UNIT - 2

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*** EIGEN VALUE AND EIGEN VECTOR OF LINEAR TRANSFERMATION:

Let $T: V \rightarrow V$ a linear operator of n dimensional vector space over the field F.
A scalar $\lambda \in F$ is called Eigen value of T if there exists a non-zero vector α in V such that $T(\alpha) = \lambda \alpha$ any $\alpha \neq 0$ of V such that $T(\alpha) = \lambda \alpha$ is called an Eigen vector of T belonging to the Eigen of λ.

The W_λ is known as collection of all Eigen value which is a subspace.

Step I: characteristic equation

$$|A - \lambda I| = 0$$

Step II: Solving given equation we get value λ known as Eigen value.

Step III: For Eigen ~~value~~ vector depend upon the value of Eigen value.

Q. $A = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$

Soln. let $\lambda \in F$ be any scalar such that characteristic polynomial are

$$[A - \lambda I] = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3-\lambda & 2 \\ -1 & 0-\lambda \end{bmatrix}$$

$\cancel{[A - \lambda I]}$

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 2 \\ -1 & 0-\lambda \end{vmatrix}$$

$$= (3-\lambda)(-\lambda) + 2$$

$$= -3\lambda + \lambda^2 + 2$$

$$= \lambda^2 - 3\lambda + 2$$

$\cancel{\Rightarrow \lambda = 1, 2}$

If $|A - \lambda I| = 0$

$\therefore \lambda^2 - 3\lambda + 2 = 0$

$\Rightarrow \lambda = 1, 2.$

Eigen value are $1, 2$.

$\lambda_1 = 1, \ \lambda_2 = 2$

Q. $A = \begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix}$

Soln let $\lambda \in F$ be any scalar such that characteristic polynomial are

$$[A - \lambda I] = \begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4-\lambda & 2 \\ 3 & 3-\lambda \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 4-\lambda & 2 \\ 3 & 3-\lambda \end{vmatrix}$$

$$= (4-\lambda)(3-\lambda) - 6$$

$$= -4\lambda + \lambda^2 - 6 - 3\lambda$$

$$= \lambda^2 - 7\lambda + 6$$

If $|A - \lambda I| = 0$

$\therefore \lambda^2 - 7\lambda - 6 = 0$

$\Rightarrow \cancel{\lambda^2 - 2\lambda}$ $\lambda_1 = 6$, $\lambda_2 = 1$

Q. $\begin{bmatrix} 5 & 4 \\ 2 & 2 \end{bmatrix}$

Soln. Let $\lambda \in F$ be any scalar such that characteristic polynomial are

$$[A - \lambda I] = \begin{bmatrix} 5 & 4 \\ 2 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5-\lambda & 4 \\ 2 & 2-\lambda \end{bmatrix}$$

$$[A - \lambda I] = \begin{vmatrix} 5-\lambda & 4 \\ 2 & 2-\lambda \end{vmatrix}$$

$$(5-\lambda)(2-\lambda) - 8$$
$$= 10 - 5\lambda - 2\lambda + \lambda^2 - 8$$
$$= \lambda^2 - 7\lambda + 2$$

If $|A - \lambda I| = 0$

$$\therefore \lambda^2 - 7\lambda + 2 = 0$$

$$\lambda_1 = \frac{1+\sqrt{41}}{2}, \quad \lambda_2 = \frac{1-\sqrt{41}}{2}$$

Q. $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

Soln. let $\lambda \in F$ be any scalar such that characteristic polynomial are

$$[A-\lambda I] = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{array}{c} \begin{array}{ccc} a_{11} & a_{12} & a_{13} \end{array} \\ \begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix} \end{array}$$

$\cancel{+(8-\lambda)\{(7-\lambda)(3-\lambda)\}}$

$8-\lambda\{(7-\lambda)(3-\lambda)-16\} + 6\{(-6(3-\lambda)+8)\} + 2\{\cancel{6}\,24 - 2(7-\lambda)\}$

$\Rightarrow 8-\lambda\{21-7\lambda-3\lambda+\lambda^2\} + 6\{-18+6\lambda+8\} + 2\{24-14+2\lambda\}$

$\Rightarrow 8-\lambda\{21-7\lambda-3\lambda+\lambda^2\} + 6\{-10+6\lambda\} + 2\{10+2\lambda\}$

$\Rightarrow -\lambda^3 + 18\lambda^2 - 45\lambda = 0$

$\Rightarrow \underline{\lambda_1 = 15}, \ \underline{\lambda_2 = 3}, \ \underline{\lambda_3 = 0}$

DATE : / /

Q. Find eigen value and eigen vector where $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$.

soln. we have characteristic polynomial

$$[A - \lambda I] = \begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix}$$

characteristic equation $|A - \lambda I| = 0$ $\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$

$(5-\lambda)(2-\lambda) - 4 = 0$

$10 - 5\lambda - 2\lambda + \lambda^2 - 4 = 0$

$\therefore \lambda^2 - 7\lambda + 6 = 0$

$\lambda^2 - 6\lambda - \lambda + 6 = 0$

$(\lambda - 6)(\lambda - 1) = 0$

$\therefore \lambda = 6, 1$

let $\lambda_1 = 6$, and $\lambda_2 = 1$

These are the eigen value of the given matrix A.

$[A - \lambda I] X = 0$ where $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Taking $\lambda = \lambda_1$ we have

$[A - \lambda_1 I] X = 0$ let $\lambda_1 = 6$

$[A - 6I] X = 0$ $\begin{bmatrix} 5-6 & 4 \\ 1 & 2-6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $R_2 \rightarrow R_1 + R_2$

$\begin{bmatrix} -1 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore \frac{x_1}{4} = \frac{x_2}{1}$

$x_1 = 4$

$x_2 = 1$

Here Rank of matrix is unity
(Rank is the no. of non-zero rows)

Now, $-x_1 + 4x_2 = 0 \Rightarrow x_1 = 4x_2$

$$X_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Q. Find eigen value and eigen vector where $A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$

P.T.O.

Soln.

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

$$[A - \lambda I] = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3-\lambda & 2 & 4 \\ 2 & 0-\lambda & 2 \\ 4 & 2 & 3-\lambda \end{bmatrix}$$

$$\Rightarrow (3-\lambda)\left[(-\lambda)(3-\lambda) - 4\right] - 2\left[(2)(3-\lambda) - 8\right] + 4\left[4 - 4\lambda\right] = 0$$

$$\Rightarrow (3-\lambda)\left[-3\lambda + \lambda^2 + 4\right] + 2\left[6 - 2\lambda - 8\right] + 4\left[4 - 4\lambda\right]$$

$$\Rightarrow -9\lambda + 3\lambda^2 + 12 + 3\lambda^2 - \lambda^3 - 4\lambda - 4\lambda - 4 + 16 - 16\lambda$$

$$\Rightarrow -\lambda^3 + 6\lambda^2 - 33\lambda + 24$$

$$\Rightarrow \lambda(-\lambda^2 + 6\lambda - 33) + 24$$

PAGE NO. $) = 0$

DATE: $(\lambda+1) = 0$

$\lambda = -1$

$\Rightarrow \cancel{3\ \text{ex}}$

On solving, we get.

$$-\lambda^3 + 6\lambda^2 + 15\lambda + 8 = 0$$

$$-(\lambda^3 - 6\lambda^2 - 15\lambda - 8) = 0$$

$$\Rightarrow \lambda^3 - 6\lambda^2 - 15\lambda - 8 = 0$$

$$\therefore (\lambda+1) = 0$$

So, $\lambda^2(\lambda+1) - 7\lambda(\lambda+1) - 8(\lambda+1) = 0$

$$\Rightarrow (\lambda+1)(\lambda^2 - 7\lambda - 8) = 0$$

$$\Rightarrow (\lambda+1)(\lambda^2 - 8\lambda + \lambda - 8) = 0$$

$$\lambda(\lambda - 8) + 1(1 - 8) = 0$$

$$(\lambda+1)(\lambda+1)(\lambda-8) = 0$$

$$\therefore \lambda = -1, -1, 8 \quad \text{(eigen value)}$$

Now, eigen vector,

$$[A - \lambda I]x = 0$$

$$[A - \lambda I]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Put, $\boxed{\lambda = 8}$

$$\begin{pmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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*** CAYLEY HAMILTON THEOREM –

* Every square matrix satisfied own characteristic Equation.

i.e., $|A - \lambda I| = 0$

where A is square matrix and λ is scalar.

~~Verify cayley hamilton~~

Q. Verify cayley hamilton theorem and find its inverse.

$$\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

Soln $[A - \lambda I] = \begin{bmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{bmatrix}$

$\therefore |A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$

$(2-\lambda)\begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} + 1\begin{vmatrix} -1 & -1 \\ 1 & 2-\lambda \end{vmatrix} + 1\begin{vmatrix} -1 & 2-\lambda \\ 2 & -1 \end{vmatrix}$

$$(2-\lambda)\{(2-\lambda)^2-1\}+(-1)(2-\lambda)+1+1-(2-\lambda)=0$$

$$(2-\lambda)\{4+\lambda^2-4\lambda-1\}-2+\lambda+\cancel{1}+\cancel{1}-\cancel{2}+\lambda=0$$

$$(2-\lambda)\{\lambda^2-4\lambda+3\}+2\lambda-2=0$$

$$2\lambda^2-8\lambda+6-\lambda^3+4\lambda^2-3\lambda+2\lambda-2=0$$

$$-\lambda^3+6\lambda^2+9\lambda+4=0$$

$$\Rightarrow \underline{\lambda^3-6\lambda^2+9\lambda-4=0}$$

By cayley hamilton theorem,

put $\lambda = A$

$$A^3-6A^2+9A-4I=0 \quad \text{———} \quad (1)$$

$$\cancel{AA = A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}}$$

$$AA = A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} \{(2\times2)+(-1\times-1)+(1\times1)\} & \{(2\times-1)+(-1\times2)+(1\times-1)\} & \{(2\times1)+(1\times-1)+(1\times2)\} \\ \{(-1\times2)+(2\times-1)+(-1\times1)\} & \{(-1\times-1)+(2\times2)+(-1\times-1)\} & \{(-1\times1)+(2\times-1)+(-1\times2)\} \\ \{(1\times2)+(-1\times-1)+(2\times1)\} & \{(1\times-1)+(-1\times2)+(2\times-1)\} & \{(1\times1)+(-1\times-1)+(2\times2)\} \end{bmatrix}$$

$$\Rightarrow A^2 = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

Taking LHS we have

$$A^3 - 6A^2 + 9A - 4I$$

$$\Rightarrow \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

~~Multiplying~~

P.T.O.

Multiplying (1) by A^{-1}

$$A^{-1}\left[A^3 - 6A^2 + 9A - 4I\right] = 0$$

$$A^3A^{-1} - 6A^2A^{-1} + 9AA^{-1} - 4IA^{-1} = 0$$

$$A^2 - 6A + 9I - 4A^{-1} = 0$$

$$\therefore\ 4A^{-1} = A^2 - 6A + 9I$$

$$A^{-1} = \frac{1}{4}\left[A^2 - 6A + 9I\right]$$

$$\Rightarrow A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

P.T.O.

Similarly

let A and B be $n \times n$ (square) matrix over the field F for which there exist ~~an~~ an Invertible Matrix P such that $B = P^{-1} A P$.

Then B is said to be similar to A over F and it is expressed by $B \sim A$.

Example: let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

Find B: Now $P^{-1} = \dfrac{Adj\ P}{|P|}$

Now $|P| = 2 - 1 = 1$, $P_{11} = 2$, $P_{12} = -1$, $P_{21} = -1$, $P_{22} = 1$

$$\text{cofactor of } P = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\therefore Adj\ A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\Rightarrow P^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

Adjoint of 2 X 2 Matrix —

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\Rightarrow \text{Adj}\, A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

change signs

exchange

Now, $P^{-1}AP = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

$P^{-1}AP = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & +4 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 0 & 1 \end{bmatrix}$

*** BILINEAR FORM -

* A polynomial of set of variable $(x_1, x_2, \ldots x_n)$ and $(y_1, y_2, \ldots y_n)$ in which the cofficient are the member and is homogeneous, linear with respect to variable in each of these set is called bilinear form

* In order to express as following -

$$A(x,y) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i y_j$$

$$= a_{11}x_1y_1 + a_{12}x_1y_2 + a_{13}x_1y_3 + \ldots + a_{21}x_2y_1 + a_{22}x_2y_2 + a_{23}x_2y_3 + \ldots + a_{31}x_3y_1 + a_{32}x_3y_2 + a_{33}x_3y_3 + \ldots \quad (1)$$

Here we study only real bilinear form in which the ~~cofi~~ cofficient are a_{ij} and variable x_i, y_j are real quantity then this system (1) can be written as

$$A(x,y) = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= X' A Y$$

where $X' = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Q. $A(x,y) = x_1y_1 + 2x_1y_2 + 3x_2y_1 + 4x_2y_2 + 5x_3y_1 + 6x_3y_2$

Ans. $A(x,y) = [x_1 \; x_2 \; x_3] \begin{bmatrix} x_1y_1 & 2x_1y_2 \\ 3x_2y_1 & 4x_2y_2 \\ 5x_3y_1 & 6x_3y_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$X' = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$, $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

P.T.O.

$z = x + iy$
$\bar{z} = x - iy$
(conjugate)

R = Real plane
C = Complex plane
$\langle \cdot \rangle$ = symbol of mapping

*** INNER PRODUCT SPACE (IPS)

Let X be linear space over the field F (R or C) then $\langle \cdot \rangle : X \times X \rightarrow R$ is called linear product.

If product is satisfied following condition

For any $x, y, z \in X$ and $\alpha \in F$

(i) $\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

(ii) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$

(iii) $\langle x, y \rangle = \overline{\langle y, x \rangle}$ complex conjugate.

(iv) $\langle x, x \rangle \geq 0$,

(v) $\langle x, x \rangle = 0 \Leftrightarrow x = 0$

The linear space X with inner product i.e., $\langle X, \langle \cdot \rangle \rangle$ is called ~~inner~~ inner product space.

P.T.O.

DATE: / /

Q. Consider linear space R^n defined over the field

$$x = (x_1, x_2, \ldots x_n)$$

$$y = (y_1, y_2, \ldots y_n)$$ such that

$$\langle x, y \rangle = \sum_{k=1}^{n} x_k y_k$$

Then show that $(\langle\ \rangle \cdot)$ is ~~linear~~ inner product space

Soln. ~~If~~ let $x, y, z \in R^n$ and let α be real no. then

① $$\langle x+y, z\rangle = \sum_{k=1}^{n} (x_k + y_k, z_k)$$

$$= \sum_{k=1}^{n} (x_k, z_k) + \sum_{k=1}^{n} (y_k, z_k)$$

$$(x, z) + (y+z)$$

② $$(\alpha x, y) = \sum_{k=1}^{n} \alpha x_k y_k = \alpha \sum_{k=1}^{n} x_k y_k$$

$$= \alpha (x, y)$$

③ $$\langle x, y \rangle = \sum_{k=1}^{n} x_k y_k = \sum_{k=1}^{n} y_k x_k = (y, x)$$

④ $$(x, x) = \sum_{k=1}^{n} x_k y_k = \sum_{k=1}^{n} y_k^2 \geq 0$$

$$= (x, x) \geq 0$$

⑤ $(x, x) = 0$

$$\sum_{k=1}^{n} x_k x_k \quad \text{if } x = 0$$

$$\sum_{k=1}^{n} |x_k|^2 =$$

$$x_1 = 0, \; x_2 = 0 \ldots, \; x_n = 0$$

$x = 0$

$= 0$

Hence we see that all 5 conditions of Inner product space are satisfied by space R^n.

Q. Prove that complex plain is IPS.

Q: Prove that $V_2(R)$ is an IPS with an inner product defined by

$\alpha = (a_1, a_2)$, $\beta = (b_1, b_2)$ $\forall \alpha, \beta \in R$

① $(\alpha, \beta) = 3a_1 b_1 + 2 a_2 b_2$

② $(\alpha, \beta) = a_1 b_1 - a_2 b_1 - a_1 b_2 + 4 a_2 b_2$

③ $(\alpha, \beta) = a_1 b_1 - a_2 b_1 - a_1 b_2 + 2 a_2 b_2$

Soln Linearity

Let $\gamma = c_1, c_2 \in V_2(R)$ then for $a, b \in R$

$$a\alpha + b\beta = a(a_1, a_2) + b(b_1, b_2)$$
$$= (aa_1, aa_2) + (bb_1, bb_2)$$
$$= \underbrace{(aa_1 + bb_1)}_{X}, \underbrace{(aa_2 + bb_2)}_{Y}$$

$$(a\alpha + b\beta, \gamma) = a(\alpha, \gamma) + b(\beta, \gamma)$$

(I) we have,

$$(a\alpha + b\beta, \gamma) = 3(aa_1 + bb_1)c_1 + 2(aa_2 + bb_2)c_2$$

($\because$ by (I)).

$$= a(3a_1 c_1 + 2a_2 c_2) + b(3b_1 c_1 + 2b_2 c_2)$$
$$= a(\alpha, \gamma) + b(\beta, \gamma)$$

(II) Symmetric property – $(\beta, \alpha) = (\alpha, \beta)$

$[(b_1, b_2)(a_1, a_2)]$

$$(\beta, \alpha) = (\alpha, \beta)$$

$3b_1 a_1 + 2b_2 a_2$
$3b_1 a_1 + 2b_2 a_2$

let $(\beta, \alpha) = 3b_1 a_1 + 2b_2 a_2$
$= 3a_1 b_1 + 2a_2 b_2 = (\alpha, \beta)$
$\therefore (\beta, \alpha) = \alpha, \beta.$

$\because a_1, a_2, b_1, b_2$ are the member of $V_2(R)$
$\therefore$ It satisfies commutative

(III) Non negative – put $\beta = \alpha$ in (1) we get,

$(\alpha, \alpha) = 3a_1 a_1 + 2a_2 a_2$
$\Rightarrow 3a_1^2 + 2a_2^2$ $\quad \therefore 3a_1^2 + 2a_2^2$
$\Rightarrow a_1 = 0, a_2 = 0$
$\Rightarrow \alpha = 0$

Q. $\langle x, y \rangle = \sum_{K=1}^{n} x_k \bar{y}_k$ is $\langle \cdot \rangle$ space over x, $(\langle \cdot \rangle, x)$

Soln. let $x, y, z \in C^n$ and α be scalar
Then,
(1) $\langle x+y, z \rangle = \sum_{K=1}^{n} (x_k + y_k)\bar{z}_k = \sum_{K=1}^{n} x_k \bar{z}_k$
$+ \sum_{K=1}^{n} y_k \bar{z}_k$
$= \langle x, z \rangle + \langle y, z \rangle$

(ii) $\langle \alpha x, y \rangle = \sum_{k=1}^{n} \alpha x_k \overline{y_k} = \alpha \sum_{k=1}^{n} (x_k \overline{y_k}) = \alpha \langle x, y \rangle$

(iii) $\overline{(y, x)} = \sum_{k=1}^{n} \overline{y_k \overline{x_k}} = \sum_{k=1}^{n} \overline{y_k} \, \overline{\overline{x_k}}$

$= \sum_{k=1}^{n} \overline{y_k} x_k = \sum_{k=1}^{n} x_k \overline{y_k} = \langle x, y \rangle$

(iv) $\langle x, x \rangle = \sum_{k=1}^{n} x_k \overline{x_k} = \sum_{k=1}^{n} |x_k|^2 \geq 0$

(v) $\langle x, x \rangle = 0$ iff $\underline{x = 0}$

we have, $\sum_{k=1}^{n} |x_k|^2 = 0$

$x_1 + x_2 + x_3 + \ldots \leq 0$

$x_1 = 0, \; x_2 = x_3 = 0$

$\therefore x = 0$

we see that the map$\langle \cdot \rangle$ defined in complex plain is an inner product space.

Q. Prove that in complex plane is an inner product space.

*** CAUCHY SCHWARZ INEQUALITY

* If X be linear space and $(x, \langle \cdot \rangle)$ be Inner product space and let $x \in X$

then $|\langle x, y\rangle| \leq \sqrt{\langle x, x\rangle} \cdot \sqrt{\langle y, y\rangle}$

If x, y are linearly independent

Proof: If $x=0$ and $y=0$ the inequality is satisfied.

let $x \neq 0$ and $y \neq 0$ for any scalar α, we have

$$0 \leq \langle \alpha x + y, \alpha x + y\rangle = \langle \alpha x, \alpha x + y\rangle + \langle y, \alpha x + y\rangle$$

or

or $0 \leq \langle \alpha x, \alpha x\rangle + \langle \alpha x, y\rangle + \langle y, \alpha x\rangle + \langle y, y\rangle$

or $0 \leq |\alpha|^2 \langle x, x\rangle + \alpha \langle x, y\rangle + \alpha \langle y, x\rangle + \langle y, y\rangle$

or $0 \leq |\alpha|^2 \langle x, x\rangle + \alpha \langle x, y\rangle + \bar{\alpha} \langle \overline{x, y}\rangle + \langle y, y\rangle$

$$0 \leq |\alpha| \langle x, x \rangle + 2 \text{ Real} \cdot \alpha \langle x, y \rangle + \langle y, y \rangle$$

let $\alpha = -\dfrac{\overline{\langle x, y \rangle}}{\langle x, x \rangle}$ then $|\alpha|^2 = \dfrac{|\overline{\langle x, y \rangle}|^2}{|\langle x, x \rangle|^2}$

$$= \frac{|\langle x, y \rangle|^2}{\langle x, x \rangle^2} \quad (\because |\bar{z}| = |z|)$$

Thus $0 \leq \dfrac{|\langle x, y \rangle|^2}{\langle x, x \rangle} - 2\dfrac{|\langle x, y \rangle|^2}{\langle x, x \rangle} + \langle y, y \rangle$

$(\because z\bar{z} = |z|^2)$

$(\because |\langle y, y \rangle| = \langle y, y \rangle = \text{real no.})$

$$0 \leq -\frac{|\langle x, y \rangle|^2}{\langle x, x \rangle} + \langle y, y \rangle$$

$$\Rightarrow \frac{|\langle x, y \rangle|^2}{\langle x, x \rangle} \leq \langle y, y \rangle$$

$$\Rightarrow |\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle$$

$$\Rightarrow |\langle x, y \rangle| \leq \sqrt{\langle x, y \rangle} \sqrt{\langle y, y \rangle} \quad \text{——— (I)}$$

Suppose the equality holds in (I), i.e.,

$$|\langle x, y \rangle| \leq \sqrt{\langle x, x \rangle} \sqrt{\langle y, y \rangle}$$

for

$$\alpha = -\frac{\overline{\langle x, y\rangle}}{\langle x, x\rangle}$$

$$\langle \alpha x + y, \alpha x + y\rangle = |\alpha|^2 \langle x, x\rangle + 2\,\mathrm{Re}\,\alpha(x,y) + \langle y, y\rangle$$

$$= -\frac{|\langle x, y\rangle|^2}{\langle x, x\rangle} + \langle y, y\rangle$$

$$= -\frac{\cancel{\langle x, x\rangle} \cdot \langle y, y\rangle}{\cancel{\langle x, x\rangle}} + \langle y, y\rangle$$

$\Rightarrow \langle \alpha x + y, \alpha x + y\rangle = 0$

$\Rightarrow \alpha x + y = 0$

$\therefore$ x, y are linearly dependent vectors

Conversely – let x, y are linearly dependent, then we can ~~create~~ write $x = \beta y$ for some scalar β then we have

$$\langle x, y\rangle = \langle \beta y, y\rangle = \beta \langle y, y\rangle$$

$\Rightarrow |\langle x, y\rangle| = |\beta| \langle y, y\rangle$ $\left[\because |\langle y, y\rangle| = \langle y, y\rangle = \text{Real no.}\right.$

Also $\langle x, x\rangle \cdot |\langle y, y\rangle| = \langle \beta y, \beta y\rangle \cdot \langle y, y\rangle$

$$= \beta\bar{\beta} \cdot \langle y, y\rangle \langle y, y\rangle$$

$$= |\beta|^2 (\langle y, y\rangle)^2$$

or $\sqrt{\langle x, x\rangle} \cdot \sqrt{\langle y, y\rangle} = |\beta| \langle y, y\rangle$

i.e. $|\langle x, y\rangle| = \sqrt{\langle x, x\rangle} \cdot \sqrt{\langle y, y\rangle}$

*** NORM OR LENGTH OF A VECTOR IN AN INNER PRODUCT SPACE-

Let V be an Inner Product space. If $\alpha \in V$, then the norm or the length of the vector α, written as $\|\alpha\|$, is defined as the positive square root of (α, α) i.e.

$$\|\alpha\| = \sqrt{(\alpha, \alpha)}.$$

Ex 1: In the inner product space $V_n(C)$, If $\alpha = (a_1, a_2, a_3, \ldots a_n)$ be any vector then

$$\|\alpha\| = \left(|a_1|^2 + |a_2|^2 + \cdots + |a_n|^2\right)^{1/2}$$

*** UNIT VECTORS-

Let V be an Inner product space. If $\alpha \in V$ is such that $\|\alpha\| = 1$ then α is called a unit vector. Thus in an inner product space a vector is called a unit vector If its length is unity.

P.T.O.

NOTE: If α is any non-zero vector in an Inner product space V, then $\frac{\alpha}{\|\alpha\|} \in V$ is a unit vector

~~If its ~~do~~ length is unity~~

Since $\alpha \neq 0$, so $\|\alpha\| \neq 0$, therefore $\frac{\alpha}{\|\alpha\|} \in V$

Now $\left(\frac{\alpha}{\|\alpha\|}, \frac{\alpha}{\|\alpha\|}\right) = \frac{1}{\|\alpha\|}\left(\alpha, \frac{\alpha}{\|\alpha\|}\right)$

$$= \frac{1}{\|\alpha\| \cdot \|\alpha\|}(\alpha, \alpha) = \frac{\|\alpha\|^2}{\|\alpha\|^2} = 1$$

$\therefore \left\|\frac{\alpha}{\|\alpha\|}\right\| = 1$ and so $\frac{\alpha}{\|\alpha\|}$ is a unit vector.

Ex. If $\alpha = (2,4,4)$ be vector in $V_3(R)$ with standard Inner Product Space. Then, $\|\alpha\| = \sqrt{(\alpha, \alpha)} = \sqrt{4+16+16} = \sqrt{36} = 6$

$\therefore \frac{\alpha}{\|\alpha\|} = \frac{(2,4,4)}{6} = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$ is a unit vector.

Eg. If V(F) be a vector space of all polynomials in x in which an Inner product is defined by $(p, q) = \int_0^1 p(x) \cdot q(x)\,dx$.

where $p = p(x)$, $q = q(x) \in V$

Then for $p(x) = x + 2$, $q(x) = x^2 - 2x - 3$

Find –

(i) (p, q) (ii) $\|p\|$ (iii) $\|q\|$ (iv) Angle between p and q.

Solu. (i) $(p, q) = \int_0^1 p(x) \cdot q(x)\,dx$

$$= \int_0^1 (x+2)(x^2 - 2x - 3)\,dx$$

$$= \int_0^1 (x^3 - 2x^2 - 3x + 2x^2 - 4x - 6)\,dx$$

$$= \frac{-37}{4}$$

(ii) $\|p\|^2 = (p, p) = \int_0^1 p(x) \cdot q(x)\,dx$

$$= \int_0^1 (x+2) \cdot (x+2)\,dx$$

$$= \int_0^1 (x^2 + 2x + 4)\,dx = 19/3$$

$\therefore \|p\| = \sqrt{\frac{19}{3}}$

(iii) $$\|q\|^2 = (q, q) = \int_0^1 q(x) \cdot q(x)\, dx$$

$$= \int_0^1 (x^2-2x-3)(x^2-2x-3)\, dx$$

$$= \int_0^1 (x^4 - 4x^3 - 2x^2 + 12x + 9)\, dx$$

$$= \frac{203}{15}$$

$$\therefore \|q\| = \sqrt{\frac{203}{15}}$$

(iv) Angle between p and q. Then,

$$\cos\theta = \frac{(p,q)}{\|p\|\,\|q\|} = \frac{-37/4}{\sqrt{\frac{19}{3} \cdot \frac{203}{15}}}$$

$$= \frac{-37\sqrt{5}}{12\sqrt{3857}}$$

$$\therefore \theta = \cos^{-1}\left\{\frac{-37\sqrt{5}}{12\sqrt{3857}}\right\}$$

DATE: / /

*** PARALLELOGRAM PROPERTY –

Q. If α and β are vector in an Inner product space V(F). Prove that $\|\alpha+\beta\|^2 + \|\alpha-\beta\|^2 = 2\|\alpha\|^2 + 2\|\beta\|^2$

Soln. we have,

$$\|\alpha+\beta\|^2 = (\alpha+\beta, \alpha+\beta)$$

$$\Rightarrow (\alpha, \alpha+\beta) + (\beta, \alpha+\beta) = (\alpha, \alpha) + (\alpha, \beta) + (\beta, \alpha) + (\beta, \beta)$$

$$= \|\alpha\|^2 + (\alpha, \beta) + (\beta, \alpha) + \|\beta\|^2 \quad \text{— (1)}$$

$$\|\alpha-\beta\|^2 = (\alpha-\beta, \alpha-\beta) = (\alpha, (\alpha-\beta)) - (\beta, \alpha-\beta)$$

$$= \langle\alpha, \alpha\rangle - \langle\alpha, \beta\rangle - \langle\beta, \alpha\rangle + \langle\beta, \beta\rangle$$

$$= \|\alpha\|^2 - (\alpha, \beta) - \langle\beta, \alpha\rangle + \|\beta\|^2 \quad \text{— (2)}$$

Adding (1) and (2) we get.

$$\|\alpha+\beta\|^2 + \|\alpha-\beta\|^2 = 2\|\alpha\|^2 + 2\|\beta\|^2$$

Q. Prove that in an Inner Product space if the vector α and β are linearly dependent if and only if

$$|\langle\alpha, \beta\rangle| = \|\alpha\| \cdot \|\beta\|$$

Soln. The condition is necessary:
let α, β be linearly dependent
i.e. one of them can be expressed
as a scalar multiple of the other.

let $\alpha = a\beta$
where $a \in F$

Then we have:

$$|\langle \alpha, \beta \rangle| = |\langle a\beta, \beta \rangle| = |a| \, \|\beta\|^2 \quad \text{--- (1)}$$

and $\|\alpha\| = \|a\beta\| = |a| \, \|\beta\|$

$$\therefore \|\alpha\| . \|\beta\| = |a| \, \|\beta\|^2 \quad \text{--- (2)}$$

From (1) and (2) we have,

$$|\langle \alpha, \beta \rangle| = \|\alpha\| \, \|\beta\|$$