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History of the Normal Distribution

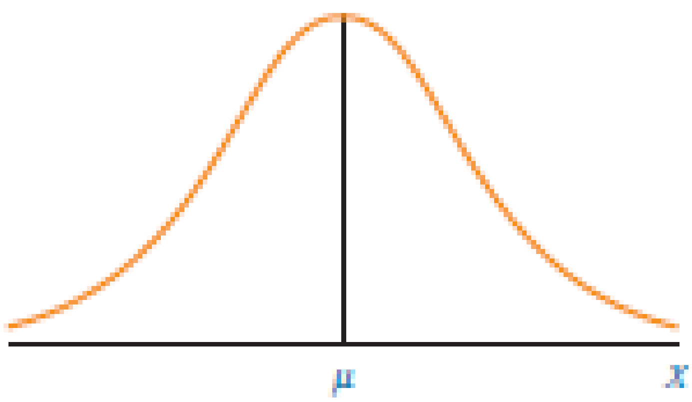
Discovery of the normal curve of errors is generally credited to mathematician and astronomer Karl Gauss (1777–1855), who recognized that the errors of repeated measurement of objects are often normally distributed.* Thus the normal distribution is sometimes referred to as the *Gaussian distribution* or the *normal curve of error.* A modern-day analogy of Gauss's work might be the distribution of measurements of machine-produced parts, which often yield a normal curve of error around a mean specification.

To a lesser extent, some credit has been given to Pierre-Simon de Laplace (1749–1827) for discovering the normal distribution. However, many people now believe that Abraham de Moivre (1667–1754), a French mathematician, first understood the normal distribution. De Moivre determined that the binomial distribution approached the normal distribution as a limit. De Moivre worked with remarkable accuracy. His published table values for the normal curve are only a few ten-thousandths off the values of currently published tables.†

The normal distribution exhibits the following characteristics.

- It is a continuous distribution.
- It is a symmetrical distribution about its mean.
- It is asymptotic to the horizontal axis.
- It is unimodal.
- It is a family of curves.
- Area under the curve is 1.

The Normal Curve



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The normal distribution is symmetrical. Each half of the distribution is a mirror image of the other half. Many normal distribution tables contain probability values for only one side of the distribution because probability values for the other side of the distribution are identical because of symmetry.

In theory, the normal distribution is asymptotic to the horizontal axis. That is, it does not touch the *x*-axis, and it goes forever in each direction. The reality is that most applications of the normal curve are experiments that have finite limits of potential outcomes. For example, even though GMAT scores are analyzed by the normal distribution, the range of scores on each part of the GMAT is from 200 to 800.

The normal curve sometimes is referred to as the *bell-shaped curve*. It is unimodal in that values *mound up* in only one portion of the graph—the center of the curve. The normal distribution actually is a family of curves. Every unique value of the mean and every unique value of the standard deviation result in a different normal curve. In addition, *the total area under any normal distribution is 1*. The area under the curve yields the probabilities, so the total of all probabilities for a normal distribution is 1. Because the distribution is symmetric, the area of the distribution on each side of the mean is 0.5.

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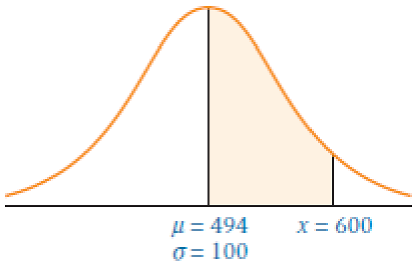
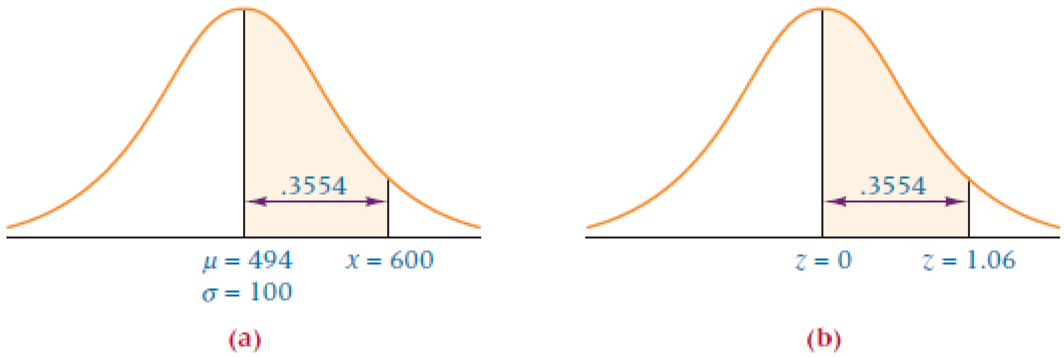
FIGURE 6.6

Graphical Depiction of the Area Between a Score of 600 and a Mean on a GMAT



z FORMULA	$z = \dfrac{x - \mu}{\sigma}, \quad \sigma \neq 0$

The Graduate Management Aptitude Test (GMAT), produced by the Educational Testing Service in Princeton, New Jersey, is widely used by graduate schools of business in



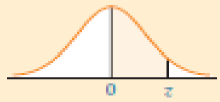
the United States as an entrance requirement. Assuming that the scores are normally distributed, probabilities of achieving scores over various ranges of the GMAT can be determined. In a recent year, the mean GMAT score was 494 and the standard deviation was about 100. What is the probability that a randomly selected score from this administration of the GMAT is between 600 and the mean? That is,

$$P(494 \leq x \leq 600 \mid \mu = 494 \text{ and } \sigma = 100) = ?$$

Figure 6.6 is a graphical representation of this problem.

The z formula yields the number of standard deviations that the x value, 600, is away from the mean.

$$z = \frac{x - \mu}{\sigma} = \frac{600 - 494}{100} = \frac{106}{100} = 1.06$$



SECOND DECIMAL PLACE IN z

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990
3.1	.4990	.4991	.4991	.4991	.4992	.4992	.4992	.4992	.4993	.4993
3.2	.4993	.4993	.4994	.4994	.4994	.4994	.4994	.4995	.4995	.4995
3.3	.4995	.4995	.4995	.4996	.4996	.4996	.4996	.4996	.4996	.4997
3.4	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4998
3.5	.4998									
4.0	.49997									
4.5	.499997									
5.0	.4999997									
6.0	.499999999									

The z value of 1.06 reveals that the GMAT score of 600 is 1.06 standard deviations more than the mean. The z distribution values in Table 6.2 give the probability of a value being between this value of x and the mean. The whole-number and tenths-place portion of the z score appear in the first column of Table 6.2 (the 1.0 portion of this z score). Across the top of the table are the values of the hundredths-place portion of the z score. For this z score, the hundredths-place value is 6. The probability value in Table 6.2 for $z = 1.06$ is .3554. The shaded portion of the curve at the top of the table indicates that the probability value given *always* is the probability or area between an x value and the mean. In this particular example, that is the desired area. Thus the answer is that .3554 of the scores on the GMAT are between a score of 600 and the mean of 494. Figure 6.7(a) depicts graphically the solution in terms of x values. Figure 6.7(b) shows the solution in terms of z values.

FIGURE 6.7
Graphical Solutions to the GMAT Problem

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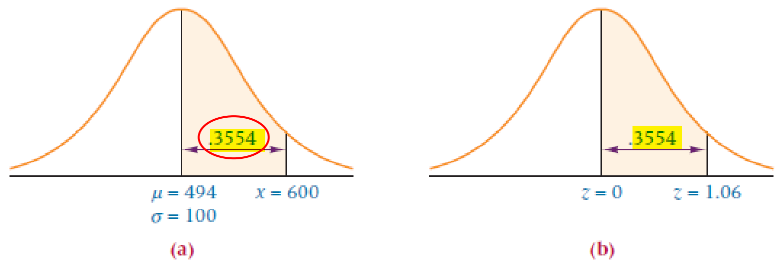
Ken Black



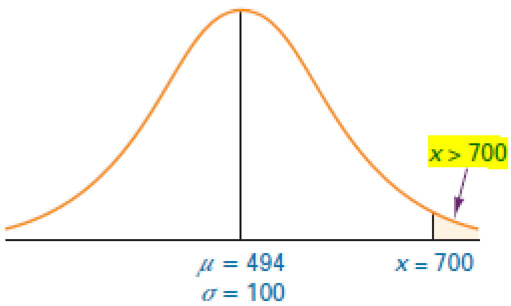
Pg 188 Prob 6.3

What is the probability of obtaining a score greater than 700 on a GMAT test that has a mean of 494 and a standard deviation of 100? Assume GMAT scores are normally distributed.

$$P(x > 700 \mid \mu = 494 \text{ and } \sigma = 100) = ?$$

Solution

Examine the following diagram.



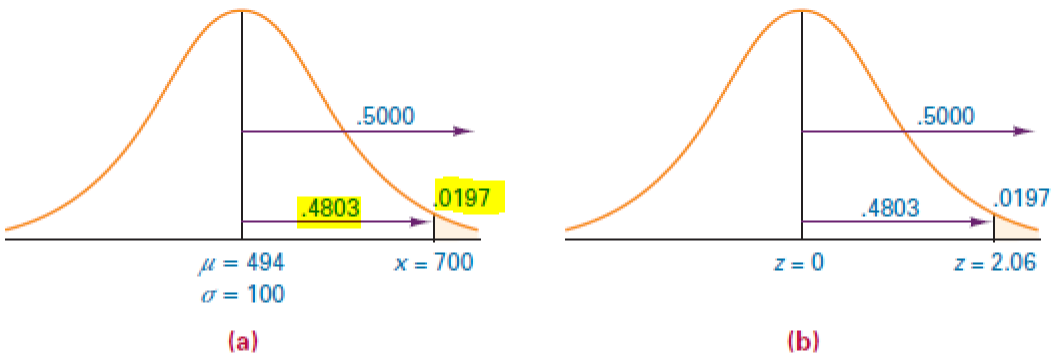
This problem calls for determining the area of the upper tail of the distribution. The z score for this problem is

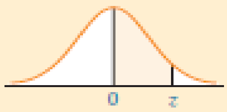
$$z = \frac{x - \mu}{\sigma} = \frac{700 - 494}{100} = \frac{206}{100} = 2.06$$

Table 6.2 gives a probability of .4803 for this z score. This value is the probability of randomly drawing a GMAT with a score between the mean and 700. Finding the probability of getting a score greater than 700, which is the tail of the distribution, requires subtracting the probability value of .4803 from .5000, because each half of the distribution contains .5000 of the area. The result is .0197. Note that an attempt to determine the area of $x \geq 700$ instead of $x > 700$ would have made no difference because, in continuous distributions, the area under an exact number such as $x = 700$ is zero. A line segment has no width and hence no area.

.5000	(probability of x greater than the mean)
−.4803	(probability of x between 700 and the mean)
.0197	(probability of x greater than 700)

The solution is depicted graphically in (a) for x values and in (b) for z values.





SECOND DECIMAL PLACE IN z

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
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1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
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1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857

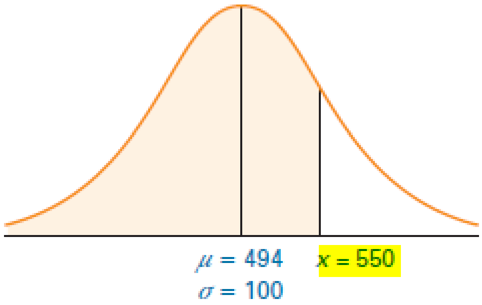
Pg 189 Prob 6.4

For the same GMAT examination, what is the probability of randomly drawing a score that is 550 or less?

$$P(x \leq 550 \mid \mu = 494 \text{ and } \sigma = 100) = ?$$

Solution

A sketch of this problem is shown here. Determine the area under the curve for all values less than or equal to 550.



The z value for the area between 550 and the mean is equal to:

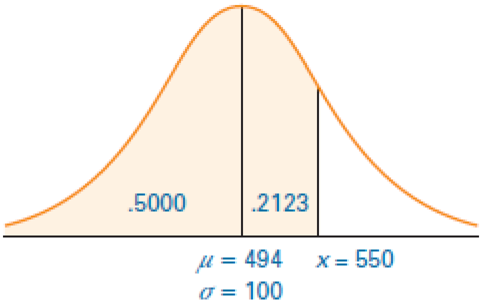
$$z = \frac{x - \mu}{\sigma} = \frac{550 - 494}{100} = \frac{56}{100} = 0.56$$

The area under the curve for $z = 0.56$ is .2123, which is the probability of getting a score between 550 and the mean. However, obtaining the probability for all values less than or equal to 550 also requires including the values less than the mean.

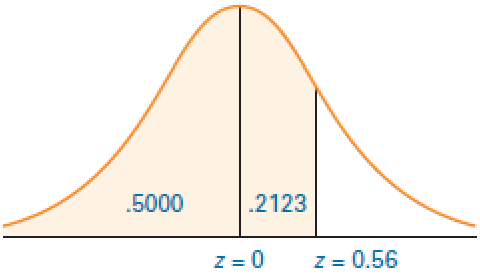
Because one-half or .5000 of the values are less than the mean, the probability of $x \leq 550$ is found as follows.

.5000	(probability of values less than the mean)
+.2123	(probability of values between 550 and the mean)
.7123	(probability of values $\leq$550)

This solution is depicted graphically in (a) for x values and in (b) for z values.

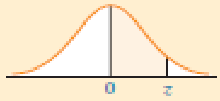


(a)



(b)

6.2 Normal Distribution 187



SECOND DECIMAL PLACE IN z

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621

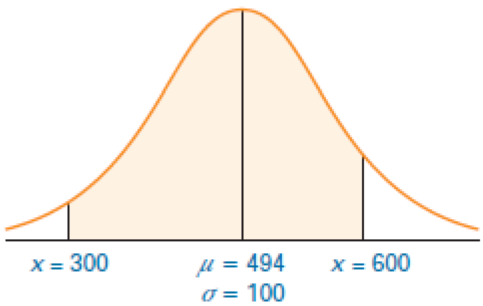
Pg 190 Problem 6.5

What is the probability of randomly obtaining a score between 300 and 600 on the GMAT exam?

$$P(300 < x < 600 \mid \mu = 494 \text{ and } \sigma = 100) = ?$$

Solution

The following sketch depicts the problem graphically: determine the area between $x = 300$ and $x = 600$, which spans the mean value. Because areas in the z distribution are given in relation to the mean, this problem must be worked as two separate problems and the results combined.



A z score is determined for each x value.

$$z = \frac{x - \mu}{\sigma} = \frac{600 - 494}{100} = \frac{106}{100} = 1.06$$

For Z = 1.06, area = 0.3554 (see page 4)

and

$$z = \frac{x - \mu}{\sigma} = \frac{300 - 494}{100} = \frac{-194}{100} = -1.94$$

Note that this z value ($z = -1.94$) is negative. A negative z value indicates that the x value is below the mean and the z value is on the left side of the distribution. None of the z values in Table 6.2 is negative. However, because the normal distribution is symmetric, probabilities for z values on the left side of the distribution are the same as the values on the right side of the distribution. The negative sign in the z value merely indicates that the area is on the left side of the distribution. The probability is always positive.

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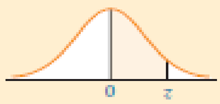

SECOND DECIMAL PLACE IN z

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857

The probability for $z = 1.06$ is .3554; the probability for $z = -1.94$ is .4738. The solution of $P(300 < x < 600)$ is obtained by summing the probabilities.

.3554	(probability of a value between the mean and 600)
+.4738	(probability of a value between the mean and 300)
.8292	(probability of a value between 300 and 600)

Graphically, the solution is shown in (a) for x values and in (b) for z values.