Note Title 26-11-2022

P1). A disk has free space of 330 MB. Is it likely to be sufficient for 300 Independent images if each image has expected size of 1 MB with a Standard deviation of 0.5 MBs?

(Hint: use CLT).

Soln: Given $n = 300$, $\mu = 1$ Mb $\sigma = 0.5$ Mb

Let $S_n = X_1 + X_2 + \cdots + X_{300}$

$$P(S_n \leq 330) \overset{CLT}{\simeq} P\left(\frac{S_n - n\mu}{\sigma\sqrt{n}} \leq \frac{330 - 300.1}{(0.5)\sqrt{300}}\right)$$

$$\simeq P(Z \leq 3.46) \approx \Phi(3.46) = 0.9997$$

P2) upgrading a certain software package requires installation of 82 new files. Files are installed consecutively. The installation time is random, but on the average, it takes 15 sec. to install one file, with a variance of 16 sec², What is the prob that the whole package is upgraded in less than 20 minutes? (Hint: CLT)

$n = 82, \quad \mu = 15 \text{ sec} \qquad \sigma^2 = 16 \text{ sec}^2 \Rightarrow \sigma = 4 \text{ sec.}$

$$P\left(S_n = X_1 + X_2 + \cdots + X_{82} \leq 20 \text{ minutes}\right)$$

$$= P(S_n < 1200 \text{ sec.}) \; ?$$

$$\simeq P\left(\frac{S_n - n\mu}{\sigma\sqrt{n}} < \frac{1200 - 82 \cdot 15}{4\sqrt{82}}\right)$$

$$\overset{CLT}{\simeq} P(Z < -0.828) \simeq \Phi(-0.828) = 1 - \Phi(0.828)$$

$$\simeq 0.2033$$

P3). A sample of 3 observations ($x_1 = 0.4$, $x_2 = 0.7$, $x_3 = 0.9$). Is collected from a continuous distl. with density

$$f_X(x) = \begin{cases} \theta x^{\theta-1}, & \text{if } 0 < x < 1. \\ 0 & \text{otherwise} \end{cases}$$

Estimate $\theta$ by (i) Moment Method, (ii) Maximum Likelihood Method

Soln:

By Moment Method:

$$E(X) = \int_0^1 x \cdot f_X(x)\, dx = \int_0^1 x^1 \theta x^{\theta-1}\, dx = \theta \int_0^1 x^\theta\, dx = \theta \cdot \left[\frac{x^{\theta+1}}{\theta+1}\right]_{x=0}^{x=1}$$

$$E(X) = \frac{\theta}{\theta+1}$$

$$M_1 = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X} \Rightarrow \bar{X} = \frac{1}{3}(0.4+0.7+0.9) = \frac{2}{3}$$

$$E(X) = M_1$$

$$\Rightarrow \frac{\theta}{\theta+1} = \bar{X} \Rightarrow \frac{\theta}{\theta+1} = \frac{2}{3} \Rightarrow \boxed{\theta_{MM} = 2} \checkmark$$

(ii) Maximum likelihood method:

Likelihood function $L(\theta) = f(x_1, x_2, x_3; \theta) = \prod_{i=1}^{3} f(x_i; \theta)$

$$= \prod_{i=1}^{3} \theta\, x_i^{\theta-1}$$

$$L(\theta) = \theta^3 \cdot \prod_{i=1}^{3} x_i^{\theta - 1}$$

Take ln on both sides,

$$\ln L(\theta) = \ln(\theta^3) + \ln\left((x_1 x_2 \cdot x_3)^{\theta-1}\right)$$

$$= 3\ln\theta + (\theta-1)\ \ln(x_1 x_2 x_3)$$

$$\ln L(\theta) = 3\ln\theta + (\theta-1)\ln(0.252)$$

$$\frac{d}{d\theta}\ln(L(\theta)) = \frac{3}{\theta} + \ln(0.252).(1)$$

$$\frac{d}{d\theta}\ln L(\theta) = 0$$

$$\boxed{\theta = 2.1765}$$

$$\left.\frac{d^2}{d\theta^2}\ln L(\theta)\right|_{\theta=2.177} = \left.-\frac{3}{\theta^2}\right|_{\theta=2.177} < 0$$

$$\boxed{\theta_{MLE} = 2.1765}$$

1). In order to ensure efficient usage of a server, it is necessary to estimate the mean number of concurrent users. According to records, the average number of concurrent users at 100 randomly selected times is 37.7, with a S.D. $\sigma = 9.2$. Construct a 90% confidence interval for the expectation '$\mu$' of the number of concurrent users?

Soln: Given $n = 100$, $\bar{x} = 37.7$ $\sigma = 9.2$

For 90% C.I, $1-\alpha = 0.9$ $\Rightarrow$ $\alpha = 0.1$

$\Rightarrow$ $\alpha/2 = 0.05$ $\Rightarrow Z_{0.05} = 1.65$

WKT $100(1-\alpha)\%$ C.I for $\mu$ is $\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$

$$\Rightarrow 37.7 - \frac{1.65 \times 9.2}{\sqrt{100}} < \mu < 37.7 + 1.65\frac{9.2}{10}$$

$$\Rightarrow \boxed{36.18 < \mu < 39.22}$$