Chairing #Definition

Basic Matrix Calculus Rule for Quadratic Forms For a scalar function $f(\mathrm{x}) = \mathrm{x}^T Aq$, its *derivative* with respect to x is:

$$\frac{\partial f}{\partial \mathrm{x}} = (A + A^T)\mathrm{x}$$

To prove this, we may expand $f(\mathrm{x})$ in component form as defined in the ***[[Quadratic Norm Representation]]***:

$$f(\mathrm{x}) = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} \mathrm{x}_i \mathrm{x}_j$$

We make the partial derivative $\frac{\partial f}{\partial \mathrm{x}_k}$, where x_k represents the k-th element of x:

$$\frac{\partial f(\mathrm{x})}{\partial \mathrm{x}_k} = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} \frac{\partial}{\partial \mathrm{x}_k} (\mathrm{x}_i \mathrm{x}_j)$$

Using the *product rule*:

$$\frac{\partial f(\mathrm{x})}{\partial \mathrm{x}_k} = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} \left(\frac{\partial \mathrm{x}_i}{\partial \mathrm{x}_k} \mathrm{x}_j + \mathrm{x}_i \frac{\partial \mathrm{x}_j}{\partial \mathrm{x}_k} \right)$$

We notice that each partial derivative equals 1 and is non-zero when the k-th element of x matches i or j (**Kronecked Delta**), so we eliminate the rest of the sum as it only matters when it matches k:

$$\frac{\partial f(\mathrm{x})}{\partial \mathrm{x}_k} = \sum_{j=1}^{n} A_{kj} \mathrm{x}_j + \sum_{i=1}^{n} A_{ik} \mathrm{x}_i$$

As the sum of both j and i go from 1 to n, they are essentially the same, thus getting the form:

$$\frac{\partial f(\mathrm{x})}{\partial \mathrm{x}_k} = (A + A_T)\mathrm{x}$$