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Punto 3 Tarea 5 Demostrar ec. 4.50

El operador laplaciano en coordenadas cilíndricas está dado por (dado que es 2D no se tiene z (y por tanto k))

$$\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 f}{\partial\varphi^2}$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial f}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2 f}{\partial\varphi^2}$$

Ahora bien, se requiere escribir cada término en diferencias finitas (reemplazando f por u) $u(\rho,\varphi,t)=u^l_{i,j}$ $i\Rightarrow\rho$ $j\Rightarrow\varphi$ $t\Rightarrow l$

$$\frac{\partial^2 u}{\partial\rho^2}=\frac{u^l_{i+1,j}-2u^l_{i,j}+u^l_{i-1,j}}{(\Delta\rho)^2} \qquad \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2}=\frac{1}{\rho^2}\left[\frac{u^l_{i,j+1}-2u^l_{i,j}+u^l_{i,j-1}}{(\Delta\varphi)^2}\right]$$

$$\frac{1}{\rho}\frac{\partial u}{\partial\rho}=\frac{1}{\rho}\left[\frac{u^l_{i+1,j}+u^l_{i-1,j}}{2\Delta\rho}\right]$$

Sin embargo, la ecuación de onda 2D en cilíndricas está dada por

$$\frac{\partial^2 u}{\partial t^2}=\alpha^2\nabla^2 u \qquad \frac{\partial^2 u}{\partial t^2}=\frac{u^{l+1}_{i,j}-2u^l_{i,j}+u^{l-1}_{i,j}}{(\Delta t)^2}$$

$$u^{l+1}_{i,j}-2u^l_{i,j}+u^{l-1}_{i,j}=\alpha^2(\Delta t)^2\left\{\frac{u^l_{i+1,j}-2u^l_{i,j}+u^l_{i-1,j}}{(\Delta\rho)^2}+\frac{1}{\rho}\left[\frac{u^l_{i+1,j}+u^l_{i-1,j}}{2\Delta\rho}\right]+\frac{1}{\rho^2}\left[\frac{u^l_{i,j+1}-2u^l_{i,j}+u^l_{i,j-1}}{(\Delta\varphi)^2}\right]\right\}$$

Factorizando $\frac{1}{\Delta\rho}$ y definiendo $v=\frac{\alpha\Delta t}{\Delta\rho}$ se tiene que

$$u^{l+1}_{i,j}=\underbrace{\left(\frac{\alpha\Delta t}{\Delta\rho}\right)^2}_{v^2}\left\{u^l_{i+1,j}-2u^l_{i,j}+u^l_{i-1,j}+\frac{\Delta\rho}{\rho[i]}\left(u^l_{i+1,j}+u^l_{i-1,j}\right)+\underbrace{\left(\frac{\Delta\rho}{\rho\Delta\varphi}\right)^2}_{\frac{\lambda^2}{\rho^2}}\left(u^l_{i,j+1}-2u^l_{i,j}+u^l_{i,j-1}\right)\right\}+2u^l_{i,j}-u^{l-1}_{i,j}$$

También defino $\lambda=\frac{\Delta\rho}{\Delta\varphi}$

$$u^{l+1}_{i,j}=v^2\left[u^l_{i+1,j}-2u^l_{i,j}+u^l_{i-1,j}+\frac{\Delta\rho}{\rho[i]}\left(u^l_{i,j}-u^l_{i-1,j}\right)+\left(\frac{\lambda}{\rho[i]}\right)^2\left(u^l_{i,j+1}-2u^l_{i,j}+u^l_{i,j-1}\right)\right]+2u^l_{i,j}-u^{l-1}_{i,j}$$