)}$, such as reflections, R^u_v, through a unit time-like vector, $s_u = -1$, or triality from a unit time-like u and space-like w, have explicit $\sqrt{s_u} = i$'s in them. This appears to contradict the fact that these are real automorphisms, but this is not the case. For example, if we consider complex f_4 as $f_{4(4)}$ with complex coefficients, then the usual anti-linear complex conjugation operator, $\sigma = K$, acting on complex f_4 produces $f_{4(4)}$ as the invariant real subspace—the $f_{4(4)}$ real form. Reflections through a unit time-like vector, $\phi = R^u_v$, which have i's in them, are complex and not real automorphisms with respect to σ because $R^u_v K \neq K R^u_v$. However, if we consider the reflection, R^0_v, through the unit octonion or space-like Clifford vector, γ_0, which is a real involutive automorphism of $f_{4(4)}$, then an alternative anti-linear complex conjugation operator on complex f_4 exists, $\sigma' = R^0_v K$, which gives the same real form, $f_{4(4)}$, and determines that $\phi = R^u_v$ with a time-like u and $s_u = -1$ *is* a real automorphism of $f_{4(4)}$, because

$$\phi\, \sigma' = R^u_v R^0_v K = R^0_v K R^u_v = \sigma'\, \phi$$

The i's in R^u_v are precisely matched with the swapping of the conjugate Q^+ and Q^- generators by R^0_v in just such a way that this works. This same argument goes through for R^u_m, R^u_p, and triality automorphisms—which can all have explicit i's in them while being real automorphisms.

The canonical triality automorphism matrix, t, for $so(8)$ or $so(4,4)$ is a 28×28 rotation matrix, satisfying $tt^T = 1$, of real coefficients,

$$t^{cd}{}_{ab} = \left(\mp\tfrac{1}{2}\bar\Gamma^c\Gamma^d\right)_{ab} = \mp\tfrac{1}{2} M^{\tilde c}{}_{ea} M^d{}_b{}^e$$

derived from the octonionic multiplication table, with the sign "$-$" for compact $f_{4(-52)}$ and otherwise "+", and the 28 basis bivectors, $\gamma_\alpha = \gamma_{ab}$, indexed by $1 \le \alpha = \frac{1}{2}(13a - a^2) + b \le 28$, with $0 \le a < b \le 7$. (The larger, 64×64 triality automorphism matrix elements are also defined for $b \geq a$, and can be deduced from anti-symmetry.) The 28 basis bivectors can be separated into 7 disjoint sets of 4 intra-commuting basis bivectors, each set spanning a Cartan subalgebra intra-rotated by triality. The triality automorphism matrix, $t_\alpha{}^\beta$, can thus be re-ordered to be block diagonal, with seven 4×4 Hadamard matrices. Typical blocks looks like:

$$t_1 = \begin{bmatrix} +1/2 & -1/2 & +1/2 & +1/2 \\ +1/2 & -1/2 & -1/2 & -1/2 \\ +1/2 & +1/2 & +1/2 & -1/2 \\ +1/2 & +1/2 & -1/2 & +1/2 \end{bmatrix} \qquad t_2 = \begin{bmatrix} -1/2 & +1/2 & +1/2 & +1/2 \\ -1/2 & +1/2 & -1/2 & -1/2 \\ -1/2 & -1/2 & +1/2 & -1/2 \\ -1/2 & -1/2 & -1/2 & +1/2 \end{bmatrix}$$

but signs vary, based on the octonionic multiplication table or if we use a non-canonical triality automorphism. For $so(4,4)$ triality automorphisms the four bivectors, γ_{ab}, spanning each Cartan subalgebra rotated by triality can be constructed from Clifford basis vectors with space-space, time-time, or space-time signature. The allowed signature sets are $\{ss, ss, tt, tt\}$ or $\{st, st, st, st\}$. If we choose one of these triality-adapted Cartan subalgebras for our f_4 Cartan-Weyl decomposition, then t is the triality matrix that rotates these Cartan basis generators, and $t^- = t^2 = t^T$ rotates the root coordinates.

There are two especially interesting f_4 Cartan subalgebra transformations we can do that emphasize the $sp(3)$ and $su(3)$ subalgebras of f_4, matching their previously described triality automorphisms:

$$c_1 = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & & \\ 1/\sqrt{2} & 1/\sqrt{2} & & \\ & & 1/\sqrt{2} & 1/\sqrt{2} \\ & & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \qquad c_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1/\sqrt{3} & -1/\sqrt{3} & -1/\sqrt{3} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{6} & -1/\sqrt{6} & \sqrt{2}/\sqrt{3} \end{bmatrix}$$

$$t_1' = c_1\, t_1\, c_1^- = \begin{bmatrix} & & 1 & \\ 1 & & & \\ & 1 & & \\ & & & 1 \end{bmatrix} \qquad t_2' = c_2\, t_2\, c_2^- = \begin{bmatrix} -1/2 & -\sqrt{3}/2 & & \\ \sqrt{3}/2 & -1/2 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

From t_1' we see that triality cycles three $su(2)'s$ in f_4, while leaving a complimentary $su(2)$ invariant, matching $sp(3)$ triality; while from t_2' we see that triality rotates in a single plane in 4-dimensional root space, matching $su(3)$ triality, while leaving a complimentary $su(3)$ invariant. The corresponding decompositions of $f_{4(-52)}$ are:

$$\begin{aligned} f_{4(-52)} &= so(8) + 8_v + 8_{s-} + 8_{s+} \\ &= su(2)_I + su(2)_{II} + su(2)_{III} + su(2)_W + (2,2,2,2) \\ &\quad + ((2,2,1,1) + (1,1,2,2))_v + ((2,1,2,1) + (1,2,1,2))_{s-} \\ &\quad + ((1,2,2,1) + (2,1,1,2))_{s+} \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} f_{4(-52)} &= so(8) + 8_v + 8_{s-} + 8_{s+} \\ &= u(1)_p + u(1)_B + su(3)_g + 3_I + \bar{3}_I + 3_{II} + \bar{3}_{II} + 3_{III} + \bar{3}_{III} \\ &\quad + (1 + \bar{1} + 3 + \bar{3})_v + (1 + \bar{1} + 3 + \bar{3})_{s-} + (1 + \bar{1} + 3 + \bar{3})_{s+} \end{aligned} \tag{4.11}$$

The f_4 roots matching these decompositions, (4.10) and (4.11), are:

f_4		ω_T	ω_s	U	V
	$\omega_I^{\wedge/\vee}$	$\mp$	$\pm$	0	0
	$\omega_{II}^{\wedge/\vee}$	$\mp$	$\mp$	0	0
	$\omega_{III}^{\wedge/\vee}$	0	0	$\mp$	$\mp$
	$W_\pm$	0	0	$\mp$	$\pm$
	$e_T^{\wedge/\vee}\ \phi_I^{/*}$	$\pm$	0	$\mp$	0
	$e_T^{\wedge/\vee}\ \phi_{II}^{/*}$	0	$\pm$	0	$\pm$
	$e_T^{\wedge/\vee}\ \phi_{III}^{/*}$	0	$\mp$	0	$\pm$
	$e_s^{\wedge/\vee}\ \phi_I^{*/}$	0	$\pm$	$\pm$	0
	$e_s^{\wedge/\vee}\ \phi_{II}^{*/}$	0	$\mp$	$\pm$	0
	$e_s^{\wedge/\vee}\ \phi_{III}^{*/}$	$\pm$	0	0	$\mp$
	$e_T^{\wedge/\vee}\ \phi^{*/}$	$\pm$	0	$\pm$	0
	$e_T^{\wedge/\vee}\ \phi_\pm$	$\pm$	0	0	$\pm$
	$\nu_{eL}^{\wedge/\vee}$	$\mp 1/2$	$\pm 1/2$	$-1/2$	$+1/2$
	$\nu_{eR}^{\wedge/\vee}$	$\pm 1/2$	$\pm 1/2$	$+1/2$	$+1/2$
	$e_L^{\wedge/\vee}$	$\mp 1/2$	$\pm 1/2$	$+1/2$	$-1/2$
	$e_R^{\wedge/\vee}$	$\pm 1/2$	$\pm 1/2$	$-1/2$	$-1/2$
	$\nu_{\mu L}^{\wedge/\vee}$	$\mp 1/2$	$\mp 1/2$	$-1/2$	$+1/2$
	$\nu_{\mu R}^{\wedge/\vee}$	$+1/2$	$-1/2$	$\pm 1/2$	$\pm 1/2$
	$\mu_L^{\wedge/\vee}$	$\mp 1/2$	$\mp 1/2$	$+1/2$	$-1/2$
	$\mu_R^{\wedge/\vee}$	$-1/2$	$+1/2$	$\pm 1/2$	$\pm 1/2$
	$\nu_{\tau L}^{\wedge}\ \ \tau_L^{\vee}$	0	0	$\mp$	0
	$\nu_{\tau L}^{\vee}\ \ \tau_L^{\wedge}$	0	0	0	$\pm$
	$\nu_{\tau R}^{\wedge}\ \ \tau_R^{\vee}$	$\pm$	0	0	0
	$\nu_{\tau R}^{\vee}\ \ \tau_R^{\wedge}$	0	$\pm$	0	0

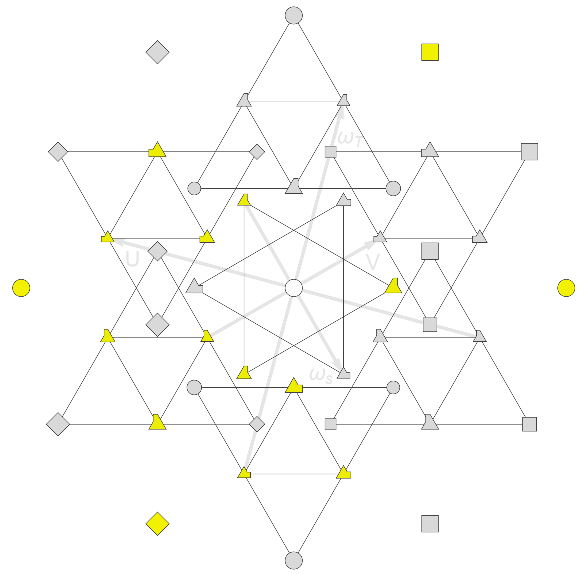


Table 3. The 48 roots of f_4, labeled as fermions of three generations, showing both the triality-invariant $su(2)_W$ and the triality mixing of three $su(2)$'s, as per (4.10).

f_4		p	x	y	z
	g	0	$(+1$	-1	$0\,)$
	$X_I^{rgb}\ \bar{X}_I^{rgb}$	0	$(\pm 1$	± 1	$0\,)$
	$X_{II}^{rgb}\ \bar{X}_{II}^{rgb}$	± 1	$(\mp 1$	0	$0\,)$
	$X_{III}^{rgb}\ \bar{X}_{III}^{rgb}$	∓ 1	$(\mp 1$	0	$0\,)$
	l_I	$-1/2$	$-1/2$	$-1/2$	$-1/2$
	$\bar{l}_I$	$+1/2$	$+1/2$	$+1/2$	$+1/2$
	$q_I^{(rgb)}$	$-1/2$	$(-1/2$	$+1/2$	$+1/2\,)$
	$\bar{q}_I^{(rgb)}$	$+1/2$	$(+1/2$	$-1/2$	$-1/2\,)$
	l_{II}	$-1/2$	$+1/2$	$+1/2$	$+1/2$
	$\bar{l}_{II}$	$+1/2$	$-1/2$	$-1/2$	$-1/2$
	$q_{II}^{(rgb)}$	$+1/2$	$(-1/2$	$+1/2$	$+1/2\,)$
	$\bar{q}_{II}^{(rgb)}$	$-1/2$	$(+1/2$	$-1/2$	$-1/2\,)$
	l_{III}	$+1$	0	0	0
	$\bar{l}_{III}$	-1	0	0	0
	$q_{III}^{(rgb)}$	0	$(-1$	0	$0\,)$
	$\bar{q}_{III}^{(rgb)}$	0	$(+1$	0	$0\,)$

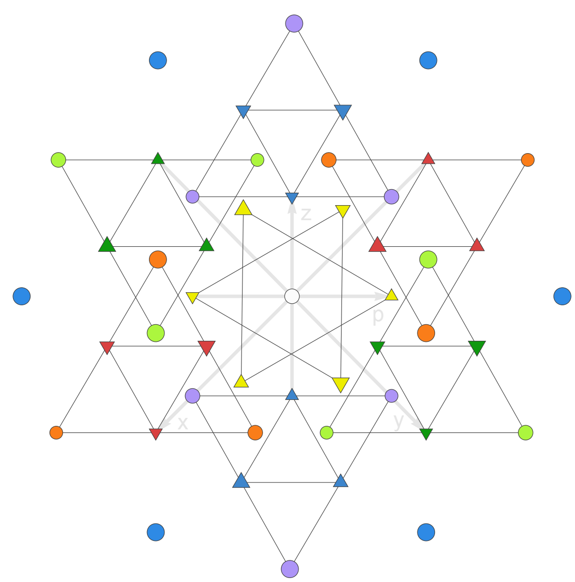


Table 4. The 48 roots of f_4, showing the triality-invariant $su(3)_g$ plane, as per (4.11). Coordinates in parenthesis are permuted over specified columns.

5 Exceptional Magic

To obtain larger Lie algebras we can consider the tensor product of division algebras or their split versions, such as the 32-dimensional vector space $\mathbb{H} \otimes \mathbb{O}$. From any two division algebras, $\mathbb{D}'$ and $\mathbb{D}$, of dimension n' and n and signature (p', q') and (p, q), we can construct chiral representations of Clifford algebras, $Cl(p'+p, q'+q)$ or $Cl(q'+p, p'+q)$, in a similar manner to the construction of Clifford division algebra representations. In a *Clifford compound division algebra representation*, $(n' + n)$-dimensional Clifford basis vectors are expressed as:

$$v = v^\alpha \gamma_\alpha = \begin{bmatrix} 0 & v_- \\ v_+ & 0 \end{bmatrix} \qquad v_- = \begin{bmatrix} 1\otimes\tilde{x} & \pm\tilde{y}'\otimes 1 \\ y'\otimes 1 & -1\otimes x \end{bmatrix} \qquad v_+ = \begin{bmatrix} 1\otimes x & \pm\tilde{y}'\otimes 1 \\ y'\otimes 1 & -1\otimes\tilde{x} \end{bmatrix}$$

which may be understood as matrices of inter-commuting division algebra elements, $x \in \mathbb{D}$ and $y' \in \mathbb{D}'$, or as $\mathbb{R}(4(n' \times n))$ matrices via their multiplication coefficients. The result of multiplying two vectors is

$$u\,v = \begin{bmatrix} u_- v_+ & 0 \\ 0 & u_+ v_- \end{bmatrix} \qquad u_- v_+ = \begin{bmatrix} \tilde{w} & \pm\tilde{z}' \\ z' & -w \end{bmatrix} \begin{bmatrix} x & \pm\tilde{y}' \\ y' & -\tilde{x} \end{bmatrix} = \begin{bmatrix} 1\otimes\tilde{w}x \pm \tilde{z}'y'\otimes 1 & \pm\tilde{y}'\otimes\tilde{w} \mp \tilde{z}'\otimes\tilde{x} \\ z'\otimes x - y'\otimes w & \pm z'\tilde{y}'\otimes 1 + 1\otimes w\tilde{x} \end{bmatrix}$$

from which we see that the result of squaring a Clifford vector is

$$v\,v = \begin{bmatrix} 1\otimes\tilde{x}x \pm \tilde{y}'y'\otimes 1 & 0 \\ 0 & \pm y'\tilde{y}'\otimes 1 + 1\otimes x\tilde{x} \end{bmatrix}$$

and so the represented Clifford algebra has signature $(p' + p, q' + q)$ or $(q' + p, p' + q)$, depending on the choice of $\pm$. The chiral bivector part of uv is a representative element of a spin Lie algebra, $so(p' + p, q' + q)$ or $so(q' + p, p' + q)$, which is

$$u_- v_+ \;\in\; \begin{bmatrix} (\mathbb{B}'_M\otimes 1) \oplus (1\otimes\mathbb{B}_M) & \mathbb{D}'\otimes\mathbb{D} \\ \mathbb{D}'\otimes\mathbb{D} & (\mathbb{B}'_P\otimes 1) \oplus (1\otimes\mathbb{B}_P) \end{bmatrix}$$

with the direct sum of bi-products on the diagonal, and the tensor product of the division algebras on the off diagonal. Expanding upon the previous description of Lie algebras using $su(3, \mathbb{D})$, we have a family of Lie algebras described heuristically as:

$$\begin{bmatrix} \mathbb{B}'_M \oplus \mathbb{B}_M & \mathbb{D}'_{\tilde{v}} \otimes \mathbb{D}_{\tilde{v}} & \mathbb{D}'_\psi \otimes \mathbb{D}_\psi \\ \mathbb{D}'_v \otimes \mathbb{D}_v & \mathbb{B}'_P \oplus \mathbb{B}_P & \mathbb{D}'_{\tilde{\chi}} \otimes \mathbb{D}_{\tilde{\chi}} \\ \mathbb{D}'_{\tilde{\psi}} \otimes \mathbb{D}_{\tilde{\psi}} & \mathbb{D}'_\chi \oplus \mathbb{D}_\chi & \mathbb{B}'_V \otimes \mathbb{B}_V \end{bmatrix} \;\sim\; su(3, \mathbb{D}'\otimes\mathbb{D})$$

This family is the *exceptional magic square of Lie algebras*,[23] shown in Table 5. Each member of the magic square has a canonical triality automorphism, t, constructed from the triality automorphisms of its constituent parts. Every member also has three $so(p' + p, q' + q) \sim su(2, \mathbb{D}'\otimes\mathbb{D})$ subalgebras, related by the canonical triality automorphism.

$\mathfrak{g}_{\mathbb{D}'\otimes\mathbb{D}}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$	$\mathbb{C}'$	$\mathbb{H}'$	$\mathbb{O}'$
$\mathbb{R}$	$su(2)$	$su(3)$	$sp(3)$	f_4	$sl(3)$	$sp(6,\mathbb{R})$	$f_{4(4)}$
$\mathbb{C}$	$su(3)$	$2\,su(3)$	$su(6)$	e_6	$sl(3,\mathbb{C})$	$su(3,3)$	$e_{6(2)}$
$\mathbb{H}$	$sp(3)$	$su(6)$	$so(12)$	e_7	$sl(3,\mathbb{H})$	$sp(6,\mathbb{H})$	$e_{7(-5)}$
$\mathbb{O}$	f_4	e_6	e_7	e_8	$e_{6(-26)}$	$e_{7(-25)}$	$e_{8(-24)}$
$\mathbb{C}'$	$sl(3)$	$sl(3,\mathbb{C})$	$sl(3,\mathbb{H})$	$e_{6(-26)}$	$2sl(3)$	$sl(6,\mathbb{R})$	$e_{6(6)}$
$\mathbb{H}'$	$sp(6,\mathbb{R})$	$su(3,3)$	$sp(6,\mathbb{H})$	$e_{7(-25)}$	$sl(6,\mathbb{R})$	$so(6,6)$	$e_{7(7)}$
$\mathbb{O}'$	$f_{4(4)}$	$e_{6(2)}$	$e_{7(-5)}$	$e_{8(-24)}$	$e_{6(6)}$	$e_{7(7)}$	$e_{8(8)}$

Table 5. The exceptional magic square Lie algebras, constructed from pairs of division algebras or their split versions.

The easiest way to construct the Lie brackets of any member, $\mathfrak{g}_{\mathbb{D}'\otimes\mathbb{D}}$, of the exceptional magic square is by suitably joining the Lie brackets of its constituent pairing of $su(2)$, $su(3)$, $sp(3)$, f_4, $sl(3)$, $sp(6,\mathbb{R})$, or $f_{4(4)}$ subalgebras, corresponding to its *compound triality decomposition*,

$$\mathfrak{g}_{\mathbb{D}'\otimes\mathbb{D}} = \mathrm{Tri}(\mathbb{D}') \,+\, \mathrm{Tri}(\mathbb{D}) \,+\, \mathbb{D}'_v \otimes \mathbb{D}_v \,+\, \mathbb{D}'_m \otimes \mathbb{D}_m \,+\, \mathbb{D}'_p \otimes \mathbb{D}_p$$

The root system of any magic square Lie algebra may also be easily constructed by suitably joining the roots of the constituent pairing. The triality matrix for its root system is a block diagonal matrix, t, constructed from the triality matrices of its constituents, such as

$$t = \begin{bmatrix} +1/2 & -1/2 & +1/2 & +1/2 & 0 & 0 & 0 & 0 \\ +1/2 & -1/2 & -1/2 & -1/2 & 0 & 0 & 0 & 0 \\ +1/2 & +1/2 & +1/2 & -1/2 & 0 & 0 & 0 & 0 \\ +1/2 & +1/2 & -1/2 & +1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1/2 & +1/2 & +1/2 & +1/2 \\ 0 & 0 & 0 & 0 & -1/2 & +1/2 & -1/2 & -1/2 \\ 0 & 0 & 0 & 0 & -1/2 & -1/2 & +1/2 & -1/2 \\ 0 & 0 & 0 & 0 & -1/2 & -1/2 & -1/2 & +1/2 \end{bmatrix} \tag{5.1}$$

for a real form of e_8. Although the root components of $sl(3)$, $sp(6,\mathbb{R})$, and $f_{4(4)}$ may be a mixture of imaginary and real, the canonical triality automorphism only rotates between all compact or between all noncompact Cartan generators, and the triality matrices for these Lie algebras can only be applied to collections of all real or all imaginary root components.

The main advantage of having explicit expressions for the structure of a Lie algebra and its triality automorphisms, over the description via roots and a triality matrix, is that the triality matrix alone doesn't determine the signs of the triality maps between root vectors or generators. We could employ tricks to find these signs, but it's usually easier to find them from a direct division algebra description and the corresponding explicit triality automorphism.

6 The Explicit Structure of e_6 and its Canonical Triality Automorphism

The e_6 Lie algebra has a compound triality decomposition from combining $su(3)$ and f_4,

$$e_6 = u(1) + u(1) + so(8) + (1+\bar{1})_v \otimes 8_v + (1+\bar{1})_m \otimes 8_{s-} + (1+\bar{1})_p \otimes 8_{s+} \;\sim\; su(3, \mathbb{C}\otimes\mathbb{H})$$

and a triality automorphism that maps between the triplet of complex octonions, $\mathbb{C}\otimes\mathbb{O}$. The corresponding set of e_6 basis elements is

$$\{\, T'_1,\, T'_2,\, \gamma_{ab},\, \gamma_{a'a},\, Q^-_{a'a},\, Q^+_{a'a} \,\}$$

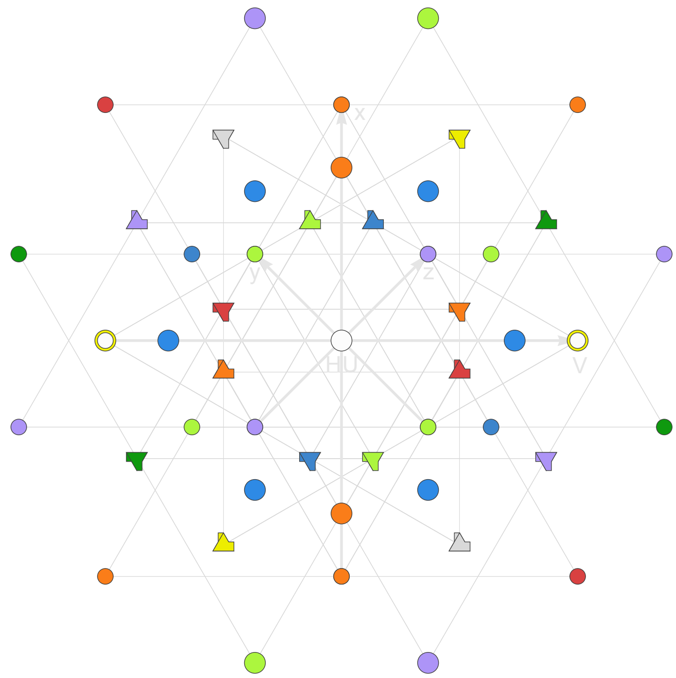
with primed index, a', ranging over complex indices, $\{0,1\}$, un-primed index, a, ranging over octonion indices, $\{0,...,7\}$, and the bivector index, ab, ranging over the 28 $so(8)$ basis generator permutations with $a<b$. The non-vanishing Lie algebra brackets between these basis elements come from combining the Lie brackets of $su(3)$ and $f_{4(-52)}$, (4.2) and (4.8):

$$\begin{array}{lll}
[\gamma_{ab},\gamma_{cd}] = 2\,\{n_{ac}\gamma_{bd} - n_{ad}\gamma_{bc} - n_{bc}\gamma_{ad} + n_{bd}\gamma_{ac}\} & & \\
[T'_1,\gamma_{0'a}] = -2\,\gamma_{1'a} & [T'_1,Q^-_{0'a}] = +Q^-_{1'a} & [T'_1,Q^+_{0'a}] = +Q^+_{1'a} \\
[T'_1,\gamma_{1'a}] = +2\,\gamma_{0'a} & [T'_1,Q^-_{1'a}] = -Q^-_{0'a} & [T'_1,Q^+_{1'a}] = -Q^+_{0'a} \\
 & [T'_2,Q^-_{0'a}] = +\sqrt{3}\,Q^-_{1'a} & [T'_2,Q^+_{0'a}] = -\sqrt{3}\,Q^+_{1'a} \\
 & [T'_2,Q^-_{1'a}] = -\sqrt{3}\,Q^-_{0'a} & [T'_2,Q^+_{1'a}] = +\sqrt{3}\,Q^+_{0'a} \\
[\gamma_{ab},\gamma_{a'c}] = 2\,\{-n_{bc}\gamma_{a'a} + n_{ac}\gamma_{a'b}\} & [\gamma_{ab},Q^-_{a'c}] = Q^-_{a'd}(-\bar{\Gamma}_a\Gamma_b)^d{}_c & [\gamma_{ab},Q^+_{a'c}] = Q^+_{a'd}(-\Gamma_a\bar{\Gamma}_b)^d{}_c \\
[\gamma_{0'a},\gamma_{1'b}] = 2\,T'_1 n_{ab} & [Q^-_{0'a},Q^-_{1'b}] = -(T_1+\sqrt{3}\,T_2)n^-_{ab} & [Q^+_{0'a},Q^+_{1'b}] = -(T_1-\sqrt{3}\,T_2)n^+_{ab} \\
[\gamma_{a'a},\gamma_{b'b}] = -2\,n'_{a'b'}\gamma_{ab} & [Q^-_{a'a},Q^-_{b'b}] = n'^-_{a'b'}\gamma_{cd}(\bar{\Gamma}^c\Gamma^d)_{ab} & [Q^+_{a'a},Q^+_{b'b}] = n'^+_{a'b'}\gamma_{cd}(\Gamma^c\bar{\Gamma}^d)_{ab} \\
[\gamma_{a'a},Q^-_{b'b}] = -M'_{\tilde{a}'\tilde{b}'}{}^{c'}(\Gamma_a)^c{}_b Q^+_{c'c} & [\gamma_{a'a},Q^+_{b'b}] = M'_{\tilde{a}'\tilde{b}'}{}^{c'}(\bar{\Gamma}_a)^c{}_b Q^-_{c'c} & [Q^-_{a'a},Q^+_{b'b}] = M'_{\tilde{a}'\tilde{b}'}{}^{c'}(\bar{\Gamma}^c)_{ab}\gamma_{c'c}
\end{array}$$

in which $M'_{\tilde{a}'\tilde{b}'}{}^{c'}$ and $(\Gamma_c)^b{}_a = M_{ca}{}^{\tilde{b}}$ are the complex and octonion multiplication tables with conjugations, and $\{n', n'^{\pm}, n, n^{\pm}\}$ are the complex and octonion metrics, also used to raise or lower indices. Different real forms of e_6 come from combining different real forms of $su(3)$ and f_4, using correspondingly different multiplication tables and metrics. The canonical triality automorphism of $e_{6(-78)}$ is

$$\begin{array}{llll}
t \;:\; & T'_1 \mapsto T''_1 = -\frac{1}{2}T'_1 - \frac{\sqrt{3}}{2}T'_2 & T'_2 \mapsto T''_2 = \frac{\sqrt{3}}{2}T'_1 - \frac{1}{2}T'_2 & \gamma_{ab} \mapsto \gamma'_{ab} = \gamma_{cd}t^{cd}{}_{ab} = \gamma_{cd}\left(-\frac{1}{2}\bar{\Gamma}^c\Gamma^d\right)_{ab} \\
 & \gamma_{a'a} \mapsto \gamma'_{a'a} = Q^-_{a'a} & Q^-_{a'a} \mapsto Q'^-_{a'a} = Q^+_{a'a} & Q^+_{a'a} \mapsto Q'^+_{a'a} = \gamma_{a'a}
\end{array}$$

from combining the triality automorphisms of the constituent $su(3)$ and $f_{4(-52)}$. For non-compact real forms of e_6, real triality automorphisms may contain explicit i's in them. It is illustrative to show the e_6 roots and canonical triality automorphism, with a triality matrix from combining triality matrices of $su(3)$ and f_4:



e_6		H	U	V	x	y	z
	g	0	0	0	$(+1$	-1	$0\,)$
	$X^{(rgb)}_{2/3}$	0	0	0	$(\pm 1$	± 1	$0\,)$
	$X^{(rgb)}_{2/3}$	0	0	± 1	$(\mp 1$	0	$0\,)$
	$X^{(rgb)}_{4/3}$	0	0	± 1	$(\pm 1$	0	$0\,)$
	$W^{\pm}$	0	∓ 1	± 1	0	0	0
	$W'^{\pm}$	0	± 1	∓ 1	0	0	0
	$X^{(rgb)}_{1/3}$	0	± 1	0	$(\pm 1$	0	$0\,)$
	$X^{(rgb)}_{1/3}$	0	± 1	0	$(\mp 1$	0	$0\,)$
	ν_L	$+\sqrt{3}/2$	$-1/2$	$+1/2$	$-1/2$	$-1/2$	$-1/2$
	ν_R	$-\sqrt{3}/2$	$+1/2$	$+1/2$	$-1/2$	$-1/2$	$-1/2$
	$\bar{\nu}_L$	$+\sqrt{3}/2$	$-1/2$	$-1/2$	$+1/2$	$+1/2$	$+1/2$
	$\bar{\nu}_R$	$-\sqrt{3}/2$	$+1/2$	$-1/2$	$+1/2$	$+1/2$	$+1/2$
	$u^{(rgb)}_L$	$+\sqrt{3}/2$	$-1/2$	$+1/2$	$(-1/2$	$+1/2$	$+1/2\,)$
	$u^{(rgb)}_R$	$-\sqrt{3}/2$	$+1/2$	$+1/2$	$(-1/2$	$+1/2$	$+1/2\,)$
	$\bar{u}^{(rgb)}_L$	$+\sqrt{3}/2$	$-1/2$	$-1/2$	$(+1/2$	$-1/2$	$-1/2\,)$
	$\bar{u}^{(rgb)}_R$	$-\sqrt{3}/2$	$+1/2$	$-1/2$	$(+1/2$	$-1/2$	$-1/2\,)$
	e_L	$+\sqrt{3}/2$	$+1/2$	$-1/2$	$-1/2$	$-1/2$	$-1/2$
	e_R	$-\sqrt{3}/2$	$-1/2$	$-1/2$	$-1/2$	$-1/2$	$-1/2$
	$\bar{e}_L$	$+\sqrt{3}/2$	$+1/2$	$+1/2$	$+1/2$	$+1/2$	$+1/2$
	$\bar{e}_R$	$-\sqrt{3}/2$	$-1/2$	$+1/2$	$+1/2$	$+1/2$	$+1/2$
	$d^{(rgb)}_L$	$+\sqrt{3}/2$	$+1/2$	$-1/2$	$(-1/2$	$+1/2$	$+1/2\,)$
	$d^{(rgb)}_R$	$-\sqrt{3}/2$	$-1/2$	$-1/2$	$(-1/2$	$+1/2$	$+1/2\,)$
	$\bar{d}^{(rgb)}_L$	$+\sqrt{3}/2$	$+1/2$	$+1/2$	$(+1/2$	$-1/2$	$-1/2\,)$
	$\bar{d}^{(rgb)}_R$	$-\sqrt{3}/2$	$-1/2$	$+1/2$	$(+1/2$	$-1/2$	$-1/2\,)$

Table 6. The 72 roots of e_6, from combining $su(3)$ and f_4, labeled as elementary particles.

7 The Explicit Structure of e_7 and its Canonical Triality Automorphism

The e_7 Lie algebra has a compound triality decomposition from combining $sp(3)$ and f_4,

$$\begin{aligned}
e_7 &= su(2)_M + su(2)_P + su(2)_V + so(8) + (2,2,1)_v \otimes 8_v + (2,1,2)_m \otimes 8_{s-} + (1,2,2)_p \otimes 8_{s+} \\
&\sim su(3, \mathbb{H} \otimes \mathbb{O})
\end{aligned}$$

and a triality automorphism that maps between the triplet of quaterni-octonions, $\mathbb{H} \otimes \mathbb{O}$. The corresponding set of e_7 basis elements is

$$\{ T^M_{A'},\, T^P_{A'},\, T^V_{A'},\, \gamma_{ab},\, \gamma_{a'a},\, Q^-_{a'a},\, Q^+_{a'a} \}$$

with primed index, a', ranging over quaternion indices, $\{0, ..., 3\}$, primed capitals, A', ranging over imaginary quaternion indices, $\{1, ..., 3\}$, un-primed index, a, ranging over octonion indices, $\{0, ..., 7\}$, and the bivector index, ab, ranging over the 28 $so(8)$ basis generator permutations with $a < b$. The non-vanishing Lie algebra brackets between these basis elements come from combining the Lie brackets of $sp(3)$ and $f_{4(-52)}$, (4.5) and (4.8):

$$\begin{array}{lll}
[\gamma_{ab}, \gamma_{cd}] = 2\{n_{ac}\gamma_{bd} - n_{ad}\gamma_{bc} - n_{bc}\gamma_{ad} + n_{bd}\gamma_{ac}\} & & \\
[T^M_{A'}, T^M_{B'}] = T^M_{C'}(2M'_{[A'B']}{}^{C'}) & [T^P_{A'}, T^P_{B'}] = T^P_{C'}(2M'_{[A'B']}{}^{C'}) & [T^V_{A'}, T^V_{B'}] = T^V_{C'}(2M'_{[A'B']}{}^{C'}) \\
[T^M_{A'}, \gamma_{b'a}] = \gamma_{c'a}(-M'_{b'A'}{}^{c'}) & [T^P_{A'}, \gamma_{b'a}] = \gamma_{c'a}(M'_{A'b'}{}^{c'}) & [T^V_{A'}, Q^-_{b'a}] = Q^-_{c'a}(-M'_{b'A'}{}^{c'}) \\
[T^M_{A'}, Q^-_{b'a}] = Q^-_{c'a}(M'_{A'b'}{}^{c'}) & [T^P_{A'}, Q^+_{b'a}] = Q^+_{c'a}(-M'_{b'A'}{}^{c'}) & [T^V_{A'}, Q^+_{b'a}] = Q^+_{c'a}(M'_{A'b'}{}^{c'}) \\
[\gamma_{ab}, \gamma_{a'c}] = 2\{-n_{bc}\gamma_{a'a} + n_{ac}\gamma_{a'b}\} & [\gamma_{ab}, Q^-_{a'c}] = Q^-_{a'd}(-\overline{\Gamma}_a\Gamma_b)^d{}_c & [\gamma_{ab}, Q^+_{a'c}] = Q^+_{a'd}(-\Gamma_a\overline{\Gamma}_b)^d{}_c
\end{array}$$

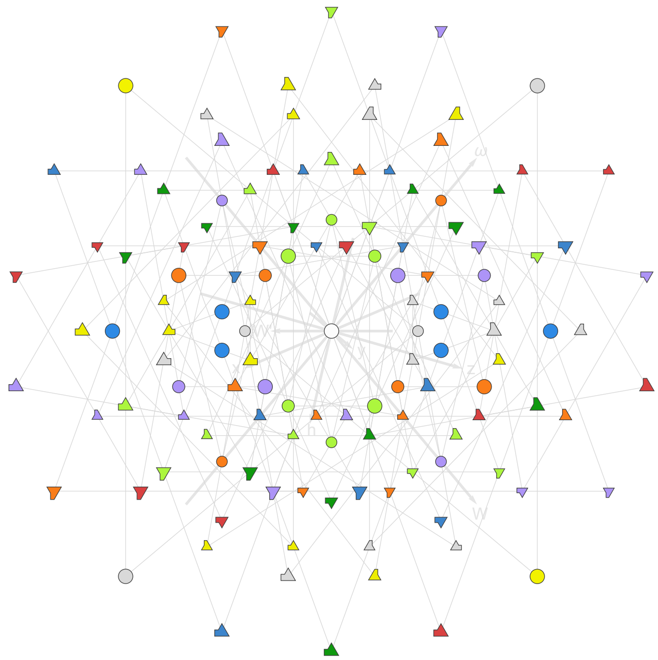
$$\begin{aligned}
[\gamma_{a'a}, \gamma_{b'b}] &= -\left(T^M_{C'}\tfrac{1}{2}(M'_{\tilde{b}'a'}{}^{C'} - M'_{\tilde{a}'b'}{}^{C'}) + T^P_{C'}\tfrac{1}{2}(M'_{b'\tilde{a}'}{}^{C'} - M'_{a'\tilde{b}'}{}^{C'})\right) n_{ab} - 2\,n'_{a'b'}\gamma_{ab} \\
[Q^-_{a'a}, Q^-_{b'b}] &= -\left(T^V_{C'}\tfrac{1}{2}(M'_{\tilde{b}'a'}{}^{C'} - M'_{\tilde{a}'b'}{}^{C'}) + T^M_{C'}\tfrac{1}{2}(M'_{b'\tilde{a}'}{}^{C'} - M'_{a'\tilde{b}'}{}^{C'})\right) n^-_{ab} + n'^-_{a'b'}\gamma_{cd}(\overline{\Gamma}^c\Gamma^d)_{ab} \\
[Q^+_{a'a}, Q^+_{b'b}] &= -\left(T^P_{C'}\tfrac{1}{2}(M'_{\tilde{b}'a'}{}^{C'} - M'_{\tilde{a}'b'}{}^{C'}) + T^V_{C'}\tfrac{1}{2}(M'_{b'\tilde{a}'}{}^{C'} - M'_{a'\tilde{b}'}{}^{C'})\right) n^+_{ab} + n'^+_{a'b'}\gamma_{cd}(\Gamma^c\overline{\Gamma}^d)_{ab}
\end{aligned}$$

$$[\gamma_{a'a}, Q^-_{b'b}] = -\tfrac{1}{\sqrt{2}}M'_{\tilde{a}'\tilde{b}'}{}^{c'}(\Gamma_a)^c{}_b Q^+_{c'c} \quad [\gamma_{a'a}, Q^+_{b'b}] = \tfrac{1}{\sqrt{2}}M'_{\tilde{a}'\tilde{b}'}{}^{c'}(\overline{\Gamma}_a)^c{}_b Q^-_{c'c} \quad [Q^-_{a'a}, Q^+_{b'b}] = \tfrac{1}{\sqrt{2}}M'_{\tilde{a}'\tilde{b}'}{}^{c'}(\overline{\Gamma}^c)_{ab}\gamma_{c'c}$$

in which $M'_{\tilde{a}'\tilde{b}'}{}^{c'}$ and $(\Gamma_c)^b{}_a = M_{ca}{}^{\tilde{b}}$ are the quaternion and octonion multiplication tables with conjugations, and $\{n', n'^{\pm}, n, n^{\pm}\}$ are the quaternion and octonion metrics, also used to raise or lower indices. Different real forms of e_7 come from combining different real forms of $sp(3)$ and f_4, using correspondingly different multiplication tables and metrics. The canonical triality automorphism of $e_{7(-133)}$ is

$$\begin{array}{llll}
t\ :\ & \gamma_{ab} \mapsto \gamma'_{ab} = \gamma_{cd}t^{cd}{}_{ab} = \gamma_{cd}\left(-\tfrac{1}{2}\bar{\Gamma}^c\Gamma^d\right)_{ab} & & \\
& T^M_{A'} \mapsto T'^M_{A'} = T^V_{A'} & T^V_{A'} \mapsto T'^V_{A'} = T^P_{A'} & T^P_{A'} \mapsto T'^P_{A'} = T^M_{A'} \\
& \gamma_{a'a} \mapsto \gamma'_{a'a} = Q^-_{a'a} & Q^-_{a'a} \mapsto Q'^-_{a'a} = Q^+_{a'a} & Q^+_{a'a} \mapsto Q'^+_{a'a} = \gamma_{a'a}
\end{array}$$

from combining the triality automorphisms of the constituent $sp(3)$ and $f_{4(-52)}$. For non-compact real forms of e_7, real triality automorphisms may contain explicit i's in them. It is illustrative to show the e_7 roots and canonical triality automorphism, with a triality matrix from combining the triality matrices of $sp(3)$ and f_4:



e_7		ω	W	W'	h	x	y	z
	$\omega^{\wedge/\vee}$	$\pm\sqrt{2}$	0	0	0	0	0	0
	$W_\pm$	0	$\pm\sqrt{2}$	0	0	0	0	0
	$W'_\pm$	0	0	$\pm\sqrt{2}$	0	0	0	0
	g	0	0	0	0	$(+1$	-1	$0)$
	$X_I^{rgb}\ \bar{X}_I^{rgb}$	0	0	0	0	$(\pm 1$	± 1	$0)$
	$X_{II}^{rgb}\ \bar{X}_{II}^{rgb}$	0	0	0	± 1	$(\mp 1$	0	$0)$
	$X_{III}^{rgb}\ \bar{X}_{III}^{rgb}$	0	0	0	∓ 1	$(\mp 1$	0	$0)$
	$\nu_{eL}^{\wedge/\vee}$	$\pm 1/\sqrt{2}$	$+1/\sqrt{2}$	0	$+1/2$	$-1/2$	$-1/2$	$-1/2$
	$\nu_{eR}^{\wedge/\vee}$	$\pm 1/\sqrt{2}$	0	$+1/\sqrt{2}$	$-1/2$	$-1/2$	$-1/2$	$-1/2$
	$e_L^{\wedge/\vee}$	$\pm 1/\sqrt{2}$	$-1/\sqrt{2}$	0	$+1/2$	$-1/2$	$-1/2$	$-1/2$
	$e_R^{\wedge/\vee}$	$\pm 1/\sqrt{2}$	0	$-1/\sqrt{2}$	$-1/2$	$-1/2$	$-1/2$	$-1/2$
	$u_L^{(rgb)\wedge/\vee}$	$\pm 1/\sqrt{2}$	$+1/\sqrt{2}$	0	$+1/2$	$(-1/2$	$+1/2$	$+1/2)$
	$\bar{u}_L^{(rgb)\wedge/\vee}$	$\pm 1/\sqrt{2}$	0	$-1/\sqrt{2}$	$+1/2$	$(+1/2$	$-1/2$	$-1/2)$
	$d_L^{(rgb)\wedge/\vee}$	$\pm 1/\sqrt{2}$	$-1/\sqrt{2}$	0	$+1/2$	$(-1/2$	$+1/2$	$+1/2)$
	$\bar{d}_L^{(rgb)\wedge/\vee}$	$\pm 1/\sqrt{2}$	0	$+1/\sqrt{2}$	$+1/2$	$(+1/2$	$-1/2$	$-1/2)$
	$\nu_{\mu L}^{\wedge/\vee}$	0	$\pm 1/\sqrt{2}$	$+1/\sqrt{2}$	-1	0	0	0
	$\nu_{\mu R}^{\wedge/\vee}$	$+1/\sqrt{2}$	$\pm 1/\sqrt{2}$	0	$-1/2$	$+1/2$	$+1/2$	$+1/2$
	$\mu_L^{\wedge/\vee}$	0	$\pm 1/\sqrt{2}$	$-1/\sqrt{2}$	-1	0	0	0
	$\mu_R^{\wedge/\vee}$	$-1/\sqrt{2}$	$\pm 1/\sqrt{2}$	0	$-1/2$	$+1/2$	$+1/2$	$+1/2$
	$c_L^{(rgb)\wedge/\vee}$	0	$\pm 1/\sqrt{2}$	$+1/\sqrt{2}$	0	$(-1$	0	$0)$
	$\bar{c}_L^{(rgb)\wedge/\vee}$	$-1/\sqrt{2}$	$\pm 1/\sqrt{2}$	0	$-1/2$	$(+1/2$	$-1/2$	$-1/2)$
	$s_L^{(rgb)\wedge/\vee}$	0	$\pm 1/\sqrt{2}$	$-1/\sqrt{2}$	0	$(-1$	0	$0)$
	$\bar{s}_L^{(rgb)\wedge/\vee}$	$+1/\sqrt{2}$	$\pm 1/\sqrt{2}$	0	$-1/2$	$(+1/2$	$-1/2$	$-1/2)$
	$\nu_{\tau L}^{\wedge/\vee}$	$+1/\sqrt{2}$	0	$\pm 1/\sqrt{2}$	$+1/2$	$+1/2$	$+1/2$	$+1/2$
	$\nu_{\tau R}^{\wedge/\vee}$	0	$+1/\sqrt{2}$	$\pm 1/\sqrt{2}$	$+1$	0	0	0
	$\tau_L^{\wedge/\vee}$	$-1/\sqrt{2}$	0	$\pm 1/\sqrt{2}$	$+1/2$	$+1/2$	$+1/2$	$+1/2$
	$\tau_R^{\wedge/\vee}$	0	$-1/\sqrt{2}$	$\pm 1/\sqrt{2}$	$+1$	0	0	0
	$t_L^{(rgb)\wedge/\vee}$	$+1/\sqrt{2}$	0	$\pm 1/\sqrt{2}$	$-1/2$	$(-1/2$	$+1/2$	$+1/2)$
	$\bar{t}_L^{(rgb)\wedge/\vee}$	0	$-1/\sqrt{2}$	$\pm 1/\sqrt{2}$	0	$(+1$	0	$0)$
	$b_L^{(rgb)\wedge/\vee}$	$-1/\sqrt{2}$	0	$\pm 1/\sqrt{2}$	$-1/2$	$(-1/2$	$+1/2$	$+1/2)$
	$\bar{b}_L^{(rgb)\wedge/\vee}$	0	$+1/\sqrt{2}$	$\pm 1/\sqrt{2}$	0	$(+1$	0	$0)$

Table 7. The 126 roots of e_7, from combining $sp(3)$ and f_4, labeled as elementary particles.

8 The Explicit Structure of e_8 and its Canonical Triality Automorphism

The e_8 Lie algebra has a compound triality decomposition from combining two f_4's,

$$\begin{aligned} e_8 &= so(8)' + so(8) + 8'_v \otimes 8_v + 8'_{s-} \otimes 8_{s-} + 8'_{s+} \otimes 8_{s+} \\ &\sim su(3, \mathbb{O} \otimes \mathbb{O}) \end{aligned}$$

and a triality automorphism that maps between the triplet of octo-octonions, $\mathbb{O} \otimes \mathbb{O}$. The corresponding set of e_8 basis elements is

$$\{ \gamma'_{ab}, \gamma_{ab}, \gamma_{a'a}, Q^-_{a'a}, Q^+_{a'a} \}$$

with primed and unprimed indices, a' and a, ranging over octonion indices, $\{0, ..., 7\}$, and the bivector indices, $a'b'$ and ab, ranging over the 28 $so(8)$ basis generator permutations with $a < b$. The non-vanishing Lie algebra brackets between these basis elements come from combining the Lie brackets of two $f_{4(-52)}$'s, (4.8):

$$\begin{aligned}
[\gamma_{a'b'}, \gamma_{c'd'}] &= 2\{n'_{a'c'}\gamma_{b'd'} - n'_{a'd'}\gamma_{b'c'} - n'_{b'c'}\gamma_{a'd'} + n'_{b'd'}\gamma_{a'c'}\} \\
[\gamma_{ab}, \gamma_{cd}] &= 2\{n_{ac}\gamma_{bd} - n_{ad}\gamma_{bc} - n_{bc}\gamma_{ad} + n_{bd}\gamma_{ac}\} \\
[\gamma_{a'b'}, \gamma_{c'a}] &= 2\{-n'_{b'c'}\gamma_{a'a} + n'_{a'c'}\gamma_{b'a}\} \quad [\gamma_{a'b'}, Q^-_{c'a}] = Q^-_{d'a}(-\overline{\Gamma}_{a'}\Gamma_{b'})^{d'}{}_{c'} \quad [\gamma_{a'b'}, Q^+_{c'a}] = Q^+_{d'a}(-\Gamma_{a'}\overline{\Gamma}_{b'})^{d'}{}_{c'} \\
[\gamma_{ab}, \gamma_{a'c}] &= 2\{-n_{bc}\gamma_{a'a} + n_{ac}\gamma_{a'b}\} \quad [\gamma_{ab}, Q^-_{a'c}] = Q^-_{a'd}(-\overline{\Gamma}_a\Gamma_b)^d{}_c \quad [\gamma_{ab}, Q^+_{a'c}] = Q^+_{a'd}(-\Gamma_a\overline{\Gamma}_b)^d{}_c \\
[\gamma_{a'a}, \gamma_{b'b}] &= -2\, n_{ab}\gamma_{a'b'} - 2\, n'_{a'b'}\gamma_{ab} \\
[Q^-_{a'a}, Q^-_{b'b}] &= n^-_{ab}\gamma_{c'd'}(\overline{\Gamma}^{c'}\Gamma^{d'})_{a'b'} + n'^-_{a'b'}\gamma_{cd}(\overline{\Gamma}^c\Gamma^d)_{ab} \\
[Q^+_{a'a}, Q^+_{b'b}] &= n^+_{ab}\gamma_{c'd'}(\Gamma^{c'}\overline{\Gamma}^{d'})_{a'b'} + n'^+_{a'b'}\gamma_{cd}(\Gamma^c\overline{\Gamma}^d)_{ab} \\
[\gamma_{a'a}, Q^-_{b'b}] &= (\Gamma_{a'})^{c'}{}_{b'}(\Gamma_a)^c{}_b Q^+_{c'c} \quad [\gamma_{a'a}, Q^+_{b'b}] = (\bar{\Gamma}_{a'})^{c'}{}_{b'} Q^-_{c'c} \quad [Q^-_{a'a}, Q^+_{b'b}] = (\overline{\Gamma}^{c'})_{a'b'}(\overline{\Gamma}^c)_{ab}\gamma_{c'c}
\end{aligned}$$

in which $(\Gamma_{c'})^{b'}{}_{a'} = M'_{c'a'}{}^{\tilde{b}'}$ and $(\Gamma_c)^b{}_a = M_{ca}{}^{\tilde{b}}$ are the octonion multiplication tables with conjugations, and $\{n', n'^{\pm}, n, n^{\pm}\}$ are the octonion metrics, also used to raise or lower indices. Different real forms of e_8 come from using one or two copies of $f_{4(4)}$, incorporating split-octonion multiplication tables and metrics, or using two $f_{4(-52)}$'s and flipping the signs to get split real $e_{8(8)}$.

The canonical triality automorphism of compact real $e_{8(-248)}$ is

$$\begin{aligned}
t \ : \ & \gamma_{ab} \mapsto \gamma'_{ab} = \gamma_{cd} t^{cd}{}_{ab} = \gamma_{cd}\left(-\tfrac{1}{2}\bar{\Gamma}^c\Gamma^d\right)_{ab} \\
& \gamma_{a'b'} \mapsto \gamma'_{a'b'} = \gamma_{c'd'} t^{c'd'}{}_{a'b'} = \gamma_{c'd'}\left(-\tfrac{1}{2}\bar{\Gamma}^{c'}\Gamma^{d'}\right)_{a'b'} \\
& \gamma_{a'a} \mapsto \gamma'_{a'a} = Q^-_{a'a} \qquad Q^-_{a'a} \mapsto Q'^-_{a'a} = Q^+_{a'a} \qquad Q^+_{a'a} \mapsto Q'^+_{a'a} = \gamma_{a'a}
\end{aligned}$$

from combining the triality automorphisms of the constituent $f_{4(-52)}$'s. For quaternionic $e_{8(-24)}$ or split real $e_{8(8)}$, real triality automorphisms—obtained by mixing triality automorphisms of f_4's—may contain explicit i's. It is illustrative to show the e_8 roots and canonical triality automorphism, with a triality matrix (5.1) from combining the triality matrices of two f_4's:

e_8		ω_T	ω_s	U	V	p	x	y	z
	$\omega_I^{\wedge/\vee}$	∓ 1	± 1	0	0	0	0	0	0
	$\omega_{II}^{\wedge/\vee}$	∓ 1	∓ 1	0	0	0	0	0	0
	$\omega_{III}^{\wedge/\vee}$	0	0	∓ 1	∓ 1	0	0	0	0
	$W_\pm$	0	0	∓ 1	± 1	0	0	0	0
	$e_T^{\wedge/\vee}\ \phi_I^{/*}$	± 1	0	∓ 1	0	0	0	0	0
	$e_T^{\wedge/\vee}\ \phi_{II}^{/*}$	0	± 1	0	± 1	0	0	0	0
	$e_T^{\wedge/\vee}\ \phi_{III}^{/*}$	0	∓ 1	0	± 1	0	0	0	0
	$e_s^{\wedge/\vee}\ \phi_I^{*/}$	0	± 1	± 1	0	0	0	0	0
	$e_s^{\wedge/\vee}\ \phi_{II}^{*/}$	0	∓ 1	± 1	0	0	0	0	0
	$e_s^{\wedge/\vee}\ \phi_{III}^{*/}$	± 1	0	0	∓ 1	0	0	0	0
	$e_T^{\wedge/\vee}\ \phi^{*/}$	± 1	0	± 1	0	0	0	0	0
	$e_T^{\wedge/\vee}\ \phi_\pm$	± 1	0	0	± 1	0	0	0	0
	g	0	0	0	0	0	$(+1$	-1	$0\,)$
	$X_I^{rgb}\ \bar{X}_I^{rgb}$	0	0	0	0	0	$(\pm 1$	± 1	$0\,)$
	$X_{II}^{rgb}\ \bar{X}_{II}^{rgb}$	0	0	0	0	± 1	$(\mp 1$	0	$0\,)$
	$X_{III}^{rgb}\ \bar{X}_{III}^{rgb}$	0	0	0	0	∓ 1	$(\mp 1$	0	$0\,)$
	$\nu_{eL}^{\wedge/\vee}$	$\mp 1/2$	$\pm 1/2$	$-1/2$	$+1/2$	$-1/2$	$-1/2$	$-1/2$	$-1/2$
	$\nu_{eR}^{\wedge/\vee}$	$\pm 1/2$	$\pm 1/2$	$+1/2$	$+1/2$	$-1/2$	$-1/2$	$-1/2$	$-1/2$
	$\bar{\nu}_{eL}^{\wedge/\vee}$	$\pm 1/2$	$\pm 1/2$	$-1/2$	$-1/2$	$+1/2$	$+1/2$	$+1/2$	$+1/2$
	$\bar{\nu}_{eR}^{\wedge/\vee}$	$\mp 1/2$	$\pm 1/2$	$+1/2$	$-1/2$	$+1/2$	$+1/2$	$+1/2$	$+1/2$
	$e_L^{\wedge/\vee}$	$\mp 1/2$	$\pm 1/2$	$+1/2$	$-1/2$	$-1/2$	$-1/2$	$-1/2$	$-1/2$
	$e_R^{\wedge/\vee}$	$\pm 1/2$	$\pm 1/2$	$-1/2$	$-1/2$	$-1/2$	$-1/2$	$-1/2$	$-1/2$
	$\bar{e}_L^{\wedge/\vee}$	$\pm 1/2$	$\pm 1/2$	$+1/2$	$+1/2$	$+1/2$	$+1/2$	$+1/2$	$+1/2$
	$\bar{e}_R^{\wedge/\vee}$	$\mp 1/2$	$\pm 1/2$	$-1/2$	$+1/2$	$+1/2$	$+1/2$	$+1/2$	$+1/2$
	$u_L^{(rgb)\wedge/\vee}$	$\mp 1/2$	$\pm 1/2$	$-1/2$	$+1/2$	$-1/2$	$(-1/2$	$+1/2$	$+1/2\,)$
	$u_R^{(rgb)\wedge/\vee}$	$\pm 1/2$	$\pm 1/2$	$+1/2$	$+1/2$	$-1/2$	$(-1/2$	$+1/2$	$+1/2\,)$
	$\bar{u}_L^{(rgb)\wedge/\vee}$	$\pm 1/2$	$\pm 1/2$	$-1/2$	$-1/2$	$+1/2$	$(+1/2$	$-1/2$	$-1/2\,)$
	$\bar{u}_R^{(rgb)\wedge/\vee}$	$\mp 1/2$	$\pm 1/2$	$+1/2$	$-1/2$	$+1/2$	$(+1/2$	$-1/2$	$-1/2\,)$
	$d_L^{(rgb)\wedge/\vee}$	$\mp 1/2$	$\pm 1/2$	$+1/2$	$-1/2$	$-1/2$	$(-1/2$	$+1/2$	$+1/2\,)$
	$d_R^{(rgb)\wedge/\vee}$	$\pm 1/2$	$\pm 1/2$	$-1/2$	$-1/2$	$-1/2$	$(-1/2$	$+1/2$	$+1/2\,)$
	$\bar{d}_L^{(rgb)\wedge/\vee}$	$\pm 1/2$	$\pm 1/2$	$+1/2$	$+1/2$	$+1/2$	$(+1/2$	$-1/2$	$-1/2\,)$
	$\bar{d}_R^{(rgb)\wedge/\vee}$	$\mp 1/2$	$\pm 1/2$	$-1/2$	$+1/2$	$+1/2$	$(+1/2$	$-1/2$	$-1/2\,)$
	$\nu_{\mu L}^{\wedge/\vee}$	$\mp 1/2$	$\mp 1/2$	$-1/2$	$+1/2$	$-1/2$	$+1/2$	$+1/2$	$+1/2$
	$\nu_{\mu R}^{\wedge/\vee}$	$+1/2$	$-1/2$	$\pm 1/2$	$\pm 1/2$	$-1/2$	$+1/2$	$+1/2$	$+1/2$
	$\bar{\nu}_{\mu L}^{\wedge/\vee}$	$-1/2$	$+1/2$	$\pm 1/2$	$\pm 1/2$	$+1/2$	$-1/2$	$-1/2$	$-1/2$
	$\bar{\nu}_{\mu R}^{\wedge/\vee}$	$\mp 1/2$	$\mp 1/2$	$+1/2$	$-1/2$	$+1/2$	$-1/2$	$-1/2$	$-1/2$
	$\mu_L^{\wedge/\vee}$	$\mp 1/2$	$\mp 1/2$	$+1/2$	$-1/2$	$-1/2$	$+1/2$	$+1/2$	$+1/2$
	$\mu_R^{\wedge/\vee}$	$-1/2$	$+1/2$	$\pm 1/2$	$\pm 1/2$	$-1/2$	$+1/2$	$+1/2$	$+1/2$
	$\bar{\mu}_L^{\wedge/\vee}$	$+1/2$	$-1/2$	$\pm 1/2$	$\pm 1/2$	$+1/2$	$-1/2$	$-1/2$	$-1/2$
	$\bar{\mu}_R^{\wedge/\vee}$	$\mp 1/2$	$\mp 1/2$	$-1/2$	$+1/2$	$+1/2$	$-1/2$	$-1/2$	$-1/2$
	$c_L^{(rgb)\wedge/\vee}$	$\mp 1/2$	$\mp 1/2$	$-1/2$	$+1/2$	$+1/2$	$(-1/2$	$+1/2$	$+1/2\,)$
	$c_R^{(rgb)\wedge/\vee}$	$+1/2$	$-1/2$	$\pm 1/2$	$\pm 1/2$	$+1/2$	$(-1/2$	$+1/2$	$+1/2\,)$
	$\bar{c}_L^{(rgb)\wedge/\vee}$	$-1/2$	$+1/2$	$\pm 1/2$	$\pm 1/2$	$-1/2$	$(+1/2$	$-1/2$	$-1/2\,)$
	$\bar{c}_R^{(rgb)\wedge/\vee}$	$\mp 1/2$	$\mp 1/2$	$+1/2$	$-1/2$	$-1/2$	$(+1/2$	$-1/2$	$-1/2\,)$
	$s_L^{(rgb)\wedge/\vee}$	$\mp 1/2$	$\mp 1/2$	$+1/2$	$-1/2$	$+1/2$	$(-1/2$	$+1/2$	$+1/2\,)$
	$s_R^{(rgb)\wedge/\vee}$	$-1/2$	$+1/2$	$\pm 1/2$	$\pm 1/2$	$+1/2$	$(-1/2$	$+1/2$	$+1/2\,)$
	$\bar{s}_L^{(rgb)\wedge/\vee}$	$+1/2$	$-1/2$	$\pm 1/2$	$\pm 1/2$	$-1/2$	$(+1/2$	$-1/2$	$-1/2\,)$
	$\bar{s}_R^{(rgb)\wedge/\vee}$	$\mp 1/2$	$\mp 1/2$	$-1/2$	$+1/2$	$-1/2$	$(+1/2$	$-1/2$	$-1/2\,)$
	$\nu_{\tau L}^{\wedge}\ \bar{\nu}_{\tau R}^{\vee}$	0	0	∓ 1	0	± 1	0	0	0
	$\nu_{\tau L}^{\vee}\ \bar{\nu}_{\tau R}^{\wedge}$	0	0	0	± 1	± 1	0	0	0
	$\nu_{\tau R}^{\wedge}\ \bar{\nu}_{\tau L}^{\vee}$	± 1	0	0	0	± 1	0	0	0
	$\nu_{\tau R}^{\vee}\ \bar{\nu}_{\tau L}^{\wedge}$	0	± 1	0	0	± 1	0	0	0
	$\tau_L^{\wedge}\ \bar{\tau}_R^{\vee}$	0	0	0	∓ 1	± 1	0	0	0
	$\tau_L^{\vee}\ \bar{\tau}_R^{\wedge}$	0	0	± 1	0	± 1	0	0	0
	$\tau_R^{\wedge}\ \bar{\tau}_L^{\vee}$	0	∓ 1	0	0	± 1	0	0	0
	$\tau_R^{\vee}\ \bar{\tau}_L^{\wedge}$	∓ 1	0	0	0	± 1	0	0	0
	$t_L^{(rgb)\wedge}\ \bar{t}_R^{(rgb)\vee}$	0	0	∓ 1	0	0	$(\mp 1$	0	$0\,)$
	$t_L^{(rgb)\vee}\ \bar{t}_R^{(rgb)\wedge}$	0	0	0	± 1	0	$(\mp 1$	0	$0\,)$
	$t_R^{(rgb)\wedge}\ \bar{t}_L^{(rgb)\vee}$	± 1	0	0	0	0	$(\mp 1$	0	$0\,)$
	$t_R^{(rgb)\vee}\ \bar{t}_L^{(rgb)\wedge}$	0	± 1	0	0	0	$(\mp 1$	0	$0\,)$
	$b_L^{(rgb)\wedge}\ \bar{b}_R^{(rgb)\vee}$	0	0	0	∓ 1	0	$(\mp 1$	0	$0\,)$
	$b_L^{(rgb)\vee}\ \bar{b}_R^{(rgb)\wedge}$	0	0	± 1	0	0	$(\mp 1$	0	$0\,)$
	$b_R^{(rgb)\wedge}\ \bar{b}_L^{(rgb)\vee}$	0	∓ 1	0	0	0	$(\mp 1$	0	$0\,)$
	$b_R^{(rgb)\vee}\ \bar{b}_L^{(rgb)\wedge}$	∓ 1	0	0	0	0	$(\mp 1$	0	$0\,)$

Table 8. The 240 roots of e_8, from combining two f_4's, labeled as elementary particles.

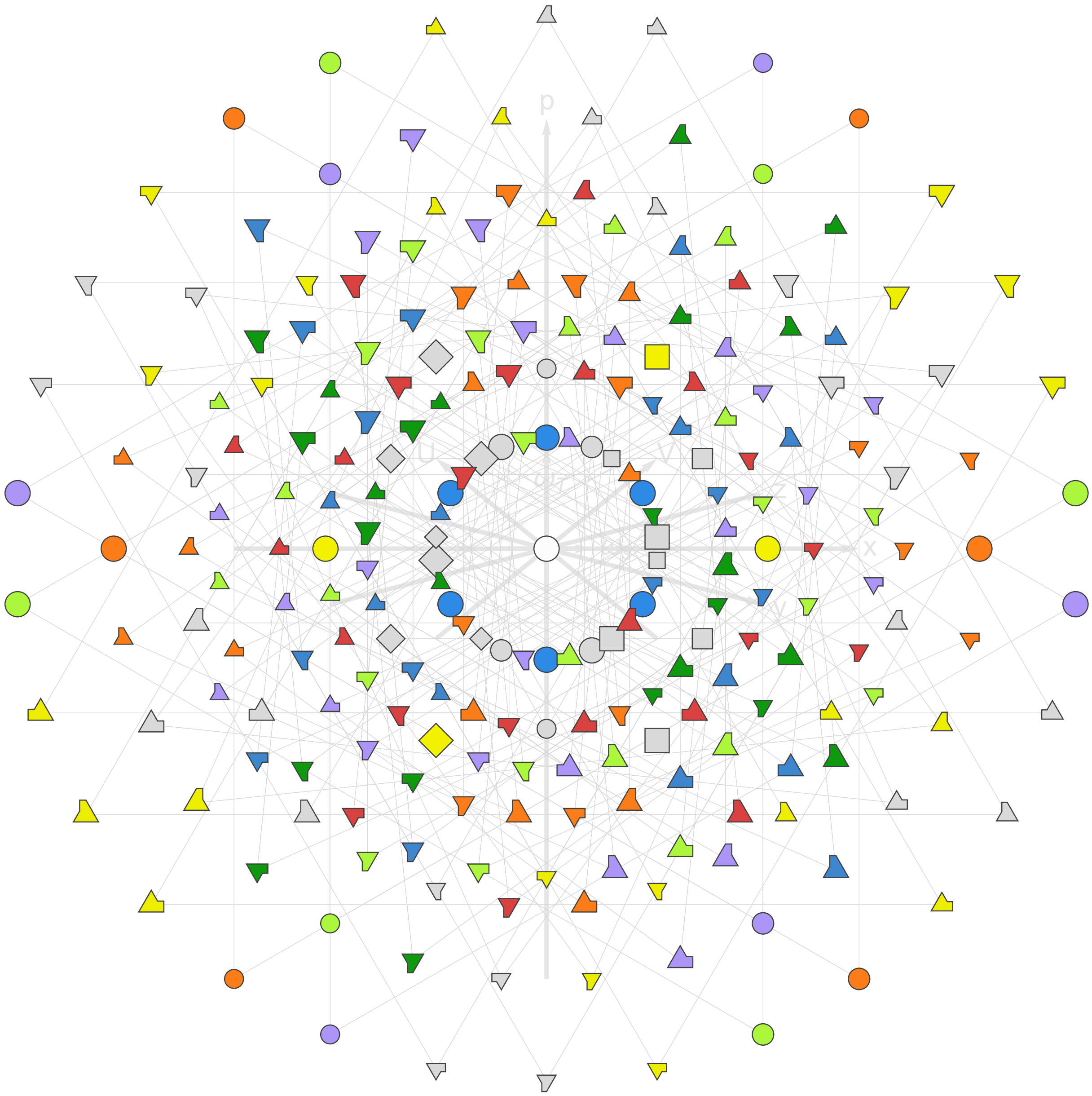


Figure 1. The 240 roots of e_8, labeled as elementary particles, with generations related by triality.

9 Exceptional Unification

The assignment of elementary particle labels to roots of e_6, e_7, and e_8 corresponds to deeper underlying theories of Exceptional Unification. Each deserves a long description, but we will discuss them briefly here.

The particle assignment within e_6, shown in Table 6, largely ignores triality and instead corresponds to the $SO(10)$ Grand Unified Theory. The five dimensional Cartan subalgebra of $so(10)$ produces the charges (U, V, x, y, z), with U and V combining to produce $su(2)$ weak charge, $W = \frac{1}{2}(-U+V)$, and $su(2)$ weaker charge, $W' = \frac{1}{2}(U+V)$, while (x, y, z) combines to give strong $su(3)$ color charges and $u(1)$ baryon minus lepton number charge,

$$g_3 = \tfrac{1}{2}(-x+y) \qquad g_8 = \tfrac{1}{2\sqrt{3}}(-x-y+2z) \qquad B = \tfrac{2}{3}(x+y+z)$$

Adding the sixth Cartan subalgebra element within e_6, for scaled helicity, H, allows the full $SO(10)$ GUT to be embedded in e_6, including one generation of fermions (without spin) as a complex 16_+ spinor of $spin(10)$. The canonical triality automorphism within e_6 transforms between up and down type leptons and weak, weaker, and $X_{\frac{1}{3}}$ bosons, which is pretty but not known to signify anything interesting.

The particle assignment within e_7, shown in Table 7, relates closely to the work of Dixon, Furey, and Hughes, with $\mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$ fermions.[1–3] However, it includes three generations of fermions, with spin, related by triality. Here, the (ω, W, W') charges each correspond to an $su(2)$ for spin, the weak force, and the weaker force, while the (h, x, y, z) $so(8)$ charges correspond to helicity, strong color, and B charge. The first generation fermion states are necessarily split into half of two different Dixon algebra, $\mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$, blocks, with triality then relating three $\mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$ blocks without overlapping states. While this works, this fitting of three fermion generations into e_7 is a bit cramped. For one thing, we need to use complex e_7 for the fit. This is somewhat fortuitous, as it means the complex spin algebra corresponding to ω becomes $su(2, \mathbb{C}) = sl(2, (C))$, which is nice for gravity, but it does not act correctly as $sl(2, \mathbb{C})_L$ should on the fermions, and using complex Lie algebras for the weak and weaker forces is problematic, as is $so(8, \mathbb{C})$. Also, relating the three generations of fermions by triality within e_7 requires that the leptons be included in a different way than the quarks, which is quirky.

The particle assignment within e_8, shown in Table 8, matches the particle assignment in “An Exceptionally Simple Theory of Everything”.[16] Each generation of particles matches to an $\mathbb{O} \otimes \mathbb{O}$ or to an $\mathbb{O}' \otimes \mathbb{O}$, depending on whether the $(\omega_T, \omega_s, U, V)$ correspond to an $so(8)$ or $so(4, 4)$, and each generation is related to the others by triality. The (p, x, y, z) $so(8)$ charges include a particle or anti-particle charge, p, which mixes with B under triality. This $so(8)$ can act on 8’s of the same signature as those of the first $so(8)$ for the compact real form, or 8’s of opposite signature for the split real form. The downside of this embedding is that it can only be the Euclidean part of a larger theory, or the physical spinors are matched $spin(4, 2)$ spinors or twistors, which introduces other complications. Several variations of the embedding of three generations of fermions within different real forms of e_8 are possible, and addressed elsewhere.[18]

10 Division algebra automorphisms

The division algebras, and their split algebras, are each invariant under transformation by elements of their *automorphism group*, $\Phi \in G_{\mathbb{D}}$. These group transformations leave multiplication invariant, $\Phi(a)\Phi(b) = \Phi(ab)$. The complex numbers are invariant under complex conjugation. The quaternions are invariant under $SO(3)$ rotations of their imaginary elements. The octonions are not invariant under $SO(7)$ rotations of their imaginary elements, but under a subgroup, G_2, that preserves octonionic non-associativity. The automorphism groups of the split algebras are similar,

$$\begin{aligned} G_{\mathbb{C}} &= \mathbb{Z}_2 & \qquad G_{\mathbb{C}'} &= \mathbb{Z}_2 \\ G_{\mathbb{H}} &= SO(3) & \qquad G_{\mathbb{H}'} &= SO(1,2) \\ G_{\mathbb{O}} &= G_{2(-14)} & \qquad G_{\mathbb{O}'} &= G_{2(2)} \end{aligned}$$

Since automorphism group elements leave division algebra multiplication invariant, these groups are subgroups of the corresponding triality group.

For the quaternions, there is an *inner triality automorphism*, Ad_t, corresponding to a 120-degree rotation around the axis formed by averaging the three unit imaginary quaternions. This rotation cycles the three imaginary unit quaternions,

$$t = -\tfrac{1}{2}(e_0 + e_1 + e_2 + e_3) \qquad t^3 = e_0 \qquad t\, e_1\, t^- = e_2 \qquad t\, e_2\, t^- = e_3 \qquad t\, e_3\, t^- = e_1$$

If we instead interpret t as an octonion, we see that Ad_t also preserves octonionic multiplication. If we represent an octonion as an 8-dimensional vector, and use the standard octonionic multiplication table, Ad_t acts on the octonions as a matrix,

$$\mathrm{Ad}_t \quad \sim \quad \begin{bmatrix} 1 &&&&&&& \\ &&1&&&&& \\ &&&1&&&& \\ &1&&&&&& \\ &&&&-1/2&+1/2&+1/2&+1/2 \\ &&&&-1/2&-1/2&-1/2&+1/2 \\ &&&&-1/2&+1/2&-1/2&-1/2 \\ &&&&-1/2&-1/2&+1/2&-1/2 \end{bmatrix}$$

If we use the same t in the split-octonions, we get the automorphism:

$$\mathrm{Ad}'_t \quad \sim \quad \begin{bmatrix} 1 &&&&&&& \\ &&1&&&&& \\ &&&1&&&& \\ &1&&&&&& \\ &&&&-1/2&-1/2&-1/2&-1/2 \\ &&&&+1/2&-1/2&-1/2&+1/2 \\ &&&&+1/2&+1/2&-1/2&-1/2 \\ &&&&+1/2&-1/2&+1/2&-1/2 \end{bmatrix}$$

This same t does not give an automorphism in the split-quaternions. However, if we have $e'_2 e'_2 = -e'_0$, we can have an inner triality automorphism from $t' = -\frac{1}{2}(e'_0 - \sqrt{3}e'_2)$, which rotates 120° in the $e'_0 - e'_2$ plane.

11 Discussion

In this work we have explored the deep relationship between division algebras, Clifford algebras, generalized reflections, triality automorphisms, triality Lie algebras, the magic square of Lie algebras, exceptional Lie algebras, root systems, and the connection with particle physics. From any of these specific subjects, the others can be understood, so a reader can make their choice of whichever mathematical starting point is more familiar. Although it is expected that current researchers will merely pull useful computational tools from this paper, it is hoped that they will also use this paper to better understand and appreciate the other ways of working with this material.

The true heart of the paper is triality—a real, cyclic, trilinear function of three elements of a vector space—which can be used to define the division algebras and their related split composition algebras. These division algebras lend themselves to the explicit matrix representation of certain Clifford algebras, which have a direct geometric interpretation. With this geometric point of view, we can describe reflections through vectors and, using triality, generalized reflections through spinors. These generalized reflections can be combined to define triality automorphisms that cycle vectors and spinors or three division algebra elements. These vectors and spinors combine with the triality algebras of division algebras to produce the triality Lie algebras, which can be understand as generalizations of $su(3)$. Within these triality Lie algebras, the relationship between vectors, chiral spinors, generalized reflections, and triality automorphisms can be described by division algebra products or more explicitly using representative Clifford algebra matrices. We encounter the surprising fact that real Lie algebra automorphisms can contain explicit i's in them, provided the automorphisms commute with the complex conjugation used to define the real form of the Lie algebra. The Cartan-Weyl description of these Lie algebras, and their root systems, can be used to visually appreciate their structure and automorphisms. Although useful, and often visually appealing, the root system description of Lie algebras and their automorphisms lacks the signs of structure constants and maps between root vectors. These signs can often be guessed or obtained algorithmically, but are easily obtained by the direct methods presented here. The triality Lie algebras combine in pairs to produce the magic square Lie algebras, which are also invariant under generalized reflections and triality automorphisms. Ultimately, building from division algebras, all exceptional Lie algebras and their automorphisms can be understood and described explicitly using these methods. The relationship to particle physics has been briefly addressed, and largely motivates this work, with triality at its center.

If Exceptional Unification is an interesting area of exploration for physics, then the explicit tools and descriptions provided in this paper are presented as the keys to the castle. It is hoped that other researchers will use this work to further their own explorations within this rich area of mathematical physics.

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