Direct kinematics: open kinematic chain robot manipulator

$$\varsigma = DK\left(q\right)$$

Pose of the end-effector

$$\varsigma = \begin{bmatrix} x \\ y \\ z \\ \varphi \\ \theta \\ \psi \end{bmatrix}$$

The velocity of the end-effector is

$$\begin{aligned} \frac{d}{dt}\varsigma(t) &= \frac{d}{dt}DK\left(q(t)\right) \\ &= \left[\nabla_q DK\left(q(t)\right)\right]\frac{d}{dt}q(t) \\ &= J(q(t))\frac{d}{dt}q(t) \end{aligned}$$

The acceleration of the end-effector is

$$\begin{aligned} \frac{d^2}{dt^2}\varsigma(t) &= \frac{d}{dt}\left(J(q(t))\frac{d}{dt}q(t)\right) \\ &= \left[\frac{d}{dt}J(q(t))\right]\frac{d}{dt}q(t) + J(q(t))\frac{d^2}{dt^2}q(t) \end{aligned}$$

Using the Euler-Lagrange formulation

$$\begin{aligned} \frac{d^2}{dt^2}\varsigma(t) &= \left[\frac{d}{dt}J(q(t))\right]\frac{d}{dt}q(t) + J(q(t))\left(M^{-1}\left(q\right)\left(\tau(t) - C(q(t),\frac{d}{dt}q(t))\frac{d}{dt}q(t) - G(q(t))\right)\right) \\ &= f(q(t),\frac{d}{dt}q(t)) + g(q(t))\tau(t) \end{aligned}$$

Problem control

Exists a reference pose trajectory $\varsigma^*(t)$ which is valid in a time range $t \in [t_k, t_{k+1})$

# 1 Lagrangian mechanics with motion restrictions

The Lagrangian function $L(\dot{q}, q) := K(\dot{q}, q) - V(q)$

According to the Euler-Lagrange theory. the dynamic motion of a mechanical system satisfies

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}L(\dot{q}, q) - \frac{\partial}{\partial q_j}L(\dot{q}, q) = \phi_{j,no-pot}(\dot{q}, q, t)$$

subjected to

$$f_k(q) < 0$$

Example:

a)

$$q_1 < \frac{\pi}{2}$$

then

$$f_1(q) = q_1 - \frac{\pi}{2}$$

b)

$$q_2 \leq \frac{\pi}{4}$$

We can obatin two equivalent conditions

$$q_2 - \frac{\pi}{4} \leq 0$$

$$q_2 - \frac{\pi}{4} + \mu < 0 \qquad \text{with } \mu > 0$$

The total variational of the proposed mechanical system is

$$V(t_1, t_2) = \int_{t_1}^{t_2} \sum_{j=1}^{n} \left( \frac{d}{dt} \frac{\partial}{\partial \dot{q}_j} L(\dot{q}, q) - \frac{\partial}{\partial q_j} L(\dot{q}, q) \right)$$

When the coordinates $q_j$ has restrictions, we need to use the Lagrange theorem that defines the Lagrange multipliers

$$L^{'}(\dot{q}, q) = L(\dot{q}, q) + \sum_{k=1}^{s} \lambda_k(t) f_k(q)$$

Then, using the Euler-Lagrange equations on $L^{'}(\dot{q}, q)$, we have

$$\frac{d}{dt} \frac{\partial}{\partial \dot{q}_j} L^{'}(\dot{q}, q, t) - \frac{\partial}{\partial q_j} L^{'}(\dot{q}, q, t) = \phi_{j,no-pot}(\dot{q}, q, t)$$

Then substituing

$$\frac{d}{dt} \frac{\partial}{\partial \dot{q}_j} \left( L(\dot{q}, q) + \sum_{k=1}^{s} \lambda_k(t) f_k(q) \right) - \frac{\partial}{\partial q_j} \left( L(\dot{q}, q) + \sum_{k=1}^{s} \lambda_k(t) f_k(q) \right) = \phi_{j,no-pot}(\dot{q}, q, t)$$

Hence

$$\frac{d}{dt} \frac{\partial}{\partial \dot{q}_j} L(\dot{q}, q) - \frac{\partial}{\partial q_j} L(\dot{q}, q) - \frac{\partial}{\partial q_j} \sum_{k=1}^{s} \lambda_k(t) f_k(q) = \phi_{j,no-pot}(\dot{q}, q, t)$$

$$\frac{d}{dt} \frac{\partial}{\partial \dot{q}_j} L(\dot{q}, q) - \frac{\partial}{\partial q_j} L(\dot{q}, q) = \phi_{j,no-pot}(\dot{q}, q, t) + \sum_{k=1}^{s} \lambda_k(t) \frac{\partial}{\partial q_j} f_k(q)$$

Noticing that

$$\begin{aligned} L(\dot{q},q) &= T(\dot{q},q) - V(q) \\ T(\dot{q},q) &= \frac{1}{2}\dot{q}^\intercal B\dot{q} \\ V(q) &= H(q) \end{aligned}$$

Hence

$$L(\dot{q},q) = \frac{1}{2}\dot{q}^\intercal B\dot{q} - H(q)$$

And the set of equations EL

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}L(\dot{q},q) - \frac{\partial}{\partial q_j}L(\dot{q},q) = \phi_{j,no-pot}(\dot{q},q,t) + \sum_{k=1}^{s}\lambda_k(t)\frac{\partial}{\partial q_j}f_k(q),\ j=1,...n$$

can be written in its equivalent vector form

$$\begin{gathered} \tfrac{d}{dt}\tfrac{\partial}{\partial \dot{q}_1}L(\dot{q},q) - \tfrac{\partial}{\partial q_1}L(\dot{q},q) = \phi_{1,no-pot}(\dot{q},q,t) + \textstyle\sum_{k=1}^{s}\lambda_k(t)\tfrac{\partial}{\partial q_1}f_k(q) \\ \vdots \\ \tfrac{d}{dt}\tfrac{\partial}{\partial \dot{q}_n}L(\dot{q},q) - \tfrac{\partial}{\partial q_n}L(\dot{q},q) = \phi_{n,no-pot}(\dot{q},q,t) + \textstyle\sum_{k=1}^{s}\lambda_k(t)\tfrac{\partial}{\partial q_n}f_k(q) \end{gathered}$$

This is equivalent to

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} \frac{\partial}{\partial \dot{q}_1}L(\dot{q},q) \\ \vdots \\ \frac{\partial}{\partial \dot{q}_n}L(\dot{q},q) \end{bmatrix} - \begin{bmatrix} \frac{\partial}{\partial q_1}L(\dot{q},q) \\ \vdots \\ \frac{\partial}{\partial q_n}L(\dot{q},q) \end{bmatrix} &= \begin{bmatrix} \phi_{1,no-pot}(\dot{q},q,t) \\ \vdots \\ \phi_{n,no-pot}(\dot{q},q,t) \end{bmatrix} + \begin{bmatrix} \sum_{k=1}^{s}\lambda_k(t)\frac{\partial}{\partial q_1}f_k(q) \\ \vdots \\ \sum_{k=1}^{s}\lambda_k(t)\frac{\partial}{\partial q_n}f_k(q) \end{bmatrix} \\ \frac{d}{dt}\nabla_{\dot{q}}L(\dot{q},q) - \nabla_q L(\dot{q},q) &= \phi_{no-pot}(\dot{q},q,t) + (\lambda(t), f(q)) \\ (\lambda(t), f(q)) \quad : &= \begin{bmatrix} \sum_{k=1}^{s}\lambda_k(t)\frac{\partial}{\partial q_1}f_k(q) \\ \vdots \\ \sum_{k=1}^{s}\lambda_k(t)\frac{\partial}{\partial q_n}f_k(q) \end{bmatrix} \end{aligned}$$

Considering $L(\dot{q},q) = \frac{1}{2}\dot{q}^\intercal B(q)\dot{q} - H(q)$

$$\frac{d}{dt}\nabla_{\dot{q}}\left(\frac{1}{2}\dot{q}^\intercal B(q)\dot{q} - H(q)\right) - \nabla_q\left(\frac{1}{2}\dot{q}^\intercal B(q)\dot{q} - H(q)\right) = \phi_{no-pot}(\dot{q},q,t) + (\lambda(t), f(q))$$

$$\frac{d}{dt}\nabla_{\dot{q}}\left(\frac{1}{2}\dot{q}^\intercal B\dot{q}\right) - \nabla_q\left(\frac{1}{2}\dot{q}^\intercal B(q)\dot{q} - H(q)\right) = \phi_{no-pot}(\dot{q},q,t) + (\lambda(t), f(q))$$

$$\nabla_{\dot{q}}\left(\frac{1}{2}\dot{q}^\intercal B\dot{q}\right) = B\dot{q}$$

$$\nabla_q\left(\frac{1}{2}\dot{q}^\intercal B(q)\dot{q}\right) = \frac{1}{2}\dot{q}^\intercal\left[\nabla_q B(q)\right]\dot{q}$$

$$\nabla_q H(q)$$

Subsituing

$$\frac{d}{dt}\left(B\left(q\right)\dot{q}\right) - \left(\frac{1}{2}\dot{q}^{\intercal}\left[\nabla_q B\left(q\right)\right]\dot{q}\right) + \nabla_q H(q) = \phi_{no-pot}(\dot{q}, q, t) + (\lambda(t), f(q))$$

$$\left(\frac{d}{dt}B\left(q\right)\right)\dot{q} + B\left(q\right)\ddot{q} - \left(\frac{1}{2}\dot{q}^{\intercal}\left[\nabla_q B\left(q\right)\right]\dot{q}\right) + \nabla_q H(q) = \phi_{no-pot}(\dot{q}, q, t) + (\lambda(t), f(q))$$

$$B\left(q\right)\ddot{q} + \underbrace{\left[\frac{d}{dt}B\left(q\right) - \frac{1}{2}\dot{q}^{\intercal}\left[\nabla_q B\left(q\right)\right]\right]}_{C(\dot{q},q)}\dot{q} + \underbrace{\nabla_q H(q)}_{G(q)} = \phi_{no-pot}(\dot{q}, q, t) + (\lambda(t), f(q))$$

Hence

$$B\left(q\right)\ddot{q} + C(\dot{q}, q)\dot{q} + G(q) = \phi_{no-pot}(\dot{q}, q, t) + (\lambda(t), f(q))$$

Considering the dissipative and XXXXXX

$$B\left(q\right)\ddot{q} + C(\dot{q}, q)\dot{q} + G(q) = \Omega\tau + \phi_{diss}(\dot{q}, q, t) + (\lambda(t), f(q))$$

Using the state variable theory, we propose

$$\begin{aligned} q_a &= q \\ q_b &= \dot{q} \end{aligned}$$

The dynamics of the proposed state variable is

$$\begin{aligned} \dot{q}_a &= \dot{q} = q_b \\ \dot{q}_b &= \ddot{q} = B^{-1}\left(q_a\right)\left[\Omega\tau + \phi_{diss}(q_b, q_a, t) + (\lambda(t), f(q_a)) - C(q_b, q_a)q_b - G(q_a)\right] \\ &= f(q_a, q_b) + g(q_a)\left(\Omega\tau + (\lambda(t), f(q_a))\right) \end{aligned}$$

where

$$\begin{aligned} g(q_a) &= B^{-1}\left(q_a\right) \\ f(q_a, q_b) &= -B^{-1}\left(q_a\right)\left(C(q_b, q_a)q_b + G(q_a)\right) \end{aligned}$$

Proposal 1.

$$f(q_a, q_b) + g(q_a)\left(\Omega\tau + (\lambda(t), f(q_a))\right) = -K_p q_a - K_D q_b$$

$$\begin{aligned} \dot{q}_a &= q_b \\ \dot{q}_b &= -K_p q_a - K_D q_b \end{aligned}$$

$$
\begin{aligned}
\frac{d}{dt}q &= \frac{d}{dt}\begin{bmatrix} q_a \\ q_b \end{bmatrix} = \begin{bmatrix} q_b \\ -K_p q_a - K_D q_b \end{bmatrix} \\
&= \begin{bmatrix} 0_{n\times n} & I_{n\times n} \\ -K_p & -K_D \end{bmatrix}\begin{bmatrix} q_a \\ q_b \end{bmatrix} \\
\frac{d}{dt}q &= Aq
\end{aligned}
$$

In the case of matrix $A$ is Hurwitz (all eigenvalues belong to $\mathbb{C}^-$), then

$$q(t) = e^{At}q(0)$$

Proposal 2.

Consider the case when

$$
\begin{aligned}
\dot{q}_a &= q_b \\
\dot{q}_b &= f(q_a, q_b) + g(q_a)\left(\Omega\tau + (\lambda(t), f(q_a))\right) + \psi(q_a, q_b, t)
\end{aligned}
$$

where $\psi(q_a, q_b, \cdot)$ represents the non-modelled sections of the robot and $\psi(\cdot, \cdot, t)$ represents the effect of all external perturbations.

The admissible class of perturbations/uncertainties

$$\psi(q_a, q_b, t) \in \Psi$$

$$
\begin{aligned}
\Psi &= \left\{\psi : Q_a \times TQ_a \times \mathbb{R}^+ \to \mathbb{R}^n \mid \|\psi\|^2 \leq \psi_0 + \psi_1 \|q_a\|^2 + \psi_2 \|q_b\|^2\right\} \\
q_a &\in Q_a \subset \mathbb{R}^n
\end{aligned}
$$

where $TQ_a$ is the tangential set of $Q_a$.

$$
\begin{gathered}
f(q_a, q_b) + g(q_a)\left(\Omega\tau + (\lambda(t), f(q_a))\right) = -K_p q_a - K_D q_b \\
\tau = \Omega^{-1}\left(g^{-1}(q_a)\left(-K_p q_a - K_D q_b - f(q_a, q_b)\right)\right) - (\lambda(t), f(q_a))
\end{gathered}
$$

Substituing this control in the dynamics of the robot under analysis:

$$
\begin{aligned}
\dot{q}_a &= q_b \\
\dot{q}_b &= f(q_a, q_b) + g(q_a)\left(\Omega\underbrace{\left[\Omega^{-1}\left(g^{-1}(q_a)\left(-K_p q_a - K_D q_b - f(q_a, q_b)\right)\right)\right] - (\lambda(t), f(q_a))}_{\tau} + (\lambda(t), f(q_a))\right) + \psi(
\end{aligned}
$$

Which is equivalent to

$$
\begin{aligned}
\dot{q}_a &= q_b \\
\dot{q}_b &= -K_p q_a - K_D q_b + \psi(q_a, q_b, t)
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dt}q &= \frac{d}{dt}\begin{bmatrix} q_a \\ q_b \end{bmatrix} = \begin{bmatrix} q_b \\ -K_p q_a - K_D q_b \end{bmatrix} + \begin{bmatrix} 0 \\ I_n \end{bmatrix}\psi(q_a, q_b, t) \\
&= \begin{bmatrix} 0_{n\times n} & I_{n\times n} \\ -K_p & -K_D \end{bmatrix}\begin{bmatrix} q_a \\ q_b \end{bmatrix} + \begin{bmatrix} 0 \\ I_n \end{bmatrix}\psi(q_a, q_b, t) \\
\frac{d}{dt}q &= Aq + B\psi(q_a, q_b, t) \\
B &= \begin{bmatrix} 0 \\ I_n \end{bmatrix}
\end{aligned}
$$

Considering that

$$
q(t) = e^{At}q(0) + \int\limits_{\tau=0}^{t} e^{A(t-\tau)}B\psi(q_a(\tau), q_b(\tau), \tau)d\tau
$$

Calculating the norm of both side

$$
\begin{aligned}
\|q(t)\|^2 &= \left\| e^{At}q(0) + \int\limits_{\tau=0}^{t} e^{A(t-\tau)}B\psi(q_a(\tau), q_b(\tau), \tau)d\tau \right\|^2 \\
&\leq (2+\varepsilon)\left\|e^{At}q(0)\right\|^2 + (2+\varepsilon^{-1})\left\| \int\limits_{\tau=0}^{t} e^{A(t-\tau)}B\psi(q_a(\tau), q_b(\tau), \tau)d\tau \right\|^2 \\
&\leq (2+\varepsilon)\left\|e^{At}q(0)\right\|^2 + (2+\varepsilon^{-1})\int\limits_{\tau=0}^{t} \left\|e^{A(t-\tau)}B\psi(q_a(\tau), q_b(\tau), \tau)\right\|^2 d\tau \\
&\leq (2+\varepsilon)\left\|e^{At}q(0)\right\|^2 + (2+\varepsilon^{-1})\int\limits_{\tau=0}^{t} \left\|e^{A(t-\tau)}B\right\|_M^2 \|\psi(q_a(\tau), q_b(\tau), \tau)\|^2 d\tau \\
&\leq (2+\varepsilon)\left\|e^{At}q(0)\right\|^2 + (2+\varepsilon^{-1})\int\limits_{\tau=0}^{t} \left\|e^{A(t-\tau)}B\right\|_M^2 \left(\psi_0 + \psi_1\|q_a\|^2 + \psi_2\|q_b\|^2\right) d\tau
\end{aligned}
$$

$$
\begin{aligned}
\|q(t)\|^2 &\leq (2+\varepsilon)\left\|e^{At}q(0)\right\|^2 + (2+\varepsilon^{-1})\int\limits_{\tau=0}^{t} \left\|e^{A(t-\tau)}B\right\|_M^2 \left(\psi_0 + \psi_1\|q_a\|^2 + \psi_2\|q_b\|^2\right) d\tau \\
&= (2+\varepsilon)\left\|e^{At}q(0)\right\|^2 + (2+\varepsilon^{-1})\int\limits_{\tau=0}^{t} \left\|e^{A(t-\tau)}B\right\|_M^2 d\tau\psi_0 + \\
&\quad (2+\varepsilon^{-1})\int\limits_{\tau=0}^{t} \left\|e^{A(t-\tau)}B\right\|_M^2 \left(\psi_1\|q_a\|^2 + \psi_2\|q_b\|^2\right) d\tau
\end{aligned}
$$

## 2 Stability theory in the Lyapunov sense

Consider a dynamic system represented by a ordinary differential equation

$$\frac{d}{dt}q(t) = f(q(t)), \quad q(0) = q_0. \quad q \in \mathbb{R}^n$$

We define a unique equilibrium point $q^{eq}$ to the state when

$$f(q^{eq}(t)) = 0$$

We call $q^{eq}(t)$ a stable equilibrium point

$$\|q(t) - q^{eq}\| \leq \alpha, \quad \forall t \geq 0$$

We call $q^{eq}(t)$ an asymptotically stable equilibrium point

$$\lim_{t\to\infty} \|q(t) - q^{eq}\| = 0$$

We call $q^{eq}(t)$ an exponential asymptotically stable equilibrium point

$$\|q(t) - q^{eq}\| \leq ae^{-\beta t}$$

Theorem. If exists a positive definite function $V : Q \to \mathbb{R}^+$ such that the following conditions fulfill
a) $V(q) \geq 0$ for all $q \in Q$
b) $V(q^{eq}) = 0$
c) $V(q)$ is decreasing in all points in $Q$ except in $q^{eq}$
then $q^{eq}$ is asymptotically stable equilibrium point.
Additionally, if
d) $V(q)$ is radially unbounded, and $q \in Q = \mathbb{R}^n$, then $q^{eq}$ is a global asymptotically stable equilibrium point.

In the case that $V(q)$ is a differentiable, then property c is equivalent to

$$\frac{d}{dt}V(q) = \left(\nabla_q V(q), \frac{d}{dt}q\right) < 0$$

Example.

$$\dot{q} = Aq$$

We want to find a Lyapunov function that characterizes the nature of the equilibrium point $q^{eq} = 0$

Lets propose

$$V = \frac{1}{2}\|q\|_P^2 = \frac{1}{2}q^\intercal Pq$$

$P \neq 0$

a) $V(q) \geq 0$ for all $q \in Q$

$$V = \frac{1}{2} q^{\intercal} P q$$

Weighted ($P$) quadratic form. Hence, $V(q) \geq 0$ if and only if $P > 0$.

b) b) $V(q^{eq}) = 0$

$$V(q^{eq}) = \frac{1}{2} \|q^{eq}\|_P^2 = \frac{1}{2} (q^{eq})^{\intercal} P q^{eq}$$

$$\begin{aligned}
\frac{d}{dt} V(q^{eq}) &= \left( \nabla_q V(q), \frac{d}{dt} q \right) \\
&= (Pq, Aq) \\
&= 0.5 (Pq, Aq) + 0.5 (Pq, Aq) \\
&= 0.5 q^{\intercal} P^{\intercal} A q + 0.5 q^{\intercal} P^{\intercal} A q \\
&= 0.5 q^{\intercal} P^{\intercal} A q + 0.5 (q^{\intercal} A^{\intercal} P q) \\
&= 0.5 q^{\intercal} (P^{\intercal} A + A^{\intercal} P) q
\end{aligned}$$

If in addition, we ask $P = P^{\intercal}$ (symmetric matrix)

Example.

$$\dot{q} = Aq + v$$

a)

$$\|v\| \leq v^{+} \|q\|$$

$$V(q) = \frac{1}{2} \|q\|_P^2 = \frac{1}{2} q^{\intercal} P q$$

The time derivative of $V(q)$ satisfies

$$\begin{aligned}
\frac{d}{dt} V(q) &= q^{\intercal} P \frac{d}{dt} q \\
&= q^{\intercal} P (Aq + v) \\
&= q^{\intercal} P A q + q^{\intercal} P v
\end{aligned}$$

C1. Multiplicative form

$$\begin{aligned}
q^{\intercal} P v &\leq \|Pq\| \|v\| \leq v^{+} \|P\|_F \|q\|^2 \\
&\leq v^{+} \|P\|_F q^{\intercal} q \\
&= q^{\intercal} \left( v^{+} \|P\| I_n \right) q
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{d}{dt} V(q) &\leq \frac{1}{2} q^{\intercal} \left( PA + A^{\top} P + Q_0 \right) q \\
Q_0 &= 2 v^{+} \|P\| I_n
\end{aligned}$$

Then if

$$PA + A^\top P + Q_0 \leq -Q, \qquad Q > 0$$

results in

$$\frac{d}{dt}V(q) \leq -\frac{1}{2}q^\intercal Q q \leq 0$$

C2. Additive form

$$\|v\| \leq v^+$$

$$\begin{aligned}
\frac{d}{dt}V(q) &= q^\intercal P \frac{d}{dt}q \\
&= q^\intercal P\left(Aq + v\right) \\
&= q^\intercal PAq + q^\intercal Pv
\end{aligned}$$

Using the Peter-Paul inequality

$$x^\top y \leq (1+\theta)\left\|x\right\|^2 + \left(1+\theta^{-1}\right)\left\|y\right\|^2, \quad x, y \in \mathbb{R}^n, \theta \in \mathbb{R}^+$$

Then

$$\begin{aligned}
\underbrace{q^\intercal P}_{x^\intercal}\underbrace{v}_{y} &\leq (1+\theta)\left\|Pq\right\|^2 + \left(1+\theta^{-1}\right)\left\|v\right\|^2 \\
&= q^\top P((1+\theta)I_n)Pq + \left(1+\theta^{-1}\right)\left(v^+\right)^2
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{d}{dt}V(q) &\leq \frac{1}{2}q^\intercal\left(PA + A^\top P\right)q + q^\top P((1+\theta)I_n)Pq + \left(1+\theta^{-1}\right)\left(v^+\right)^2 \\
&\leq \frac{1}{2}q^\intercal\left(PA + A^\top P + P((1+\theta)I_n)P\right)q + \left(1+\theta^{-1}\right)\left(v^+\right)^2
\end{aligned}$$

If

$$\frac{1}{2}\left(PA + A^\top P + P((1+\theta)I_n)P\right) \leq -Q$$

$$\frac{1}{2}\left(PA + A^\top P + P((1+\theta)I_n)P\right) + Q \leq 0$$

Then

$$\begin{aligned}
\frac{d}{dt}V(q) &\leq \frac{1}{2}q^\intercal\left(PA + A^\top P + P((1+\theta)I_n)P\right)q + \left(1+\theta^{-1}\right)\left(v^+\right)^2 \\
&\leq -\frac{1}{2}q^\intercal Q q + \left(1+\theta^{-1}\right)\left(v^+\right)^2
\end{aligned}$$

In Summary

$$\begin{aligned}
\frac{d}{dt}V(q) &\leq -\frac{1}{2}q^\intercal P^{1/2}P^{-1/2}QP^{-1/2}P^{1/2}q + (1+\theta^{-1})\left(v^+\right)^2 \\
&\leq \underbrace{-\frac{1}{2}\lambda_{\min}\left(P^{-1/2}QP^{-1/2}\right)}_{\alpha}q^\intercal P^{1/2}P^{1/2}q + \underbrace{(1+\theta^{-1})\left(v^+\right)^2}_{\beta} \\
&\leq -\alpha q^\intercal Pq + \beta \\
&\leq -\alpha V(q) + \beta
\end{aligned}$$

which has the following solution

$$V(t) \leq V(0)e^{-\alpha t} + \frac{\beta}{\alpha}\left(1 - e^{-\alpha t}\right)$$

which was obtained using the rayleigh inequality

$$\lambda_{\min}P \cdot \|x\| \leq x^\intercal Px \leq \lambda_{\max}P \cdot \|x\|$$

Barrier Lyapunov function

$$\frac{d^2}{dt^2}q = M^{-1}\left(-C(q, \frac{d}{dt}q)\frac{d}{dt}q - G(q)\right) + M^{-1}u + \eta$$

$$u = MK_pq + MK_D\frac{d}{dt}q + C(q, \frac{d}{dt}q)\frac{d}{dt}q + G(q)$$

Then

$$\frac{d^2}{dt^2}q = K_pq + K_D\frac{d}{dt}q + \eta$$

$$x = \begin{bmatrix} q \\ \frac{d}{dt}q \end{bmatrix}$$

$$\begin{aligned}
\frac{d}{dt}x &= \begin{bmatrix} 0 & I \\ K_p & K_D \end{bmatrix} x + \begin{bmatrix} 0 \\ I \end{bmatrix} \eta \\
&= Ax + B\eta
\end{aligned}$$

If

$$\|x\|_P^2 < x^+$$

then

$$\begin{aligned}
V_B &= \ln\left(\frac{x^+}{x^+ - \|x\|_P^2}\right) + \lambda_1(t)tr\left\{\tilde{K}_P\tilde{K}_P^\intercal\right\} + \lambda_2(t)tr\left\{\tilde{K}_D\tilde{K}_D^\intercal\right\} \\
\lambda_i(t) &= \lambda_{i,0}\left(\frac{x^+ - \|x\|_P^2}{x^+}\right)^r, \quad r > 0, \quad i = 1, 2 \\
\tilde{K}_P &= K_P - K_P^0 \\
\tilde{K}_D &= K_D - K_D^0
\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}V_B &= \frac{2x^\intercal P\frac{d}{dt}x}{x^+ - \|x\|_P^2} + \lambda_1(t)tr\left\{\left(\frac{d}{dt}\tilde{K}_P\right)\tilde{K}_P^\intercal\right\} + \lambda_2(t)tr\left\{\left(\frac{d}{dt}\tilde{K}_D\right)\tilde{K}_D^\intercal\right\} \\ &\quad + \frac{d}{dt}\lambda_1(t)tr\left\{\tilde{K}_P\tilde{K}_P^\intercal\right\} + \frac{d}{dt}\lambda_2(t)tr\left\{\tilde{K}_D\tilde{K}_D^\intercal\right\}\end{aligned}$$

$$\frac{d}{dt}V_B = \frac{2x^\intercal P\left(Ax+B\eta\right)}{x^+ - \|x\|_P^2} + tr\left\{\left(\lambda_1(t)\frac{d}{dt}\tilde{K}_P + \frac{d}{dt}\lambda_1(t)\tilde{K}_P\right)\tilde{K}_P^\intercal\right\} + tr\left\{\left(\lambda_2(t)\frac{d}{dt}\tilde{K}_D + \frac{d}{dt}\lambda_2(t)\tilde{K}_D\right)\tilde{K}_D^\intercal\right\}$$

Analyzing the term with

$$\begin{aligned}2x^\intercal P\left(Ax+B\eta\right) &= x^\intercal\left(P\begin{bmatrix}0 & I \\ K_P - K_P^0 + K_P^0 & K_D - K_D^0 + K_D^0\end{bmatrix} + \begin{bmatrix}0 & I \\ K_P - K_P^0 + K_P^0 & K_D - K_D^0 + K_D^0\end{bmatrix}\right. \\ &\quad 2x^\intercal PB\eta + \varepsilon\eta^\intercal\eta - \varepsilon\eta^\intercal\eta + x^\intercal Qx - x^\intercal Qx \\ &= \begin{bmatrix}x \\ \eta\end{bmatrix}^\intercal\begin{bmatrix}PA + A^\intercal P + Q & PB \\ B^\intercal P & -\varepsilon I\end{bmatrix}\begin{bmatrix}x \\ \eta\end{bmatrix} + \varepsilon\eta^\intercal\eta - x^\intercal Qx\end{aligned}$$

In view of

$$\begin{aligned}\begin{bmatrix}0 & I \\ K_P - K_P^0 + K_P^0 & K_D - K_D^0 + K_D^0\end{bmatrix} &= \begin{bmatrix}0 & I \\ K_P^0 & K_D^0\end{bmatrix} + \begin{bmatrix}0 & 0 \\ K_P - K_P^0 & K_D - K_D^0\end{bmatrix} \\ &= \begin{bmatrix}0 & I \\ K_P^0 & K_D^0\end{bmatrix} + \begin{bmatrix}0 \\ I\end{bmatrix}\begin{bmatrix}K_P - K_P^0 & K_D - K_D^0\end{bmatrix} \\ &= A^0 + \begin{bmatrix}0 \\ I\end{bmatrix}\begin{bmatrix}\tilde{K}_P & \tilde{K}_D\end{bmatrix}\end{aligned}$$

Recalling

$$\begin{aligned}2x^\intercal P\left(Ax+B\eta\right) &= x^\intercal\left(P\left(A^0 + \begin{bmatrix}0 \\ I\end{bmatrix}\begin{bmatrix}\tilde{K}_P & \tilde{K}_D\end{bmatrix}\right) + \left(A^0 + \begin{bmatrix}0 \\ I\end{bmatrix}\begin{bmatrix}\tilde{K}_P & \tilde{K}_D\end{bmatrix}\right)^\intercal P\right)x + \\ &\quad 2x^\intercal PB\eta + \varepsilon\eta^\intercal\eta - \varepsilon\eta^\intercal\eta + x^\intercal Qx - x^\intercal Qx \\ &= \begin{bmatrix}x \\ \eta\end{bmatrix}^\intercal\begin{bmatrix}PA^0 + \left(A^0\right)^\intercal P + Q & PB \\ B^\intercal P & -\varepsilon I\end{bmatrix}\begin{bmatrix}x \\ \eta\end{bmatrix} + \varepsilon\eta^\intercal\eta - x^\intercal Qx + 2x^\intercal P\begin{bmatrix}0 \\ I\end{bmatrix}\begin{bmatrix}\tilde{K}_P & \tilde{K}_D\end{bmatrix}\end{aligned}$$

If $\begin{bmatrix}PA^0 + \left(A^0\right)^\intercal P + Q & PB \\ B^\intercal P & -\varepsilon I\end{bmatrix} \leq 0$, then

$$2x^\intercal P\left(Ax+B\eta\right) \leq \varepsilon\eta^\intercal\eta - x^\intercal Qx + 2x^\intercal P\begin{bmatrix}0 \\ I\end{bmatrix}\begin{bmatrix}\tilde{K}_P & \tilde{K}_D\end{bmatrix}x$$

$$\frac{d}{dt}V_B = \frac{\varepsilon\eta^\intercal\eta - x^\intercal Qx + 2x^\intercal P\begin{bmatrix}0 \\ I\end{bmatrix}\begin{bmatrix}\tilde{K}_P & \tilde{K}_D\end{bmatrix}x}{x^+ - \|x\|_P^2} +$$
$$tr\left\{\left(\lambda_1(t)\tfrac{d}{dt}\tilde{K}_P + \tfrac{d}{dt}\lambda_1(t)\tilde{K}_P\right)\tilde{K}_P^\intercal\right\} + tr\left\{\left(\lambda_2(t)\tfrac{d}{dt}\tilde{K}_D + \tfrac{d}{dt}\lambda_2(t)\tilde{K}_D\right)\tilde{K}_D^\intercal\right\}$$

Now lets consider

$$
\begin{aligned}
2x^\intercal P \begin{bmatrix} 0 \\ I \end{bmatrix} \begin{bmatrix} \tilde{K}_P & \tilde{K}_D \end{bmatrix} x &= 2x^\intercal P \begin{bmatrix} 0 \\ I \end{bmatrix} \begin{bmatrix} I & 0 \end{bmatrix} \tilde{K}_P x + 2x^\intercal P \begin{bmatrix} 0 \\ I \end{bmatrix} \begin{bmatrix} 0 & I \end{bmatrix} \tilde{K}_D x \\
&= 2x^\intercal \tilde{K}_P^\intercal \begin{bmatrix} I \\ 0 \end{bmatrix} \begin{bmatrix} 0 & I \end{bmatrix} P x + 2x^\intercal \tilde{K}_D^\intercal \begin{bmatrix} 0 \\ I \end{bmatrix} \begin{bmatrix} 0 & I \end{bmatrix} P x \\
&= tr\left\{ 2 \begin{bmatrix} I \\ 0 \end{bmatrix} \begin{bmatrix} 0 & I \end{bmatrix} P x x^\intercal \tilde{K}_P^\intercal \right\} + tr\left\{ 2 \begin{bmatrix} 0 \\ I \end{bmatrix} \begin{bmatrix} 0 & I \end{bmatrix} P x x^\intercal \tilde{K}_D^\intercal \right\}
\end{aligned}
$$

Using this result

$$
\begin{aligned}
\tfrac{d}{dt} V_B \leq \frac{\varepsilon \eta^\intercal \eta - x^\intercal Q x}{x^+ - \|x\|_P^2} + tr\left\{ \frac{2}{x^+ - \|x\|_P^2} \begin{bmatrix} I \\ 0 \end{bmatrix} \begin{bmatrix} 0 & I \end{bmatrix} P x x^\intercal \tilde{K}_P^\intercal \right\} + tr\left\{ \frac{2}{x^+ - \|x\|_P^2} \begin{bmatrix} 0 \\ I \end{bmatrix} \begin{bmatrix} 0 & I \end{bmatrix} P x x^\intercal \tilde{K} \right. \\
tr\left\{ \left( \lambda_1(t) \tfrac{d}{dt} \tilde{K}_P + \tfrac{d}{dt} \lambda_1(t) \tilde{K}_P \right) \tilde{K}_P^\intercal \right\} + tr\left\{ \left( \lambda_2(t) \tfrac{d}{dt} \tilde{K}_D + \tfrac{d}{dt} \lambda_2(t) \tilde{K}_D \right) \tilde{K}_D^\intercal \right\}
\end{aligned}
$$

$$
\begin{aligned}
\tfrac{d}{dt} V_B \leq \frac{\varepsilon \eta^\intercal \eta - x^\intercal Q x}{x^+ - \|x\|_P^2} - \alpha \lambda_1(t) tr\left\{ \tilde{K}_P \tilde{K}_P^\intercal \right\} - \alpha \lambda_2(t) tr\left\{ \tilde{K}_D \tilde{K}_D^\intercal \right\} \\
tr\left\{ \left( \lambda_1(t) \tfrac{d}{dt} \tilde{K}_P + \left( \tfrac{d}{dt} \lambda_1(t) + \alpha \lambda_1(t) \right) \tilde{K}_P + \frac{2}{x^+ - \|x\|_P^2} \begin{bmatrix} I \\ 0 \end{bmatrix} \begin{bmatrix} 0 & I \end{bmatrix} P x x^\intercal \right) \tilde{K}_P^\intercal \right\} + \\
tr\left\{ \left( \lambda_2(t) \tfrac{d}{dt} \tilde{K}_D + \left( \tfrac{d}{dt} \lambda_2(t) + \alpha \lambda_2(t) \right) \tilde{K}_D + \frac{2}{x^+ - \|x\|_P^2} \begin{bmatrix} 0 \\ I \end{bmatrix} \begin{bmatrix} 0 & I \end{bmatrix} P x x^\intercal \right) \tilde{K}_D^\intercal \right\}
\end{aligned}
$$

Hence if

$$
\lambda_1(t) \frac{d}{dt} \tilde{K}_P + \left( \frac{d}{dt} \lambda_1(t) + \alpha \lambda_1(t) \right) \tilde{K}_P + \frac{2}{x^+ - \|x\|_P^2} \begin{bmatrix} I \\ 0 \end{bmatrix} \begin{bmatrix} 0 & I \end{bmatrix} P x x^\intercal = 0
$$

$$
\lambda_2(t) \frac{d}{dt} \tilde{K}_D + \left( \frac{d}{dt} \lambda_2(t) + \alpha \lambda_2(t) \right) \tilde{K}_D + \frac{2}{x^+ - \|x\|_P^2} \begin{bmatrix} 0 \\ I \end{bmatrix} \begin{bmatrix} 0 & I \end{bmatrix} P x x^\intercal = 0
$$

Selecting $\frac{d}{dt} \tilde{K}_P$ as

$$
\frac{d}{dt} \tilde{K}_P = -\left( \frac{\frac{d}{dt} \lambda_1(t)}{\alpha \lambda_1(t)} + 1 \right) \tilde{K}_P - \frac{1}{\lambda_1(t)} \frac{2}{x^+ - \|x\|_P^2} \begin{bmatrix} I \\ 0 \end{bmatrix} \begin{bmatrix} 0 & I \end{bmatrix} P x x^\intercal
$$

$$
\frac{d}{dt} \tilde{K}_D = -\left( \frac{\frac{d}{dt} \lambda_2(t)}{\alpha \lambda_2(t)} + 1 \right) \tilde{K}_D + \frac{1}{\lambda_2(t)} \frac{2}{x^+ - \|x\|_P^2} \begin{bmatrix} 0 \\ I \end{bmatrix} \begin{bmatrix} 0 & I \end{bmatrix} P x x^\intercal
$$

Then

$$
\frac{d}{dt} V_B \leq \frac{\varepsilon \eta^\intercal \eta - x^\intercal Q x}{x^+ - \|x\|_P^2} - \alpha \lambda_1(t) tr\left\{ \tilde{K}_P \tilde{K}_P^\intercal \right\} - \alpha \lambda_2(t) tr\left\{ \tilde{K}_D \tilde{K}_D^\intercal \right\}
$$

If

$$\|x\|^2 \geq \frac{\varepsilon \max_{t\geq 0}\{\eta^\intercal \eta\}}{\lambda_{\max}\{Q-\alpha P\}}$$

$$\frac{d}{dt}V_B \leq \frac{-\alpha x^\intercal P x}{x^+ - \|x\|_P^2} - \alpha\lambda_1(t)tr\left\{\tilde{K}_P\tilde{K}_P^\intercal\right\} - \alpha\lambda_2(t)tr\left\{\tilde{K}_D\tilde{K}_D^\intercal\right\}$$

Considering the term

$$\frac{-x^\intercal P x}{x^+ - \|x\|_P^2} = \frac{-x^\intercal P x/x^+}{1-\|x\|_P^2/x^+}$$

Defining

$$z = x^\intercal P x/x^+$$

$$\begin{aligned} \frac{-x^\intercal P x}{x^+ - \|x\|_P^2} &= -\frac{z}{1-z} \leq -\ln\left(\frac{1}{1-z}\right) = -\ln\left(\frac{1}{1-z}\right) \\ &= -\ln\left(\frac{1}{1-x^\intercal P x/x^+}\right) \\ &= -\ln\left(\frac{x^+}{x^+ - x^\intercal P x}\right) \end{aligned}$$

Substituing this term in $\frac{d}{dt}V_B$ leads to

$$\begin{aligned} \frac{d}{dt}V_B &\leq -\alpha\ln\left(\frac{x^+}{x^+ - x^\intercal P x}\right) - \alpha\lambda_1(t)tr\left\{\tilde{K}_P\tilde{K}_P^\intercal\right\} - \alpha\lambda_2(t)tr\left\{\tilde{K}_D\tilde{K}_D^\intercal\right\} \\ &\leq -\alpha V_B \end{aligned}$$