One Tailed Test

A test of any Statistical hypothesis where the alternative is one-sided, such as

$H_0 : \theta = \theta_0$ (OR) $H_0 : \theta = \theta_0$

$H_1 : \underbrace{\theta > \theta_0}_{\text{right tail alternative}}$ $\quad$ $H_1 : \underbrace{\theta < \theta_0}_{\text{Left-tail alternative.}}$

is called a One-tailed test.

Two sided (tailed) Test

$H_0 : \theta = \theta_0$

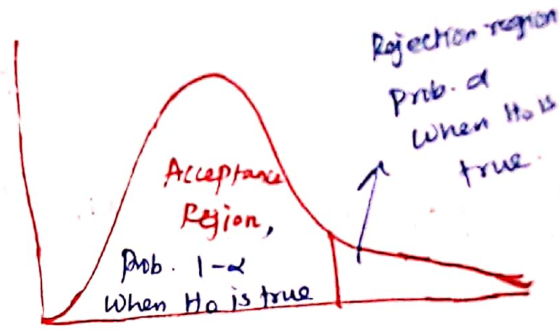
$H_1 : \theta \neq \theta_0$

$\Downarrow$

either $\theta < \theta_0$ or $\theta > \theta_0$

Our construction of the acceptance region guaranteed the desired significance level α.

However, many other regions will also have probability $(1-\alpha)$. Among them, how do we choose the best one?



To avoid type-II errors, we choose such an acceptance region that will unlikely cover the test statistic T in case if the alternative hypothesis H_1 is true. This maximizes the power of our test because we will rarely accept H_0 in this case.

Then, we look at our test statistic under the alternative.

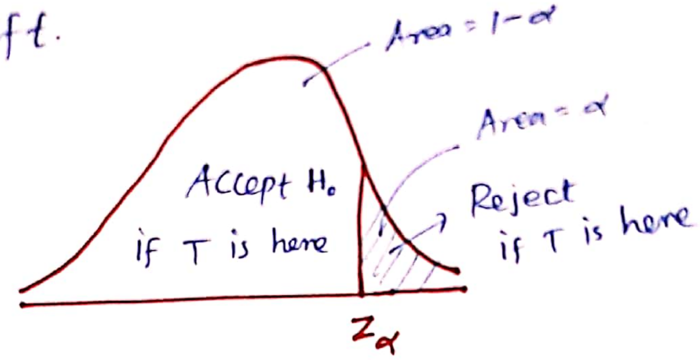
(a) a right tail alternative forces T to be large,

(b) a Left tail alternative forces T to be small

(c) a two sided alternative forces T to be either large or small

If this is the case, it tells us exactly when we should reject the null hypothesis H_0.

(a) For a right-tail alternative, the rejection region R should consists of large values of T. Choose R on the right, A on the left.

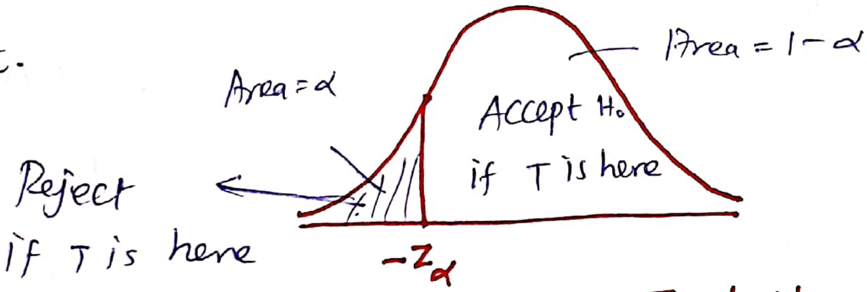


$H_0: \mu = \mu_0$

$H_1: \mu > \mu_0$

Right tail - Z-test.

(b). For a left-tail alternative, the rejection region R should consist of small values of T. Choose R on the left, A on the right.

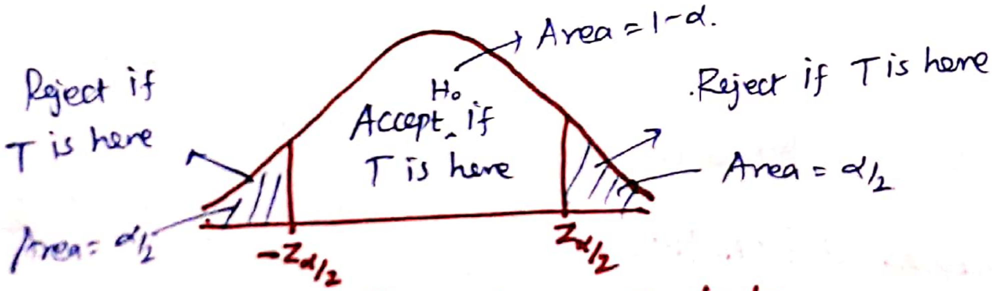


$H_0: \mu = \mu_0$

$H_1: \mu < \mu_0$

Left-tail Z-test

c). For a two sided alternative, the rejection region R should consists of very small & very large values of T. Let R consists of two extreme Regions, while A covers the middle.



$H_0: \mu = \mu_0$

$H_1: \mu \neq \mu_0$

Two sided Z-test.

Z-tests for Mean When Variance σ^2 is known:

An important case, in terms of a large number of applications, is when the sampling distribution of the test statistic is standard normal distribution.

- Sample mean has Normal distribution when the distribution of data is Normal.
- Sample means has approximately Normal distribution when they are computed from large samples (CLT). (Here the distribution of data can be arbitrary)

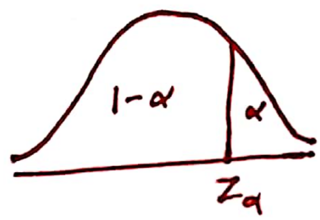
The test in this case is called a Z-test.

Null Hypothesis. $H_0: \mu = \mu_0$

Test Statistic: $\boxed{Z = \dfrac{\overline{X} - \mu_0}{\sigma/\sqrt{n}}}$

(a) A level α test with a right-tail alternative $(H_1: \mu > \mu_0)$

$$\boxed{\begin{array}{l} \text{reject } H_0 \text{ if } Z > z_\alpha \\ \text{accept } H_0 \text{ if } Z \le z_\alpha \end{array}}$$

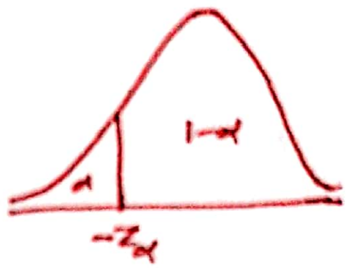


$R = (z_\alpha, +\infty)$ & $A = (-\infty, z_\alpha)$.

We accept H_0 with probability $\Phi(z_\alpha) = 1 - \alpha$, making the probability of false rejection (type-I error) equal α.

(b) With a left-tail alternative ($H_1: \mu < \mu_0$), we

$$\boxed{\begin{array}{lll} \text{reject } H_0 & \text{if} & Z < -z_\alpha \\ \text{accept } H_0 & \text{if} & Z \geq -z_\alpha \end{array}}$$



$R = (-\infty, -z_\alpha)$ & $A = [-z_\alpha, +\infty)$

(c). With a two-sided alternative ($H_1: \mu \neq \mu_0$), we

$$\boxed{\begin{array}{lll} \text{reject } H_0 & \text{if} & |Z| > z_{\alpha/2} \text{ or } (Z > z_{\alpha/2} \text{ or } Z < -z_{\alpha/2}) \\ \text{accept } H_0 & \text{if} & |Z| \leq z_{\alpha/2} \rightarrow (-z_{\alpha/2} \leq Z \leq z_{\alpha/2}) \end{array}}$$

$R = (-\infty, -z_{\alpha/2}) \cup (z_{\alpha/2}, +\infty)$ & $A = [-z_{\alpha/2}, z_{\alpha/2}]$.

This is easy to remember:

- For a two sided test, divide α by two and use $z_{\alpha/2}$
- for a one sided test, use z_α keeping in mind that the rejection region consists of just one piece.

<u>Note:</u> A widely used procedure in hypothesis testing is to use a type-I error or significance level of $\alpha = 0.05$ or $\alpha = 0.01$. These values have evolved through experience and may not be appropriate for all situations.!

(P1) (Z-test about a population mean)

The number of concurrent users for some Internet Service provider has always averaged 5000 with a standard deviation of 800. After an equipment upgrade, the average number of users at 100 randomly selected moments of time is 5200. Does it indicate at a 5% level of significance, that the mean number of concurrent users has increased? Assume that the standard deviation of the number of concurrent users has not changed.

Solution:

$H_0 : \mu = 5000$

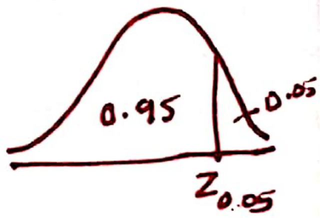
$H_1 : \mu > 5000$ (because, we are only interested to know if the mean number of users μ has increased.

Step: 1 Test Statistic.

Given : $\sigma = 800$, $\mu_0 = 5000$, $n = 100$, $\alpha = 0.05$

and $\bar{x} = 5200$. The test statistic is

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} = \frac{5200 - 5000}{800/\sqrt{100}} = 2.5$$



Step: 2 Acceptance & rejection regions:

The critical value is $z_\alpha = z_{0.05} = 1.645$
$\simeq 1.65$

$\Phi(z_{0.05}) = 0.95$

From Std normal table

With the right tail alternative

reject H_0 if $Z > 1.65$

Accept H_0 if $Z \leq 1.65$

Step: 3 Result:

Our value of the test Statistic $Z = 2.5$ belongs to the rejection region, therefore, we reject the null hypothesis H_0.

Ex. (P_2). In order to ensure efficient usage of a Server, it is necessary to estimate the mean number of concurrent users. According to records, the average number of concurrent users at 100 randomly selected times is 37.7, with a Standard deviation $\sigma = 9.2$. At the 1% significance level, do these data provide significant evidence that the mean number of concurrent users is greater than 35?

(P_3). State the null & allternative hypothesis in each case.

a). A hypothesis test will be used to potentially provide evidence that the population mean is more than 6.

(b). A hypothesis test will be used to potentially provide evidence that (i) the population mean is not equal to 8.

(ii) the population mean is less than 9.

(c)

(P4). A hypothesis will be used to test that a population mean equals 4 against the alternative that the population mean is more than 4 with known variance σ^2. What is the critical value for the test statistic Z for the following significance levels? [Assume Normal Population] Also State H_0 & H_1.

(a) 0.01 (b) 0.05 (c) 0.10

(P5). A hypothesis will be used to test that a population mean equals 8 against the alternative that the population mean is less than 8 with known variance σ^2. What is the critical value for the test statistic Z for the following significance levels (a) 0.01 (b) 0.05 (c) 0.10. [Assume Normal population] Also state the hypothesis H_0 vs H_1.

Large Sample Z-Test:-

We have developed the Z-test for the null hypothesis $H_0: \mu = \mu_0$ assuming that the population is normally distributed and σ^2 is known. In most practical situations σ^2 will be unknown. Furthermore, we may not be certain that the population is well modeled by a normal distribution. In these situations, if n is large ($n \geq 30$), the sample standard deviation s can be substituted for σ in the test procedures with little effect.

When n is large ($n \geq 30$), the test ~~Sta~~ Statistic is $Z = \dfrac{\bar{x} - \mu}{s/\sqrt{n}}$ and by Central Limit theorem it has an approximate Standard normal distribution.

Large Sample Test for mean μ when σ^2 is unknown.

& $n \geq 30$.

Null hypothesis $H_0 : \mu = \mu_0$

Test Statistic $Z = \dfrac{\bar{X} - \mu}{s/\sqrt{n}} \sim N(0,1)$

Alternative Hypothesis	Level α test Large Sample Z-test
(i) $H_1 : \mu > \mu_0$ Right tail Test	reject H_0 if $Z > Z_\alpha$ Accept H_0 if $Z \leq Z_\alpha$
(ii) $H_1 : \mu < \mu_0$ Left tail Test	reject H_0 if $Z < -Z_\alpha$ accept H_0 if $Z \geq -Z_\alpha$
(iii) $H_1 : \mu \neq \mu_0$ Two sided Test.	reject H_0 if $Z > Z_{\alpha/2}$ or $Z < -Z_{\alpha/2}$ Accept H_0 if $-Z_{\alpha/2} \leq Z \leq Z_{\alpha/2}$