#Chairing #Explanation Let's now develop a similar ***Lyapunov function*** to the one we did in [[Lyapunov no Perturbations]], but adding a **perturbator** variable v to $\dot{\mathrm{x}}$:

$$\dot{\mathrm{x}} = A\mathrm{x} + v$$

Now, the **perturbations** can be *internal* or *external.* An internal perturbation is often ***state-dependent***, which means that they are bounded and multiplicate the *state* of the *system*; While external perturbations are often *additive* and ***non-state dependent***.

We can bound our equation to these **perturbations** in two different forms in order to see the max of how these affect the system. #### Multiplicative form In this form, v is **state-dependent**, so it can be bounded as:

$$||v|| \leq v^{+}||\mathrm{x}||,\ v^{+} > 0$$

Then:

$$\begin{aligned} \frac{d}{dt}V(\mathrm{x}) &= \langle P\mathrm{x}, A\mathrm{x} + v\rangle \\ &= \mathrm{x}^{T}P(A\mathrm{x} + v) \\ &= \underbrace{\mathrm{x}^{T}PA\mathrm{x}}_{Nominal} + \underbrace{\mathrm{x}^{T}Pv}_{Perturbations} \end{aligned}$$

We now must analyze the effect of the **perturbative term** to be able to properly analyze the maximum effect the perturbations can have in our system.

We use the [[Cauchy-Schwarz Inequality]] to bound it. As $\mathrm{x}^{T}Pv = \langle (P\mathrm{x})^{T}v\rangle$:

$$|(P\mathrm{x})^{T}v| \leq ||P\mathrm{x}||\ ||v||$$

For a *symmetric matrix*, its ***induced Euclidian norm*** is defined as its largest *eigenvalue.*

$$||P|| = \lambda_{\max}(P)$$

We also note that the *product* of two norms is *equal* or *greater* than the norm of the product.

$$||P\mathrm{x}|| \leq ||P||\ ||\mathrm{x}||$$

Now we can substitute $||v||$ in our ***Cauchy-Schwarz Inequality*** with our initial bound for it.

$$|(P\mathrm{x})^{T}v| \leq ||P||\ ||\mathrm{x}||\ ||v|| \leq ||P||\ ||\mathrm{x}||v^{+}||\mathrm{x}||$$

Considering:

$$\begin{aligned} ||P||\ ||\mathrm{x}||v^{+}||\mathrm{x}|| &= ||P||v^{+}||\mathrm{x}||^{2} \\ &= ||P||v^{+}\mathrm{x}^{T}\mathrm{x} = \mathrm{x}^{T}||P||v^{+}I\mathrm{x} \end{aligned}$$

Given all this, along with the analysis of the **nominal** term done [[Lyapunov no Perturbations|here]], we can bound the whole **Lyapunov function** as:

$$\frac{d}{dt}V(\mathrm{x}) \leq -\frac{1}{2}\mathrm{x}^{T}Q\mathrm{x} + \mathrm{x}^{T}||P||v^{+}I\mathrm{x} < 0$$

$$\frac{d}{dt}V(\mathrm{x}) \leq \mathrm{x}^T\left(-\frac{Q}{2} + ||P||v^+ I\right)\mathrm{x} < 0$$

Additive Form In this form, v is **not state-dependent**, so it can be bounded as:

$$||v|| \leq v^+,\ v^+ > 0$$

Following a similar analysis as before, we get:

$$\frac{d}{dt}V(\mathrm{x}) = \underbrace{\mathrm{x}^T PA\mathrm{x}}_{Nominal} + \underbrace{\mathrm{x}^T Pv}_{Perturbations}$$

We can **bound** the perturbation using the ***[[Peter-Paul Inequality]]*** for $\underbrace{\mathrm{x}^T P}_{x}\underbrace{v}_{y}$:

$$\mathrm{x}^T Pv \leq \frac{\theta}{2}||\mathrm{x}^T P||^2 + \frac{1}{2\theta}||v||^2,\ \forall\theta > 0$$

We can then use this to bound our ***Lyapunov function***:

$$\frac{d}{dt}V(\mathrm{x}) \leq -\frac{1}{2}\mathrm{x}^T Q\mathrm{x} + \frac{\theta}{2}\mathrm{x}^T P^2\mathrm{x} + \frac{1}{2\theta}||v||^2$$

Substituting our bound for v:

$$\leq -\frac{1}{2}\mathrm{x}^T Q\mathrm{x} + \frac{\theta}{2}\mathrm{x}^T P^2\mathrm{x} + \frac{(v^+)^2}{2\theta}$$

$$\leq \frac{1}{2}\mathrm{x}^T(\underbrace{-Q + \theta P^2}_{Q_1})\mathrm{x} + \frac{(v^+)^2}{2\theta}$$

Where θP^2 is the part of Q_1 that compensates for v. In order to achieve this overcoming the perturbations we must say that if:

$$\frac{1}{2}(\underbrace{-Q + \theta P^2}_{Q_1}) \leq 0$$

Then:

$$\frac{d}{dt}V(\mathrm{x}) \leq \frac{1}{2}\mathrm{x}^T Q_1\mathrm{x} + \underbrace{\frac{(v^+)^2}{2\theta}}_{\beta}$$

Here, to make sure that our nominal term overcomes the perturbative one, we can use the ***[[Rayleigh Quotient]]*** to bound $V(\mathrm{x})$:

$$R(P,\mathrm{x}) = \frac{\mathrm{x}^T P\mathrm{x}}{\mathrm{x}^T\mathrm{x}}$$

$$\lambda_{\min}(P) \leq \frac{\mathrm{x}^T P\mathrm{x}}{\mathrm{x}^T\mathrm{x}} \leq \lambda_{\max}(P)$$

$$\lambda_{\min}(P)||\mathrm{x}||^2 \leq \mathrm{x}^T P\mathrm{x} \leq \lambda_{\max}(P)||\mathrm{x}||^2$$

$$\lambda_{\min}(P)||\mathrm{x}||^2 \leq 2V(\mathrm{x}) \leq \lambda_{\max}(P)||\mathrm{x}||^2$$

Similarly, we can bound Q_1:

$$\lambda_{\min}(Q_1)||\mathrm{x}||^2 \leq \mathrm{x}^T Q_1 \mathrm{x} \leq \lambda_{\max}(Q_1)||\mathrm{x}||^2$$

Disregarding the inequality, we can substitute $\mathrm{x}^T Q_1 \mathrm{x}$ into the $\frac{d}{dt}V(\mathrm{x})$ equation:

$$\frac{d}{dt}V(\mathrm{x}) \leq -\frac{1}{2}\lambda_{\min}(Q_1)||\mathrm{x}||^2 + \beta$$

Now we can relate $||\mathrm{x}||^2$ to $V(\mathrm{x})$ from our [[Lyapunov With Perturbations#^1f77a9|bound]] of it:

$$\lambda_{\min}(P)||\mathrm{x}||^2 \leq \mathrm{x}^T P \mathrm{x}$$

$$||\mathrm{x}||^2 \leq \frac{2V(\mathrm{x})}{\lambda_{\min}(P)}$$

Substituting we get:

$$\frac{d}{dt}V(\mathrm{x}) \leq -\underbrace{\frac{\lambda_{\min}(Q_1)}{\lambda_{\min}(P)}}_{\alpha} V(\mathrm{x}) + \beta$$

Thus we get the final equation that provides ***ultimate boundedness*** to our *system*, ensuring trajectories stay near x^{eq}, even if they do not converge to it.

$$\frac{d}{dt}V(\mathrm{x}) \leq -\alpha V(\mathrm{x}) + \beta$$