Chairing #Explanation

The ***Lyapunov*** function is part of the ***Lyapunov Stability Theory***, which describes a scalar function V that helps determine the stability of an equilibrium point.

Consider a system:

$$\mathrm{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\dot{\mathrm{x}} = A\mathrm{x}$$

To propose a ***Lyapunov function***, we need it to fulfill these conditions: . a) $V(\mathrm{x}) > 0, \forall \mathrm{x} \in \mathrm{X}, \mathrm{X} \to R^+$ b) $V(\mathrm{x}^{eq}) = 0$ c) $\frac{d}{dt}V(\mathrm{x}) < 0$ for all X except x^{eq} . A common proposal for $V(\mathrm{x})$ is:

$$V(\mathrm{x}) = \frac{1}{2}||\mathrm{x}||_P^2$$

Where $||\mathrm{x}||_P^2$ representes the **quadratic** form associated with the positive definite matrix P. We can decompose the **quadratic Euclidean norm** into its ***[[Quadratic Norm Representation|matrix notation]]*** as follows:

$$V(\mathrm{x}) = \frac{1}{2}\mathrm{x}^T P\mathrm{x}, P \neq 0$$

a) $V(\mathrm{x}) > 0, \forall \mathrm{x} \in \mathrm{X}, \mathrm{X} \to R^+$ From the definition of the ***[[Positive Definite Matrix]]***, we know that for this to be achieved, we need P to be a positive definite matrix.

Only if $P > 0$

b) $V(\mathrm{x}^{eq}) = 0$ Similarly, if the **equilibrium point** is 0, then its simple to see that this condition is achieved.

$$V(\mathrm{x}^{eq}) = \frac{1}{2}(\mathrm{x}^{eq})^T P\mathrm{x}^{eq} = 0$$

If the **equilibrium point** is different than zero, then:

c) $\frac{d}{dt}V(\mathrm{x}) < 0$ **for all** X **except** x^{eq} Because of the *chain rule*, we get the *time derivative* of $V(\mathrm{x})$ as:

$$\frac{d}{dt}V(\mathrm{x}) = \langle \nabla_x V(\mathrm{x}), \frac{d}{dt}\mathrm{x} \rangle$$

From ***[[Matrix Calculus Derivative Rules]]***, we get that the derivative or *gradient* of $V(\mathrm{x})$ is $P\mathrm{x}$, so the inner product becomes:

$$\frac{d}{dt}V(\mathrm{x}) = \langle P\mathrm{x}, A\mathrm{x} \rangle$$

Considering both $P\mathrm{x}$ & $A\mathrm{x}$ are 2×1 vectors, we need to transpose the first one to be able to apply the inner product, basically:

$$\frac{d}{dt}V(\mathrm{x}) = (P\mathrm{x}) \cdot (A\mathrm{x}) = (P\mathrm{x})^T(A\mathrm{x})$$

Considering that:

$$(P\mathrm{x})^T = \mathrm{x}^T P^T$$

$$\frac{d}{dt}V(\mathrm{x}) = \mathrm{x}^T P^T A\mathrm{x} = 0.5\mathrm{x}^T P^T A\mathrm{x} + 0.5\mathrm{x}^T P^T A\mathrm{x}$$

Because $\mathrm{x}^T P^T A\mathrm{x}$ is a **scalar**, its transpose is equal to itself, so:

$$(\mathrm{x}^T P^T A\mathrm{x})^T = \mathrm{x}^T A^T P\mathrm{x}$$

Then we can do:

$$\frac{d}{dt}V(\mathrm{x}) = 0.5\mathrm{x}^T P^T A\mathrm{x} + 0.5\mathrm{x}^T A^T P\mathrm{x}$$

$$\frac{d}{dt}V(\mathrm{x}) = 0.5\mathrm{x}^T \underbrace{(P^T A + A^T P)}_{-Q} \mathrm{x}$$

So we arrive at the conclusion that Q must be symmetric and **greater** than 0 to guarantee ***real eigenvalues***.