Note Title 14-11-2021

Sampling distribution:

- Let $X_1, X_2, \dots X_n$ be a random sample (iid) of size 'n' ($x_1, x_2 \dots x_n$) from Normal Population with mean μ & Variance σ^2.

Function Estimator: Sample Mean $\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n} = \frac{X_1 + X_2 + \dots + X_n}{n}$

$\bar{X} = h(X_1, X_2, \dots X_n)$

Value of the Estimator Estimate: $\bar{x} = \frac{x_1 + x_2 + \dots + x_n}{n}$

$X_1, X_2, \dots X_n \sim N(\mu, \sigma^2)$

population mean: μ — Population Variance: σ^2

$$\boxed{\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)}$$

Standardize $\bar{x} \Rightarrow \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1) \rightarrow$ Std. Normal Dist.

- We use the above results to estimate the population mean μ. Here we assume that $\boxed{\sigma^2 \text{ is known}}$ by using sample mean $\bar{x}$

$$\boxed{\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}}$$

→ To estimate the population variance "σ^2" using the sample variance "s^2"

1 "**Student-t-distribution**" 'Gosset', 1908

When σ^2 is unknown? How to estimate μ?

Let $x_1, x_2, \ldots x_n$ be a random sample from a Normal population with mean μ. If $\bar{x}$ denotes the sample mean and 's' denotes the sample standard deviation.

t-statistic

$$\boxed{t = \frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t_{n-1}}$$

- Note that the population variance σ^2 is unknown.
- Small sample size 'n'

i.e, $\dfrac{\bar{X} - \mu}{S/\sqrt{n}}$ has a t-distribution with n-1 degrees of freedom.

$X_1 \cdots X_n$

$\downarrow$

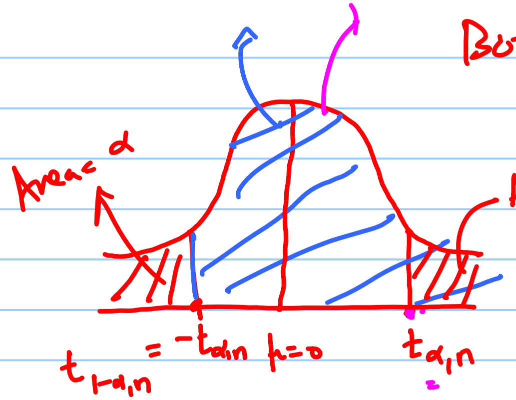
$S^2 \to S$

The pdf of the Student t-distribution is

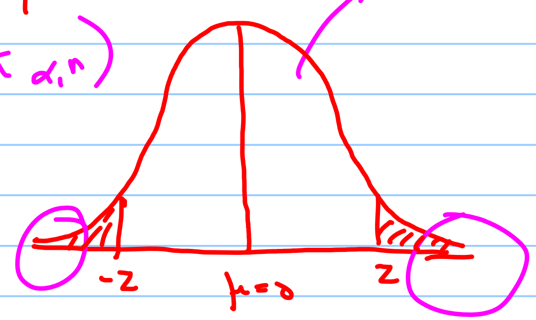
$$\Rightarrow \quad f(t) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\sqrt{\pi n}} \cdot \left(1 + \frac{t^2}{n}\right)^{-(n+1)/2} \qquad -\infty < t < \infty$$

Student 't'- distribution.

Both are Bell shaped.



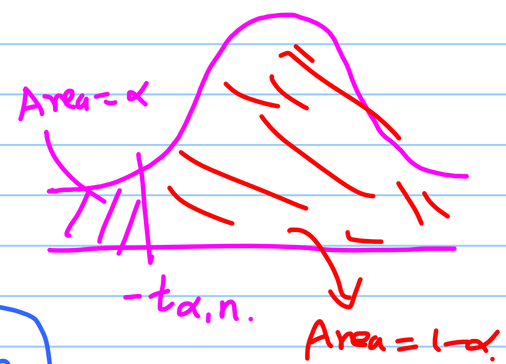
Area $= \alpha = P(t \geq t_{\alpha,n})$



Std. Normal Curve.

't - distribution has heavier tails' than the normal dist.

$$\boxed{t_{1-\alpha,n} = -t_{\alpha,n}}$$



Population Parameters, μ, σ^2, λ, n, p, p; λ $= t_{1-\alpha,n}$

$(x_1 \cdots x_n)$

Choose appropriate estimator & sampling dist.

"Point Estimation:" — Single value for the unknown population parameter

μ, σ^2

(i) Moment Method: (MM).

(ii) (Maximum Likelihood Method (MLE)).

Moment Method:

~~Let~~ $X_1, X_2, \ldots X_n$ be a random sample from a population 'X' with joint PDF $f(\underline{x}; \theta_1, \theta_2, \ldots, \theta_m)$, where $\theta_1, \theta_2, \ldots, \theta_m$ are m unknown parameters.

$X \sim Exp(\lambda)$

$X \sim N(\mu, \sigma^2)$

Population Parameter

$\mu, \lambda, \sigma^2, p, \ldots$

The k^{th} order population moment about the origin is

$$E(X^k) = \int x^k f(\underline{x}; \theta_1, \ldots \theta_m)\, dx.$$

$\mu = E(X)$

$E(X^2)$

Further, let $M_k = \frac{1}{n} \sum_{i=1}^{n} X_i^k$ be the k^{th} order sample moment about 0.

$$M_1 = \frac{1}{n} \sum_{i=1}^{n} x_i = \bar{x}$$

$\theta_1, \theta_2, \ldots \theta_m$

$M_1 = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}$ $\quad$ $E(X)$ = First Population Moment

$M_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$ $\quad$ $E(X^2)$ = Second Population Moment

$\vdots$ $\quad$ $\vdots$

$M_m = \frac{1}{n}\sum_{i=1}^{n} X_i^m$ $\quad$ $E(X^m)$ = m^{th} order population Moment about 0.

In the method of moments, we find the estimator for the parameters $\theta_1, \theta_2, \ldots, \theta_m$ by equating the first 'm' population moments to the first m sample moments about 0.

$$E(X) = M_1$$

$$E(X^2) = M_2$$

$\vdots$

$$E(X^m) = M_m.$$

P1). Exponential distribution ($\lambda > 0$

Aim: To find the moment estimator for λ. unknown parameter.

$$X \sim Exp(\lambda).$$

Suppose that $X_1, X_2 \ldots X_n$ is a random sample from an exponential dist. with parameter λ

By MM, $\boxed{E(X) = M_1}$ $\longrightarrow$ (*)

$$X \sim Exp(\lambda) \Rightarrow E(X) = \int_0^\infty x . \lambda e^{-\lambda x} dx = \frac{1}{\lambda}$$

$$M_1 = \frac{1}{n} \sum_{i=1}^{n} X_i = \bar{X}$$

$E(x) = M_1$

From ⊛, $\Rightarrow \frac{1}{\lambda} = \bar{X}$

$\Rightarrow \boxed{\hat{\lambda} = \frac{1}{\bar{X}}}$

$\hat{\theta}$ - Estimator of θ → Function

θ - Estimate ↳ Numerical value

$X_1, \dots, X_n.$

$x_1, \dots, x_n$

$\bar{x} = \frac{1}{n} \sum x_i$

$\lambda = \frac{1}{\bar{x}}$

P2). Let $X_1, \dots, X_n \sim Poi(\lambda)$

WKT $E(X) = \lambda$

$M_1 = \bar{X}$

$\Rightarrow E(X) = M_1$

$\Rightarrow \boxed{\hat{\lambda} = \bar{X}}$

$\boxed{\lambda = \bar{x}}$

P3). $X_1, X_2, \dots X_n \sim N(\overset{\theta_1}{\mu}, \overset{\theta_2}{\sigma^2})$

By the method of Moments.

$$\left.\begin{aligned} E(X) &= M_1 \\ E(X^2) &= M_2 \end{aligned}\right\} \begin{aligned} &-① \\ &-② \end{aligned}$$

$M_1 = \bar{X}$

$M_2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2$

From ①, $\boxed{\hat{\mu} = \bar{X}}$

From ②, $\sigma^2 + \mu^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2$

$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \mu^2 \Rightarrow \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \hat{\mu}^2$

- $E(X) = \mu$ ✓

$Var(X) = \sigma^2$

$\Rightarrow E(X^2) - (E(X))^2 = \sigma^2$

$\Rightarrow$ • $E(X^2) = \sigma^2 + \mu^2$

$E(X^2) = M_2$

$\mu = E(X)$

$Var(X) = E(X^2) - (E(X))^2$

$\sigma^2 = E(X^2) - \mu^2$

$\Rightarrow E(X^2) = \mu^2 + \sigma^2$

$$\boxed{\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{x}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}$$ //.

14). Let $X_1, X_2 \ldots X_n$ be a random sample of size 'n' from a population X with pdf

$$f(x;\theta) = \begin{cases} \theta x^{\theta-1}, & 0 < x < 1 \\ 0, & \text{Otherwise} \end{cases}$$

where $0 < \theta < \infty$ is an unknown parameter.

Using the Moment Method, find an estimator of θ?

If $x_1 = 0.2$, $x_2 = 0.6$, $x_3 = 0.5$, $x_4 = 0.3$ is a random sample of size 4, then what is the estimate of θ?

$$E(X) = \int_0^1 x.\underline{f(x;\theta)}\, d\theta =$$

$$\int x^n\, dx = \frac{x^{n+1}}{n+1}$$

Ans: $\boxed{\hat{\theta} = \dfrac{\bar{X}}{1-\bar{X}}}$ ✓ $\boxed{\text{Estimate of } \theta = \dfrac{2}{3}}$

$\bar{x} =$

Maximum Likelihood Estimator:

"How to find a maximum of a function."