Project 7

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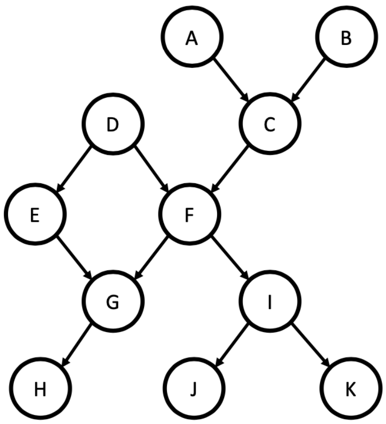
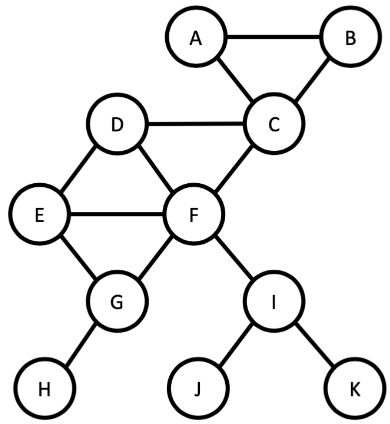
Problem 17: Junction Tree Algorithm

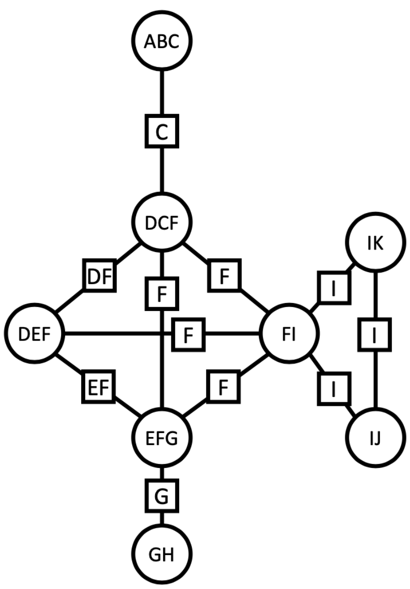
(a) Build the Junction Tree of the network.



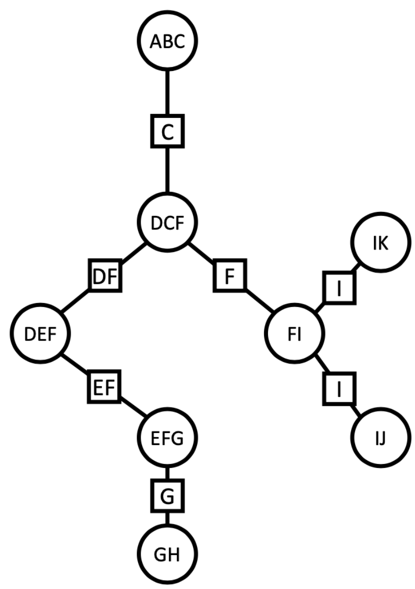
DAG



Moral graph



Junction graph



Junction tree

(b) Write the joint probability P(U) in terms of the cluster and separator potentials.

$$P(A, B, C, D, E, F, G, H, I, J) = \sum_{A,B,C,D,E,F,G,H,I,J} P(U)$$

$$= \sum_{A,B,C,D,E,F,G,H,I,J} \phi(U) = \frac{\phi(ABC)\phi(DCF)\phi(DEF)\phi(EFG)\phi(GH)\phi(FI)\phi(IS)\phi(IK)}{\phi(C)\phi(DF)\phi(EF)\phi(G)\phi(F)\phi(I)^2}$$

Problem 18: Benefit of storing messages

(a)

The formula for recursively computing the forward and backward messages is:

$$\mu_\alpha(x_n) = \sum_{x_{n-1}} \psi_{n-1,n}(x_{n-1}, x_n)\mu_\alpha(x_{n-1})$$
$$\mu_\beta(x_n) = \sum_{x_{n+1}} \psi_{n,n+1}(x_n, x_{n+1})\mu_\beta(x_{n+1})$$

Initialization:

$$\mu_\alpha(x_2) = \sum_{x_1} \psi_{1,2}(x_1, x_2)$$
$$\mu_\beta(x_4) = \sum_{x_5} \psi_{4,5}(x_4, x_5)$$

(b)

The computational complexity for computing the marginal probability $P(X_4 = 1)$ is $O(NK^2)$, where N is the chain length, here 5 and K is the number of values each node can assume, here 2. For each node that can assume K values, since $(n-1)K^2$ computations have to be done for the forward message passing and $(N-n)K^2$ computations have to be done for the backward message passing, this sums to $(N-1)K^2$ computations which is of the order $O(NK^2)$

(c)

If the messages are stored, by doing one pass in the forward and one pass in the backward, the total number of steps is reduced to $2(N-1)K^2$, and therefore the order is $O(NK^2)$. In the general case, the number of steps for N nodes is $N(N-1)K^2$ and therefore the order of computational complexity is $O(N^2K^2)$

Problem 19 (data analysis): Message passing on a chain

(a) Store clique potentials in an R object

```
clique_potentials = array(dim = c(2, 2, 4), dimnames = list(c("0", "1"), c("0", "1"),
c("Psi12", "Psi23", "Psi34", "Psi45")))
```

compute clique potentials

```r
X1 = c(1/3,2/3)
X2 = c(4/5,2/3)
X3 = c(5/7,1/3)
X4 = c(3/5,2/5)
X5 = c(1/2,7/9)
```

```r
clique_potentials = array(dim = c(2, 2, 4),
                          dimnames = list(c("0", "1"), c("0", "1"),
                                          c("Psi12", "Psi23", "Psi34", "Psi45")))
clique_potentials[1,1,"Psi12"] = (2/3)*(1/5)
clique_potentials[1,2,"Psi12"] = (2/3)*(4/5)
clique_potentials[2,1,"Psi12"] = (1/3)*(1/3)
clique_potentials[2,2,"Psi12"] = (1/3)*(2/3)
clique_potentials[,,"Psi23"] = c(1-X3,X3)
clique_potentials[,,"Psi34"] = c(1-X4,X4)
clique_potentials[,,"Psi45"] = c(1-X5,X5)
```

```r
clique_potentials[,,"Psi12"]
```

```
##           0         1
## 0 0.1333333 0.5333333
## 1 0.1111111 0.2222222
```

```r
clique_potentials[,,"Psi23"]
```

```
##           0         1
## 0 0.2857143 0.7142857
## 1 0.6666667 0.3333333
```

```r
clique_potentials[,,"Psi34"]
```

```
##     0   1
## 0 0.4 0.6
## 1 0.6 0.4
```

```r
clique_potentials[,,"Psi45"]
```

```
##           0         1
## 0 0.5000000 0.5000000
## 1 0.2222222 0.7777778
```

(b) Computing forward messages

```r
forward_messages = array(dim = c(5,2), dimnames = list(c("X1","X2", "X3","X4","X5"),c("0", "1")))
forward_messages["X1",] = c(1,1)
for(i in 2:5){
  forward_messages[i,]=forward_messages[i-1,]%*%clique_potentials[,,i-1]
}
forward_messages
```

```
##             0         1
## X1 1.0000000 1.0000000
## X2 0.2444444 0.7555556
## X3 0.5735450 0.4264550
## X4 0.4852910 0.5147090
## X5 0.3570253 0.6429747
```

(d) Computing backward messages

```
backward_messages = array(dim = c(2,5),
                          dimnames = list(c("0", "1"),
                                          c("X1","X2","X3","X4","X5")))
backward_messages[,"X5"] = c(1,1)
for(i in 4:1){
  backward_messages[,i]=clique_potentials[,,i]%*%backward_messages[,i+1]
}
backward_messages
```

```
##          X1 X2 X3 X4 X5
## 0 0.6666667  1  1  1  1
## 1 0.3333333  1  1  1  1
```

(e) Compute the marginal probability distribution for each node

```
unnormalised = array(0,dim = c(5,2),
                     dimnames=list(c("p(X1)","p(X2)","p(X3)",
                                     "p(X4)","p(X5)"),
                                   c("0","1")))
for(i in 1:5){
    unnormalised[i,] = forward_messages[i,]*backward_messages[,i]
}
Z = rowSums(unnormalised)
marginal_probability = unnormalised/Z
marginal_probability
```

```
##                0         1
## p(X1) 0.6666667 0.3333333
## p(X2) 0.2444444 0.7555556
## p(X3) 0.5735450 0.4264550
## p(X4) 0.4852910 0.5147090
## p(X5) 0.3570253 0.6429747
```