Unit 2

Eigen value and Eigen vector of linear transformation

Let $T: V \to V$ a linear operator of n dimension vector space over the field F. A scalar $\lambda \in F$ is called Eigen value of T if there exists a non-zero vector α in V such that $T(\alpha) = \lambda\alpha$ any $\alpha \neq 0$ of V, such that $T(\alpha) = \lambda\alpha$ is called an eigen vector of T belonging to the Eigen of λ. The W_λ is known as collection of all eigen value which is a subspace

$$\lambda^n + A_1\lambda^{n-1} + A_2\lambda^{n-2} \pm \quad A_n.$$

Step I: Characteristic equation $|A - \lambda I| = 0$

Step II: Solving given equation we get value λ known as Eigen value.

Step III: For Eigen vector depend upon the value of eigen value.

Ex. $A = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$

Sol^n Let $\lambda \in F$ be any scalar such that characteristic polynomial are

$$[A - \lambda I] = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3-\lambda & 2 \\ -1 & 0-\lambda \end{bmatrix}$$

$$|A-\lambda I| = \begin{vmatrix} 3-\lambda & 2 \\ -1 & -\lambda \end{vmatrix}$$

$$= (3-\lambda)(-\lambda) + 2$$
$$= -3\lambda + \lambda^2 + 2$$
$$= \lambda^2 - 3\lambda + 2$$

If $|A - \lambda I| = 0$

$$\therefore \lambda^2 - 3\lambda + 2 = 0$$
$$\lambda^2 - 2\lambda - \lambda + 2$$
$$\lambda(\lambda - 2) - 1(\lambda - 2)$$
$$(\lambda - 1)(\lambda - 2) = 0$$
$$\lambda = 1, 2$$

Q Find characteristic equation and characteristic value of the following matrix

1 $A = \begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix}$

2 $\begin{bmatrix} 5 & 4 \\ 2 & 2 \end{bmatrix}$

3 $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

1 $A = \begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix}$

Let $\lambda \in F$ be any scalar such that characteristic polynomial are

$$[A - \lambda I] = \begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$|A - \lambda I| = \begin{bmatrix} 4-\lambda & 2 \\ 3 & 3-\lambda \end{bmatrix}$$

$$|A - \lambda I| = (4-\lambda)(3-\lambda) - 2(3)$$

$$= -4\lambda + \lambda^2 - 6$$

$$= \lambda^2 - 4\lambda - 6$$

$$\lambda^2 - 4\lambda - 6 = 0$$

$$= 12 - 3\lambda - 4\lambda + \lambda^2 - 6$$

$$\lambda^2 - 7\lambda + 6$$

$$\lambda^2 - 6\lambda - \lambda + 6$$

$$\lambda(\lambda - 6) - 1(\lambda - 6)$$

$$(\lambda - 1)(\lambda - 6) = 0$$

$$\lambda = 1, 6.$$

(ii) $$|A - \lambda I| = \begin{bmatrix} 5 & 4 \\ 2 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$|A - \lambda I| = \begin{bmatrix} 5-\lambda & 4 \\ 2 & 2-\lambda \end{bmatrix}$$

$$(5-\lambda)(2-\lambda) - 4(2)$$

$$10 - 2\lambda - 5\lambda + \lambda^2 - 8.$$

$$= \lambda^2 - 7\lambda + 2$$

$$\lambda = \frac{7 \pm 2\sqrt{14}}{2}$$

0, 3, 15

(iii) $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

$$|A - \lambda I| = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix}$$

$$= 8-\lambda\left[7-\lambda(3-\lambda) - (-4)(-4)\right]$$
$$+ \left[6\left[-6(3-\lambda) - (2(-4))\right]\right.$$
$$+ \left[2\left[+24 - (14-2\lambda)\right]\right]$$

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$$= 8-\lambda\left[21 - 3\lambda - 7\lambda + \lambda^2 - 16\right]$$
$$+ \left[6\left[-18 + 6\lambda + 8\right]\right]$$
$$+ \left[2\left[24 - 14 + 2\lambda\right]\right]$$

$$= \cancel{8}\,40 - 80\lambda + 8\lambda^2 - 5\lambda + 10\lambda^2 - \lambda^3$$
$$+ -60 + 36\lambda\ \cancel{+48}\ 20 + 4\lambda$$

$$= -\lambda^3 + 18\lambda^2 - 45\lambda$$

$$\lambda^3 - 18\lambda^2 + 45\lambda = 0$$
$$\lambda\left[\lambda^2 - 18\lambda + 45\right]$$
$$(\lambda - 0)(\lambda - 3)(\lambda - 15) = 0$$

$$\lambda = 0, 3, 15.$$

1. Find Eigen value and Eigen vectors where

$$A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}.$$

Sol^n we have: Characteristic Polynomial

$$[A - \lambda I] = \begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix}$$

characteristic Equation $|A - \lambda I| = 0$ $\quad \begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$

$$(5-\lambda)(2-\lambda) - 4 = 0 \qquad 10 - 5\lambda - 2\lambda + \lambda^2 - 4 = 0$$

$$\lambda^2 - 7\lambda + 6 = 0$$

$$\lambda^2 - 6\lambda - \lambda + 6 = 0$$

$$\lambda(\lambda - 6) - 1(\lambda - 6)$$

$$(\lambda - 1)(\lambda - 6)$$

$$\lambda = 1, 6$$

Let $\lambda_1 = 6$, and $\lambda_2 = 1$, These are the Eigen values of the given matrix A.

$$[A - \lambda I] X = 0$$

where $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Taking $\lambda = \lambda_1$ we have

$[A - \lambda_1 I] X = 0$ let $\lambda_1 = 6$

$$[A - 6I] X = 0$$

$$\begin{bmatrix} 5-6 & 4 \\ 1 & 2-6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$R_2 \Rightarrow R_1 + R_2$$

$$\begin{bmatrix} -1 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Here Rank of matrix is unity.

Now $-x_1 + 4x_2 = 0$

$\Rightarrow x_1 = 4x_2$

$$\frac{x_1}{4} = \frac{x_2}{1}$$

$x_1 = 4$

$x_2 = 1$

$$X = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Again taking $\lambda = \lambda_2$ we have

$[A - \lambda_2 I] X = 0$ let $\lambda_2 = 1$

$[A - I] X = 0$

$$\begin{bmatrix} 5-1 & 4 \\ 1 & 2-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$R_2 \to R_2 - \frac{R_1}{4}$$

$$\begin{bmatrix} 4 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Hence Rank is unity.

Now $4x_1 + 4x_2 = 0$

$$\frac{x_1}{1} = \frac{-x_2}{1}$$

$x_1 = 1$

$x_2 = -1$

$$X = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Q $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

Sol^n $|A - \lambda I| = 0$

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$$(8-\lambda)\begin{vmatrix} 7-\lambda & -4 \\ -4 & 3-\lambda \end{vmatrix} + 6\begin{vmatrix} -6 & -4 \\ 2 & 3-\lambda \end{vmatrix} + 2\begin{vmatrix} -6 & 7-\lambda \\ 2 & -4 \end{vmatrix} = 0$$

$-\lambda^3 + 18\lambda^2 - 45\lambda = 0$

$\lambda^3 - 18\lambda^2 + 45\lambda = 0$

$\lambda(\lambda^2 - 18\lambda + 45) = 0$

$\lambda = 0 \quad (\lambda^2 - 15\lambda - 3\lambda + 45) = 0$

$\lambda = 0 \quad \lambda = 3, 15 \quad , \quad \lambda = 0, 3, 15.$

Let $\lambda = \lambda_1 = 0$

$\lambda = \lambda_2 = 3$

$\lambda = \lambda_3 = 15$

we have $[A - \lambda I]X_1 = 0$ and taking $\lambda = \lambda_1 = 0$

and $X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$8x_1 - 6x_2 + 2x_3 = 0$

$-6x_1 + 7x_2 - 4x_3 = 0$

$2x_1 - 4x_2 + x_3 = 0$

Find eigen value and eigen vector of

1 $A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$

2 $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

3 $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

$$|A - \lambda I| = 0$$

$$\begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$\begin{bmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{bmatrix} = 0$$

$$3-\lambda\left[-\lambda(3-\lambda) - 4\right] - 2\left[2(3-\lambda) - 8\right] + 4(4 + 4\lambda) = 0$$

$$3-\lambda(-3\lambda + \lambda^2 - 4) - 2\left[-2\lambda - 2\right] + 16 + 16\lambda = 0$$

$$-9\lambda + 3\lambda^2 - 12 + 3\lambda^2 - \lambda^3 + 4\lambda + 4\lambda + 4 + 16 + 16\lambda = 0$$

$$-\lambda^3 + 6\lambda^2 + 15\lambda + 8 = 0$$

$-1, -6+15-8$

$8, -1, -1$

$\cancel{\lambda^3 - 6\lambda^2 + 15\lambda + 8} = 0$

$\lambda^3 - 6\lambda^2 - 15\lambda - 8 = 0$

$$
\begin{array}{l}
(\lambda+1)\,\overline{)\lambda^3 - 6\lambda^2 - 15\lambda - 8}\,(\lambda^2 - 7\lambda - 8 \\
\quad -\lambda^3 \mp \lambda^2 \\
\hline
\qquad -7\lambda^2 - 15\lambda - 8 \\
\qquad \pm 7\lambda^2 \mp 7\lambda - 8 \\
\hline
\qquad\qquad -8\lambda - 8 \\
\qquad\qquad \mp 8\lambda \mp 8 \\
\hline
\qquad\qquad 0 \quad 0
\end{array}
$$

$(\lambda+1)(\lambda^2 - 7\lambda - 8)$

$(\lambda+1)(\lambda+1)(\lambda-8)$

$\lambda = -1, -1, 8$

$$
\begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

$4x_1 + 2x_2 + 4x_3 = 0$

$2x_1 + 1x_2 + 2x_3 = 0$

$4x_1 + 2x_2 + 4x_3 = 0$

Cayley Hamilton Theorem:-

Every square matrix satisfied own characteristic equation -

i.e $|A-\lambda I|=0$

where A is square matrix and λ is scalar:-

verify Cayley Hamilton Theorem and find its inverse.

$$\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$|A-\lambda I| = \begin{bmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{bmatrix}$$

$$\therefore |A-\lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix}$$

$$2-\lambda((2-\lambda)^2-1)-1(-1(2-\lambda)+1)+1(1-(2-\lambda))=0$$

$$2-\lambda(4+\lambda^2-4\lambda-1] = 2+\lambda+1+1-2+\lambda=0$$

$$2-\lambda\{4+\lambda^2-4\lambda-1\}+2\lambda-2=0$$

$$2\lambda^2-8\lambda+6-\lambda^3+4\lambda^2-3\lambda+2\lambda-2=0$$

$$-\lambda^3+6\lambda^2-9\lambda+4=0$$

$$\lambda^3-6\lambda^2+9\lambda-4=0$$

Now put $\lambda = A$

$$A^3 - 6A^2 + 9A - 4I = 0 \quad \text{—(1)}$$

$$\therefore \quad A \cdot A = A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 4+1+1 & -2-2-1 & 2+1+1 \\ -2-2-1 & 1+4+1 & -1-2-3 \\ 2+1+2 & -1-2-2 & 1+\cancel{1}+4 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} = A^2$$

$$\text{Now } A^3 = A^2 \cdot A = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$$\text{Now } A^3 - 6A^2 + 9A - 4I = 0$$

$$\begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6 \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9 \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$So = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - \begin{bmatrix} 36 & -30 & 30 \\ -30 & 36 & -30 \\ 30 & -30 & 36 \end{bmatrix}$$

$$= \begin{bmatrix} -14 & 9 & -9 \\ 9 & -14 & 9 \\ -9 & 9 & -14 \end{bmatrix}$$

$$= \begin{bmatrix} -14 & 9 & -9 \\ 9 & -14 & 9 \\ -9 & 9 & -14 \end{bmatrix} + \begin{bmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Now $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} - 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= 0$

Multiplying (1) by A^{-1}

$$A^{-1}[A^3 - 6A^2 + 9A - 4I] = 0$$

$$A^3 . A^{-1} - 6A^2 . A^{-1} + 9A . A^{-1} - 4I . A^{-1} = 0$$

$$A^2 - 6A + 9I - 4A^{-1} = 0$$

$$\therefore \quad 4A^{-1} = A^2 - 6A + 9I$$

$$A^{-1} = \frac{1}{4}(A^2 - 6A + 9I)$$

1 $\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

Evaluate A^{-2}

2 $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

3 $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

Sat similarity

Let A and B be $n \times n$ (square) matrix over the field F for which there exists an invertible matrix P such that $B = P^{-1}AP$.

then B is said to be similar A over F and it is expressed $B \sim A$

ex Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

Fixed B: Now $P^{-1} = \dfrac{\text{Adj } A}{|P|}$

Now $|P| = 2 - 1 = 1$, $P_{11} = 2$

$P_{12} = -1$ $P_{21} = -1$ $P_{22} = 1$

Adj A $\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

$P^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$ Now $P^{-1}AP = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 2 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

$P^{-1}AP = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 0 & 1 \end{bmatrix}$

Theorem - If matrix A and B are similar then $|A| = |B|$

Proof :- Since B is similar to A hence $\exists$ an inverteable matrix such that $B = P^{-1}AP$
Then $|B| = |P^{-1}AP| = |P^{-1}|\,|A|\,|P|$
$= |A|\,|P^{-1}P| = |A|\,|1| = |A| \cdot 1 = |A|$

Theorem 2 Two similar matrix have same eigenvalue.

Proof - Let A and B be the two similar matrix then $\exists$ an invertiable matrix P such that

$B = P^{-1}AP$

Now $B - \lambda I = P^{-1}AP - \lambda I = P^{-1}AP - P^{-1}\lambda IP$
$= P^{-1}(A - \lambda I)P$

$\therefore |B - \lambda I| = |P^{-1}(A - \lambda I)P|$
$|P^{-1}|\,|A - \lambda I|\,|P|$
$= |A - \lambda I|\,|P^{-1}P| = |A - \lambda I|$

Hence A and B have same characteristic Polynomials and there so eigen value are same

~~Dea~~

Diagonalization of Matrices

Def: - A matrix A over the field F is said to be diagonalizable if it is similar to a diagonal matrix over the field F. Then a matrix A is diagonalizable if there exists an invertible matrix P such that $P^{-1}AP = D$, where D is a diagonal matrix also the matrix P is then said to be diagonalize A or transform A to a diagonal form.

We know that.

(1) The eigen values of a diagonal matrix are its diagonal elements and

The eigen values of similar matrix are same:

Hence if $P^{-1}AP = D = diag(d_1, d_2, d_n)$ Then the eigen value of A are d_1, d_2, d_n

Diagonalizable operators

Def: Let V be finite dimensional vector space over field F and $T: V \to V$ a linear operator, then the operator T is said to be diaganalizable if a basis B of V be such that elements of B are eigen vectors of T

let $B = \{\alpha_1, \alpha_2 \ldots \alpha_n\}$ be a basis of an n dimension vector space V where each α_i, $1 \leq i < n$ is an eigen vector of T

$P^{-1}AP = dig(\lambda_1, \lambda_2 \dots \lambda_n)$

step 1 : characteristic equation

step 2 : Eigen value

step 3 - eigen vector

step 4 $P = [x_1, x_2]$

step 5 $P^{-1} = \frac{P, d_i P}{|P|}$

Step $P^{-1}AP = diag(\lambda_1, \lambda_2)$

Diagonization form.

Q

$$A = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & -1 & 4 \\ 3 & 2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{vmatrix} = 0$$

$$1-\lambda \begin{vmatrix} 2-\lambda & -1 \\ 1 & -1-\lambda \end{vmatrix} + 1 \begin{vmatrix} 3 & -1 \\ 2 & -1-\lambda \end{vmatrix} + 4 \begin{vmatrix} 3 & 2-\lambda \\ 2 & 1 \end{vmatrix} = 0$$

$$1-\lambda \left[(2-\lambda)(-(1+\lambda)+1\right] + \left[3(-1-\lambda)+2\right] + 4\left[3-2(2-\lambda)\right] = 0$$

$$-(2-\lambda)(1-\lambda^2)+1-\lambda-3-3\lambda+2+12-16+8\lambda = 0$$

$$\therefore \quad -2+2\lambda^2+\lambda-\lambda^3-4+4\lambda = 0$$

$$-\lambda^3+2\lambda^2+5\lambda-6 = 0$$

$$\lambda^3-2\lambda^2-5\lambda+6 = 0$$

$$\lambda^2(\lambda-1)-\lambda(\lambda-1)-6(\lambda-1) = 0$$

$$(\lambda-1)(\lambda^2-\lambda-6)=0$$
$$(\lambda-1)(\lambda^2-3\lambda+2\lambda-6)=0$$
$$(\lambda-1)(\lambda-3)(\lambda+2)=0$$
$$\lambda = 1, 3, -2$$

Now Eigen vector $[A-\lambda I]x_1 = 0$

$$\lambda = \lambda_1 = 1$$

$$\begin{bmatrix} 0 & -1 & 4 \\ 3 & +1 & -1 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$0x_1 - x_2 + 4x_3 = 0 \quad -(1)$$
$$3x_1 + x_2 - x_3 = 0 \quad -(2)$$
$$2x_1 + x_2 - 2x_3 = 0 \quad -(3)$$

From (1)

$$x_2 = +4x_3$$
$$\frac{x_2}{4} = \frac{x_3}{1}$$
$$\therefore x_2 = 4$$
$$x_3 = 1$$

$$3x_1 + 3 = 0$$
$$x_1 = -1$$

we have

$$x_1 = [-1 \quad 4 \quad 1]$$

$$\lambda = \lambda_2 = 3,$$

Let x_2.

$$\begin{bmatrix} -2 & -1 & 4 \\ 3 & -1 & -1 \\ 2 & 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 + R_1$

$$\begin{bmatrix} -2 & -1 & 4 \\ 3 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x_1 - x_2 + 4x_3 = 0$$
$$\underline{\underset{-}{3x_1} \underset{+}{-x_2} \underset{+}{-x_3} = 0}$$
$$-5x_1 + 5x_3 = 0$$
$$x_1 = x_3$$
$$\frac{x_1}{1} = \frac{x_3}{1}$$
$$x_1 = 1, \quad x_3 = 1$$
$$x_2 = 2$$

$$x_2 = [1, 2, 1]$$

put $\lambda = -2$

$$x_3 = [1, -1, -1]$$

Let $P = [x_1, x_2, x_3]$

$$P = \begin{bmatrix} -1 & 4 & 1 \\ 1 & 2 & 1 \\ 1 & -1 & -1 \end{bmatrix}$$

$|P| =$

A^{-1}

cofactor of P

Adjoin [cofactor of P]^T

$P^{-1} = \frac{Adj P}{|A|}$

$P^{-1}AP$

$P = [x_1, x_2, x_3]$

Transpose $= \begin{bmatrix} -1 & 1 & 1 \\ 4 & 2 & -1 \\ 1 & 1 & -1 \end{bmatrix}$

Adj P

Cofactor $= \begin{bmatrix} -2+1 & -[4+1] & 4-2 \\ -(-1-1) & 1-1 & -(-1-1) \\ -1-2 & -[1-4] & -2-4 \end{bmatrix}$

$= \begin{bmatrix} -1 & 3 & 2 \\ 2 & 0 & 2 \\ -3 & 3 & -6 \end{bmatrix}$

Diaganalization

1 characteristics Equation

2 Eigen value (λ_1, λ_2)

3 Eigen vector $[A - \lambda_1 I]X_1 = 0$

4 $(A - \lambda_2 I)X_2 = 0$

5 $P = [X_1 \; X_2 \; X_3]$

6 $P^{-1} = \frac{Adj P}{|P|}$

Bilinear form.

A polynomial of set of variable $(x_1, x_2 \ldots x_n)$ and (y_1, y_2, y_3) in which the coefficient are member and is homogeneous & linear with respect to variable in each of these set is called bilinear form.

In order to express as $A(X,Y) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i y_j$

$$= a_{11} x_1 y_1 + a_{12} x_1 y_2 + a_{13} x_1 y_3 + \ldots$$

$$+ a_{21} x_2 y_1 + a_{22} x_2 y_2 + a_{23} x_2 y_3 + \ldots$$

$$+ a_{31} x_3 y_1 + a_{32} x_3 y_2 + a_{33} x_3 y_3 + \ldots \quad (1)$$

Here we study ~~orally~~ only the linear form in which the coefficient are a_{ij} and variables x_i and y_j all are real quantity.

Then this system (1) can be written as

$$A(x,y) = \begin{bmatrix} x_1, & x_2, & x_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= X' A Y$$

where $x^{t} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Ex 1: change into bilinear form

$$A(x, y) = x_1 y_1 + 2x_1 y_2 + 3x_2 y_1 + 4x_2 y_2 + 5x_3 y_1 + 6x_3 y_2$$

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \\ 5 & 6 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Inner product space

Def^n: Let X be a linear space over the field F (R or C) then $\langle . , \rangle : X \times X \rightarrow R$ is called inner product if product is satisfying following condition

For any $x, y, z \in X$ and $\alpha \in F$

$\|x\| \geq 0$

(1) $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$

(2) $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$

(3) $\langle x, y\rangle = \overline{\langle y, x\rangle}$ complex conjugate

(4) $\langle x, x\rangle \geq 0$

(5) $\langle x, x\rangle = 0 \iff x = 0$

The linear space X with inner product that is $\langle X, \langle \cdot, \cdot\rangle\rangle$ is called inner product space

Q eg:- considered linear space R^n

define $x = (x_1, x_2 \ldots x_n)$
$y = (y_1, y_2 \ldots y_n)$ such that

$$\langle x, y\rangle = \sum_{k=1}^{n} x_k y_k$$

Then show that $(\langle\,\rangle.)$ is inner product space.

Sol now let $x, y, z \in R^n$ and let α be real no then

$$\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$$

$$= \sum_{k=1}^{n} (x_k + y_k, z_k)$$

$$= \sum_{k=1}^{n} (x_k, z_k) + \sum_{k=1}^{n} (y_k, z_k)$$

$$= \langle x, z\rangle + \langle y, z\rangle$$

(ii) $(\alpha x, y) = \sum_{k=1}^{n} \alpha x_k y_k = \alpha \sum_{k=1}^{n} x_k y_k$

$= \alpha (x, y)$

(iii) $\langle x, y \rangle = \sum_{k=1}^{n} x_k y_k = \sum_{k=1}^{n} y_k x_k = (y, x)$

(iv) $\langle x, x \rangle = \sum_{k=1}^{n} x_k x_k = \sum_{k=1}^{n} x_k^2 \geq 0$

$\langle x, x \rangle \geq 0$

(v) $\langle x, x \rangle = 0 = x = 0$

$\sum_{k=1}^{n} x_k^n = 0$ for $k = 1, \ldots, n$.

$\therefore x_1 = 0, x_2 = 0, x_3 = 0$

$\therefore x = 0$

Hence we see that all ~~five~~ space are in satified all 5 condition of IPS

Ex Show that linear space C^n for all n complex number

$x = (x_1, x_2 \ldots x_n)$
$y = (y_1, y_2 \ldots y_n)$

define as $\langle x, y \rangle = \sum_{k=1}^{n} x_k \bar{y}_k$

Q Prove that $V_2(R)$ is an IPS with an inner product obtained by $\alpha = (a_1, a_2)$, $\beta = (b_1, b_2)$ $\forall$ $\alpha, \beta \in R$.

(i) $(\alpha, \beta) = 3a_1b_1 + 2a_2b_2$

(ii) $(\alpha, \beta) = a_1b_1 - a_2b_1 - a_1b_2 + 4a_2b_2$

(iii) $(\alpha, \beta) = a_1b_1 - a_2b_1 - a_1b_2 + 2a_2b_2$.

Let $\gamma = c_1, c_2 \in V_2(R)$ then for $a, b \in R$

$$a\alpha + b\beta = a(a_1, a_2) + b(b_1, b_2)$$
$$= (aa_1, aa_2) + (bb_1, bb_2)$$
$$= (\underbrace{aa_1 + bb_1}_{A}), (\underbrace{aa_2 + bb_2}_{B})$$

$\therefore$ ~~$a\alpha + b\beta$~~ $\longleftarrow$ $[\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle]$

Now

$$(a\alpha, b\beta, \gamma) = a(\alpha\gamma) + b(\beta \cdot \gamma)$$

we have $(\underset{A}{a\alpha} + \underset{B}{b\beta}, \gamma) = 3(aa_1 + bb_1)c_1 + 2(aa_2 + bb_2)c_2$

$$a(\alpha, \gamma) + b(\beta, \gamma)$$

$$= a(3a_1c_1 + 2a_2c_2) + b(3b_1c_1 + 2b_2c_2)$$
$$= a(\alpha, \gamma) + b(\beta, \gamma).$$

Symmetric property : $(\beta, \alpha) = (\alpha, \beta)$

Let $(\beta, \alpha) = 3b_1a_1 + 2b_2a_2$
$$= 3a_1b_1 + 2a_2b_2 = (\alpha, \beta) \checkmark$$
$(\beta, \alpha) = \alpha, \beta$.

(b_1, b_2), (a_1, a_2)

$3b_1a_1 + 2b_2a_2$

$3b_1a_1 + 2b_2 \ldots$

a_1, a_2, b_1, b_2 are the member of $V_2(R)$

$\therefore$ it's satisfied commutative

(ii) $(\alpha x, y) = \sum_{k=1}^{n} \alpha x_k y_k = \alpha \sum_{k=1}^{n} x_k y_k$

$$= \alpha (x, y)$$

(iii) $\langle x, y \rangle = \sum_{k=1}^{n} x_k, y_k = \sum_{k=1}^{n} y_k x_k = (y, x)$

(iv) $\langle x, x \rangle = \sum_{k=1}^{n} x_k x_k = \sum_{k=1}^{n} x_k^2 \geq 0$

$$\langle x, x \rangle \geq 0$$

(v) $\langle x, x \rangle = 0 = x = 0$

$\sum_{k=1}^{n} x_k^n = 0$ for $k = 1 \ldots, n$.

$\therefore x_1 = 0, x_2 = 0, x_3 = 0$

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Hence we see that all ~~five~~ space are in satified all 5 condition of IPS

Ex Show that linear space C^n for all n complex number

$x = (x_1, x_2 \ldots x_n)$

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define as $\langle x, y \rangle = \sum_{k=1}^{n} x_k \bar{y}_k$

Q Prove that $V_2(R)$ is an IPS with an inner product obtained by $\alpha = (a_1, a_2)$, $\beta = (b_1, b_2)$ ∀ $\alpha, \beta \in R$

(i) $(\alpha, \beta) = 3a_1b_1 + 2a_2b_2$
(ii) $(\alpha, \beta) = a_1b_1 - a_2b_1 - a_1b_2 + 4a_2b_2$
(iii) $(\alpha, \beta) = a_1b_1 - a_2b_1 - a_1b_2 + 2a_2b_2$.

Let $\gamma = c_1, c_2 \in V_2(R)$ then for $a, b \in R$

$a\alpha + b\beta = a(a_1, a_2) + b(b_1, b_2)$
$= (aa_1, aa_2) + (bb_1, bb_2)$
$= \{\underset{A}{aa_1 + bb_1}), (\underset{B}{aa_2 + bb_2})$

$\leftarrow$ $[\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle]$

∴ ~~$a\alpha + b\beta$~~

Now

$(a\alpha, b\beta, \gamma) = a(\alpha\gamma) + b(\beta \cdot \gamma)$

we have $(\underset{A}{a\alpha} + \underset{B}{b\beta}, \gamma) = 3(aa_1 + bb_1)c_1 + 2(aa_2 + bb_2)c_2$

$a(\alpha, \gamma) + b(\beta, \gamma)$

$= a(3a_1c_1 + 2a_2c_2) + b(3b_1c_1 + 2b_2c_3)$
$= a(\alpha, \gamma) + b(\beta, \gamma)$.

a Symmetric property : $(\beta, \alpha) = (\alpha, \beta)$

Let $(\beta, \alpha) = 3b_1a_1 + 2b_2a_2$
$= 3a_1b_1 + 2a_2b_2 = (\alpha, \beta)$ ✓
$(\beta, \alpha) = \alpha, \beta$.

(b_1, b_2), (a_1, a_2)
$3b_1a_1 + 2b_2a_2$
$3b_1a_1 + 2b_2c_2$
a_1, a_2, b_1, b_2 are the members of $V_2(R)$

∴ its satisfied commutative

Ex. Sol

(iii) Non Negative:

we have to put $\beta = \alpha$ in eq (1)

$3a_1b_1 + 2a_2b_2$

so that we have required eqⁿ -

$(\alpha,\alpha) = (\alpha,\beta) = 3a_1a_1 + 2a_2a_2$

$= 3a_1^2 + 2a_2^2$ (non-negative)

$\Leftrightarrow 3a_1^2 = 0$ and $2a_2^2 = 0$

$a_1 = 0 \quad , a_2 = 0$ (by (1) $\langle x,x\rangle \geq 0$

$\Leftrightarrow \alpha = 0$ (As $\alpha = (a_1, a_2)$ both are zero so $\alpha = 0$)

Ex. Sol $\langle x,y\rangle = \sum_{k=1}^{n} x_k \bar{y}_k$ is $\langle\rangle$ space over x $(\langle\rangle, x)$.

Let $x,y,z \in C^n$ and α be scalar then -

(i) $\langle x+y, z\rangle = \sum_{k=1}^{n} (x_k + y_k)\bar{z}_k$

$= \sum_{k=1}^{n} x_k \bar{z}_k + \sum_{k=1}^{n} y_k \bar{z}_k$

$\langle x,z\rangle + \langle y,z\rangle$

(ii) $\langle \alpha x, y\rangle = \sum_{k=1}^{n} \alpha x_k \bar{y}_k = \alpha \sum_{k=1}^{n} (x_k \bar{y}_k)$

$= \alpha \langle x,y\rangle$

(iii) $\langle \overline{y, x} \rangle = \sum_{k=1}^{n} (\overline{y_k x_k}) = \sum_{k=1}^{n} \overline{y_k}\, \overline{\overline{x_k}}$

$$= \sum_{k=1}^{n} \overline{y_k}\, x_k$$

$$\langle x, y \rangle$$

(iv) $\langle x, x \rangle = \sum_{k=1}^{n} x_k \cdot \overline{x}_k = \sum_{k=1}^{n} |x_k|^2 \geq 0$

(v) $\langle x, x \rangle = 0$ iff $x = 0$ behave.

we see that the $\langle \cdot \rangle$ define in complex plane is an inner product space.

Q Prove that in complex plain is an IPS.

* Cauchy - Schwartzs inequality :-

If 'x' be linear space and $(X, \langle \cdot \rangle)$ be a IPS and let $x \in X$ then $|\langle x, y \rangle| \leq \sqrt{\langle x, x \rangle}, \sqrt{\langle y, y \rangle}$ iff x, y are linearly independent

$(x+iy)(x-iy) - 2x$

Proof: If $x=0$ and $y=0$ the inequality is satisfied at $x \neq 0$ and $y \neq 0$ for any scalar α, we have

$$0 \leq \langle \alpha x+y, \alpha x+y\rangle = \langle \alpha x, \alpha x+y\rangle + \langle y, \alpha x+y\rangle$$

or

$$0 \leq \langle \alpha x, \alpha x\rangle + \langle \alpha x, y\rangle + \langle y, \alpha x\rangle + \langle y, y\rangle$$

or $$0 \leq |\alpha|^2 \langle x, x\rangle + \alpha\langle x, y\rangle + \alpha\langle y, x\rangle + \langle y, y\rangle$$

or

$$0 \leq |\alpha|^2 \langle x, x\rangle + \alpha\langle x, y\rangle + \bar{\alpha}\,\overline{\langle x, y\rangle} + \langle y, y\rangle$$

$$0 \leq |\alpha| \langle x, x\rangle + 2\,\mathrm{Re}\,\alpha\langle x, y\rangle + \langle y, y\rangle$$

Let $\alpha = -\dfrac{\overline{\langle x, y\rangle}}{\langle x, x\rangle}$ then $|\alpha|^2 = \dfrac{|\overline{\langle x, y\rangle}|^2}{|\langle x, x\rangle|^2}$

$$= \frac{|\langle x, y\rangle|^2}{\langle x, x\rangle^2} \qquad (\because |\bar{z}| = |z|)$$

Then $0 \leq \dfrac{|\langle x, y\rangle|^2}{\langle x, x\rangle} - 2\dfrac{|\langle x, y\rangle|^2}{\langle x, x\rangle} + \langle y, y\rangle$

$(\because z\bar{z} = |z|^2)$

$(\because |\langle y, y\rangle| = \langle y, y\rangle = \text{real})$

$$0 \leq -\frac{|\langle x, y\rangle|^2}{\langle x, x\rangle} + \langle y, y\rangle$$

$$\frac{|\langle x, y\rangle|^2}{\langle x, x\rangle} \leq \langle y, y\rangle$$

$|\langle x, y\rangle| \leq \sqrt{\langle x, x\rangle}\ \sqrt{\langle y, y\rangle}$

$|\langle x, y\rangle|^2 \leq \langle x, x\rangle \langle y, y\rangle$ —(I)

$|\langle x, y\rangle| \leq \sqrt{(x, x)}\ \sqrt{(y, y)}$

Suppose the equality holds in (I) i.e

for $\alpha = -\dfrac{\langle x, y\rangle}{\langle x, x\rangle}$

$\langle \alpha x + y, \alpha x + y\rangle = |\alpha|^2 \langle x, x\rangle + 2 Re\, \alpha \langle x, y\rangle + \langle y, y\rangle$

$= -\dfrac{|\langle x, y\rangle|^2}{\langle x, x\rangle} + \langle y, y\rangle$

$= -\dfrac{\langle x, x\rangle \langle y, y\rangle}{\langle x, x\rangle} + \langle y, y\rangle$

$\langle \alpha x + y, \alpha x + y\rangle = 0$

$\alpha x + y = 0$

$\therefore x, y$ are linearly dependent vectors

Conversely - let x, y are linearly dependent then we can create $x = \beta y$ for some scalars then we have

$\langle x, y\rangle = \langle \beta y, y\rangle = \beta \langle y, y\rangle$

$|\langle x, y\rangle| = |\beta|^2 \langle y, y\rangle$ $\left[\because |\langle y, y\rangle| = \langle y, y\rangle\right]$
Real no

Also $\langle x, x\rangle\ |\langle y, y\rangle| = \langle \beta y, \beta y\rangle \langle y, y\rangle$

$\beta \bar{\beta} \cdot \langle y,y\rangle \langle y,y\rangle$

$|\beta|^2 (\langle y,y\rangle)^2 \langle y,y\rangle$

or $\sqrt{\langle x,x\rangle} \cdot \sqrt{\langle y,y\rangle} = |\beta| \langle y,y\rangle$

i.e $|\langle x,y\rangle| = \sqrt{\langle x,x\rangle} \cdot \sqrt{\langle y,y\rangle}$