Hypothesis Testing:

A vital role of Statistics is in verifying Statements, Claims, conjectures, and in general - testing hypotheses. Based on a random Sample, we can use Statistics to verify whether.

- the average number of concurrent users increased by 2000 this year
- the average connection speed is 54 Mbps, as claimed by the Internet Service Provider.
- the proportion of defective products is at most 3%, as promised by the manufacturer.
- etc.

Defn: A Statistical Hypothesis: is a statement about the parameters of one or more populations.

Note: The hypotheses are always Statements about the population under study, not statements about the sample.

To begin, we need to state exactly what we are testing.

These are

Null Hypothesis $\rightarrow H_0$ Notation.

Alternative Hypothesis $\rightarrow H_1$ (or H_A)

H_0 & H_1 are simply two mutually exclusive statements.
Each test results either in acceptance of H_0 or its rejection in favor of H_1.

For example, Suppose that we are interested in the burning rate of the Solid propellant. Burning rate is a r.v that can be described by a Probability distribution. Now we are interested in deciding whether or not the mean burning rate is 50 centimeters per second. We may express it as

$H_0: \mu = 50$ Cms/sec $\longrightarrow$ Null Hypothesis

$H_1: \mu \neq 50$ Cms/sec $\longrightarrow$ Alternative Hypothesis.

Here the alternative hypothesis is a statement that contradicts the null hypothesis. Because the alternative hypothesis specifies values of μ that could be either $\mu > 50$ or $\mu < 50$

It is called two-sided alternative hypothesis.

In some Situations, we may wish to formulate a One-sided alternative hypothesis, as in

$H_0: \mu = 50$ $\qquad$ or $\qquad$ $H_0: \mu = 50$

$H_1: \mu < 50$ $\qquad\qquad\quad$ $H_1: \mu > 50$.

We will always state the null hypothesis as an equality claim. A procedure leading to a decision about the null hypothesis is called a test of a hypothesis.

Hypothesis testing Procedures rely on using the information in a random Sample from the population of Interest.
If this information is Consistent with the null hypothesis, we will not reject it; however if this information is inconsistent with the null hypothesis, we will conclude that the null hypothesis is false and reject it in favor of the alternative.

Testing the hypothesis involves taking a random Sample, Computing a test Statistic from the Sample data, and then using the test Statistic to make a decision about the null hypothesis.

Tests of Statistical Hypotheses:

Let us consider the propellant burning rate problem,

we wish to test $H_0: \mu = 50$

$H_1: \mu \neq 50$.

Suppose that a Sample of $n = 10$ specimens is tested and the Sample mean burning rate $\bar{x}$ is Observed.
A value of $\bar{x}$ that falls close to the hypothesized value of $\mu = 50$ doesnot conflict with the null hypothesis that the true mean μ is really 50.

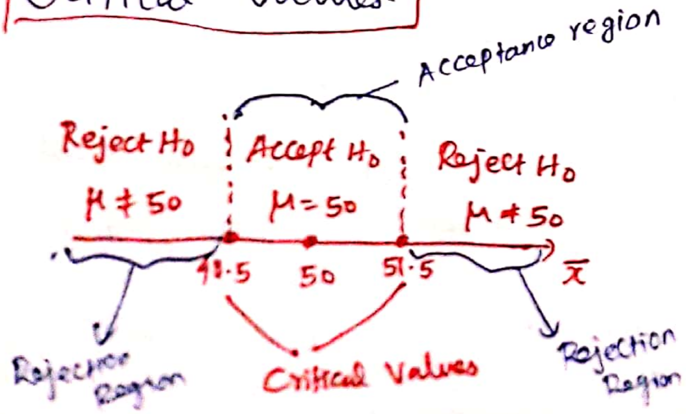
On the other hand, a sample mean that is considerably different from 50 is evidence in support of the alternative hypothesis H_1. Thus, the sample mean is the test statistic in this case.

The sample mean can take on many different values. Suppose that if $48.5 \le \bar{x} \le 51.5$, we will not reject the null hypothesis $H_0: \mu = 50$, and if either $\bar{x} < 48.5$ or $\bar{x} > 51.5$, we will reject the null hypothesis in favor of the alternative hypothesis $H_1: \mu \neq 50$.

The values of $\bar{x}$ that are less than 48.5 and greater than 51.5 form a region ~~for which~~ called **rejection region**

$48.5 \le \bar{x} \le 51.5$ form a region called **Acceptance region** for which we will fail to reject the null hypothesis.

The boundaries between the critical regions (rejection regions) and the acceptance region are called the **Critical Values.**



Decision Criteria:

We reject H_0 in favor of H_1 if the test statistic falls in the rejection region and accept H_0 otherwise.

Sampling errors:

When testing hypotheses, we realize that all we see is a random sample. Therefore, with all the best statistics skills, our decision to accept or to reject H_0 may still be wrong. This is a sampling error.

Four possible situations.

	Result of the test	
	Reject H_0	Accept H_0
H_0 is true	Type-I error	Correct
H_0 is false	correct	Type-II error

Defn: A type-I error occurs when we reject the true null hypothesis H_0.

A type-II error occurs when we accept the false null hypothesis H_0.

Each error occurs with a certain probability that we hope to keep small.

A type-I error is often considered more dangerous and undesired than a type-II error. Making a type-I error can be compared with convicting an innocent defendant or sending a patient to a surgery when he or she does not need one.

For this reason, we shall design tests that bound the probability of type-I error by a pre-assigned ~~zero~~ small number 'α'. Under this conditions, we may want to minimize the probability of type-II error.

Definition:

Probability of a type-I error is called the significance level of a test.

$$\alpha = P(\text{type-I error}) = P(\text{ reject } H_0 \text{ when } H_0 \text{ is true})$$

The power of a statistical test is the probability of rejecting the ~~null~~ false null hypothesis.

(ie, the probability of correctly rejecting a false null hypothesis)

The power is computed as $\boxed{1-\beta}$ where $\beta = P(\text{type-II error})$
$= P(\text{Accept } H_0 \text{ when } H_0 \text{ is false})$.

Level α tests: General approach:

A standard algorithm for a level α test of a null hypothesis H_0 against an alternative hypothesis H_1 consists of 3 steps.

Step: 1 Test Statistic.

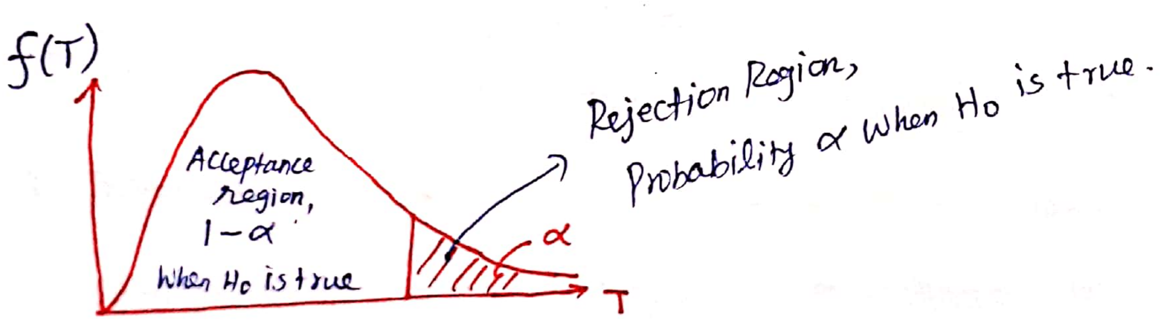
Testing hypothesis is based on a test statistic T, a quantity computed from the data that has some known sampling distribution if the null hypothesis is true.

ie we find a suitable test statistic and compute it from the data.

Step: 2 Acceptance Region & rejection region.

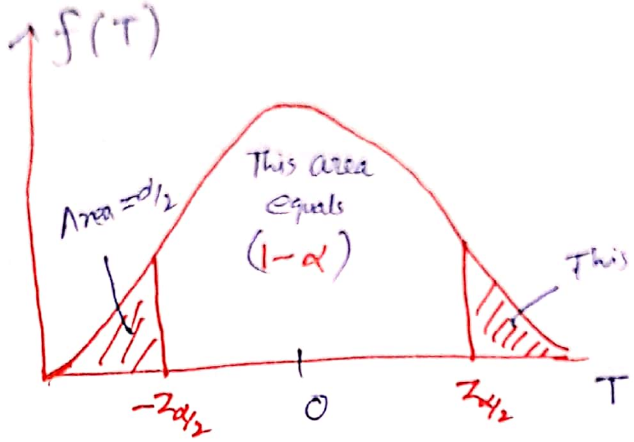
Next, we consider the sampling distribution of test statistic when H_0 is true. If it has a density function $f(T)$, then ~~the whole area is $1-\alpha$~~ the whole area under the density curve is 1, and we can always find a portion of it whose area is $(1-\alpha)$. It is called acceptance region.

The remaining part, the complement of the acceptance region, is called rejection region. By the complement rule, its area is α.



Acceptance & Rejection Regions.

As another example, consider the sampling distribution of T is standard normal, then the area between $-z_{\alpha/2}$ & $z_{\alpha/2}$ equals exactly $1-\alpha$.



The interval $A = [-z_{\alpha/2}, z_{\alpha/2}]$ can serve as a level α acceptance region, and the remaining part

$$R = A^c = (-\infty, -z_{\alpha/2}) \cup (z_{\alpha/2}, +\infty) \quad \text{is}$$

the rejection region.

Areas under the density curve are probabilities, and we conclude that

Level α Test:!

$$P(T \in \text{acceptance region} \mid H_0 \text{ is true}) = 1-\alpha$$

&

$$P(T \in \text{rejection region} \mid H_0 \text{ is true}) = \alpha$$

Significance Level

Step:3 Result & Its interpretation:

Accept the null hypothesis H_0 if the test statistic T belongs to the acceptance region. Reject H_0 in favor of the alternative hypothesis H_A if T belongs to the rejection region.