University of Helsinki
Faculty of Science
Bachelors Program in Mathematics

Bachelor's thesis

Harmonic Extension at the Boundary of Half-Space

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June 2025

Otsikko: Harmoninen laajennus puoliavaruuteen
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Kuukausi ja vuosi: Kesäkuu 2025
Sivumäärä: 28 s.

Tiivistelmä:
Tutkielma käsittelee Dirichlet-ongelmaa, jossa tavoitteena on laajentaa reuna-arvot määrittelevä funktio n-ulotteisesta Euklidisesta avaruudesta $n+1$-ulotteiseen puoliavaruuteen siten, että laajennus on harmoninen ja siten ratkaisee Laplacen-yhtälön. Tätä kohti tutkielmassa edetään esittelmällä Fourier-muunnos ja käyttämällä sitä avuksi Poisson-ytimen johtamisessa. Poisson-ydintä käytetään Poisson-integraalin määrittämiseen. Tämä muodostaa eksplisiittisen muodon Dirichlet-ongelman ratkaisulle, kun ratkottava osittaisdifferentaaliyhtälö on Laplacen-yhtälö.

Tutkielmassa on kaksi päätulosta. Näistä ensimmäinen todistaa, että Poisson-integraali toteuttaa reuna-arvo-ongelman kun reuna-arvot määräytyvät funktiosta, joka kuuluu L^p-avaruuteen. Tätä tulosta täydennetään lisätuloksella, jonka avulla voidaan näyttää, että ratkaisun suppeneminen noudattaa vahvempaa suppenemiskäsitettä silloin, kun reuna-arvot määrittävä funktio kuuluu tiukemmat säännöllisyysvaatimukset toteuttavaan funktiavaruuteen. Toinen tutkielman päätuloksista laajentaa ensimmäistä tulosta vastaavan tuloksen koskemaan myös ääreellisiä Borel-mittoja. Tämä mahdollista monipuolisemman reuna-arvodatan käyttämisen reuna-arvo-ongelmassa.

Nämä tulokset antavat eksplisiittisen integraallimuodon funktiolle, joka ratkaisee halutun Dirichlet-ongelman. Tulokset valottaa myös sitä miten harmoniset laajennukset käyttäytyvät, kun reuna-arvot määrittävän funktion säännöllisyydelle oletetaan erilaisia ehtoja. Lisäksi tämän kaltaisilla tuloksilla on sovelluksia monissa erilaisissa matematiikan ja fysiikan sovelluksissa. Tuloksia voidaan käyttää esimerkiksi osittaisdifferentaaliyhtälöiden analyysissä ja potentiaaliteoriassa. Työn lopussa käyn läpi vielä joitakin mahdollisia yleistyksiä, joihin tuloksia on mahdollista laajentaa.

Avainsanat: Harmoninen analyysi, Osittaisdifferentaaliyhtälöt, Fourier analyysi

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Month and year: June 2025
Page count: 28 pp.

Abstract:
In this thesis, I will introduce results for the Dirichlet problem in the Euclidean half-space. The goal is to extend a boundary value function from n-dimensional Euclidean space to the whole of the $n+1$-dimensional half-space, so that it solves the Laplace equation, and thus is harmonic.

I will first introduce some tools that are required for the analysis of this problem, and are also useful in the broader field of analysis. The main analytical tool used in this study is the Fourier transform. The Fourier transform is then used to derive the Poisson kernel, which provides an explicit harmonic expression for the extension of the boundary data in the form of a Poisson integral.

The thesis presents two main results. The first of these proves that the Poisson integral can be used as an extension of a boundary value function when it belongs to the L^p space. An extension of this result with stricter regularity assumptions on the function, which leads to a stronger notion of convergence, is also presented. The second main result shows that a similar result also holds for finite Borel measures. This allows an even wider variety of data to be represented at the boundary of the half-space.

These results provide an explicit solution to a boundary value problem for Laplace's equation, but also show how harmonic extensions behave under different regularity assumptions. These results have applications in a wide range of mathematical analysis, especially in partial differential equations, and in physics in the form of potential theory. In the end, I will highlight some of the possible ways in which these results can be extended to cover some more generalized or slightly modified settings.

Keywords: Harmonic analysis, Partial differential equations, Fourier analysis

Contents

Chapter 1

Introduction

This thesis is a study of results in the field of harmonic and Fourier analysis. I will be concentrating on results that prove that Poisson integrals in the case of L^p spaces and Poisson-Stieltjes integrals in the case of finite Borel measures are a valid way to extend a function describing some boundary value data at the boundary of a half space $\mathbb{R}^n \times \{0\}$, to the whole of the half-space in a way that it satisfies the Laplace equation $\Delta u = \sum_i \frac{\partial^2 u}{\partial x_i^2} = 0$, while converging towards the function at the boundary. This work will follow mostly the results given in the *Introduction to Fourier Analysis on Euclidean Spaces* by E. M. Stein and G. Weiss [9].

The investigated problem is a special case of Dirichlet's problem. Dirichlet's problem poses a question: can we find a solution for a partial differential equation (PDE), such that it converges to some specified function at the boundary of the space? In this thesis, I consider Laplace's equation as the PDE, and the space will be the Euclidean half-space $\mathbb{R}^n \times \mathbb{R}_+ = \mathbb{R}^{n+1}_+$.

This question relates to many areas of mathematical analysis of PDEs [5, 6] and is deeply connected to mathematical physics [4]. In mathematics, these types of results are frequently used in the analysis of PDEs to extend certain weaker solution concepts to boundary value problems. This allows for the embedding of less regular initial value data, which may not satisfy the boundary condition in the classical sense of functions. These are connected to physical problems through potential theory, where, for example, gravitational and electrostatic potentials can be described by harmonic functions as described in *Methods of Mathematical Physics: Volume II Partial Differential Equations* by Hilbert and Courant [4].

The rest of this introduces some of the essential tools that are used in the construction of the Poisson kernel in Chapter 2. This also includes showing some properties of the Poisson kernel that will be needed later. Chapter 3 will concentrate on the main results of this thesis. First, I will show that the Poisson integral extends the boundary value function in L^p to the half-space. The second part will extend this result to include finite Borel measures.

Chapter 2

Poisson Formula for Harmonic Functions in $\mathbb{R}^{n+1}_{+}$

This chapter introduces some necessary preliminaries to understand the main result concerning the Dirichlet problem introduced in Chapter 3. In order to understand that, one needs to be familiar with certain mathematical constructs. Most importantly, Fourier transforms and harmonic functions. These provide us with the tools that are used to define the objective of having a function that solves the PDE, in this case Laplace's equation, within the half-space in a way that it converges towards some desired functions at the boundary of the half-space.

For this purpose, I will go through the derivation of a Poisson kernel, which is a special type of function that has the property of solving Laplace's equation and acting as an approximate identity with respect to convolution.

2.1 Fourier Transform

The study of harmonic functions requires the use of the Fourier transforms in many situations. The Fourier transform has many desirable properties that can be used in the analysis. One of those is that the use of the Fourier transformation converts differential operators into algebraic expressions, and then recovers the original function via the inverse transform. This greatly eases the analysis of many PDEs, where it can be easier to manipulate algebraic expressions than differential ones. More generally, it is sometimes easier to manipulate certain expressions in the Fourier space and then transform them back to the original form to obtain the result in the desired form.

The results that prove these properties, among many others, are fundamental in the area of Fourier analysis. However, in this chapter, I will be concentrating especially on using the Fourier transform to transform specific functions that turn out to have very special properties concerning the class of harmonic functions. The class of harmonic functions—those functions that solve the Laplace equation $\Delta u = 0$—in the half-space $\mathbb{R}^{n+1}_{+} = \{(x_1, x_2, \ldots, x_n, y) | x_i \in \mathbb{R} \text{ for all } i \in \{1, 2, \ldots, n\}, y \in \mathbb{R}_+\}$ can be represented explicitly using Poisson kernels. Obtaining such an object requires the Fourier transform of another particular function. This will then allow the recovery of a function from its boundary values $f(x) = u(x, 0)$. By using the Poisson kernel, we can generate a solution for the boundary value problem by convolving it with the function f. This will yield the harmonic extension into the half-space.

Let's start by defining the Fourier transform in the L^1-space.

Definition 2.1.1 (Fourier Transform in L^1)**.** If $f \in L^1(\mathbb{R}^n)$ the Fourier transform of f is the function $\hat{f}$ defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx$$

for all $\xi \in \mathbb{R}^n$. I will also occasionally denote the Fourier transform by $\hat{f}(\xi) = \mathcal{F}[f](\xi)$.

The definition of a Fourier transformation does not yet guarantee any particular properties for the transformation $\hat{f}$. However, results exist for this purpose. One of the most prominent is called the Riemann-Lebesgue lemma, which gives a quite sharp characterization of the Fourier transform of a L^1-function. It establishes that the transformation does, in fact, enjoy very good regularity properties with regard to the continuity and the decay of the tail of the transformation.

Lemma 2.1.2 (Riemann-Lebesgue Lemma)**.** *Suppose* $f \in L^1(\mathbb{R}^n)$*. Then* $\lim_{n \to \xi} \hat{f}(\xi) = 0$ *and* $\hat{f} \in C_0(\mathbb{R}^n)$*.*

Proof. Let $\alpha(\xi) = \frac{\xi}{2|\xi|^2}$. Then by Euler's identity

$$1 = -e^{\pi i} = -e^{\pi i 2\alpha(\xi) \cdot \xi}$$

holds. Now for $\xi \neq 0$, the previous identity substituted into the Fourier transformation of $f \in L^1(\mathbb{R}^n)$ gives

$$\begin{aligned}
\hat{f}(\xi) &= \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx \\
&= \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} (-e^{\pi i 2\alpha(\xi) \cdot \xi}) dx \\
&= -\int_{\mathbb{R}^n} f(x) e^{-2\pi i (x - \alpha(\xi)) \cdot \xi} dx \\
&= -\int_{\mathbb{R}^n} f(x + \alpha(\xi)) e^{-2\pi i x \cdot \xi} dx
\end{aligned}$$

which allows for rewriting the transformation. By using identity $\hat{f}(\xi) = \frac{\hat{f}(\xi)+\hat{f}(\xi)}{2}$ the Fourier transform is now

$$\begin{aligned}\hat{f}(\xi) &= \frac{1}{2}\left(\int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}dx - \int_{\mathbb{R}^n} f(x+\alpha(\xi))e^{-2\pi i x\cdot\xi}dx\right)\\ &= \frac{1}{2}\int_{\mathbb{R}^n}(f(x) - f(x+\alpha(\xi)))e^{-2\pi i x\cdot\xi}dx.\end{aligned}$$

This gives a following bound for $\hat{f}$

$$|\hat{f}(\xi)| \leq ||f - f(\cdot + \alpha(\xi))||_1 \xrightarrow{\xi\to\infty} 0$$

concluding the proof. □

The result ensures that the integral in the Fourier transform converges, and therefore that the resulting function $\hat{f}$ behaves well enough to be integrable.

It is possible to generalize these results to other spaces as well. However, that would, in some cases, such as defining the Fourier transform in L^2, require not only coming up with a proof that allows a wider class of functions but also redefining the whole transform. Having the result in L^1 will suffice for now.

2.2 Poisson Kernel

Now we can use the Fourier transform to establish further results that are needed to analyze the problem of extending a function to $\mathbb{R}^{n+1}_+$. The first step is to derive the Fourier transform of a Gaussian function and notice that it transforms it into another Gaussian function.

Theorem 2.2.1 (Fourier Transform of a Gaussian Function)**.** *For all $\alpha > 0$ the following holds*

$$\int_{\mathbb{R}^n} e^{-\pi\alpha|y|^2}e^{-2\pi i\xi\cdot y}dy = \alpha^{-\frac{n}{2}}e^{-\pi\frac{|\xi|^2}{\alpha}}$$

This says that the Gaussian function is its own Fourier transform.

Proof. In this case, the integral can be transformed to correspond to the one-dimensional case by noticing that for $y, \xi \in \mathbb{R}^n$ holds:

$$\begin{aligned}e^{-\pi\alpha|y|^2}e^{-2\pi i\xi\cdot y} &= e^{-\pi\alpha\sum_{j=1}^n y_j^2}e^{-2\pi i\sum_{j=1}^n \xi_j y_j}\\ &= e^{-\pi\alpha y_1^2}e^{-\pi\alpha y_2^2}\dots e^{-\pi\alpha y_n^2}e^{-2\pi i\xi_1 y_1}e^{-2\pi i\xi_2 y_2}\dots e^{-2\pi i\xi_n y_n}\\ &= \prod_{j=1}^n e^{-\pi\alpha y_j^2}\prod_{j=1}^n e^{-2\pi i\xi_j y_j}\\ &= \prod_{j=1}^n e^{-\pi\alpha y_j^2 - 2\pi i\xi_j y_j}\end{aligned}$$

Now the problem can be approached as a product of one-dimensional integrals. In each of these integrals, the term $-\pi\alpha y_j^2 - 2\pi i\xi_j y_j$ can be completed as a square. The term can be written as:

$$-\pi\alpha y_j^2 - 2\pi i\xi_j y_j = -\pi\alpha\left(y_j^2 + \frac{2i\xi_j}{\alpha}\right)$$

and its completed square gives:

$$\begin{aligned}-\pi\alpha y_j^2 - 2\pi i\xi_j y_j &= -\pi\alpha\left(\left(y_j + \frac{i\xi_j}{\alpha}\right)^2 - \left(\frac{i\xi_j}{\alpha}\right)\right)\\ &= -\pi\alpha\left(y_j + \frac{i\xi_j}{\alpha}\right)^2 - \pi\frac{\xi_j^2}{\alpha}\end{aligned}$$

Therefore, we can denote the integral as $I_j(\xi_j) = e^{-\frac{\pi\xi_j^2}{\alpha}}\int_{-\infty}^{\infty} e^{-\pi\alpha\left(y_j + \frac{i\xi_j}{\alpha}\right)^2} dy_j$. Now it is possible to do a change of variable so that $u_j = y_j + \frac{i\xi_j}{\alpha}$, so the integral can be written as:

$$I_j(\xi_j) = e^{-\frac{\pi\xi_j^2}{\alpha}}\int_{-\infty}^{\infty} e^{-\pi\alpha u^2} du_j$$

The famous result on Gaussian integrals gives that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$. Applying this to $I_j(\xi_j)$ with a substitution $v^2 = \alpha u^2$ and $dv = \sqrt{\alpha} du$:

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-\pi\alpha u^2} du &= \frac{1}{\sqrt{\alpha}}\int_{-\infty}^{\infty} e^{-\pi v^2} dv\\ &= \frac{1}{\sqrt{\alpha}}\end{aligned}$$

Substituting the previous result to $I_j(\xi_j)$.

$$\begin{aligned}I_j(\xi_j) &= e^{-\frac{\pi\xi_j^2}{\alpha}}\int_{-\infty}^{\infty} e^{-\pi\alpha u^2} dy_j\\ &= \frac{e^{-\frac{\pi\xi_j^2}{\alpha}}}{\sqrt{\alpha}}\end{aligned}$$

And then using $I_j(\xi_j)$ to solve the integral given in the statement:

$$\begin{aligned}
\int_{\mathbb{R}^n} e^{-\pi\alpha|y|^2} e^{-2\pi i\xi\cdot y} dy &= \prod_{j=1}^{n} \int_{-\infty}^{\infty} e^{-\pi\alpha y_j^2 - 2\pi i \xi_j y_j} \\
&= \prod_{j=1}^{n} I_j(\xi_j) \\
&= \prod_{j=1}^{n} \frac{e^{-\frac{\pi\xi_j^2}{\alpha}}}{\sqrt{\alpha}} \\
&= \alpha^{-\frac{n}{2}} e^{-\frac{\pi}{\alpha}\sum_{j=1}^{n} \xi_j^2} \\
&= \alpha^{-\frac{n}{2}} e^{-\frac{\pi|\xi|^2}{\alpha}}
\end{aligned}$$

Which concludes the proof, as it is the claimed Fourier transformation of a Gaussian function. □

This result can now be used to derive the Poisson kernel, as a Fourier transform of the following function $e^{-\epsilon|x|}$.

Theorem 2.2.2 (Poisson Kernel). *For all $\alpha > 0$*

$$\begin{aligned}
\int_{\mathbb{R}^n} e^{-2\pi\alpha|y|} e^{-2\pi i\xi\cdot y} dy &= \frac{\Gamma\frac{(n+1)}{2}}{\pi^{\frac{n+1}{2}}} \frac{\alpha}{(\alpha^2 + |\xi|^2)^{\frac{n+1}{2}}} \\
&= c_n \frac{\alpha}{(\alpha^2 + |\xi|^2)^{\frac{n+1}{2}}}
\end{aligned}$$

Proof. This theorem holds without the loss of generality when we take $\alpha = 1$ due to the scaling property of the Fourier transform. For the Fourier transform holds $f(\alpha x) \overset{\mathcal{F}}{\Longleftrightarrow} \frac{1}{|\alpha|}\hat{f}\left(\frac{\xi}{\alpha}\right)$ so it is always possible to scale the y by $\frac{1}{\alpha}$, which gets of the alpha in the integral. Now let's establish the following auxiliary result.

$$e^{-\beta} = \frac{1}{\sqrt{\pi}} \int_0^{\infty} \frac{e^{-u}}{\sqrt{u}} e^{-\frac{\beta^2}{4u}} du \tag{2.1}$$

To prove this, let's establish some other integral identities.

$$\frac{1}{1+x^2} = \int_0^{\infty} e^{-(1+x^2)u} du$$

This follows straight from the definition of the integral for the exponential function. The second result is not as straightforward.

$$e^{-\beta} = \frac{2}{\pi} \int_0^{\infty} \frac{\cos \beta x}{1+x^2}$$

The idea is to use the residue theorem on a function $f(z) = \frac{e^{i\beta z}}{1+z^2}$. The residue theorem is a crucial result in complex analysis, providing a valuable tool for evaluating integrals of analytic functions. The statement and proof for the theorem can be found in multiple textbooks of complex analysis [1, 7, 3]. As $\frac{e^{i\beta z}}{1+z^2} = \frac{e^{i\beta z}}{(z+i)(z-i)}$, the function has two poles at $z = \pm i$. Now let $\beta > 0$ and Γ_R be the contour consisting of the real line from $[-R, R]$ and the upper semicircle C_R of radius R centered at 0. The contour integral can then be represented as a sum of these parts.

$$I_R = \oint_{\Gamma_R} f(z)dz = \int_{-R}^{R} f(z)dz + \int_{C_R} f(z)dz$$

By the residue theorem, as the only pole inside the upper contour is at $z = i$, $I_R = 2\pi i \mathrm{Res}_{z=i} f(z)$. Since it is the only pole $\mathrm{Res}_{z=i} f(z) = \lim_{z\to i}(z - i)\frac{e^{i\beta z}}{(z+i)(z-i)} = \frac{e^{-\beta}}{2i}$ and thus the integral evaluates to $I_R = 2\pi i \frac{e^{-\beta}}{2} = \pi e^{-\beta}$. The arc integral will vanish as $R \to \infty$, since on the arc C_R, $z = Re^{i\theta}$, for $\theta \in [0, \pi]$ and $|f(z)| = \frac{|e^{i\beta z}|}{|1+z^2|} \leq \frac{1}{R^2-1}$ as $|z^2+1| \geq |z|^2 - 1 = R^2 - 1$ and for the numerator, the z can be substituted to obtain a bound.

$$\begin{aligned}
|e^{i\beta z}| &= |e^{i\beta Re^{i\theta}}| \\
&= e^{i\beta R(\cos\theta + i\sin\theta)}| \\
&= |e^{i\beta R\cos\theta - \beta R\sin\theta}| \\
&= |e^{i\beta R\cos\theta}||e^{-\beta R\sin\theta}| \\
&= e^{-\beta R\sin\theta} \\
&\leq 1
\end{aligned}$$

This gives an upper bound for the arc integral as follow.

$$|\int_{C_R} f(z)dz| \leq \pi R \max_{z\in C_R} |f(z)| = \frac{\pi R}{R^2 - 1}$$

Now the limit as $R \to \infty$ is

$$\lim_{R\to\infty} \frac{\pi R}{R^2 - 1} = 0$$

Assembling the original integral back together, the result comes out as:

$$\int_{-\infty}^{\infty} \frac{e^{i\beta z}}{1+z^2} = \pi e^{\beta}$$

Since $\Re e^{i\beta z} = \cos\beta z$ the integral can be written:

$$\begin{aligned}
2\int_0^{\infty} \frac{\cos\beta x}{1+x^2}dx &= \pi e^{-\beta} \\
\Leftrightarrow e^{-\beta} &= \frac{2}{\pi}\int_0^{\infty} \frac{\cos\beta x}{1+x^2}dx
\end{aligned}$$

Using these two, the first identity can now be proven.

$$\begin{aligned}
e^{-\beta} &= \frac{2}{\pi}\int_0^\infty \frac{\cos\beta x}{1+x^2}dx \\
&= \frac{2}{\pi}\int_0^\infty \cos\beta x\left(\int_0^\infty e^{u}e^{-ux^2}du\right)dx \\
&= \frac{2}{\pi}\int_0^\infty e^{-u}\left(\int_0^\infty e^{-ux^2}\cos\beta x dx\right)du \\
&= \frac{2}{\pi}\int_0^\infty e^{-u}\left(\frac{1}{2}\int_0^\infty e^{ux^2}e^{i\beta x}dx\right)du \\
&= \frac{2}{\pi}\int_0^\infty e^{-u}\left(\pi\int_{-\infty}^\infty e^{4\pi^2uy^2}e^{-2\pi i\beta y}dy\right)du \\
&= \frac{2}{\pi}\int_0^\infty \frac{\sqrt{\pi}e^{-u}e^{-\frac{beta^2}{4u}}}{2\sqrt{u}}du \\
&= \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-u}}{\sqrt{u}}e^{-\frac{\eta^2}{4u}}du
\end{aligned}$$

Which combines the two identities.

The Poisson kernel itself is then derived using the integrals given before, and Theorem 2.2.1, we can compute the integral as follows.

$$\begin{aligned}
\int_{\mathbb{R}^n} e^{-2\pi\alpha|y|}e^{-2\pi i\xi\cdot y}dy &= \int_{\mathbb{R}^n}\left(\frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-u}}{\sqrt{u}}e^{-\frac{4\pi^2|y|^2}{4u}}du\right)e^{-2\pi i\xi\cdot y}dy \\
&= \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-u}}{\sqrt{u}}\left(\int_{\mathbb{R}^n} e^{-\frac{4\pi^2|y|^2}{4u}}e^{-2\pi i\xi\cdot y}dy\right)du \\
&= \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-u}}{\sqrt{u}}\left(\sqrt{\frac{u}{\pi}}^n e^{-u|\xi|^2}\right)du \\
&= \frac{1}{\pi^{\frac{n+1}{2}}}\int_0^\infty e^{-u}u^{\frac{n-1}{2}}e^{-u|\xi|^2} \\
&= \frac{1}{\pi^{\frac{n+1}{2}}}\frac{1}{(1+|\xi|^2)^{\frac{n+1}{2}}}\int_0^\infty e^{-s}s^{\frac{n-1}{2}}ds \\
&= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}}\frac{1}{(1+|\xi|^2)^{\frac{n+1}{2}}}
\end{aligned}$$

This gives the formulation of the Poisson kernel in the half-space $\mathbb{R}^{n+1}_+$. $\square$

The Fourier transform of the function $e^{-2\pi\alpha|y|}$ constitutes the Poisson kernel, and it will be denoted by $P(x,y) = c_n y(y^2+|x|^2)^{-\frac{n+1}{2}}$. Also, the first transformation has a name: Weierstrass kernel, which is usually denoted by $W(x,y)$, analogous to the Poisson kernel. However, the Poisson kernel is in the focus for the remainder of the thesis. Key properties it has are the harmonicity and smoothness. In addition, the Poisson is a "good kernel"

[8]. Good kernels have some good properties with respect to the convolution operation. Namely, those can be interpreted as approximate identities with respect to convolution.

Definition 2.2.3 (Convolution)**.** Let f and g be functions, then $f * g$ is defined as the following integral transform:

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-t)g(t)dt$$

This definition and some results on it will be used later in the analysis of the boundary value problem in Chapter 3.

In order for a kernel to qualify as a good kernel, it must satisfy the following conditions.

Definition 2.2.4 (Good Kernel)**.** Let $\{K_y(x)\}_{y>0} \subset L^1(\mathbb{R}^n)$ be a family of functions. This family is called the family of good kernels if it fulfills three conditions.

1. $$\int_{\mathbb{R}^n} K_y(x)dx = 1$$

2. For all $y > 0$ and all $x \in \mathbb{R}^n$, $K_y(x) > 0$.

3. For all $\delta > 0$ holds
$$\lim_{y\to 0} \int_{\{x\in\mathbb{R}^n \mid |x|>\delta\}} K_y(x) = 0$$

Good kernels are an important concept for the following analysis, as they act as an approximate identity with respect to a convolution. This means that the kernel behaves as an identity at the limit when $y \to 0$ and thus $\|f * K_y - f\| \xrightarrow{y\to 0} 0$. When a function in L^p space is convolved with a good kernel, this resembles averaging the function in the area close to a point. As the kernel integrates into one and its tails diminish the further from the origin x gets. When the area over which it is averaged converges towards the point at which the function is evaluated in the sense of condition 3. of Definition 2.2.4, the value of the convolution gets closer to the value at that point. The next lemma will show that the Poisson kernel belongs to this class of kernels.

Lemma 2.2.5 (Poisson Kernel is a Good Kernel)**.** *Poisson kernels are a family of good kernels.*

Proof. The first part can be shown to hold by integrating the kernel. First let's change to spherical coordinates so that $r = |x|$ and $dx = \omega_n r^{n-1}dr$, where ω_n is the surface are of the unit sphere in $\mathbb{R}^{n+1}$. Then the integral can be written as:

$$\int_{\mathbb{R}^n} P(x,y)dx = c_n y \omega_{n-1} \int_0^\infty \frac{r^{n-1}dr}{(y^2+r^2)^{\frac{n+1}{2}}}$$

Then letting $r = yt$ gives

$$\begin{aligned}
c_n y \omega_{n-1} \int_0^\infty \frac{r^{n-1}dr}{(y^2+r^2)^{\frac{n+1}{2}}} &= c_n y \omega_{n-1} \int_0^\infty \frac{y^n t^{n-1}dt}{(y^2+y^2t^2)^{\frac{n+1}{2}}} \\
&= c_n \omega_{n-1} \int_0^\infty \frac{t^{n-1}dt}{(t^2+1)^{\frac{n+1}{2}}}
\end{aligned}$$

and again, substituting $t = \tan\theta$ gives

$$\begin{aligned}
c_n \omega_{n-1} \int_0^\infty \frac{t^{n-1}dt}{(t^2+1)^{\frac{n+1}{2}}} &= c_n \omega_{n-1} \int_0^{\frac{\pi}{2}} \frac{\tan^{n-1}\theta \sec^2\theta d\theta}{(\tan^2\theta+1)^{\frac{n+1}{2}}} \\
&= c_n \omega_{n-1} \int_0^{\frac{\pi}{2}} \frac{\tan^{n-1}\theta \sec^2\theta d\theta}{\sec^{n+1}\theta} \\
&= c_n \omega_{n-1} \int_0^{\frac{\pi}{2}} \frac{\tan^{n-1}\theta d\theta}{\sec^{n-1}\theta} \\
&= c_n \omega_{n-1} \int_0^{\frac{\pi}{2}} \sin^{n-1}\theta d\theta
\end{aligned}$$

Now the $\omega_{n-1}\sin^{n-1}\theta$ is the surface are of a sphere with radius $\sin\theta$ in the half space, so

$$\omega_{n-1} \int_0^{\frac{\pi}{2}} \sin^{n-1}\theta d\theta = \frac{\omega_n}{2}$$

Also, the c_n^{-1} is equal to half of the surface area of a n-dimensional sphere; these two cancel out and leave 1 as a result.

The second condition follows directly from the definition of the Poisson kernel.

Lastly, the third condition also requires integrating the expression. Let's use the first part to write the integral in polar coordinates. In what follows, let $\delta > 0$.

$$\int_{\mathbb{R}^n} P(x,y)dx = c_n y \omega_{n-1} \int_0^\infty \frac{r^{n-1}dr}{(y^2+r^2)^{\frac{n+1}{2}}}$$

The following observation about the numerator inside the integral $y^2 + r^2 \geq$

r^2, enables the use of inequality for the integral.

$$\begin{aligned} c_n y \omega_{n-1} \int_\delta^\infty \frac{r^{n-1} dr}{(y^2 + r^2)^{\frac{n+1}{2}}} &\leq c_n y \omega_{n-1} \int_\delta^\infty \frac{r^{n-1} dr}{r^{n+1}} \\ &= c_n y \omega_{n-1} \int_\delta^\infty \frac{dr}{r^2} \\ &= -c_n y \omega_{n-1} \left[\frac{1}{r} \right]_\delta^\infty \\ &= \frac{c_n y \omega_{n-1}}{\delta} \end{aligned}$$

Now, taking the limit shows the claim for any $\delta > 0$.

$$\lim_{y \to 0} \frac{c_n y \omega_{n-1}}{\delta} = 0$$

□

As the Poisson kernel satisfies all of the criteria for good kernels, it can also be interpreted as a kind of an average close to a point. Moreover, it turns out that as the y goes to zero, the convolution of a function in L^p will converge towards the function against which it is convolved.

2.3 Laplace Equation and Harmonicity

The main result of this thesis deals with extending a function from Euclidean space $\mathbb{R}^n$ to the half space $\mathbb{R}^n \times \mathbb{R}_+$, and more precisely, the results consider harmonic extensions. Defining harmonic functions requires the definition of Laplace's equation, which is one of the most important partial differential equation since it appears in a wide variety of problems in mathematics and physics [5].

Definition 2.3.1 (Laplace's equation)**.** Laplace's equation is a second-order PDE defined by Laplace's operator Δ as:

$$\Delta u = \nabla \cdot \nabla u = \sum_{n=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = 0.$$

The set of functions that solve Laplace's equation has many special characteristics and are thus also named.

Definition 2.3.2 (Harmonic Functions)**.** Functions that solve Laplace's equation are called harmonic.

To check that the Poisson kernel is a harmonic function, it is possible just straightforwardly to calculate the second derivatives for it. The first derivatives of Poisson kernel are:

$$\frac{\partial P(x,y)}{\partial x_i} = -c_n \frac{(n+1)yx_i}{(y^2+|x|^2)^{\frac{n+3}{2}}} \quad \text{and}$$
$$\frac{\partial P(x,y)}{\partial y} = c_n \left(\frac{1}{(y^2+|x|^2)^{\frac{n+1}{2}}} - \frac{(n+1)y^2}{(y^2+|x|^2)^{\frac{n+3}{2}}} \right)$$

and the second derivatives are:

$$\begin{aligned}
\frac{\partial^2 P(x,y)}{\partial x_i^2} &= -c_n(n+1)y\frac{\partial}{\partial x_i}\left(\frac{x_i}{(y^2+|x|^2)^{\frac{n+3}{2}}}\right) \\
&= -c_n(n+1)y\left(\frac{1}{(y^2+|x|^2)^{\frac{n+3}{2}}} + x_i\frac{\partial}{\partial x_i}\left(\frac{1}{(y^2+|x|^2)^{\frac{n+3}{2}}}\right)\right) \\
&= -c_n(n+1)y\left(\frac{1}{(y^2+|x|^2)^{\frac{n+3}{2}}} + x_i\frac{-(n+3)x_i}{(y^2+|x|^2)^{\frac{n+5}{2}}}\right) \\
&= c_n\left(\frac{(n+1)(n+3)yx_i^2}{(y^2+|x|^2)^{\frac{n+5}{2}}} - \frac{(n+1)y}{(y^2+|x|^2)^{\frac{n+3}{2}}}\right) \quad \text{and}
\end{aligned} \tag{2.2}$$

$$\begin{aligned}
\frac{\partial^2 P(x,y)}{\partial y^2} &= c_n\frac{\partial}{\partial y}\left(\frac{1}{(y^2+|x|^2)^{\frac{n+1}{2}}} - \frac{(n+1)y^2}{(y^2+|x|^2)^{\frac{n+3}{2}}}\right) \\
&= c_n\left(\frac{-(n+1)y}{(y^2+|x|^2)^{\frac{n+3}{2}}} - (n+1)\frac{\partial}{\partial y}\left(\frac{y^2}{(y^2+|x|^2)^{\frac{n+3}{2}}}\right)\right) \\
&= c_n\left(\frac{-(n+1)y}{(y^2+|x|^2)^{\frac{n+3}{2}}} - (n+1)\left(\frac{2y}{(y^2+|x|^2)^{\frac{n+3}{2}}} + y^2\frac{\partial}{\partial y}\left(\frac{1}{(y^2+|x|^2)^{\frac{n+3}{2}}}\right)\right)\right) \\
&= c_n\left(\frac{-(n+1)y}{(y^2+|x|^2)^{\frac{n+3}{2}}} - (n+1)\left(\frac{2y}{(y^2+|x|^2)^{\frac{n+3}{2}}} + y^2\left(\frac{(n+3)y}{(y^2+|x|^2)^{\frac{n+5}{2}}}\right)\right)\right) \\
&= c_n\left(\frac{(n+1)(n+3)y^3}{(y^2+|x|^2)^{\frac{n+5}{2}}} - \frac{3(n+1)y}{(y^2+|x|^2)^{\frac{n+3}{2}}}\right)
\end{aligned} \tag{2.3}$$

Now, the equation 2.2 and 2.3 can be combined to obtain the Laplacian of the Poisson kernel.

$$
\begin{aligned}
\Delta_{x,y}P(x,y) &= \sum_{i=1}^{n} \frac{\partial^2 P(x,y)}{\partial x_i^2} + \frac{\partial^2 P(x,y)}{\partial y^2} \\
&= \sum_{i=1}^{n} c_n \left(\frac{(n+1)(n+3)yx_i^2}{(y^2+|x|^2)^{\frac{n+5}{2}}} - \frac{(n+1)y}{(y^2+|x|^2)^{\frac{n+3}{2}}} \right) + \\
&\qquad c_n \left(\frac{(n+1)(n+3)y^3}{(y^2+|x|^2)^{\frac{n+5}{2}}} - \frac{3(n+1)y}{(y^2+|x|^2)^{\frac{n+3}{2}}} \right) \\
&= c_n \left(\frac{(n+1)(n+3)y|x|^2}{(y^2+|x|^2)^{\frac{n+5}{2}}} - \frac{n(n+1)y}{(y^2+|x|^2)^{\frac{n+3}{2}}} + \right. \\
&\qquad \left. \frac{(n+1)(n+3)y^3}{(y^2+|x|^2)^{\frac{n+5}{2}}} - \frac{3(n+1)y}{(y^2+|x|^2)^{\frac{n+3}{2}}} \right) \\
&= c_n \left(\frac{(n+1)(n+3)y(y^2+|x|^2)}{(y^2+|x|^2)^{\frac{n+5}{2}}} - \frac{(n+3)(n+1)y}{(y^2+|x|^2)^{\frac{n+3}{2}}} \right) \\
&= c_n \left(\frac{(n+1)(n+3)y}{(y^2+|x|^2)^{\frac{n+5}{2}-1}} - \frac{(n+3)(n+1)y}{(y^2+|x|^2)^{\frac{n+3}{2}}} \right) \\
&= c_n \left(\frac{(n+1)(n+3)y}{(y^2+|x|^2)^{\frac{n+3}{2}}} - \frac{(n+3)(n+1)y}{(y^2+|x|^2)^{\frac{n+3}{2}}} \right) \\
&= 0
\end{aligned}
$$

This shows that the Poisson kernel satisfies the Laplace equation with respect to x and y, since $\Delta_{x,y}P(x,y) = 0$.

Chapter 3

Harmonic functions in $\mathbb{R}^{n+1}_+$

This chapter will consider the main result of the Dirichlet problem in the half-space for Laplace's equation. I will only consider here the tangential convergence, where the limit is taken tangentially towards the boundary value. All of this chapter follows closely the results by Stein and Weiss [9].

I will start the chapter by first introducing some new, necessary constructs that are used in the main result: Theorem 3.2.3. This result, together with Theorem 3.1.5, provides a way to lift the problem from the boundary $\mathbb{R}^n$ to the half space $\mathbb{R}^{n+1}_+$, where the extension is harmonic; consequently, the results from harmonic functional analysis apply to the problem. First, Theorem 3.1.5 will show that L^p functions behave as desired, and the Poisson integral converges towards the function at the boundary $\mathbb{R}^n \times \{0\}$. Then Theorem 3.2.3 will extend this result to apply to a broader class of finite Borel measures.

These results need some additional concepts to be understood. Therefore, in the second Section definition of the total variational measure and the norm induced by that are defined. Also, to prove Theorem 3.2.3, a weaker notion of convergence is needed, and for this purpose, the weak* convergence is introduced.

3.1 The Characterization of Poisson Integral for L^p-space

In this section, I will concentrate on the version of Dirichlet's problem where the data—meaning the function towards which the harmonic function converges at the boundary of a half space—belongs to the L^p space. Main results of this section, Theorems 3.1.5 and 3.1.6 will be concerned with recovering the boundary value function by extending it to the half space through the Poisson integral. These also establish a regularity for the Poisson integral and show that it also belongs to the L^p space.

Proofs for these theorems require some preliminary results. These are introduced before the main results. These results show that the convolutions of L^p functions with functions that resemble good kernels as approximate identities will converge, in some cases pointwise and in other cases in the L^p sense, to the function against which it is convoluted.

The first of these results shows convergence in the L^p sense.

Lemma 3.1.1. *Suppose $\phi \in L^1(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \phi(x)dx = 1$ and for $\epsilon > 0$ let $\phi_\epsilon(x) = \epsilon^{-n}\phi(x/\epsilon)$. If $f \in L^p(\mathbb{R}^n)$ for some $1 \leq p < \infty$, or $f \in C_0 \subset L^\infty(\mathbb{R}^n)$, then $\|f * \phi_\epsilon - f\|_p \to 0$ as $\epsilon \to 0$. In particular, $u(x,\epsilon) = \int_{\mathbb{R}^n} f(t)P(x-t,\epsilon)dt$ and $s(x,\epsilon) = \int_{\mathbb{R}^n} f(t)W(x-t,\epsilon)dt$ converge to f in the L^p norm as $\epsilon \to 0$.*

Proof. By change of variables

$$\int_{\mathbb{R}^n} \phi_\epsilon(t)dt = \int_{\mathbb{R}^n} \epsilon^{-n}\phi(t/\epsilon)dt = \int_{\mathbb{R}^n} \phi(t)dt = 1$$

and therefore

$$(f * \phi_\epsilon)(x) - f(x) = \int_{\mathbb{R}^n} (f(x-t) - f(x))\phi_\epsilon(t)dt.$$

Then by Minkowski's integral inequality

$$\begin{aligned}\|f * \phi_\epsilon - f\|_p &\leq \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x-t) - f(x)|^p dx\right)^{\frac{1}{p}} \epsilon^{-n}|\phi(t/\epsilon)|dt \\ &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x-\epsilon t) - f(x)|^p dx\right)^{\frac{1}{p}} |\phi(t)|dt\end{aligned}$$

where $\omega(h) = \left(\int_{\mathbb{R}^n} |f(x-h) - f(x)|^p dx\right)^{\frac{1}{p}}$ is the L^p modulus of continuity of f. Since $\omega(h) \leq 2\|f\|_p$ the modulus is bounded. Additionally, as $h \to 0$ also $\omega(h) \to 0$ when $f \in C_0$. If $f \in L^p(\mathbb{R}^p)$, since C_0 is dense in L^p, the function can be approximated by some f_c that belongs to C_0, and therefore

$$\|f * \phi_\epsilon - f\|_p \leq \int_{\mathbb{R}^n} \omega(-\epsilon t)|\phi(t)|dt.$$

By dominated convergence, the right-hand side of the equation tends to zero as $\epsilon \to 0$, as $\omega(-\epsilon t)|\phi(t)| \leq 2\|f\|_p|\phi(t)|$, proving the lemma. ☐

For the purposes of the second result, the concept of Lebesgue point and Lebesgue set is needed. These provide a generalization of continuity to L^p spaces where we do not have pointwise control of a function, and therefore, continuity as such cannot be enforced.

Definition 3.1.2. Let f be locally integrable. Then the set that is formed by x's such that

$$\frac{1}{r^n}\int_{|t|<r}|f(x-t)-f(x)|dt \to 0$$

as $r \to 0$, is called the Lebesgue set of f.

Lebesgue sets are now used in the next result to show that pointwise convergence of the Poisson integral can also be established under certain conditions for almost every $x \in \mathbb{R}^n$.

Lemma 3.1.3. *Suppose $\phi \in L^1(\mathbb{R}^n)$. Let $\psi(x) = \text{ess.sup}_{|t|\geq|x|}|\phi(t)|$ and, for $\epsilon > 0$, let $\phi_\epsilon(x) = \epsilon^{-n}\phi(x/\epsilon)$. If $\psi \in L^1(\mathbb{R}^n)$ and $f \in L^p(\mathbb{R}^n)$ for some $1 \leq p \leq \infty$, then $\lim_{\epsilon\to 0}(f * \phi_\epsilon)(x) = f(x)\int_{\mathbb{R}^n}\phi(t)dt$ whenever x belongs to the Lebesgue set of f. In particular, the Poisson and Gauss-Weierstrass integrals of f, $\int_{\mathbb{R}^n} f(t)P(x-t,\epsilon)dt$ and $\int_{\mathbb{R}^n} f(t)W(x-t,\epsilon)dt$, converge to $f(x)$ as $\epsilon \to 0$ for almost every $x \in \mathbb{R}^n$.*

Proof. Let x be a fixed point in the Lebesgue set of f and $\delta > 0$. Then for some $\eta > 0$ holds

$$r^{-n}\int_{|t|<r}|f(x-t)-f(x)|dt < \delta \tag{3.1}$$

whenever $r \leq \eta$.

Let $a = \int_{\mathbb{R}^n}\phi_\epsilon(t)dt$. Then for all $\epsilon > 0$

$$\begin{aligned}
|(f*\phi_\epsilon)(x) - af(x)| &= \left|\int_{\mathbb{R}^n}(f(x-t)-f(x))\phi_\epsilon(t)dt\right| \\
&\leq \left|\int_{|t|<\eta}(f(x-t)-f(x))\phi_\epsilon(t)dt\right| + \\
&\quad \left|\int_{|t|\geq\eta}(f(x-t)-f(x))\phi_\epsilon(t)dt\right| \\
&= I_1 + I_2
\end{aligned}$$

This splits the analysis into two cases. The first one considers points that are close to the x and the second gives a tail estimate for values that are further.

To estimate I_1 it is possible to use the assumption that x belongs to the Lebesgue set. Let $\psi_0(r) = \psi(|x|)$ be the radial function generated from ψ. It is a decreasing function of r. Thus for Ω_n, denoting the volume of solid unit sphere of $\mathbb{R}^n$

$$\frac{\Omega_n(2^n-1)}{2^n}r^n\psi_0(r) \leq \int_{r/2\leq|x|\leq r}\psi(x)dx \to 0$$

as $r \to 0$, or $r \to \infty$. Thus there exists a constant A such that for $0 < r < \infty$ $r^n \psi_0(r) \le A$. Let Σ_{n-1} denote the surface of the unit sphere in $\mathbb{R}^n$ and $g(r) = \int_{\Sigma_{n-1}} |f(x - r\sigma) - f(x)| d\sigma$, where σ is the surface are element of Σ_{n-1}. Then, by a change of variable, Equation 3.1 is equivalent to

$$G(r) = \int_0^r s^{n-1} g(s) ds \le \delta r^n$$

when $r \le \eta$. By using $G(r)$ we can further analyze the integral I_1.

$$\begin{aligned}
I_1 &= \left| \int_{|t|<\eta} (f(x-t) - f(x)) \phi_\epsilon(t) dt \right| \\
&\le \int_{|t|<\eta} |f(x-t) - f(x)| \epsilon^{-n} \phi(t/\epsilon) dt \\
&= \int_0^\eta r^{n-1} g(r) \epsilon^{-n} \psi_0(r/\epsilon) dr \\
&= [G(r)\epsilon^{-n}\psi_0(r/\epsilon)]_0^\eta - \int_0^\eta G(r) \epsilon^{-n} d\psi_0(r/\epsilon) \qquad (3.2) \\
&\le [\delta r^n \epsilon^{-n} \psi_0(r/\epsilon)]_0^\eta - \int_0^{\frac{\eta}{\epsilon}} G(\epsilon s) \epsilon^{-n} d\psi_0(s) \\
&\le \delta A - \int_0^{\frac{\eta}{\epsilon}} \delta s^n d\psi_0(s) \\
&\le \delta \left(A - \int_0^\infty s^n d\psi_0(s) \right)
\end{aligned}$$

Then by doing the change of variables from radial ψ_0 back to the Euclidean space, as $dx = r^{n-1} dr d\sigma$ the last row of equation 3.2 can be integrated by using integration by parts and the fact that $\psi_0(s) \to 0$ as $s \to 0$ or $s \to \infty$.

$$\begin{aligned}
-\int_0^\infty s^n d\psi_0(s) &= n \int_0^\infty s^{n-1} \psi_0(s) ds - [s^n \psi_0(s)]_0^\infty \\
&= n \int_0^\infty s^{n-1} \psi_0(s) ds \\
&= \frac{n}{\omega_{n-1}} \int_{\mathbb{R}^n} \psi(x) dx
\end{aligned}$$

This shows that I_1 is bounded by a constant δB.

The tail estimate I_2 can be analyzed by using Hölder's inequality for

$1/p + 1/p' = 1$.

$$
\begin{aligned}
I_2 &= \left| \int_{|t|\geq\eta} (f(x-t) - f(x))\phi_\epsilon(t)dt \right| \\
&\leq \int_{|t|\geq\eta} |f(x-t) - f(x)|\phi_\epsilon(t)dt \\
&\leq \int_{|t|\geq\eta} (|f(x-t)|\phi_\epsilon(t) + |f(x)|\phi_\epsilon(t))dt \\
&\leq \|f\|_p \|\phi_\epsilon \chi_{\{|t|\geq\eta\}}\|_{p'} + |f(x)| \|\phi_\epsilon \chi_{\{|t|\geq\eta\}}\|_1
\end{aligned}
\tag{3.3}
$$

Where

$$
\|\phi_\epsilon \chi_{\{|t|\geq\eta\}}\|_1 = \int_{|t|\geq\eta} \phi_\epsilon(t)dt = \int_{|t|\geq\eta/\epsilon} \phi(t)dt
$$

so the second term in the equation 3.3 tends to zero at the same time as ϵ.

The same is also true for the second term. For p and p' holds the following relationship: $p' = 1 + p'/p$. Then, by Hölder's inequality applied to the first term gives

$$
\begin{aligned}
\|\phi_\epsilon \chi_{\{|t|\geq\eta\}}\|_{p'} &= \left(\int_{|t|\geq\eta} \phi_\epsilon(t)^{p'} dt \right)^{\frac{1}{p'}} \\
&= \left(\int_{|t|\geq\eta} \phi_\epsilon(t)^{\frac{p'}{p}} \phi_\epsilon(t)dt \right)^{\frac{1}{p'}} \\
&\leq \|\phi_\epsilon \chi_{\{|t|\geq\eta\}}\|_\infty^{\frac{1}{p}} \|\phi_\epsilon \chi_{\{|t|\geq\eta\}}\|_1^{\frac{1}{p'}}
\end{aligned}
$$

However, $\|\phi_\epsilon \chi_{\{|t|\geq\eta\}}\|_\infty = \epsilon^n \psi(\eta/\epsilon)$, which goes to zero as ϵ goes zero.

Therefore, as ϵ tends to zero, I_2 also tends to zero and I_1 is bounded by some constant δB where the δ depends on ψ. Consequentially also the difference $|(f * \phi_\epsilon)(x) - f(x)| \to 0$. $\square$

Now we have all of the results that are needed for the convergence of the Poisson integral. The last preliminary result for the main theorems of this section provides one more inequality that is used to bound the norm of the Poisson integral with respect to the boundary value function.

Lemma 3.1.4. *If $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, and $g \in L^1(\mathbb{R}^n)$ then $h = f * g = \int_{\mathbb{R}^n} f(x-t)g(t)dt$ is well defined and belongs to $L^p(\mathbb{R}^n)$. Also,*

$$
\|h\|_p \leq \|f\|_p \|g\|_1.
$$

Proof. By Minkowski's integral inequality, we have

$$
\begin{aligned}
\|h\|_p &= \left(\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} f(x-t)g(t)dt \right|^p dx \right)^{\frac{1}{p}} \\
&\leq \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x-t)|^p dx \right)^{\frac{1}{p}} |g(t)|dt \\
&= \int_{\mathbb{R}^n} \|f(\cdot - t)\|_p g(t)dt \\
&\leq \|f\|_p \|g\|_1
\end{aligned}
$$

so also $h \in L^p(\mathbb{R}^n)$. □

Finally, all the required results are established, and it is possible to prove the main theorems in this section. These results will show that the Poisson integral acts as a natural harmonic extension for a function in L^p into the half space. It gives the conditions under which a function f that is defined over $\mathbb{R}^n$ can be recovered from the boundary of a harmonic function defined over one dimension higher half space $\mathbb{R}^{n+1}_+$.

Theorem 3.1.5. *If $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$ and*

$$u(x,y) = \int_{\mathbb{R}^n} f(t)P(x-t,y)dt$$

is the Poisson integral of f then u is harmonic in $\mathbb{R}^{n+1}_+$, converges towards the function f at the boundary of half space $\lim_{y\to 0} u(x,y) = f(x)$ for almost every $x \in \mathbb{R}^n$ and

$$\|u(\cdot,y)\|_p \leq \|f\|_p \tag{3.4}$$

for all $y > 0$. If $1 \leq p < \infty$ then

$$\|u(\cdot,y) - f\|_p \to 0$$

as $y \to 0$. That is, $u(\cdot,y)$ converges to f in the L^p norm as $y \to 0$.

Proof. The Poisson integral can be shown to be harmonic by showing that $\Delta_{x,y}u(x,y) = 0$. Since the Poisson kernel is harmonic, this could be shown by taking the Laplace operator inside the integral. Let $y > 0$, then from the equation 2.2 the absolute value of the second derivative with respect to x_i

can be bounded as:

$$\begin{aligned}
\left|\frac{\partial^2 P(x,y)}{\partial x_i^2}\right| &= \left|c_n(n+1)y\left(\frac{(n+3)x_i^2}{(y^2+|x|^2)^{\frac{n+5}{2}}} - \frac{1}{(y^2+|x|^2)^{\frac{n+3}{2}}}\right)\right| \\
&= \left|c_n(n+1)y\left(\frac{(n+3)x_i^2}{(y^2+|x|^2)^{\frac{n+5}{2}}} - \frac{y^2+|x|^2}{(y^2+|x|^2)^{\frac{n+5}{2}}}\right)\right| \\
&\leq c_n(n+1)y\frac{(n+2)(y^2+|x|^2)}{(y^2+|x|^2)^{\frac{n+5}{2}}} \\
&= \frac{C_n^x}{(y^2+|x|^2)^{\frac{n+3}{2}}}.
\end{aligned}$$

The second derivative of P with respect to y from the equation 2.3 can be bounded similarly.

$$\left|\frac{\partial^2 P(x,y)}{\partial y^2}\right| \leq \frac{C_n^y}{(y^2+|x|^2)^{\frac{n+3}{2}}}$$

Therefore, the dominated convergence applies to the Poisson integral and allows for the change of order in integration and differentiation. By using this and the Poisson kernels' harmonicity, the integral is shown to be harmonic.

$$\begin{aligned}
\Delta_{x,y}u(x,y) &= \Delta_{x,y}\int_{\mathbb{R}^n} P(x-t,y)f(t)dt \\
&= \int_{\mathbb{R}^n} \Delta_{x,y}P(x-t,y)f(t)dt \\
&= \int_{\mathbb{R}^n} 0\cdot f(t)dt \\
&= 0
\end{aligned}$$

The limit of the Poisson integral $\lim_{y\to 0} u(x,y) = f(x)$ for almost every x, follows from the Lemma 3.1.3.

Similarly, L^p convergence $\|u(\cdot,y) - f\|_p$ follows from the Lemma 3.1.1.

Finally, the inequality 3.4 is a direct consequence of Lemma 3.1.4, where $g(x) = P(x,y)$ and the convolution corresponds to the Poisson integral $u(x,y)$.

These prove the lemma. □

Figure 3.1 shows Theorem 3.1.5 in action. In the example, the domain is not exactly the half space, but a subspace of it, as the integration is done only in the $[0,2\pi]$ interval and therefore the Poisson integral is only showing a subset of $[0,2\pi]\times\mathbb{R}_+$.

The next result will strengthen the result of Theorem 3.1.5 when stricter regularity assumptions are applicable for the boundary value function f. It shows that in specific cases where it is assumed to be continuous and bounded, the convergence of the Poisson integral accepts a stronger convergence result, as it is shown to converge uniformly in this case.

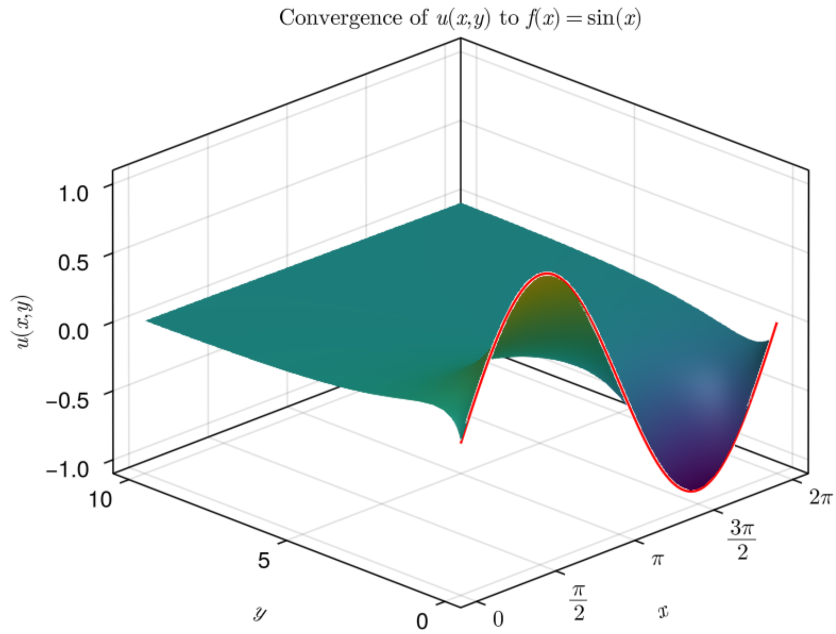

Figure 3.1: Convergence of Poisson integral $u(x, y) = \int_0^{2\pi} P(x-t, y) \sin(t)dt$ to $f(x) = \sin(x)$, which is shown as the red line, in 2-dimensional subspace $[0, 2\pi] \times \mathbb{R}_+$.

Theorem 3.1.6. *If $f \in C_0 \subset L^p(\mathbb{R}^n)$ then the Poisson integral of f, $u(x, y)$, converges to f uniformly: $\|u(\cdot, y) - f\|_\infty = \sup_{x\in\mathbb{R}^n} |u(x, y) - f(x)| \to 0$ as $y \to 0$.*

If the function f is only assumed to be continuous and bounded, the convergence is uniform on compact subsets of $\mathbb{R}^n$. In either case, we can exted $u(x, y)$ to $\overline{\mathbb{R}}^{n+1}_+ = \mathbb{R}^{n+1}_+ \cup \mathbb{R}^n$ by letting $u(x, 0) = f(x)$ and we obtain a continuous function.

Proof. If $f \in C_0 \subset L^\infty$, the Poisson integral $u(x, y)$ converges to $f(x)$ as $y \to 0$, as every point belongs to the Lebesgue set of f and Lemma 3.1.3 holds.

To establish uniform convergence in a compact subset $K \subset \mathbb{R}^n$, let $\epsilon > 0$, then there exists $\delta > 0$ such that $|f(x-t) - f(x)| < \epsilon$ if $x \in K$ and $|t| < \delta$.

Therefore

$$\begin{aligned}
|u(x,y)-f(x)| &= \left| \int_{\mathbb{R}^n} f(x-t)P(t,y)dt - f(x) \right| \\
&\le \int_{|t|\le\delta} |f(x-t)-f(x)|\, P(x,t)dt + \\
&\quad \int_{|t|>\delta} |f(x-t)-f(x)|\, P(x,t)dt \\
&\le \epsilon \int_{\mathbb{R}^n} P(t,y) + 2\|f\|_\infty \int_{|t|>\delta} P(t,y)dt \\
&= \epsilon + 2\|f\|_\infty \int_{|t|>\delta} P(t,y)dt
\end{aligned}$$

For fixed δ, $\int_{|t|>\delta} P(x,y)dt \le c_n y \int_{|t|>\delta} |t|^{-n-1}dt \to 0$ as $y \to 0$. Thus $|u(x,y)-f(x)| \le \epsilon$ when y gets close to zero, for all $x \in K$. If $f \in C_0$, then f is also uniformly continuous and for δ exists $|f(x-t)-f(x)| < \epsilon$ for all $x \in \mathbb{R}^n$ as $|t| < \delta$. Therefore, u converges uniformly towards f as $y \to 0$, and the claim of the lemma is proved. □

This section showed that the Poisson integral characterizes the harmonic function in the half space $\mathbb{R}^{n+1}_+$ that converges towards the boundary value function. Theorem 3.1.6 additionally showed that by assuming more regularity for the boundary data, it is also possible to get stronger convergence results.

These results could be used, for example, as a tool to solve certain problems in PDEs. Most straightforward applications consider, of course, the Laplace equation in the Euclidean half space as if one is trying to solve a problem of the form:

$$\Delta u = 0 \text{ in } \mathbb{R}^{n+1}_+, \text{ and } \lim_{y\to 0} u(x,y) = f(x).$$

These results show that the Poisson integral will give the solution that is sought.

However, the result is applicable to a wider range of problems than this. The natural next step from here could be to consider the heat equation $u_t = \Delta u$, which has the Laplace's equation as a steady state equilibrium. To solve it for an initial data $u(x,0) = f(x)$, the techniques that were developed in the theorems of this section could also be used.

3.2 The Characterization of Poisson-Stiltjes Integral for the Space of Finite Borel Measures $\mathcal{M}$

After considering L^p functions in Section 3.1 and proving the Poisson integral converges to a L^p function at the boundary of a half space, it is possible to widen the scope and consider a similar problem in other spaces. In this section, L^p functions are replaced by measures, especially finite Borel measures. The convergence result with finite Borel measures is otherwise very similar to Theorem 3.1.5, but some definitions change due to the move from L^p functions to finite Borel measures $\mathcal{M}$.

First of all, the measure $\mu \in \mathcal{M}(\mathbb{R}^n)$ requires a new definition for the norm. For this purpose, we need the total variation. The total variation of a measure is a positive measure defined by another measure as follows.

Definition 3.2.1 (Total Variation Measure)**.** Let μ be a measure on measurable space $(X, \mathcal{S})$. The total variation measure is the function $|\mu| : \mathcal{S} \to [0, \infty]$ that is defined by

$$\begin{aligned}|\mu|(E) = \sup\{&|\mu|(E_1) + \ldots + |\mu|(E_m)| \\ &m \in \mathbb{N}, E_1, \ldots, E_m \text{ are disjoint and } E_1, \ldots, E_m \subset E\}.\end{aligned}$$

for $E \subset X$. The total variation norm $\|\mu\| = |\mu|(X)$ is the total variation measure throughout the space.

The total variation is a measure, and the total variation norm is a norm as provided in Axler's *Measure, Integration & Real Analysis* [2]. The total variation of a measure can be seen as an extension of the concept of absolute value to measures from functions. It attempts to encode the "size" of a measure by disregarding the sign.

As the convergence concept used in theorem 3.1.5 was L^p convergence in the norm, that is also out of question now. For that reason, a new type of convergence is introduced.

Definition 3.2.2 (Weak* convergence)**.** Let X be a normed linear space, and suppose that $\mu_n, \mu \in X^*$, where X^* is the dual space of the normed space X. Then μ_n is said to converge weak* if

$$\forall x \in X, \quad \lim_{n\to\infty} \langle x, \mu_n \rangle = \langle x, \mu \rangle.$$

These are enough to acquire a similar result for finite Borel measures as for L^p functions. The next theorem shows this result.

Theorem 3.2.3. *If $\mu \in \mathcal{M}(\mathbb{R}^n)$ and $u(x, y) = \int_{\mathbb{R}^n} P(x - t, y) d\mu(t)$ is the Poisson-Stiltjes integral of μ then u is harmonic in $\mathbb{R}^{n+1}_+$ and*

$$||u(\cdot, y)||_1 = \int_{E_n} |u(x, y)| dx \leq ||\mu|| \tag{3.5}$$

where $||\mu||$ is the total variation, $|\mu|(\mathbb{R}_n)$, of μ. Furthermore,

$$\lim_{y\to 0} \int_{E_n} u(x, y)\varphi(x) dx = \int_{E_n} \varphi(x) d\mu(x) \tag{3.6}$$

for all $\varphi \in C_0$; that is, $u(\cdot, y)$ converges to μ in the weak topology.*

Proof. The Poisson-Stiltjes integral can be shown to be harmonic in an analogous way to the Poisson integral. The analysis concerning Poisson kernels' second derivatives does not change, and therefore, the dominated convergence can also be used in this instance. Then the result is acquired by the same change of order in integration and differentiation.

$$\begin{aligned}
\Delta_{x,y} u(x,y) &= \Delta_{x,y} \int_{\mathbb{R}^n} P(x-t,y)d\mu(t) \\
&= \int_{\mathbb{R}^n} \Delta_{x,y} P(x-t,y)d\mu(t) \\
&= \int_{\mathbb{R}^n} 0 \cdot d\mu(t) \\
&= 0
\end{aligned}$$

The inequality 3.5 now follows from the Lemma 3.1.4 by changing the L^1 function g to a Borel measure $\mu \in \mathcal{M}(\mathbb{R}^n)$ and applying the total variation to the expression $h(x) = \int_{\mathbb{R}^n} f(x-t)d\mu(t)$. Minkowski's integral inequality can be applied to the norm to obtain that $\int_{\mathbb{R}^n} |u(x,y)|dx \leq \int_{\mathbb{R}^n} \|u(\cdot,y)\|_1 d\mu(t)$. Since $\int_{\mathbb{R}^n} P(x,y)dx = 1$, this leads to the inequality 3.5.

The weak* convergence still remains to be proven. In order to prove that, suppose $v(x,y) = \int_{\mathbb{R}^n} P(x-t,y)\varphi(x)dx$ i.e., $v(x,y)$ is $\varphi(x)$s Poisson integral. Then the equation 3.6 can be written as

$$\begin{aligned}
\int_{\mathbb{R}^n} u(x,y)\varphi(x)dx &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} P(x-t,y)d\mu(t) \right) \varphi(x)dx \\
&= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} P(x-t,y)\varphi(x)dx \right) d\mu(t) \\
&= \int_{\mathbb{R}^n} v(t,y)d\mu(t)
\end{aligned}$$

which allows the use of Theorem 3.1.6 for the C_0 function φ and its Poisson integral v.

$$\begin{aligned}
\left| \int_{\mathbb{R}^n} u(x,y)\varphi(x)dx - \int_{\mathbb{R}^n} \varphi(t)d\mu(t) \right| &= \left| \int_{\mathbb{R}^n} (v(t,y) - \varphi(t))d\mu(t) \right| \\
&\leq \int_{\mathbb{R}^n} |v(t,y) - \varphi(t)|d\mu(t) \\
&\leq \|v(\cdot,y) - \varphi\|_\infty |\mu|(\mathbb{R}^n) \xrightarrow{y \to 0} 0
\end{aligned}$$

Since Theorem 3.1.6 provides that $\|v(\cdot,y) - \varphi\|_\infty \to 0$ and μ is assumed to belong in the space of finite Borel measures, the whole expression in the last line must vanish as $y \to 0$. This proves the weak convergence and concludes the last step of the proof. $\square$

This generalization to measures allows a broader range of boundary data to be applicable for the boundary value problem. This broadens the use of the Poisson kernel to a new class of problems where the initial data is not necessarily a well-behaved function. It also allows techniques that are needed when analyzing PDEs, such as the weak solutions.

Chapter 4

Conclusion

This concludes the results in the upper half space with tangential convergence to a function or measure. Together, theorems 3.1.5 & 3.2.3 show that there exists an extension of a function $L^p(\mathbb{R}^n)$ or a Borel measure $\mathcal{M}(\mathbb{R}^n)$ to the half-space $\mathbb{R}^{n+1}_+$ that solves the Laplace equation and thus is harmonic. Theorem 3.1.6 also shows that the result can be strengthened by having stronger assumptions about the regularity of the boundary data.

These results are only a first step to the results that could be further generalized. One possibility is to consider other paths to the boundary that do not necessarily approach it tangentially. Stein and Weiss are also considering this scenario in their book *Introduction to Fourier Analysis on Euclidean Spaces* [9]. With a nontangential convergence path, the analysis becomes more complicated and requires some tools that were not used for the theorems introduced in this thesis. In particular, this would require the use of maximal functions to obtain certain bounds to establish the result in the non-tangential case. Other possible extensions could include changing the domain in which the problem is set. Also, the differential equation could be changed from the Laplace equation to some other partial differential equation.

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