Project 6

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Problem 15: Monte Carlo estimation of an expected value

We start by showing that $Var(\hat{g}(X)) = \frac{Var(g(X))}{N}$. Bienaymé's identity states that:

$$Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathrm{Var}(X_i) + \sum_{\substack{i,j=1\\ i\neq j}}^{n} \mathrm{Cov}(X_i, X_j) = \sum_{i,j=1}^{n} \mathrm{Cov}(X_i, X_j)$$

Where $X_1, \ldots, X_n$ are pairwise independent integrable random variables with finite second moments. Since the covariance between any pair of independent random variables is zero we get the following.

$$Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathrm{Var}(X_i)$$

specifically for our proof we have that:

$$Var(\hat{g}(X)) = Var(\frac{\sum_{i=1}^{n} g(X_i)}{N})$$

Using one of the properties of the variance we can pull out the constant N and obtain the following:

$$Var(\frac{\sum_{i=1}^{n} g(X_i)}{N}) = \frac{1}{N^2} Var(\sum_{i=1}^{n} g(X_i))$$

Then using Bienaymé's identity we have that:

$$\frac{1}{N^2} Var(\sum_{i=1}^{n} g(X_i)) = \frac{1}{N^2} \sum_{i=1}^{n} Var(g(X_i))$$

Finally since all the X_i's are i.i.d with variance equal to $Var(g(X))$ we arrive at the expression we had seek to proof.

$$\frac{1}{N^2} \sum_{i=1}^{n} Var(g(X_i)) = \frac{N}{N^2} Var(g(X)) = \frac{Var(g(X))}{N}$$

To prove that $\mathbb{E}[\hat{g}(X)] = \mathbb{E}[g(X)]$ we assume (as we did for the last proof) that the RV $g(X_i)$ has finite variance $Var(g(X))$. Then by way of Chebyshev's inequality on $\hat{g}(X)$ we have that:

$$\mathrm{P}(|\hat{g}(X) - \mathbb{E}[g(X)]| \geq \varepsilon) \leq \frac{Var(g(X))}{N\varepsilon^2}$$

Then the probability that this inequality does not hold is 1 minus this quantity sucht that

$$P(|\hat{g}(X) - \mathbb{E}[g(X)]| < \varepsilon) = 1 - \mathrm{P}(|\hat{g}(X) - \mathbb{E}[g(X)]| \geq \varepsilon) \geq 1 - \frac{Var(g(X))}{N\varepsilon^2}$$

Then by taking $N \to \infty$ we can see that the second term goes to 0, which implies that as the sample size increases the $\hat{g}(X)$ converges to $\mathbb{E}[g(X)]$ with probability 1.