[signature]

3) d) $\sum_{n=1}^{\infty} \frac{a^n}{(n+2)(n+a)\,5^n} = S_n$

Se observa q-e $b_n = \frac{a^n}{(n+2)(n+a)5^n} = \frac{1}{(n+2)+(n+a)} \frac{a^n}{5^n} \leq \left(\frac{a}{5}\right)^n$

$b_n \leq \left(\frac{a}{5}\right)^n$. entonces, $S_n \leq \sum_{n=1}^{\infty} \left(\frac{a}{5}\right)^n = \underbrace{\sum_{n=0}^{\infty} \left(\frac{a}{5}\right)^n}_{\substack{\text{geométrica} \\ \text{con } r<1 \\ \text{si } a<5}} - 1$

Si $a<5$:

$\sum_{n=0}^{\infty} \left(\frac{a}{5}\right)^n - 1 = \frac{1}{1-\frac{a}{5}} - 1$, es decir Converge

Como $S_n \leq \sum_{n=0}^{\infty} \left(\frac{a}{5}\right)^n - 1$, S_n Converge por criterio de comparación

$a = 5$: $S_n = \sum_{n=1}^{\infty} \frac{5^n}{(n+2)(n+a)5^n} = \sum_{n=1}^{\infty} \frac{1}{(n+2)(n+5)}$

busco a y b / $S_n = \sum_{n=1}^{\infty} \frac{a}{n+2} + \frac{b}{n+5}$

$[(n+5)a + b(n+2) = 1] \Leftrightarrow an + 5a + bn + 2b = 1$

$\Leftrightarrow an = -bn$ y $5a + 2b = 1$

$\Leftrightarrow \begin{cases} an - bn = 0 \\ a5 + 2b = 1 \end{cases} \Leftrightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 5 & 2 & 1 \end{array}\right] \Leftrightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & -3 & 1 \end{array}\right] \Leftrightarrow \begin{cases} b = -\frac{1}{3} \\ a = \frac{1}{3} \end{cases}$

y así, $S_n = \sum_{n=1}^{\infty} \frac{1}{3(n+2)} - \frac{1}{3(n+5)} = \frac{1}{3} \sum_{n=1}^{\infty} \frac{1}{n+2} - \frac{1}{n+5} + \frac{1}{n+3} - \frac{1}{n+3} + \frac{1}{n+4} - \frac{1}{n+4}$

$= \frac{1}{3} \sum_{n=1}^{\infty} \underbrace{\left(\frac{1}{n+2} + \frac{1}{n+3} + \frac{1}{n+4}\right)}_{b_n} - \underbrace{\left(\frac{1}{n+3} + \frac{1}{n+4} + \frac{1}{n+5}\right)}_{b_{n+1}}$

es telescópica, entonces

$S_n = \frac{1}{3}\left(b_1 + \lim_{n\to\infty} b_{n+1}\right) = \frac{1}{3}\left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} - 0\right) \Rightarrow \boxed{S_n \text{ Converge}}$

$a > 5$:

$$\sum_{n=1}^{\infty} \frac{a^n}{(n+2)(n+a)5^n} > \sum_{n=1}^{\infty} \frac{a^n}{3n\,2n\,5^n} = \sum_{n=1}^{\infty} \frac{a^n}{6n^2 5^n} = \sum_{n=1}^{\infty} \frac{1}{6n^2}\left(\frac{a}{5}\right)^n$$

$$b_n = \frac{1}{6n^2}\left(\frac{a}{5}\right)^n \quad , \quad \lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{6n^2}\left(\frac{a}{5}\right)^n$$

$$c = e^{\ln\left(\frac{1}{6n^2}\left(\frac{a}{5}\right)^n\right)} = e^{\ln\left(\frac{1}{6n^2}\right) + \ln\left(\frac{a}{5}\right)^n} = e^{\overbrace{(\ln(1) - 2\ln(n)) + n\ln\left(\frac{a}{5}\right)}^{d}}$$

$$\lim_{n\to\infty} c = e^{\lim_{n\to\infty} d} \overset{\text{L'H}}{=} e^{\lim_{n\to\infty} -\frac{2}{n} + \ln\left(\frac{a}{5}\right)} = e^{\ln\left(\frac{a}{5}\right)} = \lim_{n\to\infty} b_n$$

como es distinto a 0, entonces S_n diverge