Here $n = 7$ $\boxed{s^2 = \frac{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}{n(n-1)}}$

Sample mean $\bar{x} = 10.0$ & Sample standard deviation $s = 0.283$

$95\% = 100(1-\alpha)\%$

$\Rightarrow 1-\alpha = 0.95 \Rightarrow \alpha = 0.05 \Rightarrow \frac{\alpha}{2} = 0.025$

$$t_{\alpha/2, n-1} = t_{0.025, 6} = 2.447$$

$100(1-\alpha)\%$ Confidence interval for μ when σ^2 is unknown

$$\bar{x} - t_{\alpha/2}\frac{s}{\sqrt{n}} < \mu < \bar{x} + t_{\alpha/2}\frac{s}{\sqrt{n}}$$

95% confidence interval for μ is

$$10 - (2.447)\frac{(0.283)}{\sqrt{7}} < \mu < 10 + (2.447)\left(\frac{0.283}{\sqrt{7}}\right)$$

$$\Rightarrow \boxed{9.74 < \mu < 10.26}$$

Concept of a Large-Sample Confidence interval.

Often Statisticians recommend that even when normality cannot be assumed, σ is unknown, and $n \geq 30$, s can replace σ and the confidence interval

$$\boxed{\bar{x} - z_{\alpha/2}\frac{s}{\sqrt{n}} < \mu < \bar{x} + z_{\alpha/2}\frac{s}{\sqrt{n}}}$$

may be used.

This is often referred to as a large-sample confidence interval. It should be emphasized that this is only an approximation and the quality of the result becomes better as the sample size grows larger.

(P1). Scholastic Aptitude Test (SAT) mathematics scores of a random sample of 500 high school seniors in the State of Texas are collected, and the sample mean and standard deviation are found to be 501 & 112, respectively. Find a 99% confidence interval on the mean SAT mathematics score for seniors in the State of Texas.

Soln:

Here $n = 500$, $\bar{x} = 501$, $s = 112$

Since the sample size is large, it is reasonable to use normal approximation.

$\Rightarrow$ $100(1-\alpha)\%$ confidence interval for μ when σ is unknown

$$\bar{x} - z_{\alpha/2}\frac{s}{\sqrt{n}} < \mu < \bar{x} + z_{\alpha/2}\frac{s}{\sqrt{n}}.$$

$\Rightarrow$ 99% confidence interval for μ is

$$\bar{x} - z_{0.005}\frac{s}{\sqrt{n}} < \mu < \bar{x} + z_{0.005}\frac{s}{\sqrt{n}}$$

Using std normal table, we find $z_{0.005} = 2.58$

$\therefore$ 99% confidence interval for μ is

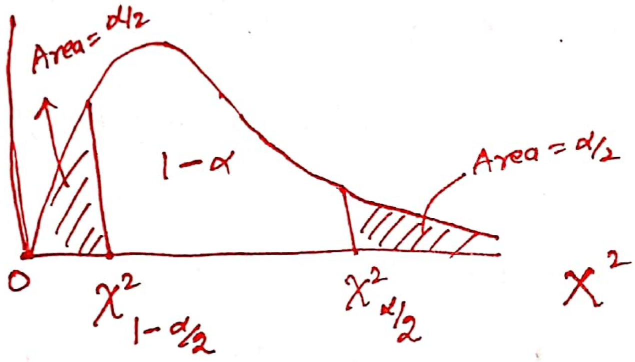
$$501 - (2.58)\frac{(112)}{\sqrt{500}} < \mu < 501 + (2.58)\frac{(112)}{\sqrt{500}}$$

$$\Rightarrow \quad 488.1 < \mu < 513.9.$$

Estimating the population Variance:

Let $X_1, X_2, \ldots, X_n$ be a random sample from a normal population with unknown variance σ^2, and let s^2 be the sample variance.

WKT. The statistic $\chi^2 = \frac{(n-1)s^2}{\sigma^2}$ has a Chi square (χ^2) distribution with $(n-1)$ degrees of freedom.



$$P\left(\chi^2_{1-\alpha/2} < \chi^2 < \chi^2_{\alpha/2}\right) = 1-\alpha \quad \text{—} \ (1)$$

Where $\chi^2_{1-\alpha/2}$ and $\chi^2_{\alpha/2}$ are values of the Chi-squared distribution with $n-1$ degrees of freedom, leaving areas of $1-\alpha/2$ and $\alpha/2$, respectively, to the right

Now substituting $\chi^2 = \frac{(n-1)s^2}{\sigma^2}$ in (1),

$$P\left(\chi^2_{1-\alpha/2} < \frac{(n-1)s^2}{\sigma^2} < \chi^2_{\alpha/2}\right) = 1-\alpha$$

Dividing each term in the inequality by $(n-1)s^2$ and then inverting each term (thereby changing the sense of the inequalities) we obtain.

$$\boxed{P\left[\frac{(n-1)s^2}{\chi^2_{\alpha/2}} < \sigma^2 < \frac{(n-1)s^2}{\chi^2_{1-\alpha/2}}\right] = 1-\alpha}$$

Confidence interval for σ^2:

If s^2 is the variance of a random sample of size 'n' from a normal population, a $100(1-\alpha)\%$ confidence interval for σ^2 is

$$\boxed{\frac{(n-1)s^2}{\chi^2_{\alpha/2}} < \sigma^2 < \frac{(n-1)s^2}{\chi^2_{1-\alpha/2}}}$$

Where $\chi^2_{\alpha/2}$ & $\chi^2_{1-\alpha/2}$ are χ^2-values with $n-1$ degrees of freedom, leaving areas of $\alpha/2$, $1-\alpha/2$, respectively, to the right.

Note: An approximate $100(1-\alpha)\%$ confidence interval for σ is obtained by taking the square root of each endpoint of the interval for σ^2.

(P1). The following are the weights of 10 packages of grass seed distributed by a certain company: 46.4, 46.1, 45.8, 47.0, 46.1, 45.9, 45.8, 46.9, 45.2, and 46.0. Find a 95% confidence interval for the variance of the weights of all such packages of grass seed distributed by this company, assuming a normal population.

Soln: First we find sample variance $s^2 = \frac{1}{n-1}\left(\sum_{i=1}^{n} x_i^2 - n(\bar{x})^2\right)$

(OR)

$$= \frac{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}{n(n-1)}$$

x_i	x_i^2	
$\vdots$	$\vdots$	
$\Sigma x_i =$	$\Sigma x_i^2 =$	

$$\Rightarrow s^2 = \frac{(10)(21{,}273.12) - (461.2)^2}{(10)(9)}$$

$$s^2 = 0.286$$

To obtain 95% confidence interval,

$$1-\alpha = 0.95$$

$$\alpha = 0.05 \Rightarrow \alpha/2 = 0.025$$

$$n-1 = 10-1 = 9.$$

$$\chi^2_{\alpha/2,\,n-1} = \chi^2_{0.025,9} = 19.023$$

$$\chi^2_{1-\alpha/2,\,n-1} = \chi^2_{1-0.025,9} = \chi^2_{0.975,9} = 2.700$$

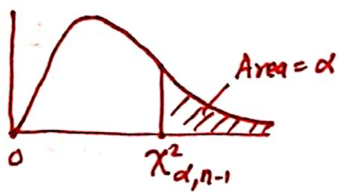
$100(1-\alpha)\%$ confidence interval for σ^2 is

$$\frac{(n-1)s^2}{\chi^2_{\alpha/2}} < \sigma^2 < \frac{(n-1)s^2}{\chi^2_{1-\alpha/2}}$$

$\therefore$ 95% confidence interval for σ^2 is

$$\frac{9(0.286)}{19.023} < \sigma^2 < \frac{9(0.286)}{2.7}$$

$$\Rightarrow \boxed{0.135 < \sigma^2 < 0.953}$$



One sided Confidence bounds on the variance:

The $100(1-\alpha)\%$ lower and upper confidence bounds on σ^2 are $\boxed{\frac{(n-1)s^2}{\chi^2_{\alpha,n-1}} \le \sigma^2}$ & $\boxed{\sigma^2 \le \frac{(n-1)s^2}{\chi^2_{1-\alpha,n-1}}}$ respectively.

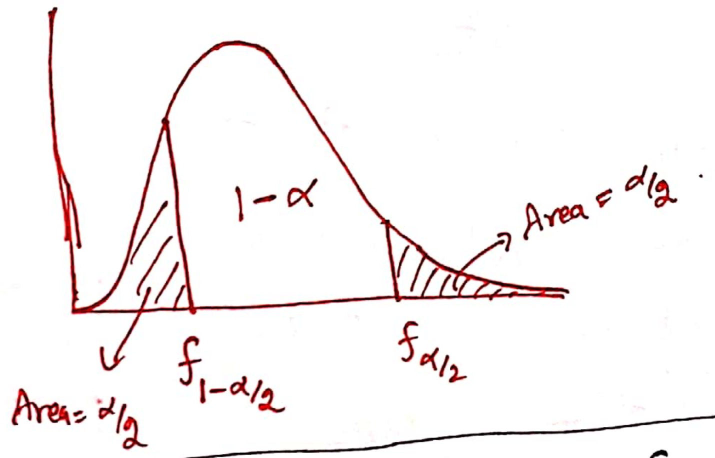
Estimating the Ratio of Two Variances:

A Point estimate of the ratio of two population Variances $\frac{\sigma_1^2}{\sigma_2^2}$ is given by the ratio $\frac{s_1^2}{s_2^2}$ of the Sample Variances. Hence, the Statistic $\frac{S_1^2}{S_2^2}$ is called an estimator of $\frac{\sigma_1^2}{\sigma_2^2}$.

If σ_1^2 & σ_2^2 are the variances of normal populations, we can establish an interval estimate of $\frac{\sigma_1^2}{\sigma_2^2}$ by using the F-statistic, i.e,

$$\boxed{F = \frac{\sigma_2^2 S_1^2}{\sigma_1^2 S_2^2}} \quad \text{— (1)}$$

W.K.T r.v. F has an F-distribution with $v_1 = n_1 - 1$ & $v_2 = n_2 - 1$ degrees of freedom.



We write,

$$\boxed{P\left(f_{1-\alpha/2}(v_1, v_2) < F < f_{\alpha/2}(v_1, v_2)\right) = 1 - \alpha} \quad \text{— (2)}$$

Where $f_{1-\alpha/2}(v_1, v_2)$ and $f_{\alpha/2}(v_1, v_2)$ are the Values of the F-distribution with v_1 and v_2 degrees of freedom, leaving areas of $1 - \alpha/2$, & $\alpha/2$, respectively, to the right.

Substituting ① in ②, we get

$$P\left[f_{1-\alpha/2}(v_1, v_2) < \frac{\sigma_2^2 S_1^2}{\sigma_1^2 S_2^2} < f_{\alpha/2}(v_1, v_2)\right] = 1-\alpha.$$

Multiplying each term in the inequality by $\frac{S_2^2}{S_1^2}$ and then inverting each term, we obtain

$$P\left[\frac{S_1^2}{S_2^2}\frac{1}{f_{\alpha/2}(v_1,v_2)} < \frac{\sigma_1^2}{\sigma_2^2} < \frac{S_1^2}{S_2^2}\cdot\frac{1}{f_{1-\alpha/2}(v_1,v_2)}\right] = 1-\alpha.$$

Also, WKT. $f_{1-\alpha/2}(v_1, v_2) = \frac{1}{f_{\alpha/2}(v_2, v_1)}$

$$\Rightarrow P\left[\frac{S_1^2}{S_2^2}\frac{1}{f_{\alpha/2}(v_1,v_2)} < \frac{\sigma_1^2}{\sigma_2^2} < \frac{S_1^2}{S_2^2} f_{\alpha/2}(v_2, v_1)\right] = 1-\alpha.$$

For any two independent random samples of sizes n_1 & n_2 selected from two normal populations, the ratio of the sample variances $\frac{s_1^2}{s_2^2}$ is computed. Then a $100(1-\alpha)\%$ confidence interval for $\frac{\sigma_1^2}{\sigma_2^2}$ is.

$$\boxed{\frac{s_1^2}{s_2^2}\frac{1}{f_{\alpha/2}(v_1, v_2)} < \frac{\sigma_1^2}{\sigma_2^2} < \frac{s_1^2}{s_2^2} f_{\alpha/2}(v_2, v_1)}$$

where $f_{\alpha/2}(v_1, v_2)$ is an f-value with $v_1 = n_1 - 1$ & $v_2 = n_2 - 1$ degrees of freedom, leaving an area of $\alpha/2$ to the right, and $f_{\alpha/2}(v_2, v_1)$ is a similar f-value with $v_2 = n_2 - 1$ & $v_1 = n_1 - 1$ degrees of freedom.

Note:

An approximate $100(1-\alpha)\%$ confidence interval for $\frac{\sigma_1}{\sigma_2}$ is obtained by taking the square root of each end point of the interval for $\frac{\sigma_1^2}{\sigma_2^2}$.

(P1). The amounts of the chemical orthophosphorus measured at two different stations on the James River. Orthophosphorus was measured in milligrams per liter. Fifteen samples were collected from station 1 with a standard deviation of 3.07 milligrams per liter, while 12 samples were collected from station 2 with a standard deviation of 0.80 mg/liter. Find a 98% confidence intervals for $\frac{\sigma_1^2}{\sigma_2^2}$ and for $\frac{\sigma_1}{\sigma_2}$, where σ_1^2, σ_2^2 are the variances of the populations of orthophosphorus contents at station 1 & station 2, respectively. Assuming that the observations came from Normal populations.

Soln: Given $n_1 = 15$, $n_2 = 12$

$s_1 = 3.07$, $s_2 = 0.80$

For 98% confidence interval, $1-\alpha = 0.98 \Rightarrow \alpha = 0.02 \Rightarrow \frac{\alpha}{2} = 0.01$

$f_{\alpha/2}(v_1, v_2) = f_{0.01}(15-1, 12-1) = f_{0.01}(14, 11) \simeq 4.30$

$f_{\alpha/2}(v_2, v_1) = f_{0.01}(12-1, 15-1) = f_{0.01}(11, 14) \simeq 3.87$.

From F-dist. table

$\therefore$ 98% confidence interval for $\frac{\sigma_1^2}{\sigma_2^2}$

$$\frac{(3.07)^2}{(0.80)^2}\left(\frac{1}{4.30}\right) < \frac{\sigma_1^2}{\sigma_2^2} < \frac{(3.07)^2}{(0.80)^2}.(3.87)$$

$$\Rightarrow \boxed{3.425 < \frac{\sigma_1^2}{\sigma_2^2} < 56.991}$$

$100(1-\alpha)\%$ confidence interval for $\frac{\sigma_1^2}{\sigma_2^2}$ is

$$\frac{s_1^2}{s_2^2}\cdot\frac{1}{f_{\alpha/2}(v_1,v_2)} < \frac{\sigma_1^2}{\sigma_2^2} < \frac{s_1^2}{s_2^2} f_{\alpha/2}(v_2,v_1)$$

Taking square roots of the confidence limits,

we find that a 98% confidence interval for $\frac{\sigma_1}{\sigma_2}$ is

$$\boxed{1.851 < \frac{\sigma_1}{\sigma_2} < 7.549}.$$

Exercises

(P1). A research engineer for a tire manufacturer is investigating tire life for a new rubber compound and has built 16 tires and tested them to end of life in a road test. The sample mean and standard deviation are 57, 389.6 & 3645.94 km. Find a 95% confidence interval on mean tire life.

(P2). An Izod impact test was performed on 30 specimens of PVC pipe. The sample mean is $\bar{x} = 1.25$ and the sample standard deviation is $s = 0.25$. Find a 99% lower confidence bound on Izod impact strength.

(P3). A random sample of 9 observations from a normal population with $\mu = 5$ yields the observed statistics $\frac{1}{8}\sum_{i=1}^{9} x_i^2 = 39.125$ & $\sum_{i=1}^{9} x_i = 45$. What is the 95% confidence interval for σ^2?

(P4). The sugar content of the syrup in canned peaches is normally distributed. A random sample of $n = 20$ cans yields a sample standard deviation of $s = 4.8$ mg. Calculate 95% two sided confidence interval for σ.

(P5). An experiment reported in Popular Science compared fuel economies for two types of similarly equipped diesel mini-trucks. Let us suppose that 12 Volkswagen & 10 Toyota trucks were tested in 90-kmph steady-paced trials. If the 12 Volkswagen trucks averaged 16 kilometers per liter with std deviation of 1.0 kmpl & the 10 Toyota trucks averaged 11 kmpl with a standard deviation of 0.8 kmpl. Construct a 98% confidence interval for σ_1/σ_2, where σ_1 & σ_2 are respectively, the standard deviations for the distances traveled per liter of fuel by the Volkswagen & Toyota mini trucks.

Scanned with CamScanner