Bayesian Theory and Data Analysis
Assignment #1

Due on February 8, 2016 at 1:00pm

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Problem 1

Hoff 2.1 Marginal and conditional probability: The social mobility data from Section 2.5 gives a joint probability distribution on (Y1,Y2)= (fathers occupation, sons occupation). Using this joint distribution, calculate the following distributions:
a) the marginal probability distribution of a fathers occupation;
b) the marginal probability distribution of a sons occupation;
c) the conditional distribution of a sons occupation, given that the father is a farmer;
d) the conditional distribution of a fathers occupation, given that the son is a farmer.

Solution:-
As we know, $MarginalDensity$ can be computed from the $JointDensity$ as

$$p(y_1) \equiv \sum_{y_2 \in Y_2} p(y_1, y_2) \ we \ get$$

	$farm$	$operations$	$craftmen$	$sales$	$professional$
$a) \ son'sdistribution$	0.023	0.26	0.24	0.125	0.352
$b) \ father'sdistribution$	0.11	0.279	0.277	0.099	0.235

We get the Conditional Probability value in the following way:

$$Pr(son_{occ} \mid father_{farm}) = \frac{Pr(son_{occ} \ \cap \ father_{farm})}{Pr(father_{farm})}$$

	$farm$	$operations$	$craftmen$	$sales$	$professional$
$c) \ Pr(son_{occupation} \mid father_{farm})$	0.163	0.318	0.281	0.072	0.163

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Similarly for d)

$$Pr(father_{occ} \mid son_{farm}) = \frac{Pr(father_{occ} \ \cap \ son_{farm})}{Pr(son_{farm})}$$

	$farm$	$operations$	$craftmen$	$sales$	$professional$
$d) \ Pr(father_{occupation} \mid son_{farm})$	0.782	0.086	0.043	0.043	0.043

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Problem 2

Hoff 2.3 Full conditionals: Let X, Y, Z be random variables with joint density (dis- crete or continuous)

$$p(x, y, z) \propto f(x, z) \ g(y, z) \ h(z)$$

. Show that:
a) $p(x|y, z) \propto f(x, z)$, i.e. $p(x|y, z)$ is a function of x and z;
b) $p(y|x, z) \propto g(y, z)$, i.e. $p(y|x, z)$ is a function of y and z;
c) X and Y are conditionally independent given Z.

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Where f, g and h are just functions. They are not necessarily densities (this is, they dont need to sum to one).

Solution:-

(a) We need to prove that

$$p(x|y,z) \propto f(x,z)$$

We know that

$$p(y_1|y_2) = \frac{p(y_1,y_2)}{p(y_2)} \quad from\ Joint\ Distribution\ property\ (Hoff - page23)$$

$$\therefore \qquad p(x|y,z) = \frac{p(x,y,z)}{p(y,z)}$$

Given $p(x,y,z) \propto f(x,z)\ g(y,z)\ h(z)$

$$p(x|y,z) = \frac{c\ f(x,z)\ g(y,z)\ h(z)}{p(y,z)}\ where\ c\ is\ a\ constant$$

$$integrating\ the\ above\ equation\ \int_{x \in X} \frac{c\ f(x,z)\ g(y,z)\ h(z)}{p(y,z)} \mathrm{d}x\ \ gives:\ k\ f(x,z),\ k\ is\ a\ constant$$

$$\implies p(x|y,z) \propto f(x,z)\ Hence\ Proved.$$

(b) We need to prove that

$$p(y|x,z) \propto f(y,z)$$

We know that

$$p(y_1|y_2) = \frac{p(y_1,y_2)}{p(y_2)} \quad from\ Joint\ Distribution\ property\ (Hoff - page23)$$

$$\therefore \qquad p(y|x,z) = \frac{p(x,y,z)}{p(x,z)}$$

Given $p(x,y,z) \propto f(x,z)\ g(y,z)\ h(z)$

$$p(y|x,z) = \frac{c\ f(x,z)\ g(y,z)\ h(z)}{p(x,z)}\ where\ c\ is\ a\ constant$$

$$integrating\ the\ above\ equation\ \int_{y \in Y} \frac{c\ f(x,z)\ g(y,z)\ h(z)}{p(x,z)} \mathrm{d}x\ \ gives:\ k\ f(y,z),\ k\ is\ a\ constant$$

$$\implies p(y|x,z) \propto f(y,z)\ Hence\ Proved.$$

(c) We need to prove that X and Y are conditionally independent given Z

$$that\ is:\quad Pr(X \cap Y|Z) = Pr(X|Z)\ \ Pr(Y|Z)$$

More generally,

$$p(x_1, x_2, ..., x_n) = \prod_i (\theta_{x_i})$$

We know that

$$Pr(X \cap Y|Z)\ = \text{Pr}(Y|Z)\ \ Pr(X|Y \cap Z)\ \ from\ Axioms\ of\ Probability:\ P3(Hoff - page14)$$

$$and\ Pr(X|Y \cap Z) = Pr(X|Z)\ \ since\ we\ proved\ in\ Problem\ 2.3\ a)\ that\ p(x|y,z) \propto f(x,z)$$

$$\implies Pr(X \cap Y|Z) = Pr(X|Z)\ \ Pr(Y|Z)$$

Therefor we come to an assertion that Y gives no additional information about X and vice-versa. Hence proved that X and Y are conditionally independent given Z based on our previous results and Rule P3

Problem 3

(Gelman et al., 2003) Discuss the following statement.
The probability of event E is considered subjective if two rational persons A and B can assign unequal probabilities to E, PrA(E) and PrB(E). These probabilities can also be interpreted as conditional: $Pr_A(E) = Pr(E|I_A)$ and $Pr_B(E) = Pr(E|I_B)$, where I_A and I_B represent the knowledge available to persons A and B, respectively. Apply this idea to the following examples:
(a) The probability that a 6 appears when a fair die is rolled, where A observes the outcome of the die roll and B does not.
(b) The probability that Brazil wins the next World Cup, where A is ignorant of soccer and B is a knowledgeable sports fan.

Solution:- These above statements can be discussed with respect to Baye's Rule,

$$\textbf{Baye's Rule:}\ \ P(H|E) = \frac{P(H)_{[belief]}\ P(E \mid H)_{[Likelihood]}}{P(E)_{[norm]}}$$

So $P(E \mid H)$ is significantly different from the P(E) because, $P(E \mid H)$ is the conditional probability or the probability measure, given a set of beliefs/prior knowledge.

(a) When A observes the outcome, he obviously has knowledge about the event and hence his probability is $Pr_A(E) = 1$
Since B is ignorant of the event, he has no prior knowledge on the event so he assigns equal probability to all possible outcomes for the event, so probability is $Pr_B(E) = \frac{1}{6}$

(b) When A makes his prediction, he tend to give equal probability to each team because he is ignorant of any conditions.
Since B is a knowledgeable sports fan, his version of prediction is weighted based on his knowledge, so his probability is calculated by multiplying his Belief with normalized likelihood of the event: Brazil Wins!

$$Pr_B(E) = \ Pr(E|I_B) \ = \frac{P(I_B)_{[belief]} \ P(E \mid I_B)_{[Likelihood]}}{P(E)_{[norm]}}$$

REFERENCES: **Peter D. Hoff -** A First Course in Bayesian Statistical Methods.