Q. 1). Given, $E_{avg} = \frac{1}{M} \sum_{i=1}^{M} E(\varepsilon_i(x)^2)$

$$E_{agg}(x) = E\left(\left\{\frac{1}{M}\sum_{i=1}^{M} h_i(x) - f(x)\right\}^2\right)$$

To PROVE:

$$E_{agg} = \frac{1}{M} E_{avg}$$

Solution: $E_{agg}(x) = E\left(\left\{\frac{1}{M}\sum_{i=1}^{m}(h_i(x) - f(x))\right\}^2\right)$

$$= E\left(\left\{\frac{1}{M}\sum_{i=1}^{m}(-f(x) + h_i(x))\right\}^2\right)$$

$$= E\left(\left\{\frac{-1}{M}\sum_{i=1}^{M}(f(x) - h_i(x))\right\}^2\right)$$

$$= \frac{1}{M^2} \cdot E\left(\left\{\sum_{i=1}^{M}(f(x) - h_i(x))\right\}^2\right)$$

But, $\varepsilon_i(x) = f(x) - h_i(x)$

$$\therefore \quad E_{agg}(x) = \frac{1}{M^2} \cdot E\left(\left\{\sum_{i=1}^{M} \varepsilon_i(x)\right\}^2\right) \quad \text{—— (1)}$$

Now, This can be broken down using sq. of summation as,

$$\left(\sum_{i=1}^{N} x_i\right)^2 = \sum_{i=1}^{N} x_i^2 + 2\sum_{i=1}^{N}\sum_{j=1}^{i-1} x_i x_j$$

$$E_{agg}(x) = \frac{1}{M^2} \cdot E\left(\sum_{i=1}^{M} \varepsilon_i(x)^2 + 2\sum_{i=1}^{M}\sum_{j=1}^{i-1} \varepsilon_i(x)\varepsilon_j(x)\right)$$

$$= \frac{1}{M^2}\left\{E\left(\sum_{i=1}^{M} \varepsilon_i(x)^2\right) + 2E\left(\sum_{i=1}^{M}\sum_{j=1}^{i-1} \varepsilon_i(x)\varepsilon_j(x)\right)\right\} \quad \text{—— (2)}$$

Substituting the following assumption in (2),

$$E(\varepsilon_i(x) \cdot \varepsilon_j(x)) = 0 \text{ for } i \neq j$$

$$\therefore E_{agg}(x) = \frac{1}{M^2}\left\{E\left(\sum_{i=1}^{M} \varepsilon_i(x)^2\right) + 0\right\}$$

$$= \frac{1}{M} \cdot \frac{1}{M}\left\{\sum_{i=1}^{M} E(\varepsilon_i(x)^2)\right\}$$

But $E_{avg}(x) = \frac{1}{M}\sum_{i=1}^{M} E(\varepsilon_i(x)^2)$

$$\therefore \boxed{E_{agg}(x) = \frac{1}{M} \cdot E_{avg}(x)}$$

Q.2) Using the eqⁿ for $E_{agg}(x)$ and $E_{avg}(x)$ from Q.1), we have,

$$E_{agg}(x) = \frac{1}{M^2} \cdot E\left(\left\{\sum_{i=1}^{M} \varepsilon_i(x)\right\}^2\right)$$

$$= \cancel{\frac{1}{M}}\, E\left(\left\{\sum_{i=1}^{M} \frac{1}{M} \varepsilon_i(x)\right\}^2\right) \quad \text{——— (1)}$$

$$E_{avg}(x) = \sum_{i=1}^{M} \frac{1}{M} \cdot E(\varepsilon_i(x))^2 \quad \text{——— (2)}$$

Given Jenson's inequality as,

$$f\left(\sum_{i=1}^{\cancel{M}} \lambda_i x_i\right) \leq \sum_{i=1}^{M} \cancel{f(x_i)}\, \lambda_i f(x_i) \quad \text{——— (3)}$$

Using (1), (2) and (3), we can assume that we have the following values for λ_i, x_i and $f(x)$

$$\lambda_i = \frac{1}{M}, \quad f(x) = x^2, \quad x_i = \varepsilon_i(x)$$

Using these values in Jenson's inequality,
we have

$$\Rightarrow \left(\left\{\sum_{i=1}^{M} \frac{1}{M} \varepsilon_i(x)\right\}^2\right) \leq \sum_{i=1}^{M} \frac{1}{M} \left\{\varepsilon_i(x)\right\}^2$$

$$\Rightarrow E\left(\left\{\sum_{i=1}^{M} \frac{1}{M} \varepsilon_i(x)\right\}^2\right) \leq E\left(\sum_{i=1}^{M} \frac{1}{M} \left\{\varepsilon_i(x)\right\}^2\right)$$

$$\Rightarrow E\left(\left\{\sum_{i=1}^{M} \frac{1}{M} \varepsilon_i(x)\right\}^2\right) \leq \sum_{i=1}^{M} \frac{1}{M} E\left(\left\{\varepsilon_i(x)\right\}^2\right)$$

These are nothing but values of $E_{agg}(x)$ and $E_{avg}(x)$

$$\Rightarrow \boxed{E_{agg}(x) \leq E_{avg}(x)}$$