Second Mid Term Exam Nov. 2022 Solution Discussion.

Note Title 13-11-2022

1. Consider a web server with an average rate of requests $\lambda = 0.2$ jobs per second. Assuming that the number of arrivals per unit time is poisson distributed and the inter-arrival time X is exponentially distributed with a parameter λ. Find the probability that an interval of 5 seconds elapses without requests.

$$\lambda = 0.2 \text{ Jobs / seconds}$$

Two ways we can solve this problem:

(i) Poisson Distribution:

Number of arrival.

$$\lambda' = 0.2 \times 5 = 1 \text{ jobs / 5 seconds.}$$

$$P(X = 0) = \frac{e^{-\lambda'}(\lambda')^0}{0!} = e^{-1} \longrightarrow 1 - P(T \leq 5) = 1 - (1 - e^{-(0.2)5}) = e^{-1}$$

(ii) Exponential dist.:

$$P(T \leq t) = F_T(t) = 1 - e^{-\lambda t}$$

$$P(T > 5) = \int_5^{\infty} (0.2) e^{-(0.2)t} \, dt = (0.2) \frac{e^{-(0.2)t}}{-0.2} \Big|_{t=5}^{t=\infty} = e^{-(0.2)5} = e^{-1}$$

The number of telephone calls that arrive at a phone exchange is often modeled as a Poisson random variable. Assume that on the average there are 10 calls per hour. $\lambda = 10$ calls per hour

(a) What is the probability that there are exactly 5 calls in one hour?

(b) What is the probability that there are 3 or fewer calls in one hour?

(c) What is the probability that there are exactly 15 calls in two hours?

WKT. X is Poisson r.v with $\lambda = 10$ calls per hour

P.M.F of X is

$$P(X=x) = \frac{e^{-\lambda} \cdot \lambda^{x}}{x!}$$

a) $P(X=5) = \dfrac{e^{-10} \cdot 10^{5}}{5!}$ //

(b) $P(X \le 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$

$= \dfrac{e^{-10} \cdot 10^{0}}{0!} + \dfrac{e^{-10} \cdot 10}{1!} + \dfrac{e^{-10} \cdot 10^{2}}{2!} + \dfrac{e^{-10} \cdot 10^{3}}{3!}$

(c) $\lambda' = 20$ calls per two hours

$P(X=15) = e^{-20} \cdot 20^{15} / 15!$ //

$E(X) = \lambda$

The lifetime of an electric bulb follows an exponential distribution with a mean life of 180 days.

$E(T) = 180 \rightarrow E(T) = \frac{1}{\lambda} = 180$

$\boxed{\lambda = \frac{1}{180}}$

(a) Suppose you bought one bulb. Find the probability it will burn out within 160 days?

(b) Given that the bulb you bought is still working after 90 days, Find the probability that it still working after another 60 days?

a) $P(T < 160) = 1 - e^{-\frac{1}{180}(160)} = 1 - e^{-8/9}$

$= \int_0^{160} \frac{1}{180} e^{-\frac{1}{180}t}\,dt = \frac{1}{180}\left[\frac{e^{-1/180 t}}{-1/180}\right]_{t=0}^{t=160} = -e^{-\frac{160}{180}} + 1$

WKT CDF of exponential r.v

$F_T(t) = P(T \leq t) = 1 - e^{-\lambda t}$

(b). Memoryless property (i) $P(T \leq x+t \mid T > t) = P(T \leq x)$

(ii) $P(T > s+t \mid T > t) = P(T > s)$ for all $s, t \geq 0$

$$P(T>60+90 \mid T>90) = P(T>60) = \int_{60}^{\infty} \frac{1}{180} e^{-\frac{1}{180}t}\,dt = \left[-e^{-\frac{1}{180}t}\right]_{t=60}^{\infty}$$
$$= e^{-1/3}$$

An analog signal received at a detector (measured in micro volts) may be modelled as a Gaussian random variable $N(200, 16^2)$ at a fixed point in time.
μ, σ^2

(i) What is the probability that the signal will exceed 240 micro volts?

(ii) What is the probability that the signal is larger than 240 micro volts given that it is greater than 210 micro volts.

(i) Here $\mu = 200$, $\sigma^2 = 16 \rightarrow \sigma = 16$.

$x > 210$ $\quad x > 240$

WKT $\quad Z = \frac{X-\mu}{\sigma} = \frac{X-200}{16}$

$$P(X>240) = P\left(\frac{X-200}{16} > \frac{240-200}{16}\right) = P(Z>2.5)$$
$$= 1 - P(Z \le 2.5)$$
$$= 1 - \Phi(2.5)$$

(ii) $$P(X>240 \mid X>210) = \frac{P(\{X>240\}\cap\{X>240\})}{P(X>210)}$$
$$= P(X>240) / P(X>210)$$

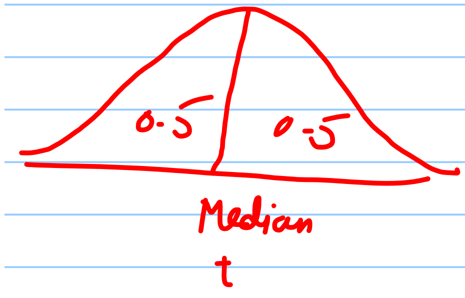
$$= \frac{P\left(\frac{X-200}{16} > \frac{240-200}{16}\right)}{P\left(\frac{X-200}{16} > \frac{210-200}{16}\right)} = \frac{P(Z > 2.5)}{P(Z > \frac{5}{8})} = \frac{1-\Phi(2.5)}{1-\Phi(5/8)}$$

Let T denote the life time of a bulb which is exponentially distributed with parameter $\lambda > 0$.

(a) Find the median of T that is the point t such that $P(T \leq t) = P(T > t) = 0.5$.

(b) Find the probability that T survives longer than its average life $E(T)$.

(c) Find the probability that T survives another average life given that it already survived its average life to begin with.



(a). $P(T \leq t) = 0.5 \Rightarrow 1 - e^{-\lambda t} = 0.5 = 1/2$

$\Rightarrow e^{-\lambda t} = 1/2$

$E(T) = \frac{1}{\lambda}$

$\Rightarrow \ln(e^{-\lambda t}) = \ln(1/2)$

$\Rightarrow -\lambda t = \cancel{\ln 1}^{0} - \ln 2 \Rightarrow \boxed{t = \frac{\ln 2}{\lambda}}$

(b) $P(T > E(T)) = P(T > \frac{1}{\lambda}) = 1 - P(T \leq \frac{1}{\lambda}) = 1 - (1 - e^{-\lambda \frac{1}{\lambda}}) = e^{-1}$

(c) $P(T > 2E(T) \mid T > E(T)) = P(T > E(T)) = e^{-1}$

Buses arrival randomly at a bus stop follows Poisson distribution with a rate of 2 per hour. Let us assume that you arrive at the bus stop at 6 AM but unfortunately you are still waiting for a bus at 7AM. Find the probability that a bus will arrive within the next half an hour.

$\lambda = 2$ per hour. $X \sim \text{Poisson}(\lambda = 2)$

T — waiting time — Exponential r.v with $\lambda = 2$

$P(T \leq 1.5 \text{ hour} \mid T > 1 \text{ hour}) = P(T \leq 0.5 \text{ hour}) = 1 - e^{-2(0.5)} = \underline{\underline{1 - e^{-1}}}$

Memoryless Property : $P(T \leq x+t \mid T > t) = P(T \leq x)$

A fair six-sided die is rolled 420 times. What is the probability that the sum of the rolls lies between 1400 and 1550? (Hint: Use CLT for the sum $X_1 + X_2 + \ldots + X_{420}$)

Prob. dist.

$X = x$	1	2	3	4	5	6
$P(X=x)$	1/6	1/6	1/6	1/6	1/6	1/6

$$P(1400 < X_1 + X_2 + \cdots + X_{420} < 1550) = ?$$

CLT. $P\left(\frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \le \frac{x - n\mu}{\sigma\sqrt{n}}\right) \approx \Phi\left(\frac{x-n\mu}{\sigma\sqrt{n}}\right)$ as $n \to \infty$

$E(X_1 + \cdots + X_{420}) = 420 \cdot E(X_i) = 420 \cdot \frac{7}{2} = 1470$

$Var(X_1 + \cdots + X_{420}) = 420 \cdot Var(X_i) = 420\left(\frac{35}{12}\right) = 1225$

$SD(X_1 + \cdots + X_{420}) = \sqrt{1225} = 35$

$E(X_i) = \sum_{x=1}^{6} x \cdot P(x)$

$= \frac{1}{6}(1+2+3+4+5+6)$

$= \frac{6 \times 7}{2 \times 6} = \frac{7}{2}$

$Var(X_i) = E(X_i^2) - (E(X_i))^2$

$= \frac{91}{6} - \frac{49}{4} = \frac{35}{12}$

$E(X_i^2) = \sum_{x=1} x^2 P(x)$

$= \frac{1}{6}(1^2 + 2^2 + \cdots + 6^2)$

$= \frac{1}{6}\left(\frac{6 \cdot 7 \cdot 13}{6}\right)$

$= \frac{91}{6}$

$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

$n=64$

You have invited 64 guests to a party. You need to make sandwiches for the guests. You believe that a guest might need 0,1, or 2 sandwiches with probabilities $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$ respectively. You assume that the number of sandwiches each guest needs is independent of other guests. How many sandwiches should you make so that you are 95% sure that there is no shortage? Hint: Use CLT for $X = X_1 + X_2 + \ldots + X_{64}$ and find x s.t $P(X \leq x) = 0.95$)

x	0	1	2
$p(x)$	$1/4$	$1/2$	$1/4$

$$E(x_i) = \sum_{x=0}^{2} x \cdot p(x) = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = 1$$

$$Var(x_i) = E(x_i^2) - E(x_i)^2 = 3/2 - 1 = 1/2$$

$$E(x_i^2) = \sum_{x=0}^{1} x^2 \cdot p(x) = 1 \cdot \frac{1}{2} + 4 \cdot \frac{1}{4} = 3/2$$

$$P(1400 < x_1 + x_2 + \cdots + x_{420} < 1550)$$

$$\simeq P\left(\frac{1400-1470}{35} < Z < \frac{1550-1470}{35}\right) \simeq P(-2.01 < Z < 2.3)$$

$$= \Phi(2.3) - \Phi(-20.01)$$

$$= \Phi(2.3) - (1 - \Phi(2.01))$$

$$E(X_1 + \cdots + X_{64}) = 64 \cdot E(X_i) = 64.$$

$$Var(X_1 + \cdots + X_{64}) = 64 \cdot Var(X_i) = 64 \cdot 1/2 = 32, \quad SD(X_1 + \cdots + X_{64}) = \sqrt{32} = 4\sqrt{2}$$

By CLT $P\left(X_1 + \cdots + X_{64} \le x \right) = 0.95$

$$\Rightarrow P\left(\frac{X_1 + \cdots + X_{64} - 64}{4\sqrt{2}} \le \frac{x - 64}{4\sqrt{2}} \right) = 0.95$$

$$\Rightarrow P\left(Z \le \frac{x - 64}{4\sqrt{2}} \right) = 0.95$$

$$\Rightarrow \Phi\left(\frac{x - 64}{4\sqrt{2}} \right) = 0.95$$

$$\Rightarrow \frac{x - 64}{4\sqrt{2}} = \Phi^{-1}(0.95)$$