Project 6

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Problem 15: Monte Carlo estimation of an expected value

Proof 1: $\mathbb{E}[\hat{g}(X)] = \mathbb{E}[g(X)]$

Using the properties of the expectation and the fact that $\mathbb{E}[g(X_i)] = \mathbb{E}[g(X)]$:

$$\mathbb{E}[\hat{g}(X)] = \mathbb{E}[\frac{1}{N}\sum_{i=1}^{N} g(X_i)] = \frac{1}{N}\mathbb{E}[\sum_{i=1}^{N} g(X_i)] = \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}[g(X_i)] = \frac{N}{N}\mathbb{E}[g(X)] = \mathbb{E}[g(X)]$$

Proof 2: $Var(\hat{g}(X)) = \frac{Var(g(X))}{N}$

Bienaymé's identity states that:

$$Var\left(\sum_{i=1}^{N} X_i\right) = \sum_{i=1}^{N}\mathrm{Var}(X_i) + \sum_{\substack{i,j=1\\ i\neq j}}^{N}\mathrm{Cov}(X_i, X_j) = \sum_{i,j=1}^{N}\mathrm{Cov}(X_i, X_j)$$

Since the covariance between any pair of independent random variables is zero we get the following.

$$Var\left(\sum_{i=1}^{N} X_i\right) = \sum_{i=1}^{N}\mathrm{Var}(X_i)$$

Specifically for our proof we have that:

$$Var(\hat{g}(X)) = Var(\frac{\sum_{i=1}^{N} g(X_i)}{N})$$

Using one of the properties of the variance we can pull out the constant N and obtain the following:

$$Var(\frac{\sum_{i=1}^{N} g(X_i)}{N}) = \frac{1}{N^2}Var(\sum_{i=1}^{N} g(X_i))$$

Then using Bienaymé's identity we have that:

$$\frac{1}{N^2}Var(\sum_{i=1}^{N} g(X_i)) = \frac{1}{N^2}\sum_{i=1}^{N} Var(g(X_i))$$

Finally since all the X_i's are i.i.d with variance equal to $Var(g(X))$ we arrive at the expression we had seek to proof.

$$\frac{1}{N^2}\sum_{i=1}^{N} Var(g(X_i)) = \frac{N}{N^2} Var(g(X)) = \frac{Var(g(X))}{N}$$

Problem 16: Sampling in the Rain Network

a) Derive the expressions for $P(C = T|R = T, S = T, W = T)$, $P(C = T|R = F, S = T, W = T)$, $P(R = T|C = T, S = T, W = T)$ and $P(R = T|C = F, S = T, W = T)$ and compute their values.

a.i)

$$P(C = T|R = T, S = T, W = T) = P(C = T|R = T, S = T)$$

$$= \frac{P(C = T)P(S = T, R = T|C = T)}{P(S = T, R = T)}$$

$$= \frac{P(C = T)P(S = T|C = T)P(R = T|C = T)}{P(C = T)P(S = T|C = T)P(R = T|C = T) + P(C = F)P(S = T|C = F)P(R = T|C = F)}$$

$$= \frac{0.5 * 0.1 * 0.8}{0.53 * 0.1 * 0.8 + 0.5 * 0.5 * 0.2} = 0.4444$$

a.ii)

$$P(C = T|R = F, S = T, W = T) =$$

$$= \frac{P(C = T)P(S = T|C = T)P(R = F|C = T)}{P(C = T)P(S = T|C = T)P(R = F|C = T) + P(C = F)P(S = T|C = F)P(R = F|C = F)}$$

$$= \frac{0.5 * 0.1 * 0.2}{0.5 * 0.1 * 0.2 + 0.5 * 0.5 * 0.8} = 0.04761905$$

a.iii)

$$P(R = T|C = T, S = T, W = T) =$$

$$\frac{P(R = T|C = T, S = T)P(W = T|R = T, C = T, S = T)}{P(W = T|C = T, S = T)} =$$

$$\frac{P(R = T|C = T)P(W = T|R = T, S = T)}{P(W = T|C = T, S = T)} =$$

$$\frac{P(R = T|C = T)P(W = T|R = T, S = T)}{P(R = T|C = T)P(W = T|R = T, S = T) + P(R = F|C = T)P(W = T|R = F, S = T)}$$

$$= \frac{0.8 * 0.99}{0.8 * 0.99 + 0.2 * 0.9} = 0.8148$$

a.iv)

$$P(R = T|C = F, S = T, W = T) =$$

$$\frac{P(R = T|C = F)P(W = T|R = T, S = T)}{P(R = T|C = F)P(W = T|R = T, S = T) + P(R = F|C = F)P(W = T|R = F, S = T)}$$

$$= \frac{0.2 * 0.99}{0.2 * 0.99 + 0.8 * 0.9} = 0.2156863$$

b) Implement the Gibbs sampler for the Bayesian network

```
#returns 1 with probability p, and 0 with probability 1-p
rbernoulli=function(p){return(1*runif(1)<p)}
# sample from distribution C given R above
sample_CgivenR = function(R){
 if(R==0){
   C = rbernoulli(0.0476) # returns 1 with probability 0.0476; otherwise 0
 } else {
   C = rbernoulli(0.4444)
 }
  return(C)
}
#' sample from distribution R given C above
sample_RgivenC = function(C){
  if(C==0){
    R = rbernoulli(0.2157)
  } else {
    R = rbernoulli(0.8148)
  }
  return(R)
}
set.seed(101)
niter = 100
C = rep(0,niter)
R = rep(0,niter)
C[1]=1
R[1]=1 # start from (1,1)
for(i in 2:niter){
  C[i] = sample_CgivenR(R[i-1])
  R[i] = sample_RgivenC(C[i])
}
res = data.frame(C=C,R=R)
```

```
#Display a 2-by-2 table for the sampled R and C.
#1=True, 0=False
table=table(res)/niter
table
```

```
##    R
## C      0    1
##   0 0.69 0.13
##   1 0.04 0.14
```

c) Estimate the marginal probability of rain, given that the sprinkler is on and the grass is wet

The marginal probability of R given the defined conditions can be obtained by marginalizing over C in the following way:

$$P(R|S=T,W=T) = \sum_{c} P(R,C|S=T,W=T)$$

```
# Marginalize over C
colSums(table)
```

```
##    0    1
## 0.73 0.27
```

Therefore, in other words:

$$P(R = T|S = T, W = T) = \sum_c P(R = T, C|S = T, W = T) = 0.27$$

$$P(R = F|S = T, W = T) = \sum_c P(R = F, C|S = T, W = T) = 0.73$$

d) Draw 50,000 samples instead of 100 using the Gibbs sampler

```
niter = 50000
C = rep(0,niter)
R = rep(0,niter)
C[1]=1
R[1]=1 # start from (1,1)
for(i in 2:niter){
  C[i] = sample_CgivenR(R[i-1])
  R[i] = sample_RgivenC(C[i])
}
res50000run1 = data.frame(C=C,R=R)
#Print the table
table50000run1=table(res50000run1)/niter
table50000run1
```

```
##    R
## C         0       1
##   0 0.64812 0.17842
##   1 0.03242 0.14104
```

```
# Marginalize over C
colSums(table50000run1)
```

```
##       0       1
## 0.68054 0.31946
```

e) Plot the relative frequencies of R = T and C = T up to each iteration t against t, for two independent runs of the sampler. Suggest a burn-in time based on this plot.

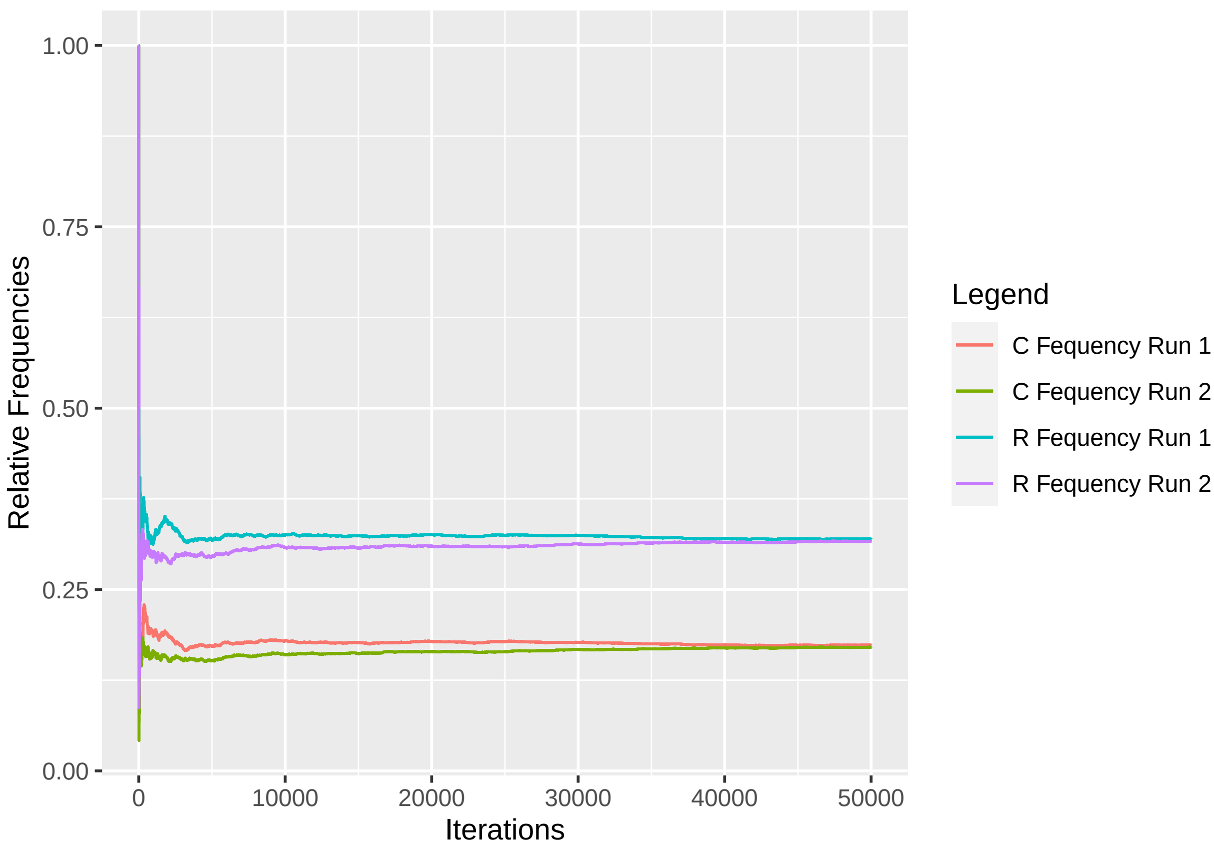
```
#First Run of Gibbs sampler
# initialise frequency vectors
freqC<-c()
freqR<-c()
for(i in 1:niter){
  freqC<-append(freqC,sum(res50000run1$C[1:i])/i)
  freqR<-append(freqR,sum(res50000run1$R[1:i])/i)
}

#Second Run of Gibbs sampler
niter = 50000
C = rep(0,niter)
R = rep(0,niter)
C[1]=1
R[1]=1 # start from (1,1)
for(i in 2:niter){
  C[i] = sample_CgivenR(R[i-1])
```

```
  R[i] = sample_RgivenC(C[i])
}
res50000run2 = data.frame(C=C,R=R)

# initialise frequency vectors
freqC2<-c()
freqR2<-c()
for(i in 1:niter){
  freqC2<-append(freqC2,sum(res50000run2$C[1:i])/i)
  freqR2<-append(freqR2,sum(res50000run2$R[1:i])/i)
}

# plot relative frequencies
df <- data.frame((1:niter),freqC,freqR,freqC2,freqR2)
ggplot(df, aes(1:niter)) +
  geom_line(aes(y=freqC, colour="C Fequency Run 1")) +
  geom_line(aes(y=freqR, colour="R Fequency Run 1")) +
  geom_line(aes(y=freqC2, colour="C Fequency Run 2")) +
  geom_line(aes(y=freqR2, colour="R Fequency Run 2")) + ylab("Relative Frequencies") + xlab("Iterations
```



An appropriate burn-in time is that at which the relative frequency reaches its stationary phase. Therefore, around ~5000 iterations would be a suitable choice.

f) Apply the Gelman and Rubin test

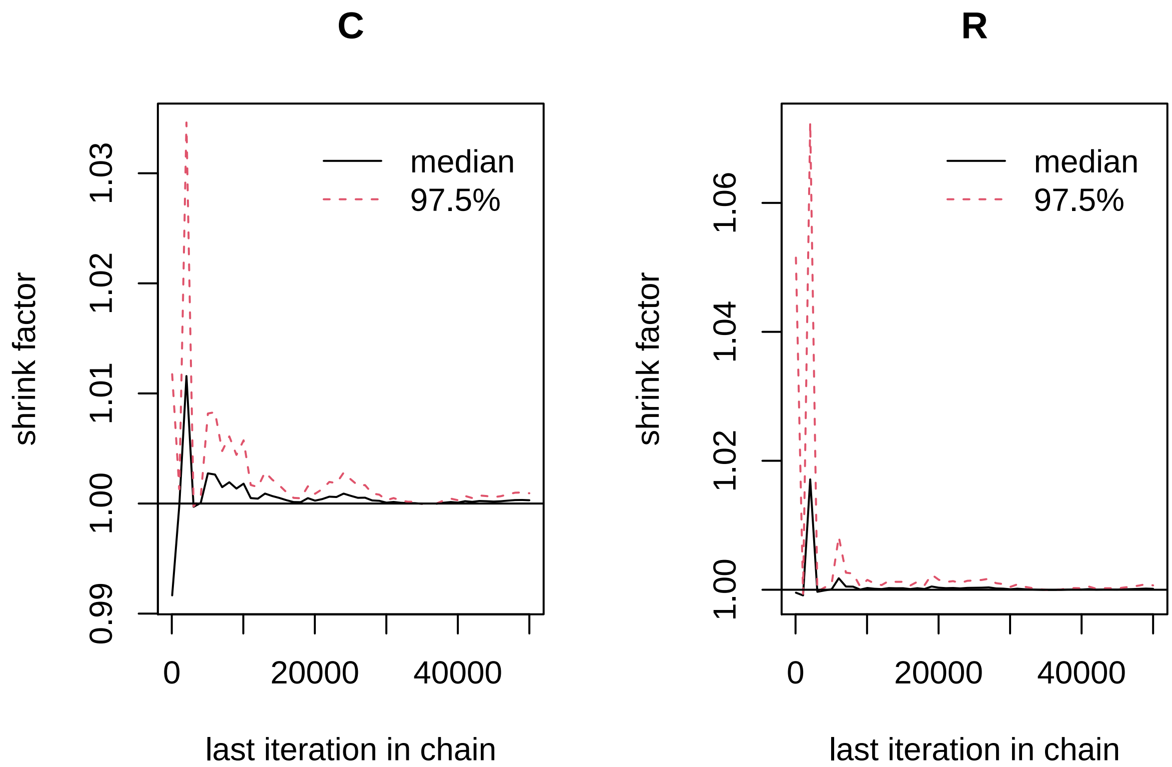
```
# Generate 2 independent chains of observations
mcmc1<-mcmc(data=res50000run1 , start = 1, end = niter)
```

```
mcmc2<-mcmc(data=res50000run2 , start = 1, end = niter)
combinedchains = mcmc.list(mcmc1, mcmc2)

# Apply Gelman and Rubin test
gelman.diag(combinedchains)
```

```
## Potential scale reduction factors:
##
##   Point est. Upper C.I.
## C          1          1
## R          1          1
##
## Multivariate psrf
##
## 1
```

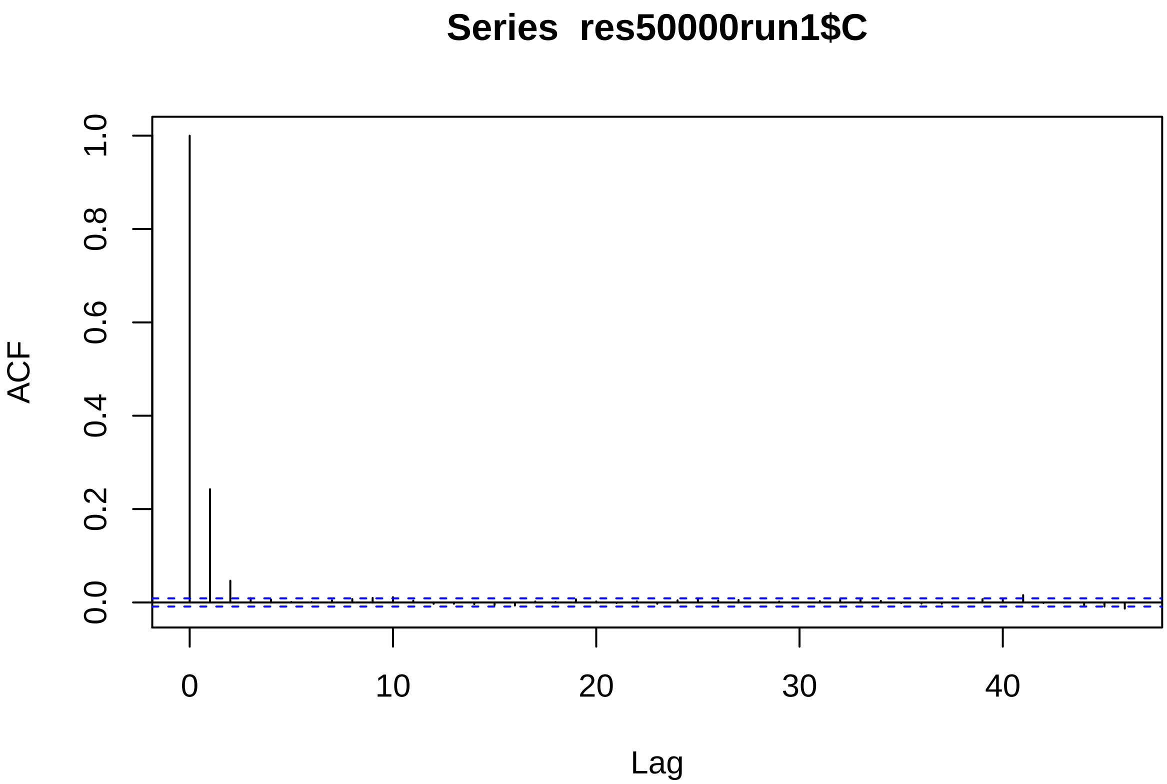
```
gelman.plot(combinedchains)
```



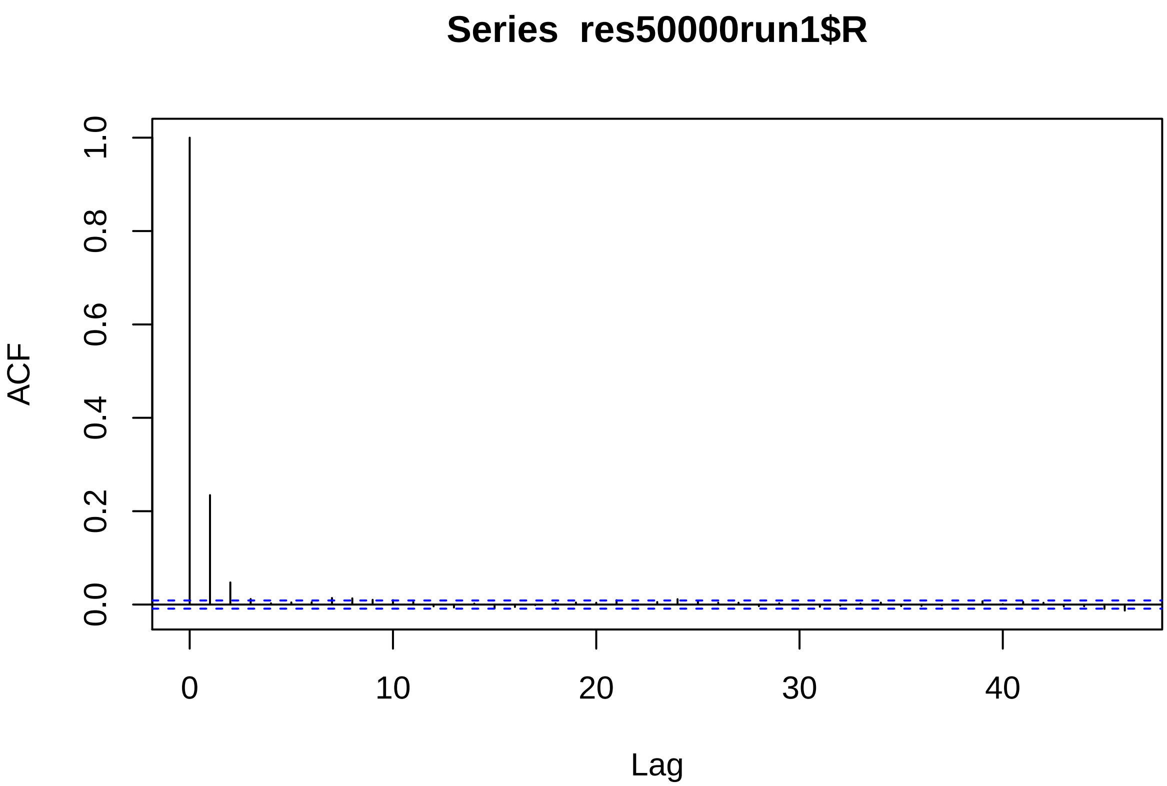
Following the same criteria that was stated above, the suggested burn-in time is about ~10000 iterations. Note that small oscillations can be observed up to the ~15000th iteration, and thus a larger burn-in time might also be considered.

g) Provide plots for both variables Rain and Cloudy and suggest an interval for drawing approximately independent samples.

```
acf(res50000run1$C,type="correlation")
```




```
acf(res50000run1$R,type="correlation")
```



The correlation between two random variables X_1 and X_2 is defined as:

$$\rho(X_1, X_2) = \frac{Cov(X_1, X_2)}{\sqrt{Var(X_1)Var(X_2)}} = \frac{1}{\sqrt{Var(X_1)Var(X_2)}}\left(E[X_1, X_2] - E[X_1]E[X_2]\right)$$

If X_1 and X_2 are independent, then $E[X_1, X_2] = E[X_1]E[X_2]$, thus

$$\rho(X_1, X_2) = \frac{1}{\sqrt{Var(X_1)Var(X_2)}}\left(E[X_1]E[X_2] - E[X_1]E[X_2]\right) = 0$$

Overall, the safe interval for taking independent samples that for which the drawn samples have a correlation of $\rho(X_1, X_2) = 0$. Therefore, given the plot above, independent samples should be drawn from the lag interval $[5,)$.

h)Re-estimate P (R = T | S = T, W = T) based on samples obtained after the suggested burn-in time and thinning.

```
# Only consider samples from burn-in time on
dd<-res50000run1[10000:50000,]

# Apply thinning by taking every 4th sample
dd_thin <- dd[c(TRUE,rep(FALSE,3)), ]

# Compute Joint Probability table
tabledd=table(dd_thin)/dim(dd_thin)[1]

# Re-estimate P (R = T | S = T, W = T)
colSums(tabledd)
```

```
##         0         1
## 0.6808319 0.3191681
```

The marginal probability of R given the defined conditions is therefore

$$P(R = T|S = T, W = T) = \sum_c P(R = T, C|S = T, W = T) = 0.32$$

$$P(R = F|S = T, W = T) = \sum_c P(R = F, C|S = T, W = T) = 0.68$$

i) Compute the probability P (R = T | S = T, W = T) analytically. Compare with (c) and (h)

$$P(R = T|S = T, W = T) = \frac{P(R = T, S = T, W = T)}{P(S = T, W = T)} = \frac{\sum_{c\in\{F,T\}} P(R = T, C = c, S = T, W = T)}{\sum_{c\in\{F,T\}}\sum_{r\in\{F,T\}} P(R = r, C = c, S = T, W = T)}$$

$$P(R = T, S = T, W = T) = \sum_{c\in\{F,T\}} P(R = T, C = c, S = T, W = T)$$

$$= \sum_{c\in\{F,T\}} P(C = c)P(R = T|C = c)P(S = T|C = c)P(W = T|R = T, C = c)$$

$$P(C = T)P(R = T|C = T)P(S = T|C = T)P(W = T|R = T, C = T) = 0.5 \times 0.8 \times 0.1 \times 0.99 = 0.0396$$

$$P(C = F)P(R = T|C = F)P(S = T|C = F)P(W = T|R = T, C = F) = 0.5 \times 0.2 \times 0.5 \times 0.99 = 0.0495$$

$$P(R=T,S=T,W=T) = 0.0396 + 0.0495 = 0.0891$$

$$P(S=T,W=T) = \sum_{c\in\{F,T\}} \sum_{r\in\{F,T\}} P(R=r,C=c,S=T,W=T)$$
$$= \sum_{c\in\{F,T\}} P(R=T,C=c,S=T,W=T) + \sum_{c\in\{F,T\}} P(R=F,C=c,S=T,W=T)$$

$$\sum_{c\in\{F,T\}} P(R=F,C=c,S=T,W=T) = \sum_{c\in\{F,T\}} P(C=c)P(R=F|C=c)P(S=T|C=c)P(W=T|R=F,C=c)$$

$$P(C=T)P(R=F|C=T)P(S=T|C=T)P(W=T|R=F,C=T) = 0.5\times 0.2\times 0.9\times 0.1 = 0.009$$

$$P(C=F)P(R=F|C=F)P(S=T|C=F)P(W=T|R=F,C=F) = 0.5\times 0.8\times 0.9\times 0.5 = 0.18$$

$$P(S=T,W=T) = 0.0891 + 0.009 + 0.18 = 0.2781$$

$$P(R=T|S=T,W=T) = \frac{P(R=T,S=T,W=T)}{P(S=T,W=T)} = \frac{0.0891}{0.2781} = 0.3203883$$

Compared with (c) and (h) the analytically computed probability seems to be very close to the one obtained from the Gibbs sampler. There is a larger difference with the values obtained in c) which is reasonable since we only sampled 100x and did neither consider the Burn in time nor the thinning.