Project 6

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Problem 15: Monte Carlo estimation of an expected value

Proof 1: $\mathbb{E}[\hat{g}(X)] = \mathbb{E}[g(X)]$

Using the properties of the expectation and the fact that $\mathbb{E}[g(X_i)] = \mathbb{E}[g(X)]$:

$$\mathbb{E}[\hat{g}(X)] = \mathbb{E}[\frac{1}{N}\sum_{i=1}^{N} g(X_i)] = \frac{1}{N}\mathbb{E}[\sum_{i=1}^{N} g(X_i)] = \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}[g(X_i)] = \frac{N}{N}\mathbb{E}[g(X)] = \mathbb{E}[g(X)]$$

Proof 2: $Var(\hat{g}(X)) = \frac{Var(g(X))}{N}$

Bienaymé's identity states that:

$$Var\left(\sum_{i=1}^{N} X_i\right) = \sum_{i=1}^{N}\mathrm{Var}(X_i) + \sum_{\substack{i,j=1\\ i\neq j}}^{N}\mathrm{Cov}(X_i, X_j) = \sum_{i,j=1}^{N}\mathrm{Cov}(X_i, X_j)$$

Since the covariance between any pair of independent random variables is zero we get the following.

$$Var\left(\sum_{i=1}^{N} X_i\right) = \sum_{i=1}^{N}\mathrm{Var}(X_i)$$

Specifically for our proof we have that:

$$Var(\hat{g}(X)) = Var(\frac{\sum_{i=1}^{N} g(X_i)}{N})$$

Using one of the properties of the variance we can pull out the constant N and obtain the following:

$$Var(\frac{\sum_{i=1}^{N} g(X_i)}{N}) = \frac{1}{N^2}Var(\sum_{i=1}^{N} g(X_i))$$

Then using Bienaymé's identity we have that:

$$\frac{1}{N^2}Var(\sum_{i=1}^{N} g(X_i)) = \frac{1}{N^2}\sum_{i=1}^{N} Var(g(X_i))$$

Finally since all the X_i's are i.i.d with variance equal to $Var(g(X))$ we arrive at the expression we had seek to proof.

$$\frac{1}{N^2}\sum_{i=1}^{N} Var(g(X_i)) = \frac{N}{N^2}Var(g(X)) = \frac{Var(g(X))}{N}$$

Problem 16: Sampling in the Rain Network