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General Properties of vector space:

Theorem 1: Let $V(F)$ be a vector space, 0 be the zero vector V and 0 be the zero scalar of F. then,

(i) $a0 = 0 \quad \forall\, a \in F$
(ii) $0\alpha = 0 \quad \forall\, \alpha \in V$
(iii) $a(-\alpha) = -(a\alpha) \quad \forall\, a \in F, \ \forall\, \alpha \in V.$
(iv) $(-a)\alpha = -(a\alpha) \quad \forall\, a \in F, \quad \forall\, \alpha \in V$
(v) $a(\alpha - \beta) = a\alpha - a\beta \quad \forall\, a \in F, \ \forall\, \alpha, \beta \in V$
(vi) $a\alpha = 0 \Rightarrow a = 0$ or $\alpha = 0$

Vector subspace:

let $V(F)$ be a vector space over the filed $(F, +, \cdot)$ and $W \subseteq V$ such that $W \neq \phi$. Then $W(F)$ is called a subspace of $V(F)$ if W itself vector space over the field $(F, +, \cdot)$ with respect to the operations of vector addition and scalar multiplication in V.

Since $W \subseteq V$, much of the algebraic structure of $W(F)$ is inherited from $V(F)$ the minimum condition that $W(F)$ must satisfy to the subspace of $V(F)$ are:

(i) $(W, +)$ is a subgroup of $(V, +)$
(ii) W is closed under scalar multiplication.

Trival Subspace:

If V is a vector space over the field F, then V itself is a subspace of V for operations in V. similarly $\{0\}$ consisting of zero vector alone is a subspace of V, called the zero subspace: of V these both V and $\{0\}$ are called trival subspace of V. other subspace are called proper subspace of V(F).

Th^m 1 The necessarey and sufficents condition for a non-empty subset W of a vector space V(F) be a subspace of V is that W is closed under vector addition and scaler multiplication in V. i.e.

$$\alpha, \beta \in W \Rightarrow \alpha+\beta \in W \quad \forall\ a \in F, \alpha \in W.$$

and $a \in F, \alpha \in W \Rightarrow a\alpha \in W \quad \forall\ a \in F, \alpha \in W$

Th^m 2: The necessary and sufficents condition for a non-empty subset W of vector space V(F) to be a subspace of V are

(i) $\alpha \in W, \beta \in W \Rightarrow \alpha - \beta \in W$

and (ii) $a \in F, \alpha \in W \Rightarrow a\alpha \in W$

Th^m 3: The necessary and sufficents condition for a non-empty subset W of vector space V(F) to be a vector subspace of V is

$$a, b \in F \text{ and } \alpha, \beta \in W \Rightarrow a\alpha + \beta b \in W.$$

A subset W of a vector space V(F) is a subspace of V(F) if and only if

$$\underline{\forall\ \alpha, \beta \in W} \text{ and } \underline{a, b \in F} \Rightarrow \underline{a\alpha + b\beta \in W}$$

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Proof: The condition is necessary: If W is a subspace of $V(F)$ then we have to prove that $a, b \in F$ and $\alpha, \beta \in W \Rightarrow a\alpha + b\beta \in W$

Since W is a subspace of $V(F)$ and so W must be closed under vector Addition and scaler multiplication.

Consequently

$$a \in F, \alpha \in W \Rightarrow a\alpha \in W$$

and $b \in F, \beta \in W \Rightarrow b\beta \in W$

Now $a\alpha \in W, b\beta \in W \Rightarrow a\alpha + b\beta \in W$

($\because$ W is closed under vector Addition).

i.e. $a, b \in F$ and $\alpha, \beta \in W \Rightarrow a\alpha + b\beta \in W$

<u>The condition is sufficient</u>:- Now suppose that W is non-empty subset of V such that the given condition i.e.

$$a, b \in W \text{ and } \alpha, \beta \in W \Rightarrow a\alpha + b\beta \in W \quad \text{--(1)}$$

is satisfied in W. to prove W is a subspace of $V(F)$. taking $a = 1$, $b = 1$ in (1), then we have

$1, 1 \in F$ and $\alpha, \beta \in W \Rightarrow 1\cdot\alpha + 1\cdot\beta \in W$

$\Rightarrow \alpha + \beta \in W$ [$\because \alpha \in W \Rightarrow \alpha \in V$ and $1\alpha = \alpha$ in V]

$-1, 0 \in F$ and $\alpha, \alpha \in W \Rightarrow (-1)\alpha + 0, \in W$

$\Rightarrow -\alpha + 0 \in W \Rightarrow -\alpha \in W$.

$\therefore$ the additive inverse of each Element of W Exists also in W.

Taking $a = 0$, $b = 0$ in (1), we see that

$0, 0 \in F$, and $\alpha, \beta \in W \Rightarrow 0\alpha + 0\beta \in W$

$\Rightarrow 0 + 0 \in W \Rightarrow 0 \in W$.

$\therefore$ The zero vector of V is included in W and it is also the zero vector of W. since $W \subseteq V$,

consequently the vector addition is ~~consequently~~ commutative as well as associative in W.
therefore $(W, +)$ is an abelian group.
Again taking $b = 0$, we have

$$a, 0 \in F \text{ and } \alpha, \beta \in W \Rightarrow a\alpha + 0\beta \in W$$
$$\Rightarrow a\alpha + 0 \in W$$
$$\Rightarrow a\alpha \in W$$

Therefore, W is closed with respect to scaler multiplication, the remaining postulates of vector space will hold in W as they hold in $V \supseteq W$, thence W itself is a vector space.
thus $W(F)$ is a subspace of $V(F)$.

Ex 1: Let W denote the collection of all Elements from the space $M_2(F)$ of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ Show that W is subspace of $M_2(F)$.

Sol^n: let $\alpha = \begin{pmatrix} a_1 & b_1 \\ -b_1 & a_1 \end{pmatrix}$ and $\beta = \begin{pmatrix} a_2 & b_2 \\ -b_2 & a_2 \end{pmatrix}$ be any two Elements of W. then

$$\alpha + \beta = \begin{pmatrix} a_1 & b_1 \\ -b_1 & a_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ -b_2 & a_2 \end{pmatrix} = \begin{pmatrix} a_1+a_2 & b_1+b_2 \\ -(b_1+b_2) & a_1+a_3 \end{pmatrix} \in W$$

Again if $c \in F$ and $\gamma = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in W$ then

$$c\gamma = c\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{pmatrix} ca & cb \\ -cb & ca \end{pmatrix} \in W$$

Hence W is Subspace of $M_2(F)$.

Ex 2. Show that the set $W = \{(a, b, 0) : a, b \in F\}$ is a subspace of $V_3(F)$

Sol^n. let $\alpha = \{a_1, b_1, 0\}$ and $\beta = \{a_2, b_2\, 0\}$ be any two elements of W and a, b be any two Element of field F. then we see that

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$a\alpha + b\beta = a(a_1, b_1, 0) + b(a_2, b_2, 0)$
$= (aa_1, ab_1, 0) + (ba_2, bb_2, 0)$
$= (aa_1 + ba_2, ab_1 + bb_2, 0) \in W$
$\because$ $aa_1 + ba_2, ab_1 + bb_2 \in F$ and the least co-ordinate of this traid is zero.
Hence W is proper subspace of $V_3(F)$.

Ex 3: let $W = \{(a_1, a_2, a_3) ; a_1, a_2, a_3 \in F$ and $a_1 + a_2 + a_3 = 0\}$
show that W is a subspace of $V_3(F)$.

Sol^n: let $\alpha = (a_1, a_2, a_3)$ and $\beta = (b_1, b_2, b_3)$ be any two Elements of W therefore

$a_1 + a_2 + a_3 = 0$ — (1)

$b_1 + b_2 + b_3 = 0$ — (2)

Again let a, b be any two element of F, then

$a\alpha + b\beta = a(a_1, a_2, a_3) + b(b_1, b_2, b_3)$
$= (aa_1, aa_2, aa_3) + (bb_1, bb_2, bb_3)$
$= (aa_1 + bb_1, aa_2 + bb_2, aa_3 + bb_3)$

Now $aa_1 + bb_1 + aa_2 + bb_2 + aa_3 + bb_3$
$= a(a_1 + a_2 + a_3) + b(b_1 + b_2 + b_3)$
$= a \cdot 0 + b \cdot 0 = 0$ using (1) and (2)
$\therefore$ $a, b \in F, \alpha, \beta \in W \Rightarrow a\alpha + b\beta \in W$
Hence W is subspace of $V_3(F)$.

Ex 4: If a_1, a_2, a_3 are fixed Elements of a field F. then the set W of all ordered triads (x_1, x_2, x_3) of element of F, such that is $a_1x_1 + a_2x_2 + a_3x_3 = 0$ is a subspace of $V_3(F)$.

Sol^n let $\alpha = (x_1, x_2, x_3)$ and $\beta = (y_1, y_2, y_3) \in W$
then x_1, x_2, x_3 y_1, y_2, y_3 are Elements of F.

Such that $a_1x_1 + a_2x_2 + a_3x_3 = 0$ —— (1)

$a_1y_1 + a_2y_2 + a_3y_3 = 0$ —— (2)

Now $\alpha + \beta = (x_1, x_2, x_3) + (y_1, y_2, y_3)$

$= (x_1 + y_1, x_2 + y_2, x_3 + y_3)$

$(\because a_1(x_1 + y_1) + a_2(x_2 + y_2) + a_3(x_3 + y_3)$

$= (a_1x_1 + a_2x_2 + a_3x_3) + (a_1y_1 + a_2y_2 + a_3y_3)$

$= 0 + 0 = 0$

Again $c \in F$ be arbitrary then

$c\alpha = c(x_1, x_2, x_3) = (cx_1, cx_2, cx_3) \in W$

$(\because a_1(cx_1) + a_2(cx_2) + a_3(cx_3) = c(a_1x_1 + a_2x_2 + a_3x_3)$

$= c \cdot 0 = 0$

Hence W is a subspace $V_3(F)$.

Ex 5: Let $M_n(F)$ be the vector space of all $n \times n$ matrices over the field F, let W be the subset of $M_n(F)$ consisting all symmetric matrices. Show that W is a subspace $M_n(F)$.

Solⁿ: Let $A = [a_{ij}]_{n \times n}$, $B = [b_{ij}]_{n \times n} \in W$

Since A and B are symmetric matrices, we have

$\therefore a_{ij} = a_{ji}$ and $b_{ij} = b_{ji}$

If $a, b \in F$ be any two scalars, then

$aA + bB = a[a_{ij}]_{n \times n} + b[b_{ij}]_{n \times n}$

$= [a a_{ij}]_{n \times n} + [b b_{ij}]_{n \times n}$

$= [a a_{ij} + b b_{ij}]_{n \times n}$

Hence $aA + bB$ is also a symmetric matrix

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of the order $n \times n$

$\therefore$ $a, b \in F$ and $\alpha, \beta \in W \Rightarrow a\alpha + b\beta \in W$

Hence W is a subspace of $M_n(F)$.

Ex:6 let R be the field of real numbers, which of the following are subspace of $V_3(R)$?

(i) $W_1 = \{(x, 2y, 3z) : x, y, z \in R\}$

(ii) $W_2 = \{(x, x, x) : x \in R\}$

(iii) $W_3 = \{(x, y, z) : x, y, z,$ are rational number$\}$.

Soln. – (i) Here $W_1 = \{(x, 2y, 3z) : x, y, z \in R\}$

let $\alpha = (x_1, 2y_1, 3z_1)$ and $\beta = (x_2, 2y_2, 3z_2)$ be any two arbitrary Elements of W_1, then $x_1, y_1, z_1, x_2, y_2, z_2 \in R$. If $a, b \in R$ be any two real number, then we have

$$a\alpha + b\beta = a(x_1, 2y_1, 3z_1) + b(x_2, 2y_2, 3z_2)$$

$$= (ax_1, 2ay_1, 3az_1) + (bx_2, 2by_2, 3bz_2)$$

$$= [ax_1 + bx_2, 2(ay_1 + by_2), 3(az_1 + bz_2)] \in W_1$$

$$\because ax_1 + bx_2 \in R.$$

$\therefore$ $a, b \in R$ and $\alpha, \beta \in W_1 \Rightarrow a\alpha + b\beta \in W_1$.

(ii) Here $W_2 = \{(x, x, x) : x \in R\}$. let $\alpha = (x_1, x_1, x_1)$ and $\beta = (x_2, x_2, x_2)$ be any two elements of W_2, then $x_1, x_2 \in R$. If $a, b \in R$ be any two real number then we have

$$a\alpha + b\beta = a(x_1, x_1, x_1) + b(x_2, x_2, x_2)$$

$$= (ax_1 + bx_2, ax_1 + bx_2, ax_1 + bx_2)$$

$$\in W_2 \quad (\because ax_1 + bx_2 \in W_2)$$

$\therefore$ W_2 is a subspace of $V_3(R)$.

(iii) Here $W_3 = \{ (x, y, z) : x, y, z$ are rational number$\}$.

Let $\alpha = (4, 6, 7)$ be any element W_3. If $a = \sqrt{5}$ is an element of R, then $a\alpha = \sqrt{5}(4, 6, 7) = (4\sqrt{5}, 6\sqrt{5}, 7\sqrt{5}) \notin W_3$ since $4\sqrt{5}, 6\sqrt{5}, 7\sqrt{5}$ are not rational number.

Consequently W_3 is not closed with respect to scaler multiplication

Hence W_3 is not subspace of $V_3(R)$.

Ex: Let V be the vector space of all real valued continuous function over R. then show that the solution set W of the differential Equation:

$$2\frac{d^2y}{dx^2} - 9\frac{dy}{dx} + 2y = 0$$

where $y = f(x)$ is subspace of V.

Sol^n Here $W = \left\{ y : 2\frac{d^2y}{dx^2} - 9\frac{dy}{dx} + 2y = 0 \right\}$

where $y = f(x)$ clearly $0 \in W$ since $y = 0$ satisfies the given differential Equation: Let $y_1, y_2 \in W$ be any two Element, then

$$2\frac{d^2y_1}{dx^2} - 9\frac{dy_1}{dx} + 2y_1 = 0 \quad \text{--- (i)}$$

$$2\frac{d^2y_2}{dx^2} - 9\frac{d^2y_2}{dx} + 2y_2 = 0 \quad \text{--- (ii)}$$

Now $a, b \in R$ then

$$2\frac{d^2}{dx^2}(ay_1 + by_2) - 9\frac{d}{dx}(ay_1 + by_2)$$

$$+ 2(ay_1 + by_2)$$

$$2\frac{d^2}{dx^2}(ay_1) + 2b\frac{d^2}{dx^2}(by_2) - 9a\frac{d}{dx}y_1 - 9b\frac{dy_2}{dx} + 2ay_1 + 2by_2$$

$$a\left(\frac{d^2}{dx^2}y_1 - 9\frac{dy_1}{dx} + y_1\right) + b\left(2\frac{d^2}{dx^2}y_2 - 9\frac{dy_2}{dx} + y_2\right)$$

$a \cdot 0 + b \cdot 0 = 0$ from ① and ②

Thus $ay_1 + by_2$ is also a given solution of differential Equation, so $ay_1 + by_2 \in W$

$\therefore$ $a, b \in R$ and $y_1, y_2 \in W \Rightarrow ay_1 + by_2 \in W$

Hence W is subspace of V.

ALGEBRA OF SUBSPACE

Thm 1: The intersection of two subspace of vector space $V(F)$ is also a subspace of $V(F)$.

Solⁿ: let W_1 and W_2 be any two subspace of vector space $V(F)$. since additive identity of 0 of V belongs to every subspace of V. i.e. $0 \in W_1$ and $0 \in W_2$. consequently $0 \in W_1 \cap W_2$ and so $W_1 \cap W_2$ non-empty.

let $\alpha, \beta \in W_1 \cap W_2$ be any two elements also let $a, b \in F$

Now $\alpha \in W_1 \cap W_2 \Rightarrow \alpha \in W_1$ and $\alpha \in W_2$

$\beta \in W_1 \cap W_2 \Rightarrow \beta \in W_1$ and $\beta \in W_2$

Since W_1, W_2 are subspace, so

$a, b \in F$ and $\alpha, \beta \in W_1 \Rightarrow a\alpha + b\beta \in W_1$ —— ①

$a, b \in F$ and $\alpha, \beta \in W_2 \Rightarrow a\alpha + b\beta \in W_2$ —— ⑪

from ① and ⑪

$a\alpha + b\beta \in W_1$ and $a\alpha + b\beta \in W_2$

$\Rightarrow a\alpha + b\beta \in W_1 \cap W_2$

$\therefore$ $a, b \in F$ and $\alpha, \beta \in W_1 \cap W_2 \Rightarrow a\alpha + b\beta \in W_1 \cap W_2$

Th^{m} 2: An arbitrary intersection of two subspace i.e. the intersection of any number of subspace of vector space is a subspace:

Sol^n: let $V(F)$ be a vector space and let $W = \bigcap_{i \in I} W_i$ be the intersection of any family of subspace of V, I being any index set. Since $0 \in W_i$ $\forall$ $i \in I$ and so

$$0 \in \bigcap_{i \in I} W_i$$

Hence $\bigcap_{i \in I} W_i$ i.e. $W \neq \phi$

let $a, b \in F$ and $\alpha, \beta \in \bigcap_{i \in I} W_i$, then $\alpha, \beta \in W_i$ $\forall$ $i \in I$ since every W_i is a subspace of V.

Consequently: $a, b \in F$ and $\alpha, \beta \in W_i$

$$\Rightarrow a\alpha + b\beta \in W_i \quad \text{for each } i \in I$$

$$\Rightarrow a\alpha + b\beta \in \bigcap_{i \in I} W_i$$

$\therefore$ $a, b \in F$ and $\alpha, \beta \in \bigcap_{i \in I} W_i \Rightarrow a\alpha + b\beta \in \bigcap_{i \in I} W_i$

Hence $\bigcap_{i \in I} W_i$ is a subspace of $V(F)$.

SMALLEST SUBSPACE CONTAINIG ANY SUBSET OF $V(F)$:

Let S be subset of vector space $V(F)$. Let T be a subspace of $V(F)$ containg S and itself is contained in every subspace of V. containg S, Then T is said to be the smallest subspace of V containg V. containg S. T is also called the subspace of V spanned or generated by S and is denoted by $T = [S]$

$$[S] = \cap \{W_n \mid S \subseteq W_n,\ W_n(F) \text{ is a subspace of } V(F)\}.$$

Thm: The union of two subspace is a subspace if and only if one is contained in the other.

Solⁿ: let $V(F)$ be a vector ~~sub~~space and let W_1 and W_2 be two subspace of $V(F)$ firstly let us assume $W_1 \subset W_2$ or $W_2 \subset W_1$, then

$$W_1 \cup W_2 = W_2 \text{ or } W_1$$

$\therefore$ $W_1 \cup W_2$ is also a subspace of $V(F)$, since W_1, W_2 are subspace.

Conversely: let $W_1 \cup W_2$ be subspace of $V(F)$. then we are to prove $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$ we shall prove it by contradiction

suppose that W_1 is not a subspace of W_2 and W_2 is not a subspace of W_1.

since $W_1 \nsubseteq W_2 \Rightarrow \exists\, \alpha \in W_1,\ \alpha \notin W_2$ — (1)

$W_2 \nsubseteq W_1 \Rightarrow \exists\, \beta \in W_2,\ \beta \notin W_1$ — (2)

Again since $W_1 \cup W_2$ is a subspace and so we have $\alpha, \beta \in W_1 \cup W_2 \Rightarrow \alpha + \beta \in W_1 \cup W_2$

$\Rightarrow \alpha + \beta \in W_1$ or $\alpha + \beta \in W_2$

If $\alpha + \beta \in W_1$, then

$(\alpha + \beta) - \alpha = \beta \in W_2$ (since W_1 is subspace and $\beta \in W_1$)

But from (2), we see that $\beta \notin W_1$ which is contradiction: Hence either W_1 is a subset of W_2 or W_2 is a subset of W_1.

Introduction To Linear Algebra

- Algebric Structure

Group (G, *)
Feild (f, +, ·)
Ring (R, +, ·)
Integral domain (I, +, ·)
Vector Space V(F) → Vector over the feild F
↳ composition of group and feild

(V, +) abalian group Satisfy all 5 condition
(V, ·) Semi group satisfy 2 properties
(clouser, associativity)

Vector Space
Feild → An algebric structure which contain more than one binary operation and having four property of group is called feild

It is denoted by (f, +, ·)
(Clouser, associativity, ideantity, inverse)
→ Properties of algebric structure.

- Boolean algebra have 3 opreations (A', +, ·)
- An element of F is called scalar
- Internal composition (+) both element are vector
- External composition (·) one element is scalar and one element is vector.

→ The element of V is defined by α, β, γ
→ The element of F is denoted by a, b, c

→ Additive identity is 0 (null or zero vector)

→ Multiplicative identity is 1.

Vector Space

Let $(F, +, \cdot)$ is a feild whose element are called scalar and let V be non empty set. The element V is vector. Then $V(F)$ is called vector space over the feild F if $V(F)$ satisfy follow condition.

(A) $(V, +)$ is abelian group

(i) Closure property → Let $\alpha, \beta, \gamma \in V \Rightarrow \alpha + \beta \in V$
$\therefore$ $+$ is closed

(ii) Associativity Property → Let $\alpha, \beta, \gamma \in V$
$\alpha + (\beta + \gamma) = (\alpha + \gamma) + \beta = (\beta + \gamma) + \alpha$

(iii) Identity Property

$\forall \ \alpha \in V, \exists \ 0 \in V$ st, $\alpha + (-\alpha) = 0 = (-\alpha) + \alpha$

(iv) Commutative Property

Let $\alpha, \beta, \gamma \in V$
$\alpha + \beta = \beta + \gamma$

(B) V is closed with respect to external composition i.e $a.\alpha \in V \quad \forall a \in f, \alpha \in V$

(C) Vector addition and scalar multiplication (distributive law)

1 $a(\alpha+\beta) = a\alpha + a\beta \quad \forall \alpha, \beta \in V, a \in F$

2 $(a+b)\cdot\alpha = a\alpha + b\alpha \quad \forall a, b \in F, \alpha \in V$

3 $(ab)\alpha = a(b\alpha)$

4 $1\cdot\alpha = \alpha \quad \forall 1 \in F, \alpha \in V.$

Ex Prove that n-tuple of element f with respect to addition and multiplication defined by $(a_1, a_2 - a_n) + (b_1, b_2 - b_n) = (a_1+b_1, a_2+b_2, a_3+b_3 - a_n+b_n)$ and $K(a_1, a_2, a_3 - a_n) = (Ka_1, Ka_2 - Ka_n)$ where each $a_i, b_i \in f$ in vector space

Let V be the set of all n-tuple over the field f

$V = \{(a_1, a_2, - a_n) : a_1, a_2 - a_n \in f\}$

Then we have to show that V(f) is vector space.

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Now V- satisfy : (A) (V+) is an albalian group.

(1) Clouser Property
Let $\alpha = (a_1, a_2 ---- a_n)$
$\beta = (b_1, b_2 ---- b_n)$

$\therefore \alpha + \beta = (a_1, a_2 ---- a_n) + (b_1, b_2 ---- b_n)$
$= (a_1 + b_1, a_2 + b_2 ---- a_n + b_n)$
$\alpha + \beta \in V$

V is closed under ~~~~ Addition n- tuple.

(2) Associative propetry
Let $\alpha, \beta, \gamma \in V$ then

$\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$

Let $\gamma = (c_1, c_2 ---- c_n)$
Taking LHS we have $\alpha + (\beta + \gamma) =$

$(a_1, a_2 ---- a_n) + \{(b_1, b_2 ---- b_n) + (c_1, c_2 ---- c_n)\}$

$(a_1, a_2 ---- a_n) + \{(b_1 + c_1, b_2 + c_2 ---- b_n + c_n)\}$

$a_1 + (b_1 + c_1), a_2 + (b_2 + c_2) = a_n + (b_n + c_n)$
$(a_1 + b_1) + c_1, (a_2 + b_2) + c_2 = (a_n + b_n) + c_n$

Such that a_i, b_i and $c_i \in F$ wher
$i = 1, 2, ---, n$

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$= \{(a_1, a_2 \cdots a_n) + (b_1, b_2 \cdots b_n)\} + (c_1, c_2 \cdots c_n)$

$(\alpha + \beta) + \gamma$

③ Commutative Property

$\forall\ \alpha, \beta \in V \ \therefore\ \alpha + \beta = \beta + \gamma$

$LHS = \alpha + \beta$

$(a_1, a_2 \cdots a_n) + (b_1, b_2 \cdots b_n)$

$(a_1 + b_1,\ a_2 + b_2,\ \cdots\ a_n + b_n)$

$\therefore$ by defination of n-tuple $\forall$ each $a_1 b_1 \in F$

$(b_1 + a_1,\ b_2 + a_2\ \cdots\ b_n + a_n)$

$(b_1 + b_2 \cdots b_n)(a_1, a_2 \cdots a_n)$ $\therefore \alpha + \beta = \beta + \gamma$

$= \beta + \gamma$

Linear functional

Let $V(F)$ be the vector space defined over the feild F and a mapping

$f: V \rightarrow$ is called linear functional

$$f(a\alpha + b\beta) = a \cdot f(\alpha) + b \cdot f(\beta)$$

$$\forall \; a, b \in F$$
$$\alpha, \beta \in V$$

function

So $\alpha \in V$ $f(\alpha) \in F$ and $f(\alpha)$ being

feild

scalar linear functional

Theorem

If $V(F)$ is a vector space then show that a mapping $f: V \rightarrow F$ defined as.

$f(\alpha) = 0, \; \forall \; \alpha \in V$ is linear functional

Solⁿ Let $\alpha, \beta \in V$ and $a, b \in F$ such that

$$f(a\alpha + b \cdot \beta) = 0$$
$$= 0 + 0$$
$$= a \cdot 0 + b \cdot 0$$

$$f(a \cdot \alpha + b \cdot \beta) = a \cdot f(\alpha) + b \cdot f(\beta)$$

hence f is linear functional on V

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Exp. Let $V(F)$ be the vector space and mapping $F: V \to f$ defined as
$(-F)\alpha = -F(\alpha)$ $\forall \alpha \in V$ is
-ve linear functional linear functional

Solⁿ Since f is linear functional on V, $\alpha \in V$.
then $= -f(\alpha) \in f$; then we have to show that $-f$ is a mapping from V into f is linear functional.

Let $\alpha, \beta \in V$ and $a, b \in f$ such that,

$$(-f)(a.\alpha + b\beta) = \cancel{a(f)\alpha} \; -f(a.\alpha + b.\beta)$$
$$= a(-f)\alpha + b.(-f)\beta$$
$$= -a.f(\alpha) - b(f)\beta$$

$\therefore$ $-f$ is linear functional

Linear Transformation

Let $V(F)$ and $U(F)$ be the two vector space defined over the feild (F) then mapping $T: U \to V$ is known as linear tranformation if $\forall$ $\alpha, \beta \in U$ and $a, b \in f$

Such that (i) $T(\alpha + \beta) = T(\alpha) + T(\beta)$
~~(ii) $T(\alpha\beta) = \alpha.T(\beta)$~~
(ii) $T(a\alpha) = a.T(\alpha)$

Now we consider LHS.

$f(a\alpha + b\beta) = f[a(a_1, b_1, c_1) + b(a_2, b_2, c_2)]$

$= f[(aa_1, ab_1, ac_1) + (ba_2 + bb_2, bc_2)]$

$= f[\underset{A}{aa_1 + ba_2}, \underset{B}{ab_1 + bb_2}, \underset{C}{ac_1 + bc_2}].$

$f(a\alpha + b\beta) = [(ac_1 + bc_2), (aa_1 + ba_2 + ab_1 + bb_2)]$

$\therefore$ by def of f

$= ac_1, a(a_1 + b_1) + bc_2, b(a_2 + b_2)$

$= a(a_1 + b_1 + c_1) + b(a_2 + b_2 + c_2)$

$\therefore$ by rev def of f

$f(a\alpha + b\beta) = a\, f(a_1 + b_1 + c_1) + b \cdot f(a_2 + b_2 + c_2)$

$f(a\alpha + b\beta) = a \cdot f(\alpha) + b \cdot f(\beta)$

LHS = RHS

Q Ex:- Show that transformation $T: V_2 \rightarrow V_2$ defined by $T(x, y) = (2x + 3y, 3x - 4y)$ is linear transformation

Sol^n Let $\alpha, \beta \in V$ and $a, b \in f$ such that

$T(a\alpha + b\beta) = a\,T(\alpha) + b\,T(\beta)$

Again let $\alpha = x_1, y_1$

$\beta = x_2, y_2$

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then taking LHS.

$T(a\alpha + b.\beta) = T(a(x_1, y_1) + b(x_2, y_2)]$

$= T[(ax_1, ay_1) + (bx_2, by_2)]]$

$T[(ax_1 + bx_2) + (ay_1 + by_2)]$ ∴ by def of f

$2(ax_1 + bx_2) + 3(ay_1 + by_2), 3(ax_1 + bx_2) - 4(ay_1 + by_2)$

$(2ax_1 + 3ay_1, 3ax_1 - 4ay_1), (2bx_2 + 3by_2, 3bx_2 - 4by_2)$

$a(2x_1 + 3y_1, 3x_1 - 4y_1) + b(2x_2 + 3y_2, 3x_2 - 4y_2)$

$a\,T(x_1, y_1) + b\,T(x_2, y_2)$

$T(a\alpha + b\beta) = aT(\alpha) + bT(\beta).$

DUAL SPACE

Let V be the vector space defined over the feild ~~of~~ F

The set of all linear functional on V is also vector space over the feild F
The dual space denoted by V^*
Some times V' is used for dual space

Remark: Let V(F) be the n-dimensional vector space defined over the basis of V_s then there exist unlikely expressed $B' = \{f_1, f_2, f_3\}$ such that

$$f_i(\alpha_j) = \delta_{ij} \text{ (cronical delta)}$$

Ex1 Find dual basis where
$$B = \{(1,-2,3), (1,-1,1), (2,-4,7)\}$$
for $V_3(\mathbb{R})$

Sol^n Let
$$\alpha_1 = (1,-2,3)$$
$$\alpha_2 = (1,-1,1)$$
$$\alpha_3 = (2,-4,7)$$

Again let $B' = \{f_1, f_2, f_3\}$ be the dual basis for B.

Such that
$$f_1(a,b,c) = a_1 a + a_2 b + a_3 c$$
$$f_2(a,b,c) = b_1 a + b_2 b + b_3 c$$
$$f_3(a,b,c) = c_1 a + c_2 b + c_3 c$$

$f_i(\alpha)_j = \delta_{ij} \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$

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We have by defination

$f_i(\alpha)_j = \delta_{ij}$

We have $i = 1$ $j = 1, 2, 3$

$f_1(\alpha_1) = 1$

$f_1(\alpha_2) = 0$

$f_1(\alpha_3) = 0.$

We have $i = 2$ $j = 1, 2, 3 ..$

$f_2(\alpha_1) = 0$

$f_2(\alpha_2) = 1$

$f_3(\alpha_2) = 0.$

We have $i = 3$ $j = 1, 2, 3.$

$f_3(\alpha_1) = 0$

$f_3(\alpha_2) = 0$

$f_3(\alpha_3) = 1.$

Now

$f_1(\alpha_1) = 1 = f_1(1, -2, 3) = 1a_1 - 2a_2 + 3a_3 = 1 -(i)$

$f_1(\alpha_2) = 0 = f_1(1, -1, 1) = a_1 - a_2 + a_3 = 0 -(ii)$

$f_1(\alpha_3) = 0 = f_1(2, -4, \cancel{7}) = 2a_1 - 4a_2 + \cancel{7}a_3 = 0 -(iii)$

~~from eq (ii)~~

$\cancel{a_1 - a_2 + a_3 = 0}$

$\boxed{a_1 = a_2 - a_3}$ $-(iv)$.

Putting value of a_1 into ~~eq(1)~~ and eq (iii).

$\cancel{a_2 - a_3 - 2a_2 + 3a_3 = 1.}$

$-a_2 + 2a_3 = 1.$

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$a_1 = 2a_2 - 3a_3$ — (iv)
$a_1 = a_2 - a_3$ — (v)
$a_1 = 4a_2 - 7a_3$ — (vi)

(i) x(ii) and subtract by 3 we get
$6a_3 - 7a_3 = 2$
$\boxed{a_3 = -2}$

Again subtract (i) and (ii). subtract by 3 we get

$-a_2 + 2a_3 = 1$
$-a_2 - 4 = 1$
$-a_2 = 5.$
$\boxed{a_2 = -5}$

$\boxed{a_1 = -3}$

$f_1(a, b, c) = -3a - 5b - 2c.$

Now for f_2
$f_2(\alpha_1) = 0 \Rightarrow f_2(1, -2, 3) = b_1 - 2b_2 + 3b_3 = 0$. — (i)
$f_2(\alpha_2) = 1 \Rightarrow f_2(1, -1, 1) = b_1 - b_2 + b_3 = 1$ — (ii)
$f_2(\alpha_3) = 0 \Rightarrow f_3(2, -4, 7) = 2b_1 - 4b_2 + 7b_3 = 0$. — (iii)

from (i), (ii) & (iii)
$b_1 - 2b_2 + 3b_3 = 0.$
$b_1 - b_2 + b_3 = 0.$
$2b_1 - 4b_2 + 7b_3 = 0.$

Multiplying (i) and (ii) and sub from (iii)

$6b_3 - 7b_3 = 0$

$\boxed{b_3 = 0}$

Again sub (ii) from (i).

$-b_2 + 2b_3 = -1$

$\boxed{b_2 = 1}$

Putting b_2 and b_3 in (i) we get $\boxed{b_1 = 2}$

$f_2(a, b, c) = 2a + b.$

Now from (3) we have

$f_3(\alpha_1) = 0 \Rightarrow f_3(1, -2, 3) = c_1 - 2c_2 + 3c_3 = 0$ — (i)

$f_3(\alpha_2) = 0 \Rightarrow f_3(1, -1, 1) = c_1 - c_2 + c_3 = 0$ — (ii)

$f_3(\alpha_3) = 0 \Rightarrow f_3(2, -4, 7) = 2c_1 - 4c_2 + 7c_3 = 1$ — (iii)

Multiplying (i) with (ii) and sub from (iii)

$6c_3 - 7c_3 = -1$

$\boxed{c_3 = 1}$

Again sub (ii) from (i)

$-c_2 + c_3 = 0$

$\boxed{c_2 = 0}$

Putting c_2 and c_3 in eq (ii) we get

$\boxed{c_1 = 1}$

$f_3(a, b, c) = a + 2b + c$

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Ex find dual basis of set $B = \{(1,-1,3), (0,1,-1)$
$(0,3,-2)\}$

Sol^n Let $\alpha_1 = (1,-1,3)$
$\alpha_2 = (0,1,-1)$.
$\alpha_3 = (0,3,-2)$

Let $B' = \{f_1, f_2, f_3\}$ be dual basis.

Then $f_1(a,b,c) = a_1a + a_2a + a_3a$
$f_2(a,b,c) = b_1a + b_2b + b_3c$.
$f_3(a,b,c) = c_1a + c_2b + c_3c$

We know that def^n we have
$f_i(\alpha_j) = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

Now Case I :- $i=1$, $j=1,2,3$
~~Eq~~ ~~E,eq~~
$f_1(\alpha_1) = \delta_{11} = 1$
$f_1(\alpha_2) = \delta_{12} = 0$
$f_1(\alpha_3) = \delta_{13} = 0$.

$i=2$, $j=1,2,3$
$f_2(\alpha_1) = \delta_{21} = 0$
$f_2(\alpha_2) = \delta_{22} = 1$
$f_{\cancel{3}}(\alpha_3) = \delta_{23} = 0$.

$i = 3 \qquad j = 1, 2, 3$

$f_3(\alpha_1) = \delta_{31} = 0$

$f_3(\alpha_2) = \delta_{32} = 0$

$f_3(\alpha_3) = \delta_{33} = 1$

Now we have

$f_1(\alpha_1) = 1 = f_1(1,-1,3) = a_1 - a_2 + 3a_3 = 1$ — (i)

$f_1(\alpha_2) = 0 = f_1(0,1,-1) = 0 + a_2 - a_3 = 0$ — (ii)

$f_1(\alpha_3) = 0 = f_1(0,3,-2) = 0 + 3a_2 - 2a_3 = 0$ — (iii)

from (i), (ii) & (iii)

$a_1 = 1, \quad a_2 = 0, \quad a_3 = 0$

$f_1(a,b,c) = 0.$

New for f_2 :-

$f_2(\alpha_1) = 0 \Rightarrow f_2(1,-1,3) = b_1 - b_2 + 3b_3 = 0$ — (iv)

$f_2(\alpha_2) = 1 \Rightarrow f_2(0,1,-1) = b_2 - b_3 = 1$ — (v)

$f_2(\alpha_3) = 0 \Rightarrow f_2(0,3,-2) = 3b_2 - 2b_3 = 0$ — (vi)

from (iv), (v) and (vi)

$b_1 = 7, \quad b_2 = -2, \quad b_3 = -3$

$f_2(a,b,c) = 7a - 2b - 3c$

for f_3 :-

$f_3(\alpha_1) = 0 \Rightarrow f_3(1,-1,3) = c_1 - c_2 + 3c_3 = 0$ — (7)

$f_3(\alpha_2) = 0 \Rightarrow f_3(0,1,-1) = c_2 - c_3 = 0$ — (8)

$f_3(\alpha_3) = 1 \Rightarrow f_3(0,3,-2) = 3c_2 - 3c_3 = 1$ — (9)

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from ⑦, ⑧ & ⑨

$c_1 = -2$

$c_2 = 1$

$c_3 = 1$

$f(a,b,c) = -2a+b+c$

Ex If $\alpha_1 = (1,1,1)$

$\alpha_2 = (1,1,-1)$

$\alpha_3 = (1,-1,-1)$

from basis of V_3 if $(f_1, f_2 f_3)$ are the dual basis of $V_3(\mathbb{R})$ and $\alpha = 0,1,0$ of ~~concept~~ then find

$f_1(\alpha), f_2(\alpha), f_3(\alpha)$

Sol^n We know that if $\alpha_1, \alpha_2, \alpha_3$ are the basis of $V_3(\mathbb{R})$ and $\alpha = 0,1,0$ ute can be expressed as foll linear combination

If $f_1(\alpha_1) = a_1$

$f_2(\alpha_2) = a_2$

$f_3(\alpha_3) = a_3$ then we have

$\alpha = a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3$

$(0,1,0) = a_1(1,1,1) + a_2(1,1,-1) + a_3(1,-1,-1)$

$(0,1,0) = a_1, a_1, a_1 + a_2, a_2, -a_2 + a_3, -a_3, -a_3$

$(0,1,0) = (a_1+a_2+a_3), (a_1+a_2-a_3), (a_1-a_2-a_3)$

comparing both side

$$a_1 + a_2 + a_3 = 0 \quad \text{—(i)}$$
$$a_1 + a_2 - a_3 = 1 \quad \text{—(ii)}$$
$$a_1 - a_2 - a_3 = 0 \quad \text{—(iii)}$$

from (i), (ii) and (iii)

$a_1 = 0, \quad a_2 = 1/2 \quad a_3 = -1/2$

$f_1(\alpha_1) = 0$

$f_2(\alpha_2) = 1/2$

$f_3(\alpha_3) = -1/2$

Q. Let $f_1 : V_2(R) \to R$ and $f_2 : V_2(R) \to R$ be the two mapping defined over the field R such that $f_1(a, b) = a + 2b$ and $f_2(a, b) = 3a - b$ $\forall\, a, b \in R$

find the value of

$(f_1 + f_2)(a, b)$

$4f_1(a, b)$

$(2f_1 - 5f_2)(a, b)$

Sol Here f_1, f_2 are linear transformation $\therefore$ thier sum will also be linear transformation

(i) $(f_1 + f_2)(a, b)$

$a + 2b + 3a - b = 4a + b.$

(ii) $4f(a, b) = 4(a + 2b)$

$= 4a + 8b.$

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$(2f_1 - 5b_2)(a,b) = -13a + 9b.$

Q Let V be the vector space of all polynomial in t over R of degree ≤ 1

i.e $V = \{a + bt : a, b \in R\}$

also $f_1 : V \to R$ and $f_2 : V \to R$ defined by

$f_1(p(t)) = \int_0^2 p(t) \cdot dt.$

$f_2(p(t)) = \int_0^2 p(t) \cdot dt$ are linear function.

find the basis of $B = (\alpha_1, \alpha_2)$, dual basis $B' = (f_1, f_2)$

Sol Let $\alpha_1 = p_1(t) = a + bt$ and $\alpha_2 = p_2(t) = c + dt$ where $a, b, c, d \in F$

Since B' is the basis of B.
∴ By defination $f_i(\alpha_j) = \delta_{ij} \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

$f_1(\alpha_1) = 1 \qquad f_1(\alpha_2) = 0$
$f_2(\alpha_1) = 0 \qquad f_2(\alpha_2) = 1$

~~$f_1(\alpha_1) = \int_0^1 a + bt \cdot dt.$~~

$$f_1(\alpha_1) = \int_0^1 p(t).\,dt$$

$$\int_0^1 a + bt \,.\, dt$$

$$1 = a + \frac{b}{2} \longrightarrow (1)$$

$$f_2(\alpha_1) = \int_0^2 (a + bt).\,dt$$

$$a + b = 0. \quad (2)$$

$$f_1(\alpha_2) = \int_0^1 (c + dt)\,dt = c + \frac{d}{2} = 0 \longrightarrow (3)$$

Solving (2) & (3)

$$\frac{b}{2} - b = 1$$

$$-b = 2$$

$$\boxed{b = -2} \quad \boxed{a = 2}$$

$$f_1 = 2 - 2t.$$

$$f_2(\alpha_2) = \int_0^2 (c + dt)\,dt.$$

$$2c + 2d = 1$$

$$c + \frac{d}{2} = \frac{1}{2} \longrightarrow (4)$$

Condition for subspace
$\omega \subseteq V$
$\omega \subset V$
subspace of $a\alpha + b\beta \in \omega$
$af + bg \in S^\circ$.

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Solving 2 & 4

$$c + \frac{d}{2} = 0.$$

$$c + d = \frac{1}{2}$$

$$\frac{d}{2} = \frac{1}{2}$$

$$d = 1$$

$$c = \frac{-1}{2}$$

$$F_2 = \frac{-1}{2} + 1t$$

Anhilator

Let V be the vector space defined over the field f. Let S be the subset of V then anhilator of is denoted by S' and is expressed as following

$$S^\circ = \{ f \in V^* : f(\alpha) = 0 \}$$
$$\alpha \in V$$

Prove that: Anhilator is a vector subspace
Since we know that by defination Anhilator
Let

$f, g \in S^\circ$ then $f(\alpha) = 0$
$g(\alpha) = 0$

$\forall \alpha \in S.$

Now let $a, b \in F$ we have.

$(af + bg)\alpha = (af)\alpha + (bg)\alpha$

$= af(\alpha) + bg(\alpha) \therefore f(a\alpha) = af(\alpha)$

$= a.0 + b.0$

$= 0 + 0 = 0 \in S^\circ$

Since $(\alpha) = 0.$ $\forall \alpha \in S$ so that $0 \in S^\circ.$

Then $f, g \in S^\circ$ and $a, b \in F.$

$af + bg \in S^\circ$

$\Rightarrow S^\circ$ is a vector subspace

Theorem Dimension of Anhilator.

Let V be the infenite vector.

Let W be the vector subspace

$$\dim W + \dim W^\circ = \dim V$$

If W is 0 subspace of V

I[st] case i.e $W = \{0\} \Rightarrow \dim W = \dim 0 = 0.$

Then $W^\circ = V$

$\dim W = \dim V^* = \dim V$

$\Rightarrow \dim W + \dim W^* = \dim V$

II case If $W = V$

then $W^\circ = \{0\}.$

$\dim W^\circ = 0.$

$\dim W + \dim W^\circ = \dim V.$

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Now suppose that W is proper subspace of V
Let $\dim V = n$ and $\dim W = n$ then $0 < m < n$

Let $B_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ be basis of W

Since B_1 is linearly independent subset of V. Also therefore is extended to form a basis for V
Let $B = \{\alpha_1, \alpha_2 — \alpha_m, \alpha_{m+1} — \alpha_n\}$ be basis for V

Let $B^* = \{f_1, f_2 — f_m, f_{m+1} — f_n\}$ be the dual basis for B then B^* is also base for V^* such that

$$f_i(\alpha_j) = \delta_{ij}$$

We claim that $S = \{f_{m+1}, f_{m+2}, — f_n\}$ is a basis for W^0. Since $S \subseteq B^*$, therefore S is linearly independent because B^* is linearly independent so S will be a basis for W^0 if W^0 is equal to the subspace of V^* spaned by S that is if $W = L(S)$

First we have to show that $W^0 \subseteq L(S)$
then $f \in W^0$ then $f \in V^*$ so let

$$f = \sum_{i=1}^{n} a_i f_i$$

Now $f \in W^0 = f(\alpha) = 0 \ \forall \alpha \in W$.

$$f(\alpha_i) = 0 \quad \text{for each } i = 1, 2 — m$$

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($\because \alpha_1, \alpha_2 \ldots \alpha_m$ are in W).

$$\left(\sum_{i=1}^{n} a_i f_i\right) \alpha_j = 0 \Rightarrow \sum_{i=1}^{n} a_i f_i (\alpha_j) = 0.$$

$\cancel{\sum_{i=1}^{n} a_i \delta_{ij} = 0}$. $\sum_{i=1}^{n} a_i \delta_{ij} = 0$ where $\delta_{ij} \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

$a_j = 0$ for each $j = 1, 2 \ldots m$

Now putting $a_1 = 0$, $a_2 = 0$ -- $a_m = 0$. in (i) we get

$$f = a_{m+1} f_{m+1} + a_n f_n$$

a linear combination element of f

$f \in L(S) \Rightarrow f \in W^0 \Rightarrow f \in L(S)$

$\therefore W^0 \subseteq L(S)$

Now we shall show that $L(S) \subseteq W^0$

Let $g \in L(S)$. Then g is a linear combination of $f_{m+1} \ldots f_n$. Let $g = \sum_{k=m+1}^{n} b_k f_k$ —(2)

Let $\alpha \in W$. Then α is linear combination of α_1, α_2 of α_m. Let

$$\alpha = \sum_{j=1}^{m} c_j \alpha_j$$

$$g(\alpha) = g\left(\sum_{k=m+1}^{n} c_j \alpha_j\right) = \sum_{j=1}^{m} c_j g(\alpha_j) \because (g \text{ is linear function})$$

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$$= \sum_{j=1}^{m} c_j \left(\sum \sum_{k=m+1}^{n} b_k f_k \right) \alpha_j$$

$$= \sum_{j=1}^{m} c_j \left(\sum_{k=m+1}^{n} b_k f_k \right)(\alpha_j) = \sum_{j=1}^{m} c_j \sum_{k=m+1}^{n} b_k f_{kj}$$

$$= \sum_{j=1}^{n} c_j \times 0 = 0 \quad (\because \delta_{kj} = 0 \text{ if } k \neq j \text{ which is so for each } k = m+1)$$

for each $j = 1, 2 \text{---} m$

Thus $g(\alpha) = 0 \ \forall \ \omega$ therefore $g \in \omega^{\circ}$

Thus $g \in L(S) \Rightarrow g \in \omega^{\circ} \quad \therefore L(S) \subseteq \omega^{\circ}$

Hence $\omega^{\circ} = L(S)$ and S is a basis for ω°

$$\dim \omega^{\circ} = n - m = \dim V - \dim \omega$$

$$\boxed{\dim V = \dim \omega + \dim \omega^{\circ}}$$

If V is finite dimensional vector space and ω is sub space of V then $\omega^* = V^* / \omega^{\circ}$.

Proof: We have $\dim (V^*/\omega^{\circ}) = \dim V^* - \dim \omega^{\circ}$

$$= \dim V - \dim \omega^{\circ} \ (\because \dim V^{\circ} = \dim V)$$

$$= \dim V - (\dim V - \dim \omega) \ (\because \dim \omega^{\circ} = \dim V - \dim \omega)$$

$$= \dim \omega = \dim \omega^*$$

Then we have $\dim \omega^* = \dim (V^* / \omega^{\circ})$

Hence by theorem we have

$$\omega^* = V^* / \omega^{\circ}.$$

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Direct sum of subspace :-

Let ω_1 and ω_2 be the subspace over the vector $V(F)$. Then V is said to be direct sum of two subspace ω_1 and ω_2. If element of r of V is uniquely express as $r = \alpha + \beta$. where $\alpha \in \omega_1$ and $\beta \in \omega_2$. Then sum is denoted by $\oplus$ i.e $V = \omega_1 \oplus \omega_2$

Disjoint Subspace.

Let $V(F)$ be the vector space of the field (F) ~~and vectorspace $V(F)$ then V is said to be direct sum of $\omega_1, \omega_2 \ldots \omega_n$ if for each $r \in V$~~

ω_1 & ω_2 are the two subspace of V. Then they are disjoint (i.e ω_1 & ω_2 are disjoint) If $\omega_1 \cap \omega_2 = \{0\}$

Extension of the direct sum.

Let $\omega_1, \omega_2 \ldots \omega_n$ be the subspace of vector $V(F)$. then V is said to be direct sum of $\omega_1, \omega_2 \ldots \omega_n$ if for each $r \in V$ then

$r = \alpha_1 + \alpha_2 + \alpha_3 \ldots \alpha_n$ where

$\alpha_1 \in \omega_1, \alpha_2 \in \omega_2 \ldots \alpha_n \in \omega_n$

i.e $V = \omega_1 \oplus \omega_2 \oplus \ldots \oplus \omega_n$

* The vector space of $V(F)$ is direct sum of two subspace ω_1 and ω_2 if (i) $V = \omega_1 + \omega_2$

(ii) $\omega_1 \cap \omega_2 = \{0\}$

We know that ~~vector~~ the x, y plane, yz plane, zx plane are the subspace of $V_3(R)$ and also the coordinate of $V_3(R)$ can be expressed uniquely therefore the direct sum of x, y plane, z axis. similarly $V_3(R)$ is the direct sum of yz plane in x axis and zx plane in x axis.

2nd case

$v_1 \rightarrow$ xy plane $= \{(x, y, 0) \quad x, y \in R\}$
and $w_1 \rightarrow$ z axis $\{(0, 0, z) \quad z \in R\}$
$v_2 \rightarrow$ yz plane $\{(0, y, z) \quad y, z \in R\}$
$w_2 \rightarrow$ x plane $\{(x, 0, 0) \quad x \in R\}$
$v_3 \rightarrow$ zx plane $\{(x, 0, z) \quad x, z \in R\}$
$w_3 \rightarrow$ y plane $\{(0, y, 0) \quad y \in R\}$

We know that a ~~space~~ vector space $V_3(R)$ is vector direct sum iff. $v_1 + w_1 = V_3(R)$
$w_1 \cap w_1 = \{0\}$

Now $v_1 + w_1 = \{(x, y, 0) \quad x, y \in R\} + \{(0, 0, z) \quad z \in R\}$

$= \{x+0, y+0, 0+z, \quad x, y, z \in R\}$

$= \{(x, y, z) \quad x, y, z \in R\}$

$v_1 + w_1 = V_3(R)$

Again ~~$w_1 \cap v_1$~~ =
$v_1 \cap w_1 = \{\cancel{0}\}$

$\{(x, y, 0) \quad x, y \in R\} \cap \{(0, 0, z) \quad z \in R\}$
$= \{0\}$

$U_1 \cap W_1 = 0$

2^{nd} case Now we know that vector space $V_3(R)$ is vector direct sum if (i) $U_2 + W_2 = V_3(R)$
(ii) $U_2 \cap W_2 = \{0\}$

Now $U_2 + W_2 = \{(0, y, z) \quad y, z \in R\} + \{(\cancel{x}, 0, 0) \quad x \in R\}$
$\{(x, 0, 0) \quad x \in R\}$.

$= \{(0 + x, \; y + 0, \; z + 0) \quad x, y, z \in R)\}$

$= \{(x, y, z) \in R\}$

$U_2 + W_2 = V_3(R)$

Again $U_2 \cap W_2$

$\{(0, y, z) \in R \quad y, z \in R)\} \cap \{(x, 0, 0) \quad x \in R\}$

$= \{0\}$

$U_2 W_2 = \{0\}$.

3^{rd} case Now we know that vector space $V_3(R)$ vector direct sum if (i) $U_3 + W_3 = V_3(R)$
(ii) $U_3 \cap W_3 = \{0\}$

Now $U_3 + W_3 = \{(x, 0, z) \quad x, z \in R)\} + \{(0, y, 0) \in R\}$

$= \{(x+0, 0+y, z+0) \quad x, y, z \in R\}$
$= \{(x, y, z) \quad x, y, z \in R\}$

$U_3 + W_3 = V_3(R)$

Again
$U_3 \cap W_3$

$\{(x, 0, z) \quad x, z \in R\} \cap \{(0, y, 0)\} \quad y \in z\}$

$\{0\}$

$U_3 \cap W_3 = \{0\}$

———— b ————

Theorem Suppose V is finite dimensional vector space and W_1, W_2 are the subspace of V then prove that (i) $(W_1 + W_2)^\circ = W_1^\circ \cap W_2^\circ$
(ii) $(W_1 \cap W_2)^\circ = W_1^\circ + W_2^\circ$

$S^\circ = \{f \in V^* : f(\alpha) = 0 \quad \alpha \in V\}$

Two sets are equal. If they contain
$A = B$ iff $A \subseteq B$
$B \subseteq A$

$r = \alpha + \beta, \quad \alpha \in W_1, \quad \beta \in W_2$

Sol^n

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~~Sol^n Now to prove it we have to show~~

~~$(W_1+W_2)^\circ \subseteq W_1^\circ \cap W_2^\circ$ — ①~~

~~$W_1^\circ \cap W_2^\circ \subseteq (W_1+W_2)^\circ$ — ②~~

~~Let α be any vector in W_1+W_2~~

~~Let $f \in W_1 \cap W_2 \Rightarrow f \in W_1, f \in W_2$~~

~~$\alpha = \alpha_1 + \alpha_2$ we have $\alpha_1 \in W_1, \alpha_2 \in W_2$~~

~~$f(\alpha) = \cancel{f(\alpha_1)+f(\alpha_2)}\ f(\alpha_1+\alpha_2) = f(\alpha_1)+f(\alpha_2)$~~

~~$= 0+0$~~

~~$(\because \alpha \in V, f \in W$ so that $f(\alpha_1) = 0$ similarly $f(\alpha_2) = 0)$~~

Sol^n Now to prove it we have to show

$(W_1+W_2)^\circ \subseteq W_1^\circ \cap W_2^\circ$

$W_1^\circ \cap W_2^\circ \subseteq (W_1+W_2)^\circ$

Let $f \in W_1^\circ \cap W_2^\circ \Rightarrow f \in W_1^\circ, f \in W_2^\circ$

Let α be any vector in W_1+W_2

$f(\alpha) = f(\alpha_1+\alpha_2) = f(\alpha_1)+f(\alpha_2)$

$= 0+0$ $\quad (\alpha \in V, f \in W_1$ so that $f(\alpha_1) = 0$ similarly $f(\alpha_2) = 0)$

Therefore $f(\alpha) = 0$.

$\alpha = W_1 + W_2$

$f \in (W_1+W_2)^\circ$

$f \in (W_1 \cap W_2) \Rightarrow f \in (W_1+W_2)^\circ$

$W_1^\circ \cap W_2^\circ \subseteq (W_1+W_2)^\circ$

$(W_1+W_2)^0 \subseteq W_1^0 \cap W_2^0$

We know set is subset of itself.

$W_1 \subseteq W_1+W_2$

$(W_1+W_2)^0 \subseteq W_1^0$ —— (3)

Again $W_2 \subseteq W_1+W_2$

$(W_1+W_2)^0 \subseteq W_2^0$ —— (4)

$(W_1+W_2)^0 \cap (W_1+W_2)^0 \subseteq W_1^0 \cap W_2^0$

$(W_1+W_2)^0 \subseteq W_1^0 \cap W_2^0$

Combine both set $(W_1+W_2)^0 = W_1^0 \cap W_2^0$

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Quotient Space — Let $V(F)$ be the Vector Space over the field F and let W be the Subspace of V.

Let $\alpha \in V$ be any vector Then
$W+\alpha = \{w+\alpha : w \in W\}$ is Called right ~~Subset~~ Coset of W is V

Any Subset of V is known as Quotient space if it Can be defined as follows. $\frac{V}{W} = \{W+V = \alpha \in V\}$

Here we define Scaler multiplication and addition

$$(W+\alpha)+(W+\beta) = W+(\alpha+\beta)$$

$$a\,(W+\alpha) = W+a\alpha$$

Thm Let V be the Vector Space and W be the ~~Spe~~ Subspace of V
Then $\dim\left(\frac{V}{W}\right) = \dim V - \dim W$

Thm Let W_1 and W_2 be the two Subspace of Vector $V(F)$ Then $\dim(W_1 \cap W_2) = \dim W_1 + \dim W_2 - \dim(W_1+W_2)$

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combine together we have
$T(a.\alpha + b.\beta) = a.T(\alpha) + b.T(\beta)$

Ex: Show that the mapping
$T: V_3(R) \longrightarrow V_2(R)$ defined as
$T(x_1, x_2, x_3) = (x_1 - x_2, x_1 + x_2)$
is linear transformation.

Sol^n Now let $\alpha = (x_1, x_2, x_3)$ and
$\beta = (y_1, y_2, y_3)$ then we have to show that,

(i) $T(\alpha+\beta) = T(\alpha) + T(\beta)$
(ii) $T(a\alpha) = a.T(\alpha)$

$\therefore T(\alpha+\beta) = T[(x_1, x_2, x_3) + (y_1, y_2, y_3)]$

$T[\underbrace{(x_1+y_1)}_{X_1}, \underbrace{(x_2+y_2)}_{X_2} + \underbrace{(x_3+y_3)}_{X_3}]$

$T(\alpha+\beta) = x_1 + y_1 - (x_2+y_2), (x_1+y_1, x_3+y_3)$

$T(\alpha+\beta) = (x_1 - x_2 + y_1 - y_2), (x_1 + x_3 + y_1 + y_3)$ —(1)

Now
$T(\alpha) + T(\beta) = T(x_1, x_2, x_3), T(y_1, y_2, y_3)$

$= (x_1 - x_2, x_1 + x_3) + (y_1 - y_2, y_1 + y_3)$

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$T(\alpha) + T(\beta) = (x_1 - x_2 + y_1 - y_2, x_2 + x_3 + y_1 + y_3)$ — (ii)

from (i) and (ii) we get.

$T(\alpha + \beta) = T(\alpha) + T(\beta)$

$= T[a(x_1, x_2, x_3)]$

$= T[ax_1, ax_2, ax_3)$

$= ax_1 - ax_2, ax_1 + a_3.$

$= a(x_1 - \cancel{x_2}, x_1 + x_3)$ by rev. defination.

$= a\,T(\alpha_1 \cancel{+ \alpha_2}, \alpha_3)$

$a\cancel{\alpha} T \; a \cdot T(\alpha)$ //

Ex: Show that the mapping :-

$T: V_3(F) \to V_2(F)$ defined as.

$f(\cancel{x}, y, z) = (y, z)$

is linear transformation

Solⁿ Let $\alpha = (x_1, y_1, z_1)$ and

$\beta = (x_2, y_2, z_2)$ then we have to show that

(i) $T(\alpha + \beta) = T(\alpha) + T(\beta)$

(ii) $T(a \cdot \alpha) = a \cdot T(\alpha)$.

Now

$T(\alpha + \beta) = T[(x_1, y_1, z_1) + (x_2 + y_2 + z_2)]$

$(x_1 + x_2 + z_1 + z_2)$ — (1)

$T(\alpha + \beta) = (x_1 - y_1 + x_2 - y_2), (x_1 + x_2 + z_1 + z_2)$

$\cancel{T(\alpha)} + T(\beta)$

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$$= T[(x_1+x_2),(y_1+y_2),(z_1+z_2)]$$

$$= ((y_1+y_2), z_1+z_2) \quad \text{— (i)}$$

by def.

$$T(\alpha)+T(\beta) = T(x_1, y_1, z_1) + T(x_2, y_2, z_2)$$

$$= (y_1, z_1) + (y_2 + z_2) \quad \text{By def}$$

$$= (y_1 + y_2), (z_1 + z_2) \quad \text{— (ii)}$$

from (i) and (ii)

$$T(\alpha+\beta) = T(\alpha) + T(\beta)$$

$$T(a\alpha) = a \cdot T(\alpha)$$

$$= T[a(x_1, y_1, z_1)]$$

$$= (ay_1, az_1)$$

$$= a(y_1, z_1) \quad \text{(By rev def)}$$

$$a \cdot T(x_1, y_1, z_1)$$

$$a \cdot T(\alpha)$$

Ex. Show that the mapping $f: V_2(R) \to V_3 R$ in linear transformation defined as follows.

$$f(a, b) = (a, b, 0)$$

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Sol^n Let $\alpha = a_1, b_1$
$\beta = a_2, b_2$
if $\alpha . \beta \in V_2$ and $a, b \in F$.
then we have to show that,
$f(a.\alpha + b.\beta) = a.f(\alpha) + b.f(\beta)$

Now considering LHS.
$f(a.\alpha + b.\beta) = f[a(a_1, b_1) + b(a_2, b_2)]$.
$= f[(aa_1 + ab_1) + (ba_2 + bb_2)]$
$= f\underbrace{(aa_1 + ba_2)}_{x} + \underbrace{(ab_1 + bb_2)}_{y}]$ [by def of f]
$= (aa_1 + ba_2, ab_1 + bb_2, 0)$ $(x, y, 0)$

$= (aa_1 + ab_1 + 0)(ba_2 + bb_2 + 0)$
[$\because$ by rev def of f]

$= a f(a_1 b_1) + b.f(a_2 b_2)$
$= a f(\alpha) + b f(\beta)$

LHS = RHS//

Ex Show that the mapping $f: V_3(R) \to V_2(R)$ is linear transformation is defined as.
$f(a, b, c) = (0, a+b)$

Sol^n Let $\alpha = a_1, b_1, c_1$
$\beta = a_2, b_2, c_2$

if $\alpha, \beta \in V$ and $a, b \in F$ then we have to show that
$f(a\alpha + b\beta) = a.f(\alpha) + b.f(\beta)$

Ex- Let W_1 be the subspace of $V_4(F)$ generated by the set of Vector

$$S_1 = \{(1,1,0,-1),(1,2,3,0)\ (2,3,3,-1)\}$$

and W_2 is the Subspace of $V_4(F)$ generates by the Set of Vectors.

$$S_2 = \{(1,2,2,-2),(2,3,2,-3),(1,3,4,-3)\}$$

find ① $\dim(W_1+W_2)$

② $\dim(W_1 \cap W_2)$

Soln- We know that $L(W_1 \cap W_2) = W_1+W_2$ Therefore W_1+W_2 is a subspace generated by the vector

$$S_1 \cup S_2 = \left\{ \begin{matrix} (1,1,0,-1),(1,2,3,0),(2,3,3,-1) \\ (1,2,2-2),(2,3,2,-3),(1,3,4,-3) \end{matrix} \right\}$$

There we make above as in term of an Echlon form

$$\begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 3 & -1 \\ 1 & 2 & 2 & -2 \\ 2 & 3 & 2 & -3 \\ 1 & 3 & 4 & -3 \end{bmatrix}_{4\times 6} \begin{matrix} \rightarrow R_1 \\ \rightarrow R_2 \\ \rightarrow R_3 \\ \rightarrow R_4 \\ \rightarrow R_5 \\ \rightarrow R_6 \end{matrix}$$

$R_3 \to R_3 - 2R_1$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - 2R_1$

$R_4 \to R_4 - R_1$,

$R_6 \to R_6 - R_1$

$$\begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 2 & -1 \\ 0 & 2 & 4 & -2 \end{bmatrix}$$

$R_3 \to R_3 - R_2$, $R_5 \to R_5 - R_4$, $R_6 \to R_6 - 2R_5$

$$\begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 \leftrightarrow R_4$

$$A = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here we see that 3 non zero rows, Hence rank of matrix is 3

$\therefore \dim(W_1 + W_2) = 3$ Ans

~~Such that~~

(2) We know that

$$\dim(W_1 \cap W_2) = \dim W_1 + \dim W_2 - \dim(W_1 + W_2)$$

$$\text{let } A_1 = \begin{bmatrix} 1, & 1, & 0, & -1 \\ 1, & 2, & 3, & 0 \\ 2, & 3, & 3, & -1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1, & 2, & 2, & -2 \\ 2, & 3, & 2, & -3 \\ 1, & 3, & 4, & -3 \end{bmatrix}$$

$R_2 \to R_2 - R_1$ $\qquad R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1, & 1, & 0, & -1 \\ 0, & 1, & 3, & 1 \\ 1, & 1, & 0, & -1 \end{bmatrix}$$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1, & 1, & 0, & -1 \\ 0, & 1, & 3, & 1 \\ 0, & 0, & 0, & 0 \end{bmatrix}$$

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$$\dim(W_1 \cap W_2) = \dim W_1 + \dim W_2 - \dim(W_1 + W_2)$$

$$= 2 + 2 - 3 = 1$$

$$\dim(W_1 \cap W_2) = 1$$