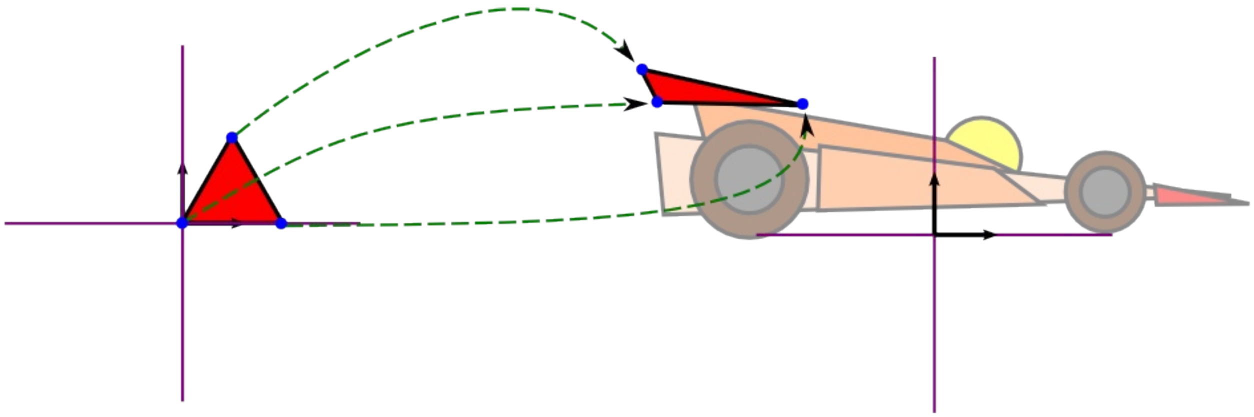
Unidad 4

Espacios y Transformaciones

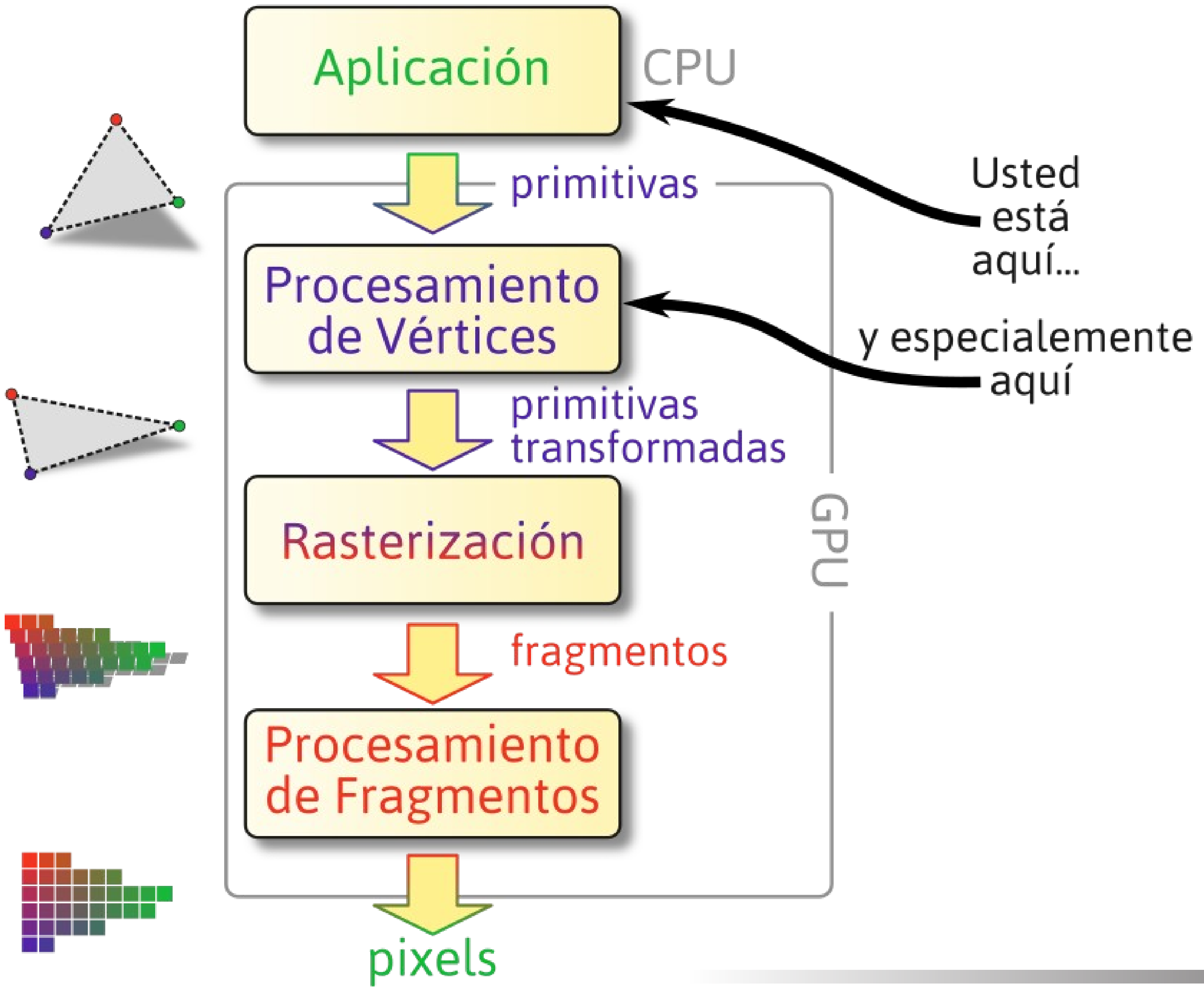
UNL FICH
Universidad Nacional del Litoral
Facultad de Ingeniería y Ciencias Hídricas

Transformación: función o mapeo que hace corresponder cada punto del espacio con otro punto del mismo espacio.

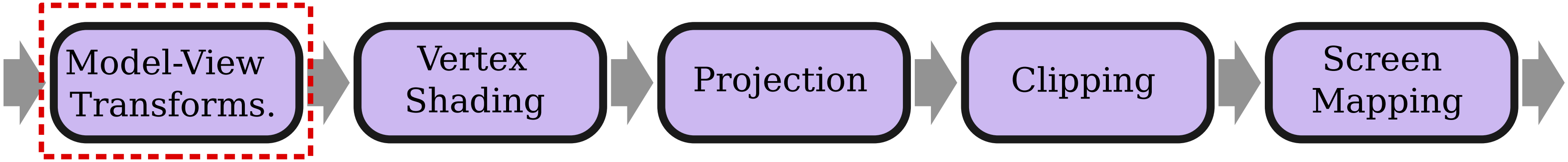
$$\underline{\boldsymbol{P}} = T\left(\boldsymbol{P}\right)$$

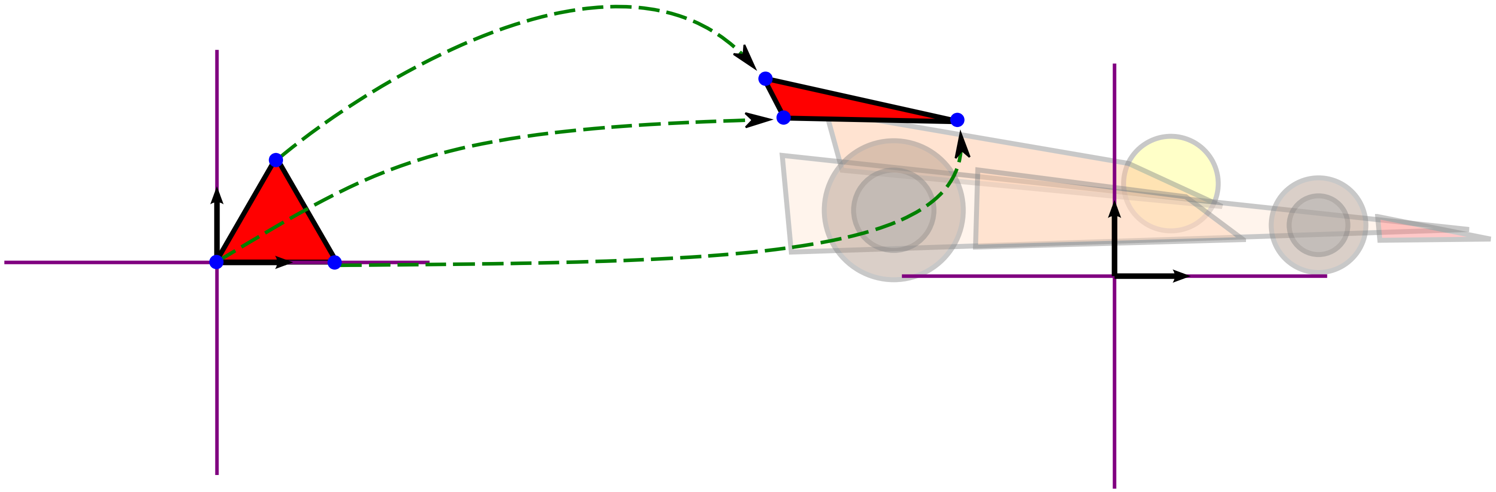


Transformaciones en el Pipeline

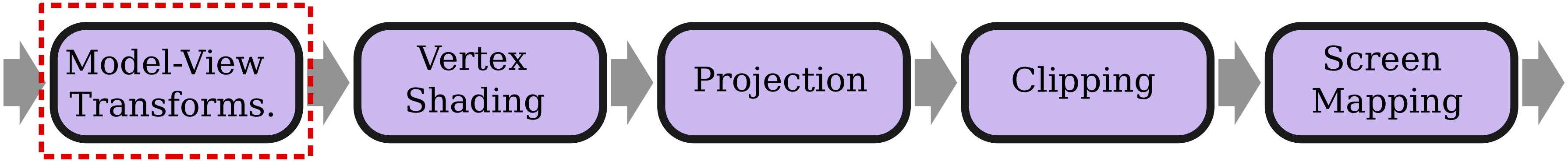


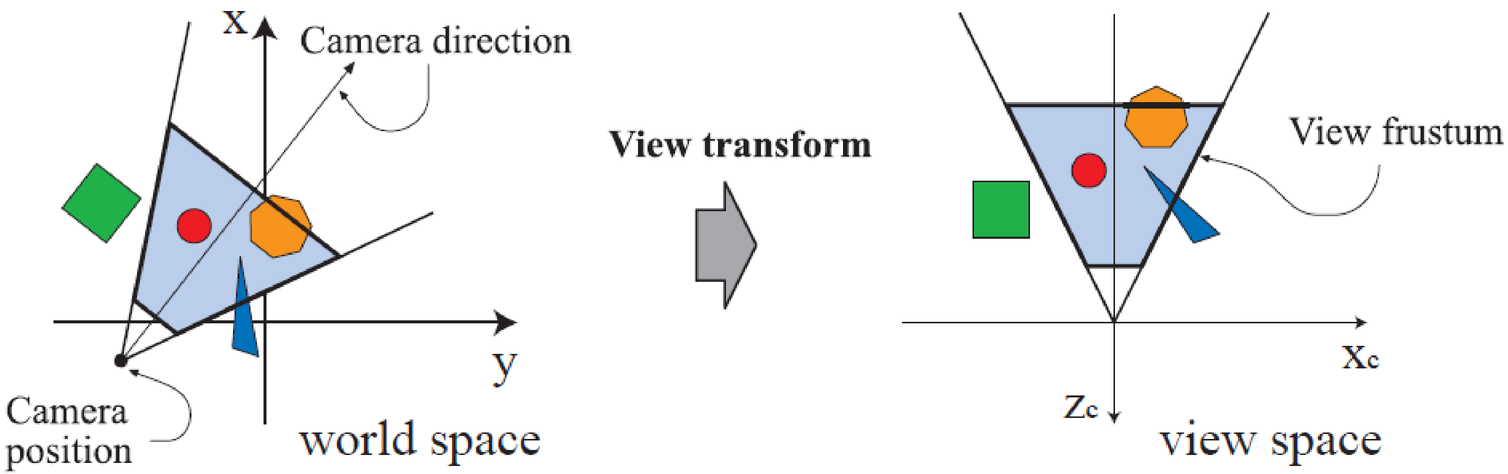
Procesamiento de Vértices: Model Matrix



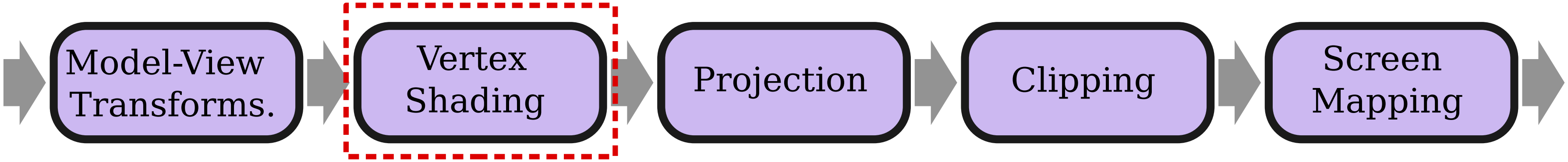


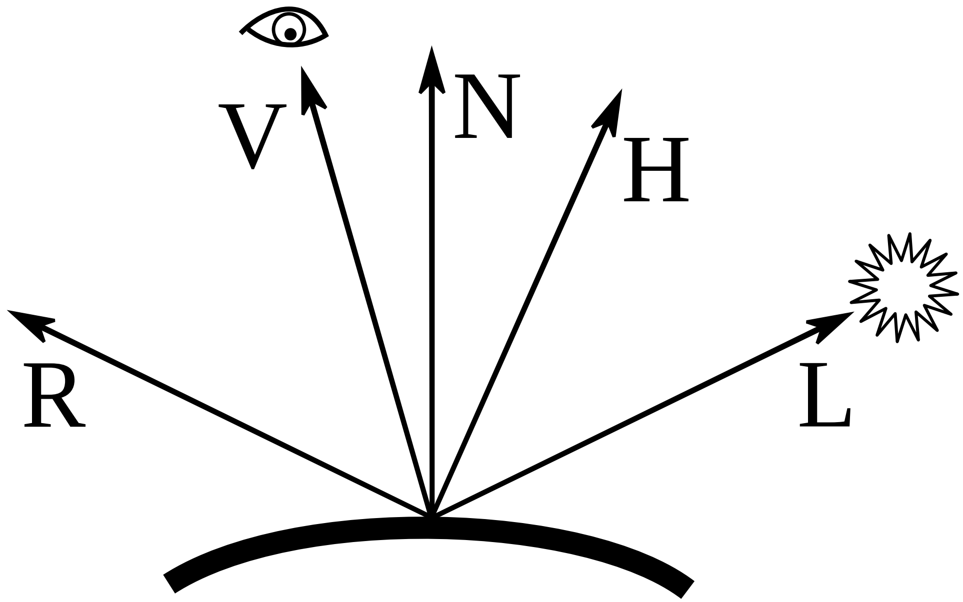
Procesamiento de Vértices: View Matrix





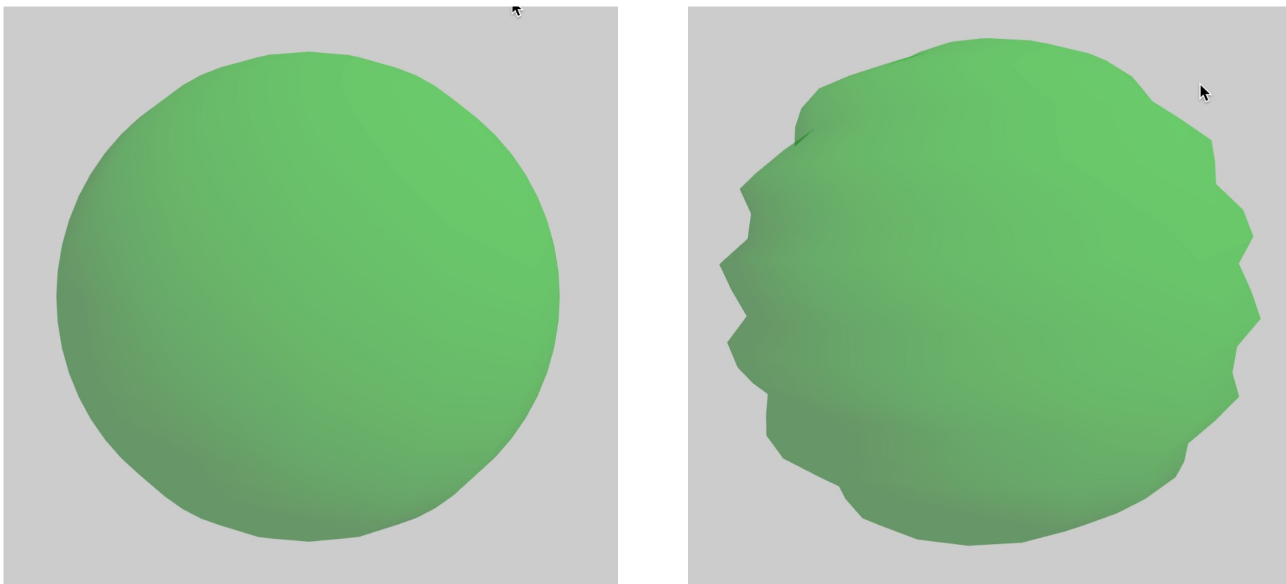
Procesamiento de Vértices: Vertex Shading



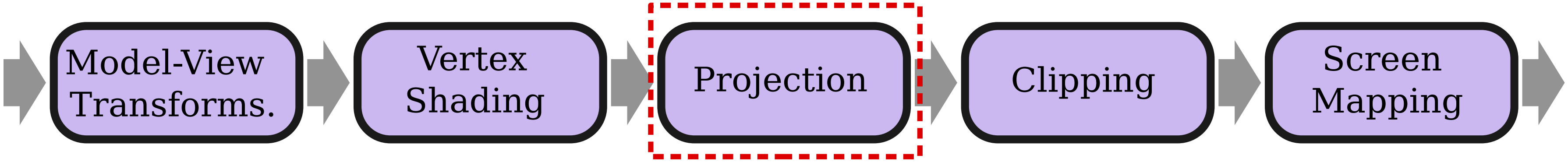


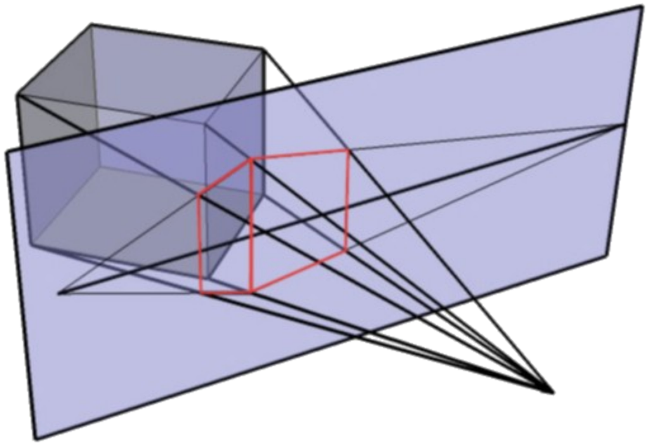
Procesamiento de Vértices: Vertex Shader

Model-View Transforms. → Vertex Shading → Projection → Clipping → Screen Mapping

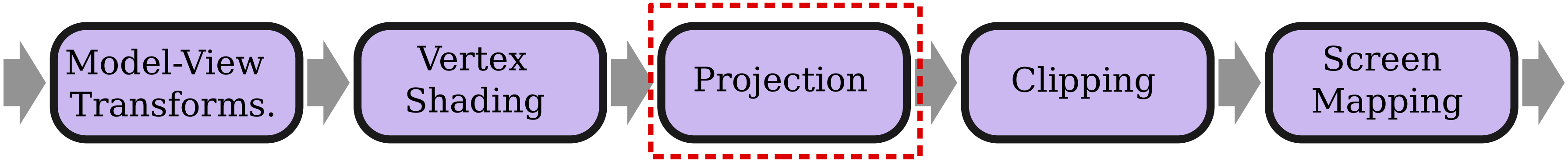


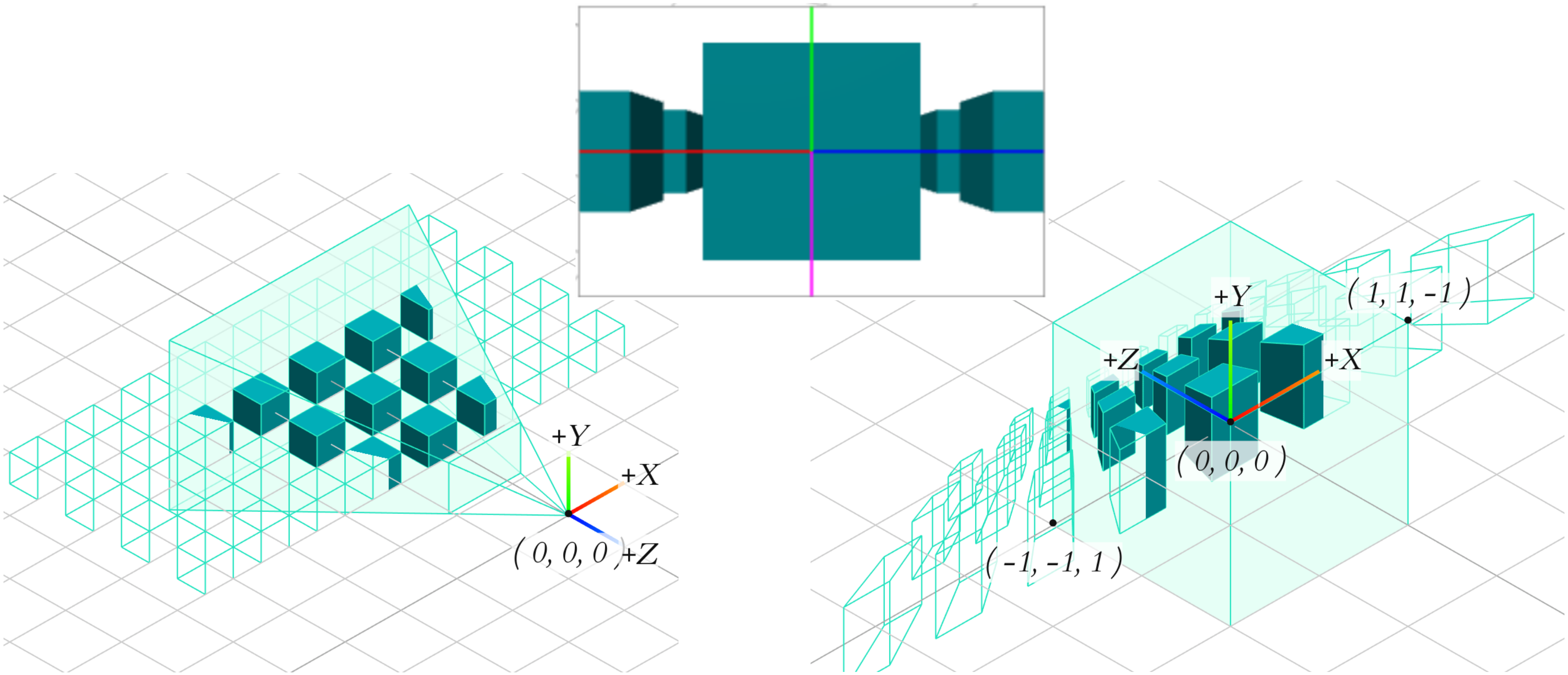
Procesamiento de Vértices: Projection



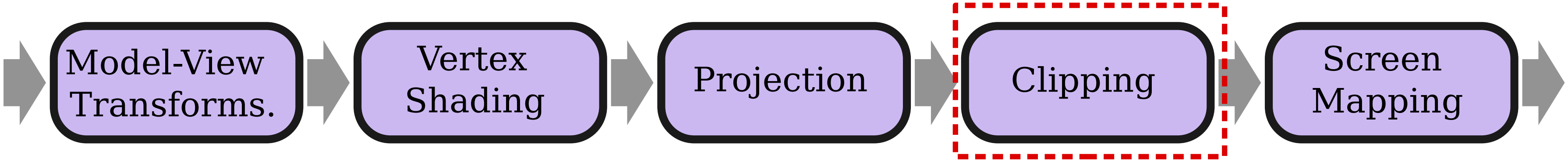


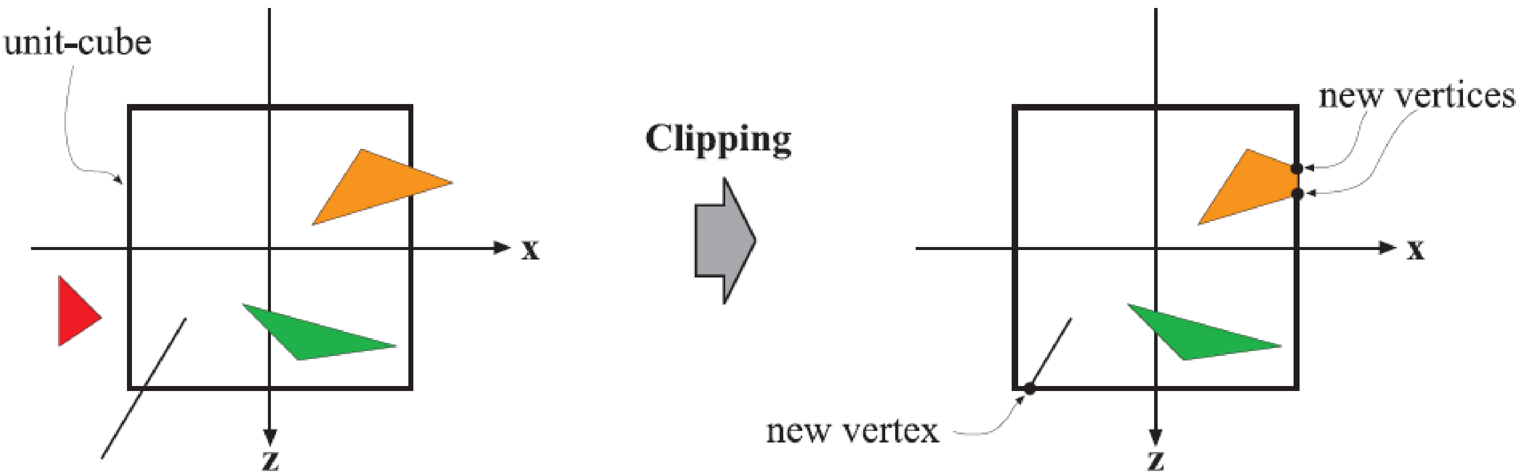
Procesamiento de Vértices: Projection Matrix



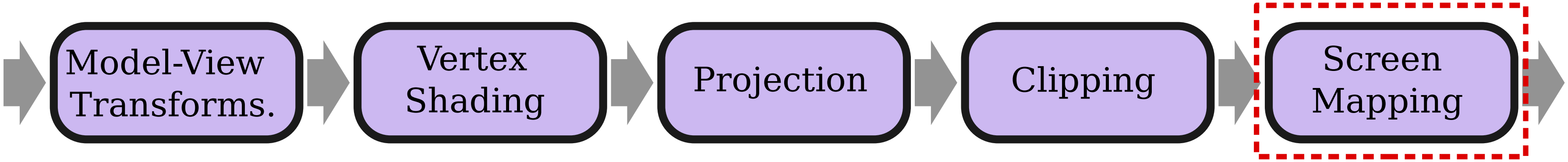


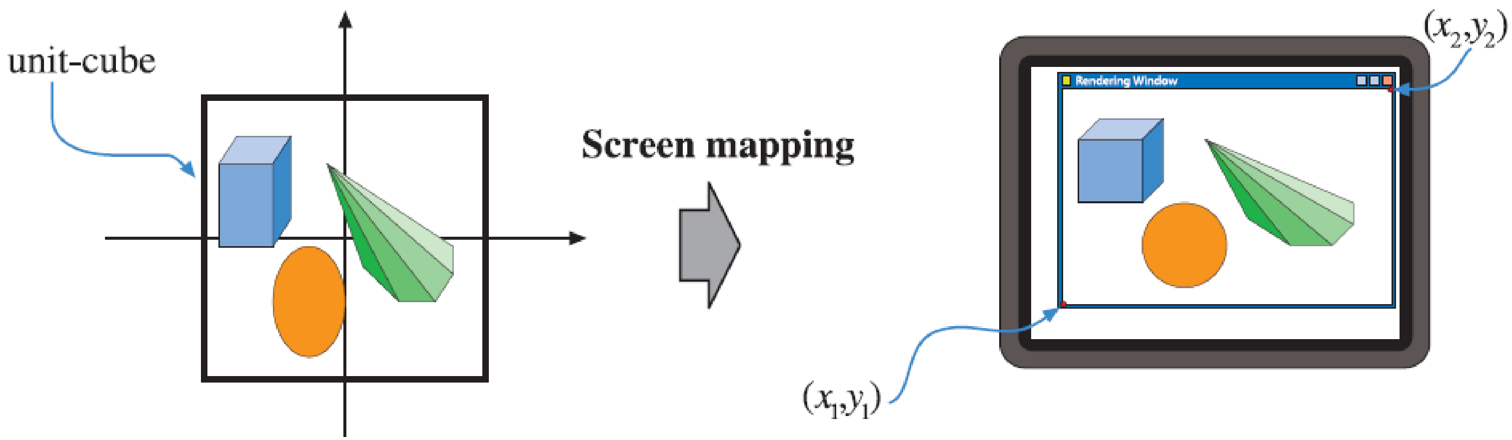
Procesamiento de Primitivas: Clipping





Procesamiento de Vértices y Primitivas





Transformación Lineal

Es **Lineal** sii preserva la combinación lineal:

$$T\left(\alpha \boldsymbol{P}_1 + \beta \boldsymbol{P}_2\right) = \alpha T\left(\boldsymbol{P}_1\right) + \beta \boldsymbol{T}\left(\boldsymbol{P}_2\right)$$

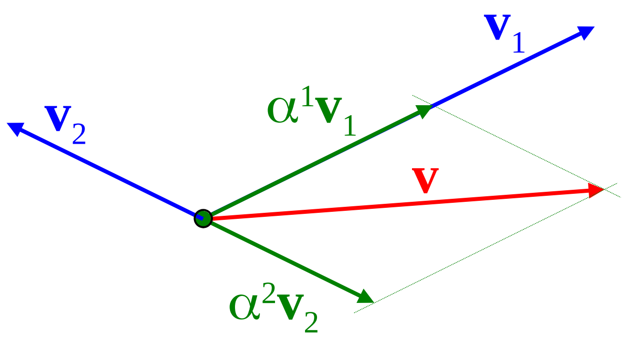
$$\Downarrow$$

$$\underline{\boldsymbol{P}} = \underline{\underline{M}}\, \boldsymbol{P}$$

Puede representarse como matriz. Aplicar la transformación equivale a premultiplicar por la matriz correspondiente.

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} Ax + By \\ Cx + Dy \end{bmatrix}$$

Espacio Vectorial – Combinación Lineal



$$\boldsymbol{v}_{\mathbf{1}} = \begin{vmatrix} v_1^1 \\ v_1^2 \\ \vdots \\ v_1^n \end{vmatrix} \in \Re^n$$

n componentes en *n* filas

$\{\boldsymbol{v}_{\boldsymbol{i}}\} = \{\boldsymbol{v}_{\mathbf{1}}, \boldsymbol{v}_{\mathbf{2}}, \ldots, \boldsymbol{v}_{\boldsymbol{m}}\}$ Conjunto de m vectores

$\boldsymbol{u} = \sum_{i=1}^{m} \alpha^i \boldsymbol{v}_{\boldsymbol{i}}$ Combinación lineal de m vectores = vector

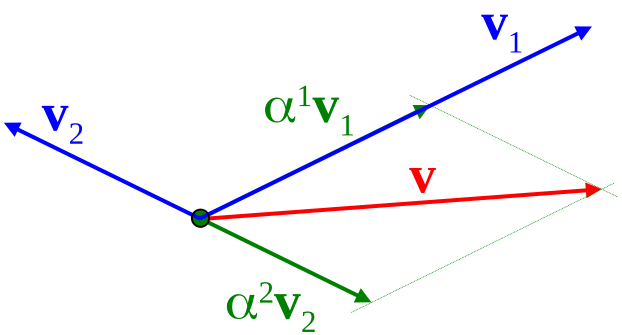
superíndice = fila

$$\begin{vmatrix} \alpha^1 \\ \alpha^2 \\ \alpha^3 \\ \vdots \\ \alpha^m \end{vmatrix}$$ m factores

m vectores

subíndice = columna o ítem $\longrightarrow$ $\begin{vmatrix} \boldsymbol{v}_{\mathbf{1}} & \boldsymbol{v}_{\mathbf{2}} & \boldsymbol{v}_{\mathbf{3}} & \ldots & \boldsymbol{v}_{\boldsymbol{m}} \end{vmatrix}$ $\begin{vmatrix} \sum_{i=1}^{m} \alpha^i \boldsymbol{v}_{\boldsymbol{i}} \end{vmatrix}$

Espacio Vectorial – Combinación Lineal



$$\{\boldsymbol{v_i}\}\ \text{es L.I} \Leftrightarrow \left(\sum \alpha^i \boldsymbol{v_i} = \mathbf{0} \Rightarrow \alpha^i = 0\ \forall\, i\right)$$

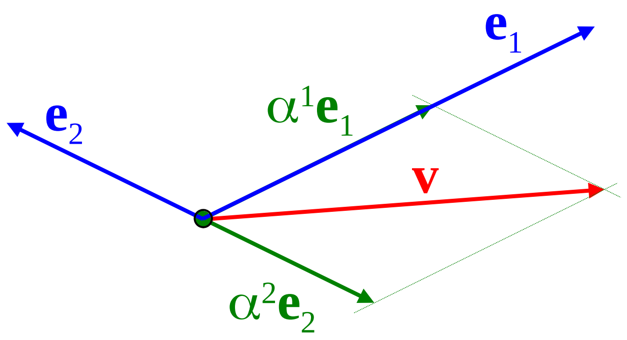
Independencia Lineal (LI)

$$\Downarrow$$

$$\cancel{\alpha^1 \boldsymbol{v_1} + \alpha^2 \boldsymbol{v_2} = \mathbf{0} \Rightarrow \boldsymbol{v}2 = -\alpha^1 / \alpha^2\, \boldsymbol{v_1}}$$

no se puede despejar uno en función del resto

Espacio Vectorial – Combinación Lineal



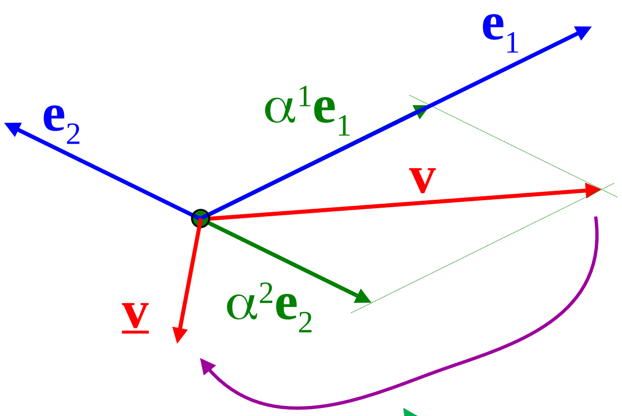
$$\{\boldsymbol{e}_i\} = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$$

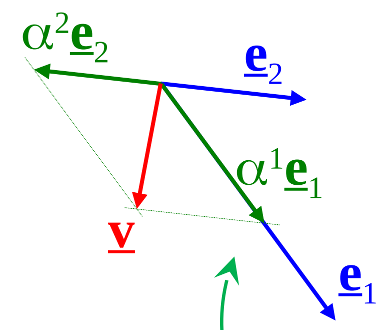
base = n vectores LI

$$\boldsymbol{v} = \sum_{i=1}^{n} \alpha^i \boldsymbol{e}_i$$

vector = combinación lineal de n vectores base

Espacio Vectorial – Combinación Lineal





$$\underline{\boldsymbol{v}} = L(\boldsymbol{v}) = L\left(\sum_{i=1}^{n} \alpha^i \boldsymbol{e}_i\right)$$

L transforma **v** en <u>**v**</u>

Por ser L lineal: $$L\left(\sum_{i=1}^{n} \alpha^i \boldsymbol{e}_i\right) = \sum_{i=1}^{n} \alpha^i \underbrace{L(\boldsymbol{e}_i)}_{\underline{\boldsymbol{e}}_i}$$

$$\underline{\boldsymbol{v}} = \sum_{i=1}^{n} \alpha^i L(\boldsymbol{e}_i) = \sum_{i=1}^{n} \alpha^i \underline{\boldsymbol{e}}_i$$

<u>**v**</u> en función de $\underline{\mathbf{e}}_i$, se mantienen los α^i

Dos formas de ver una transformación:

Vector original: $\boldsymbol{v} = \sum v_i \boldsymbol{e_i}$

(1) Componentes transformadas: $\underline{\boldsymbol{v}} = \sum \underline{v}_i \boldsymbol{e_i}$

(2) Base transformada: $\underline{\boldsymbol{v}} = \sum v_i \underline{\boldsymbol{e}}_{\boldsymbol{i}}$

$$\underline{\underline{\boldsymbol{M}}} = \begin{bmatrix} \underline{e}_x^1 & \underline{e}_x^2 \\ \underline{e}_y^1 & \underline{e}_y^2 \end{bmatrix} \quad \Rightarrow \quad \underline{\boldsymbol{v}} = \underline{\underline{\boldsymbol{M}}}\, \boldsymbol{v}$$

Aplicación de sucesivas transformaciones. Ejemplo:

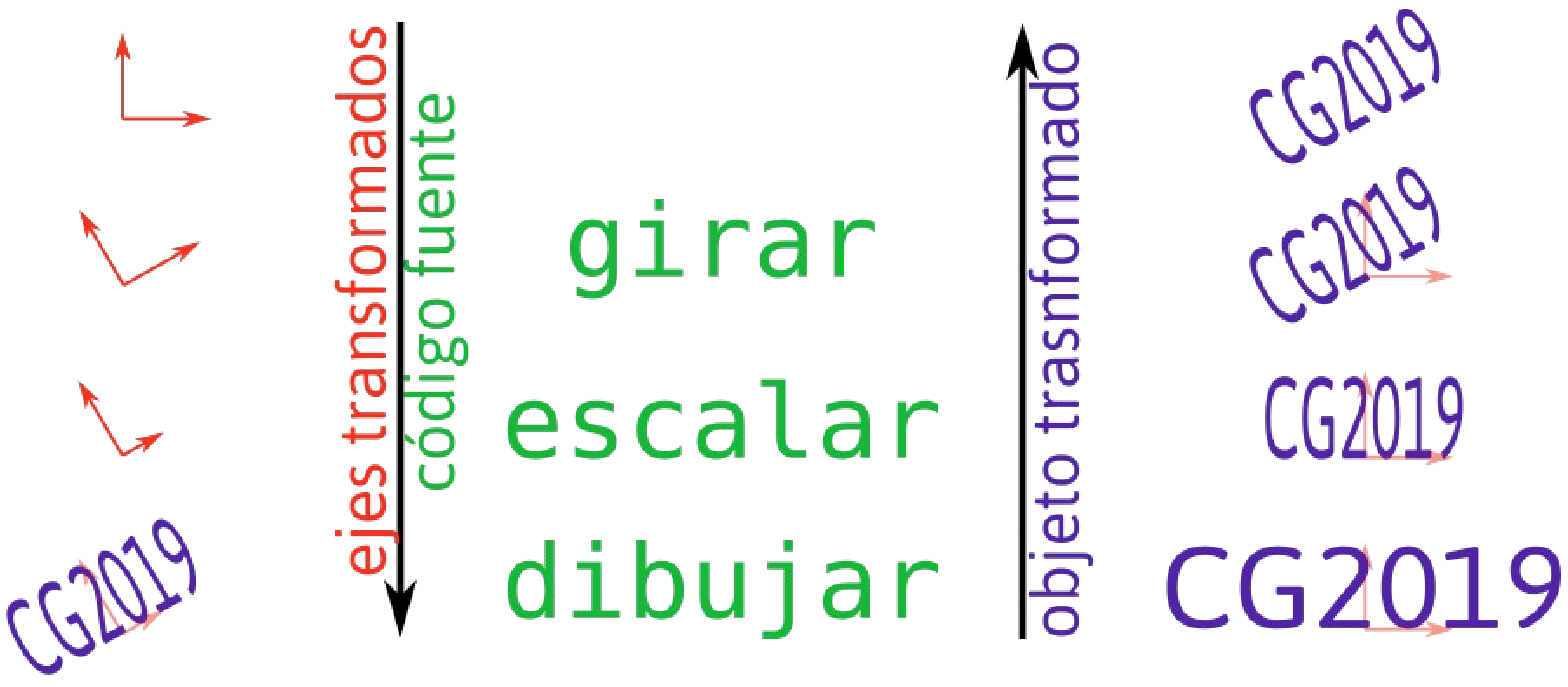
1. Desplazar: $\underline{\boldsymbol{P}}^1 = T^1(\boldsymbol{P})$
2. Escalar: $\underline{\boldsymbol{P}}^2 = T^2(\underline{\boldsymbol{P}}^1) = T^2(T^1(\boldsymbol{P}))$
3. Rotar: $\underline{\boldsymbol{P}}^3 = T^3(\underline{\boldsymbol{P}}^2) = T^3(T^2(T^1(\boldsymbol{P})))$

Combinación: Utilizando las matrices asociadas:

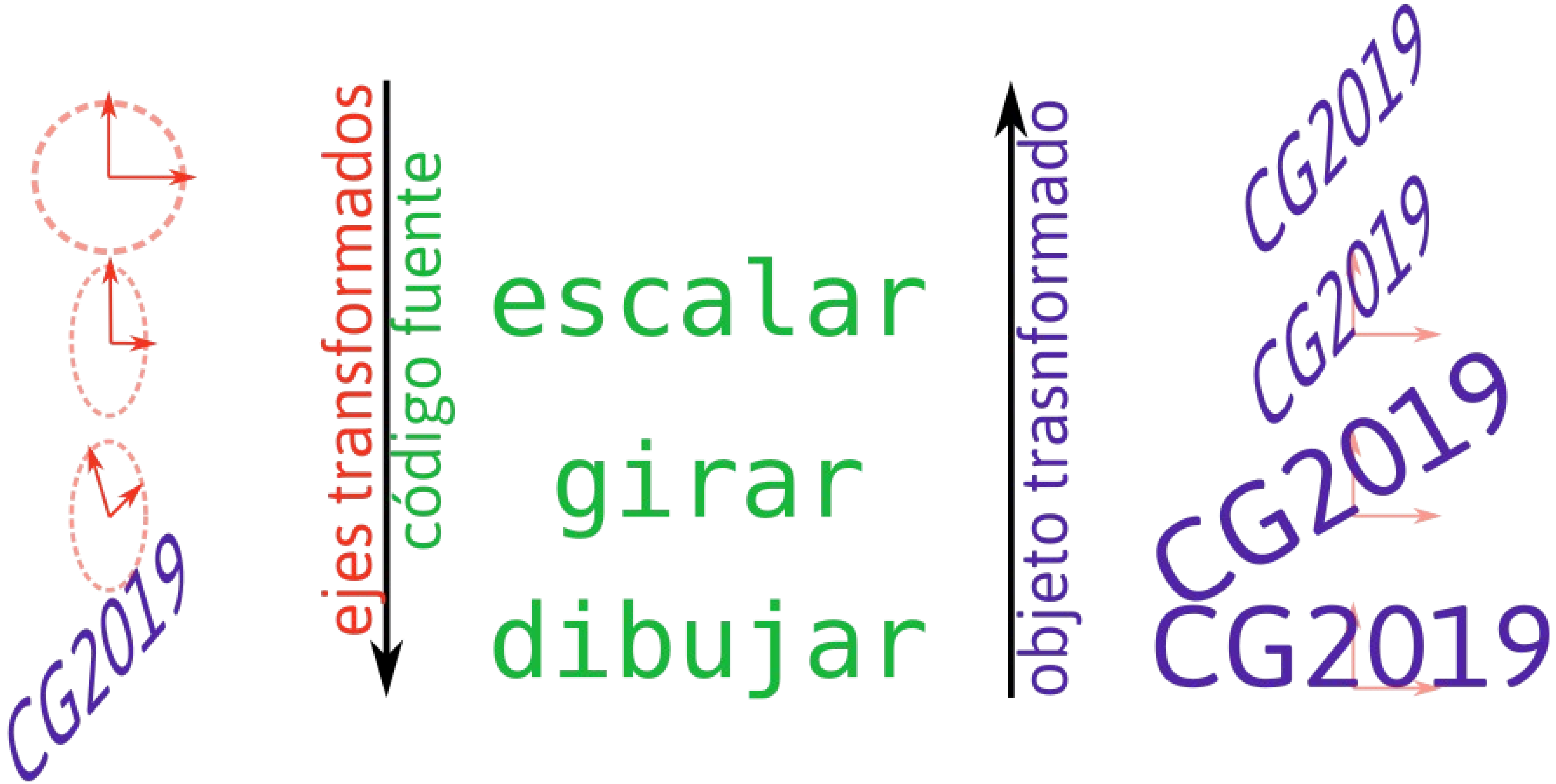
$$\begin{aligned}\underline{\boldsymbol{P}}^3 &= T^3(T^2(T^1(\boldsymbol{P}))) = \\ &= \underline{\underline{M}}^3 \cdot (\underline{\underline{M}}^2 \cdot (\underline{\underline{M}}^1 \cdot \boldsymbol{P})) = \\ &= \underbrace{(\underline{\underline{M}}^3 \cdot \underline{\underline{M}}^2 \cdot \underline{\underline{M}}^1)}_{\underline{\underline{\hat{M}}}} \cdot \boldsymbol{P} = \underline{\underline{\hat{M}}}\, \boldsymbol{P}\end{aligned}$$

Notar que el orden altera el resultado

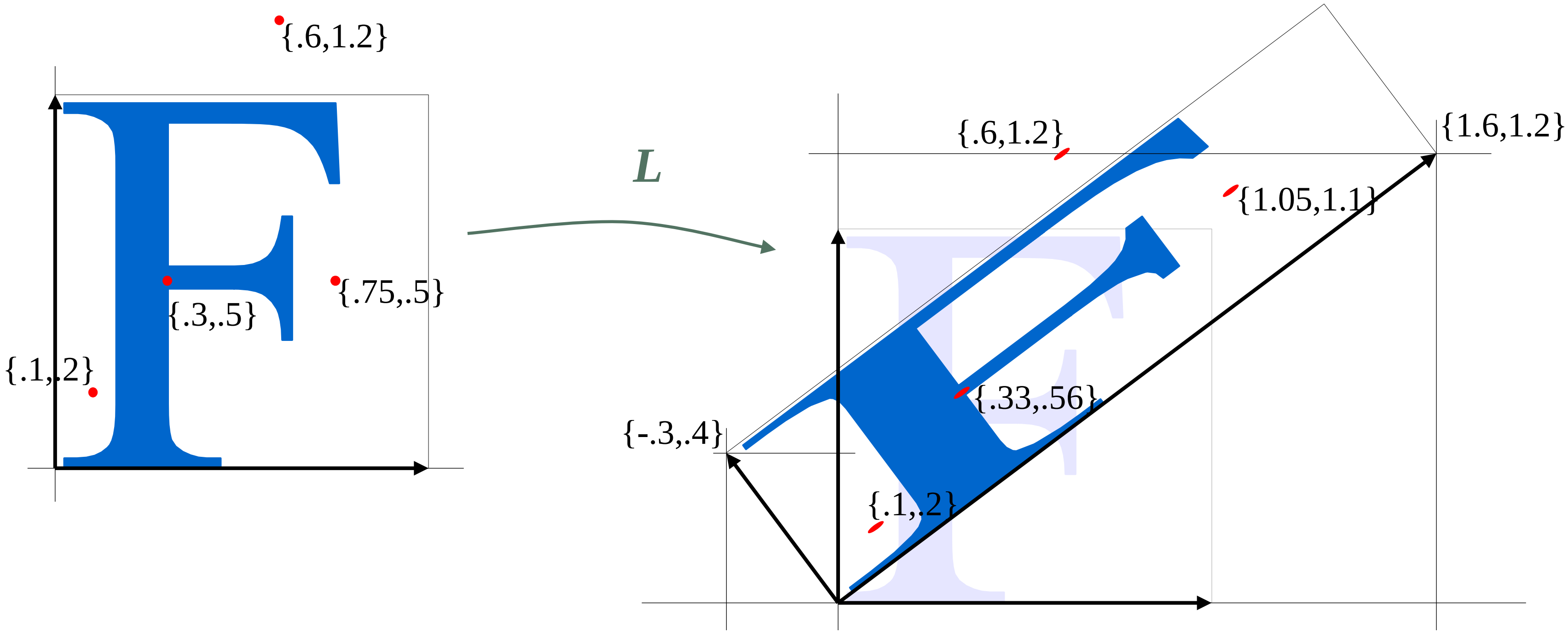
Orden de Interpretación



Orden de Interpretación



Ejemplo

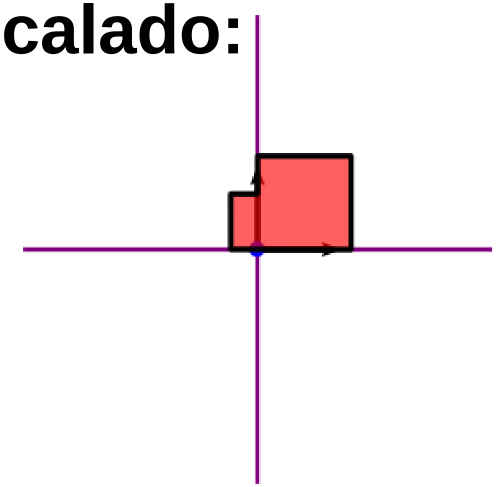


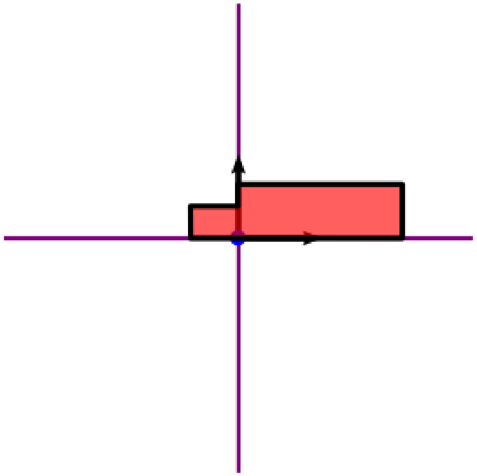
$$\begin{vmatrix} 0.1 \\ 0.2 \end{vmatrix} \Leftarrow \mathbf{v}$$

$$\begin{array}{cc} \underline{\mathbf{e}}_1 & \underline{\mathbf{e}}_2 \\ \Downarrow & \Downarrow \end{array}$$

$$\begin{vmatrix} 1.6 & -0.3 \\ 1.2 & 0.4 \end{vmatrix} \begin{vmatrix} 0.1 \\ 0.2 \end{vmatrix} \Leftarrow \underline{\mathbf{v}}$$

Escalado:

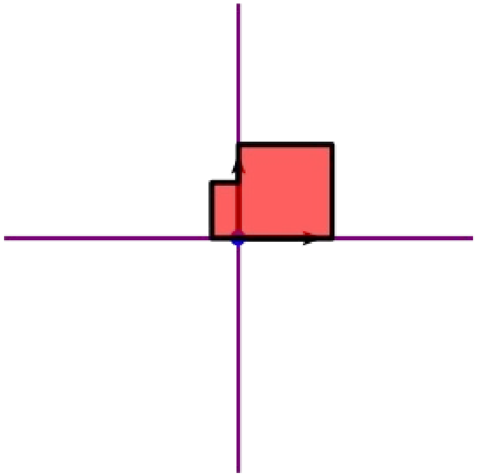


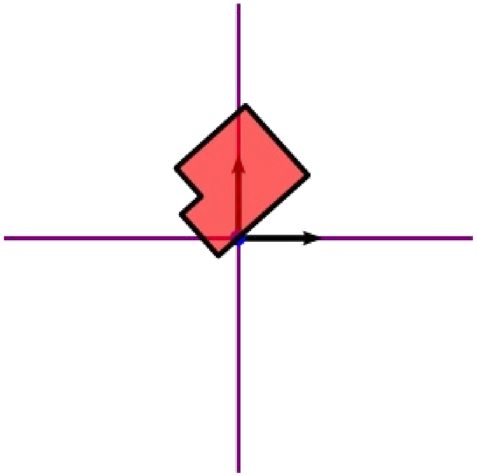


$$\underline{\underline{M}} = \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix}$$

Rotación:

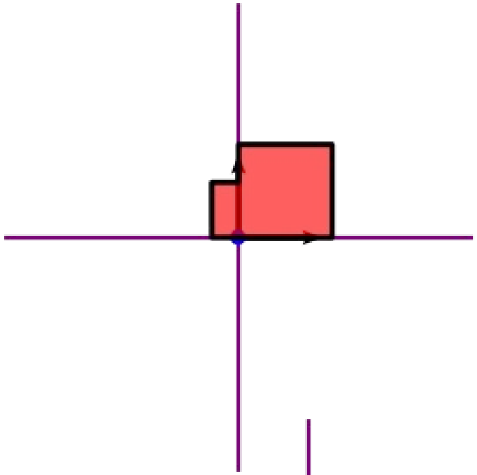
$$\underline{\underline{M}} = \begin{bmatrix} \cos\theta & -\operatorname{sen}\theta \\ \operatorname{sen}\theta & \cos\theta \end{bmatrix}$$

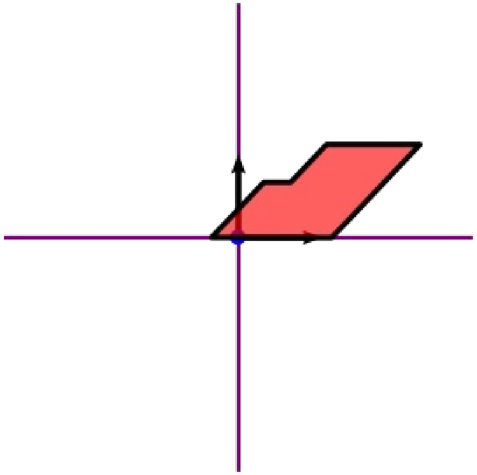


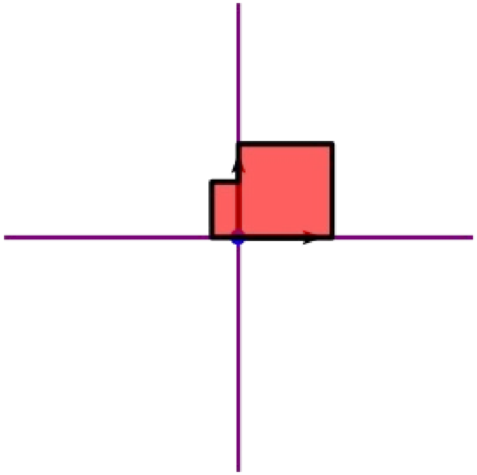


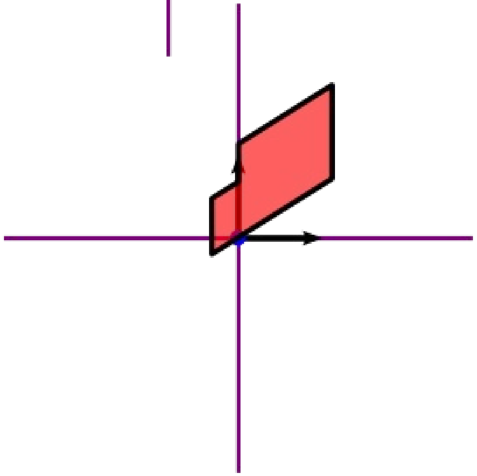
Shear/Deslizamiento:

$$\underline{\underline{\boldsymbol{M}}} = \begin{bmatrix} 1 & S_x \\ 0 & 1 \end{bmatrix}$$



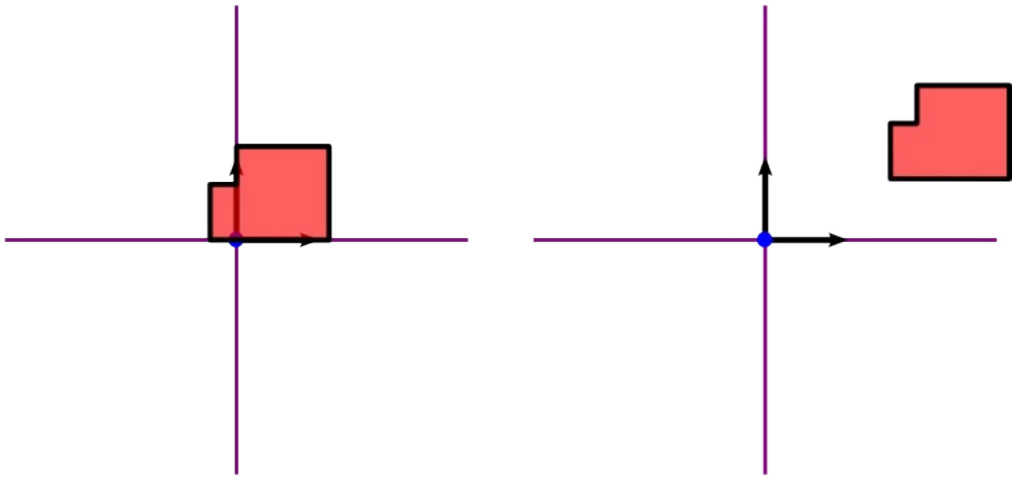






$$\underline{\underline{\boldsymbol{M}}} = \begin{bmatrix} 1 & 0 \\ S_y & 1 \end{bmatrix}$$

Traslación/Desplazamiento:



$$\underline{\boldsymbol{P}} = \boldsymbol{P} + \boldsymbol{D} \quad \Rightarrow \quad \underline{\underline{M}} = ? \;\; (\text{NO es lineal en } R^N)$$

Peeero….

Transformación Afín

Es **Afín** sii preserva la combinación afín:

$$T\left(\alpha \boldsymbol{P}_1+\beta \boldsymbol{P}_2\right)=\alpha T\left(\boldsymbol{P}_1\right)+\beta T\left(\boldsymbol{P}_2\right), \qquad \alpha+\beta=1$$

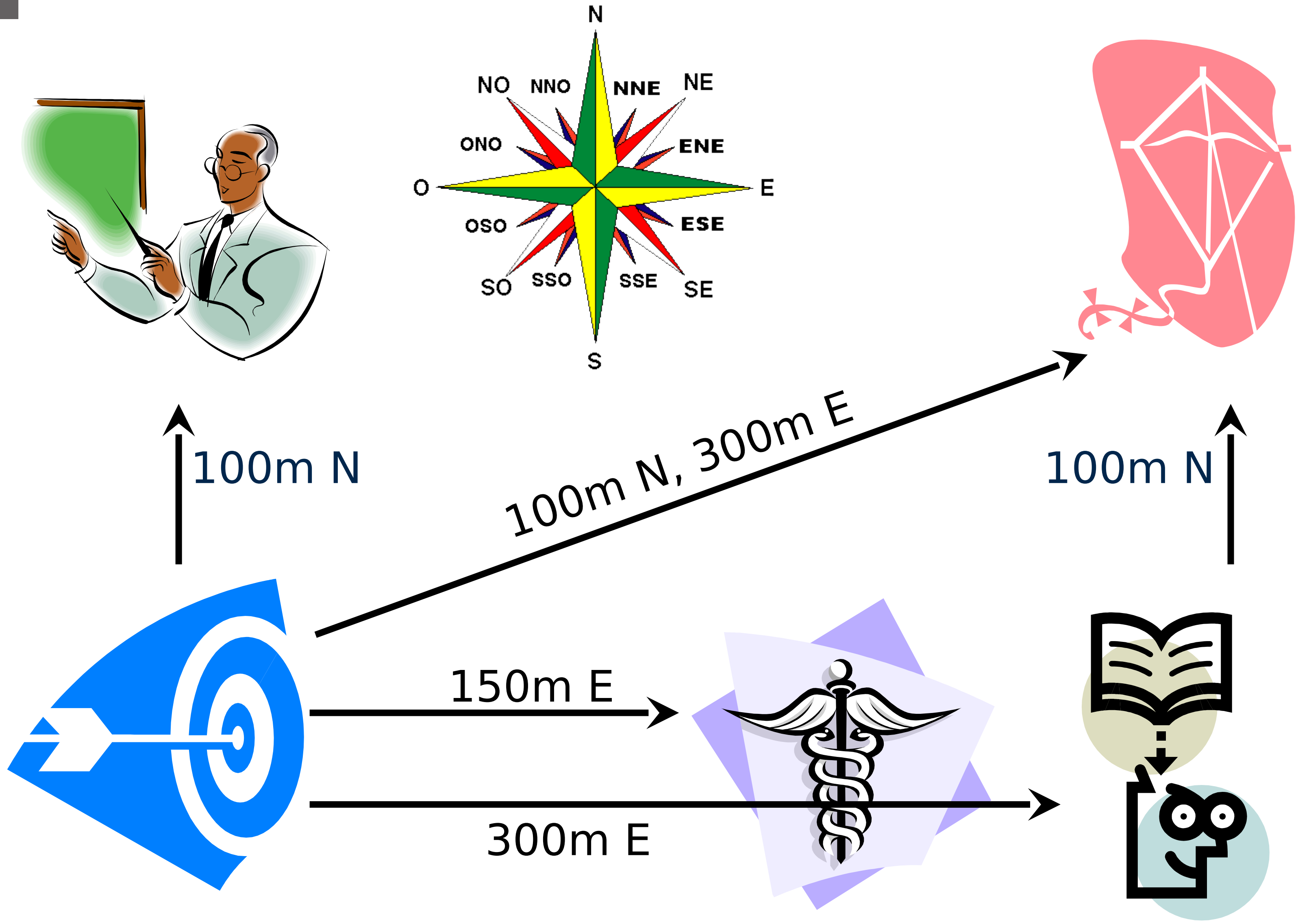
$$\Downarrow$$

Una **traslación** sí preserva la combinación afín

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} A\,0+B\,0 \\ C\,0+D\,0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

No Puede representarse como matriz en R^N.

Espacio Afín: Puntos vs. Vectores



Es **Afín** sii preserva la combinación afín:

$$T(\alpha \boldsymbol{P}_1 + \beta \boldsymbol{P}_2) = \alpha T(\boldsymbol{P}_1) + \beta T(\boldsymbol{P}_2), \qquad \alpha + \beta = 1$$

$$\Downarrow$$

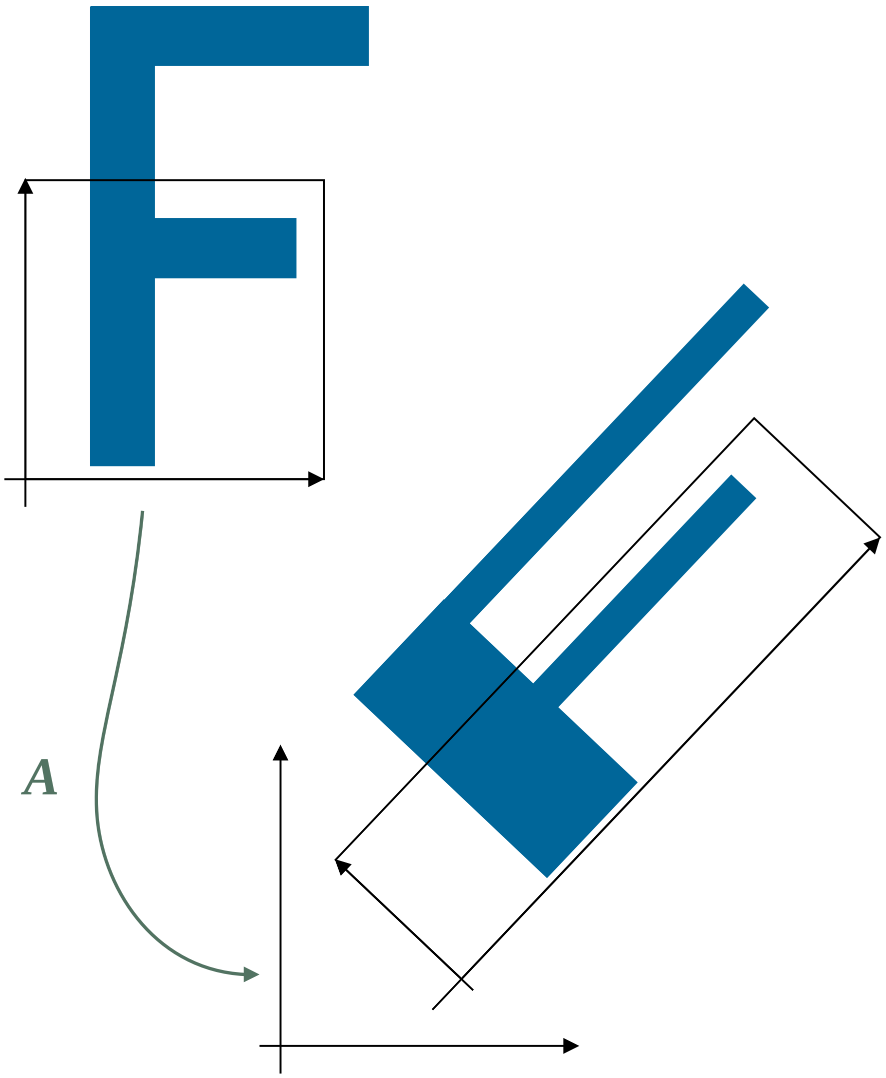
$$\underline{\boldsymbol{P}} = \underline{\underline{M}}\, \boldsymbol{P}$$

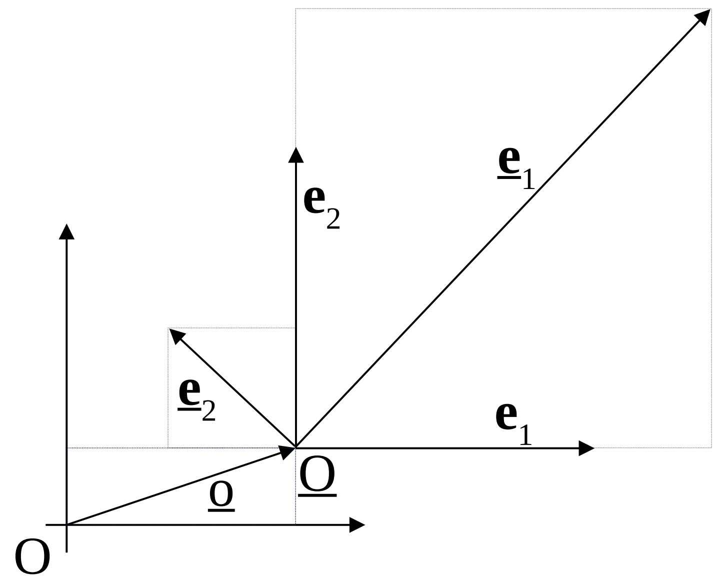
Sí puede representarse como matriz en "R^{N+1}".

$$\begin{bmatrix} A & B & T_x \\ C & D & T_y \\ \boxed{0 \quad 0 \quad 1} \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} Ax + By + 1\,T_x \\ Cx + Dy + 1\,T_y \\ 0\,x + 0\,y + 1 \end{bmatrix}$$

Esto no es Z!

Ejemplo

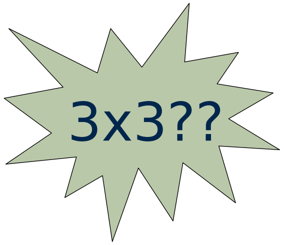




$$\begin{vmatrix} v^1 \\ v^2 \\ 1 \end{vmatrix}$$

$$\begin{array}{ccc} \underline{e}_1 & \underline{e}_2 & \underline{o} \\ \Downarrow & \Downarrow & \Downarrow \end{array}$$

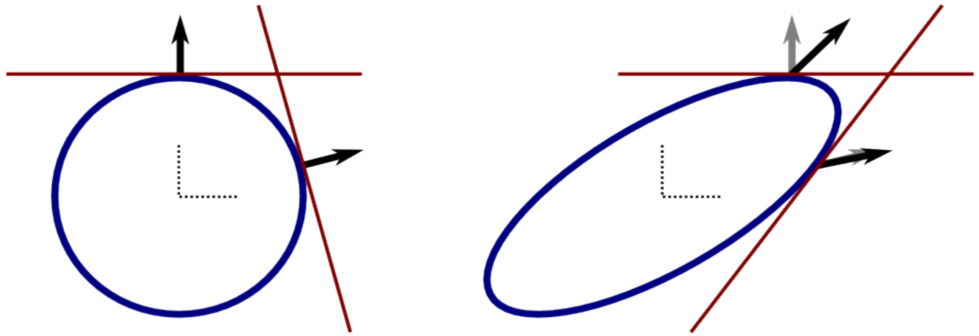
$$\begin{vmatrix} 1.3 & -0.5 & 0.6 \\ 1.5 & 0.4 & 0.1 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} \underline{v}^1 \\ \underline{v}^2 \\ 1 \end{vmatrix}$$



Vector Normal: vector ortogonal a todos los vectores tangente

$$\underline{t}(\underline{P}) = \lim_{Q \to \underline{P}} (\underline{Q} - \underline{P}) \qquad \underline{n} \cdot \underline{t} = 0$$

En una transformación afín, la transformación de la normal original no es igual a la normal de la curva transformada



Transformación de Tangentes y Normales

Para una transformación afín ***A***:

$$\boldsymbol{n}\cdot\boldsymbol{t}=0 \qquad \hat{\boldsymbol{t}}=\boldsymbol{A}\,\boldsymbol{t}$$

$$\boldsymbol{n}^T\,\boldsymbol{t}=0 \;\Rightarrow\; \boldsymbol{n}^T\underbrace{\left(\boldsymbol{A}^{-1}\hat{\boldsymbol{t}}\right)}_{\boldsymbol{t}}=0 \;\Leftarrow\; \boldsymbol{t}=\boldsymbol{A}^{-1}\hat{\boldsymbol{t}}$$

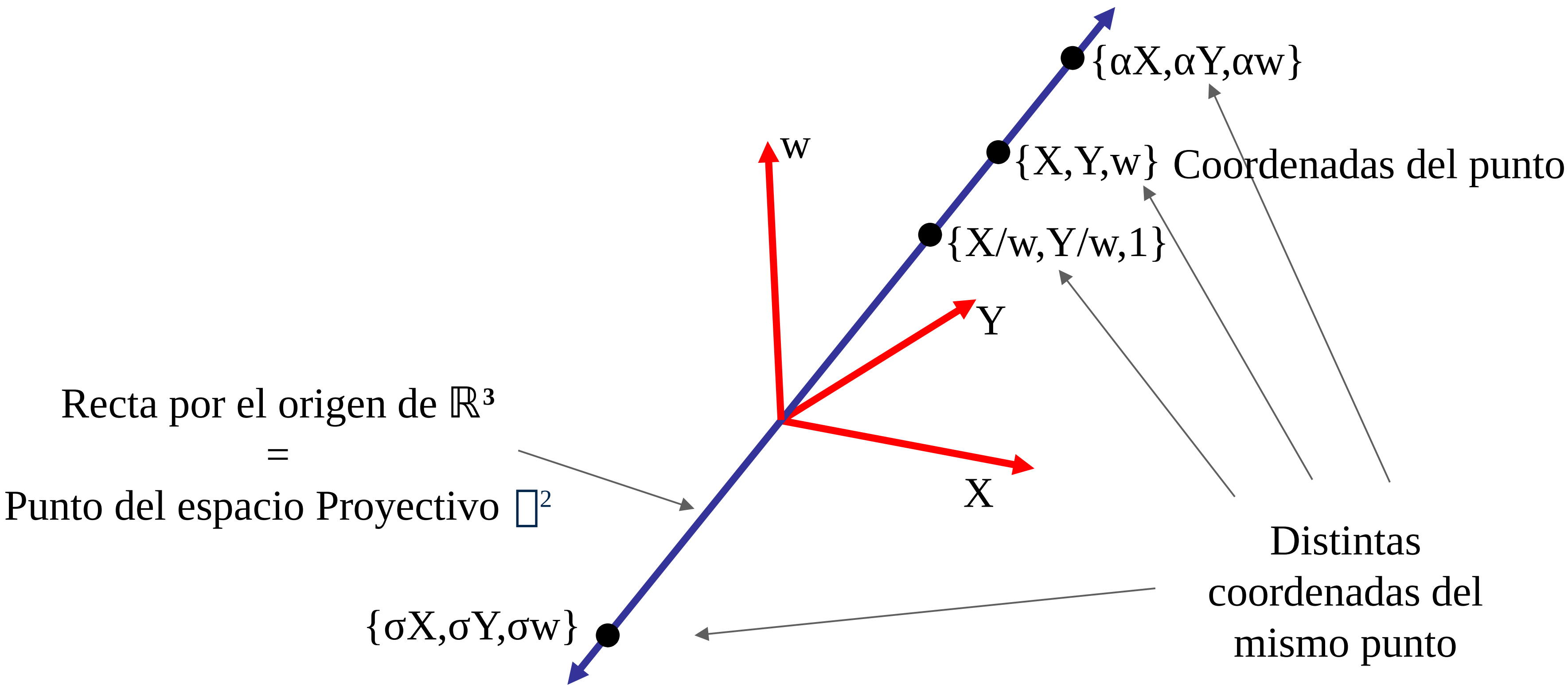
$$\left(\boldsymbol{n}^T\boldsymbol{A}^{-1}\right)\hat{\boldsymbol{t}}=0$$

$$\hat{\boldsymbol{n}}^T\,\hat{\boldsymbol{t}}=0 \;\Leftarrow\; \underbrace{\left(\left(\boldsymbol{A}^{-1}\right)^T\boldsymbol{n}\right)^T}_{\hat{\boldsymbol{n}}^T}\hat{\boldsymbol{t}}=0 \;\Rightarrow\; \boxed{\hat{\boldsymbol{n}}=\left(\boldsymbol{A}^{-1}\right)^T\boldsymbol{n}}$$

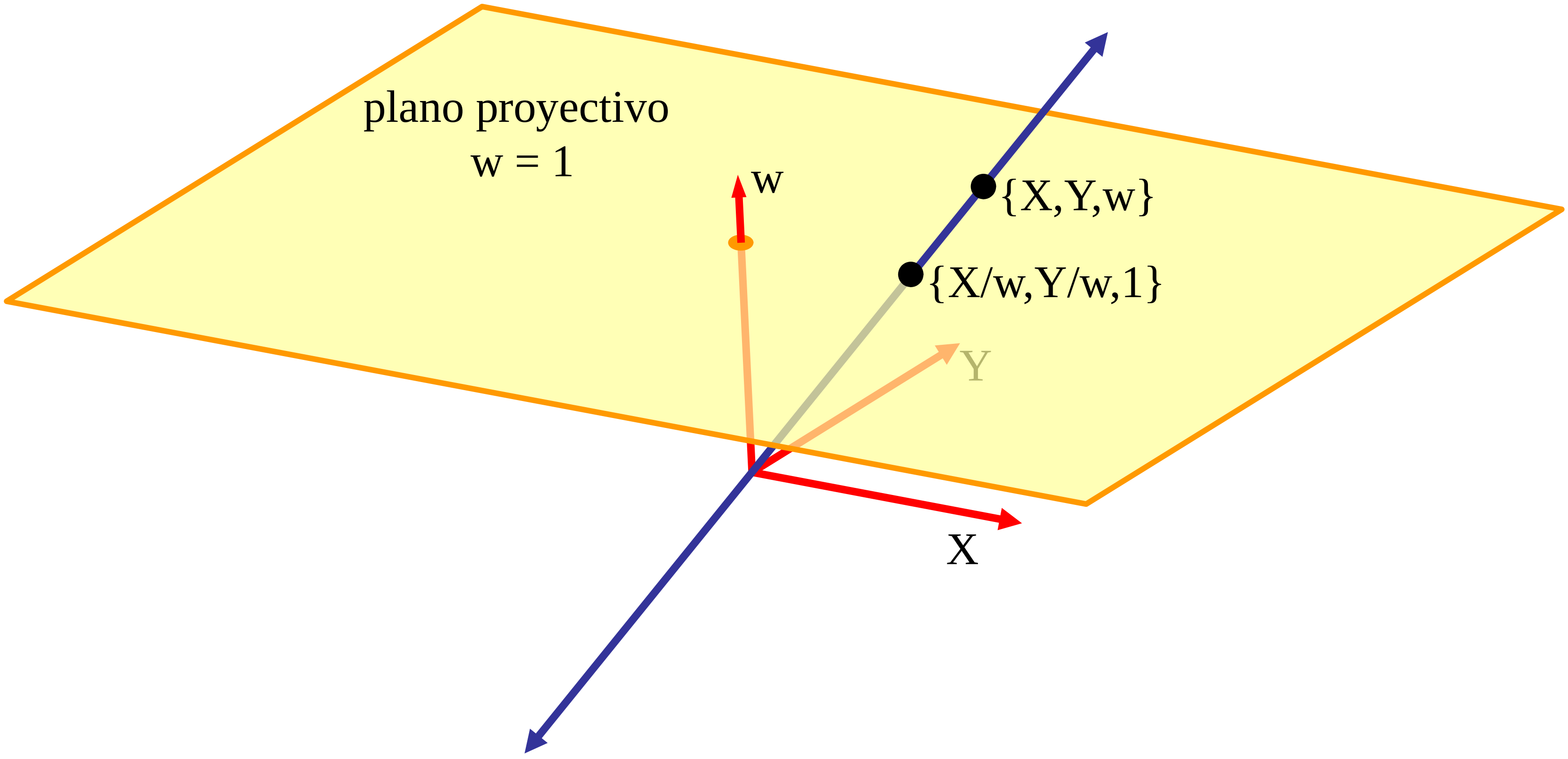
$$\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{t}}=0$$

Espacio Proyectivo

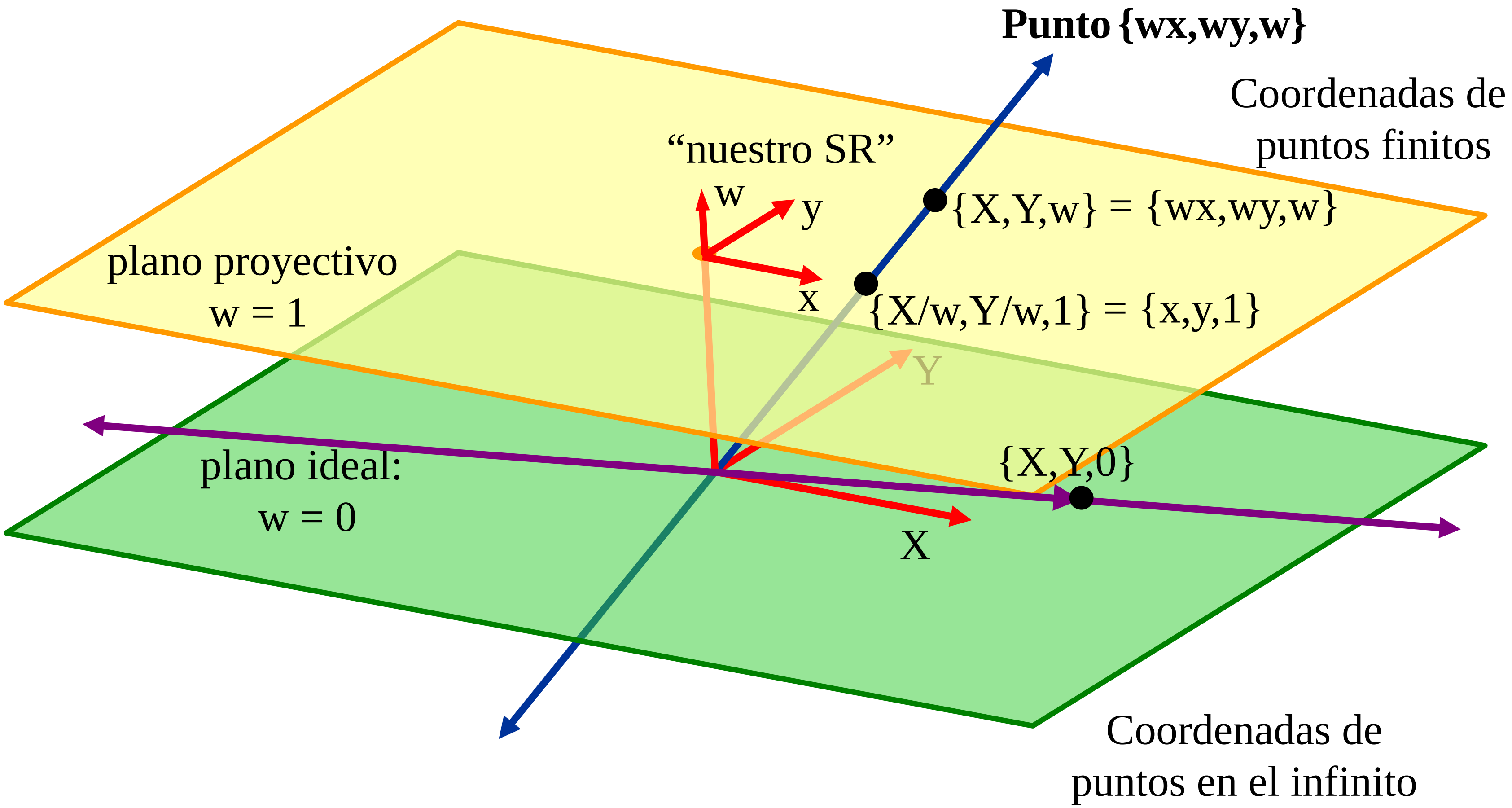
Espacio Proyectivo



Espacio Proyectivo



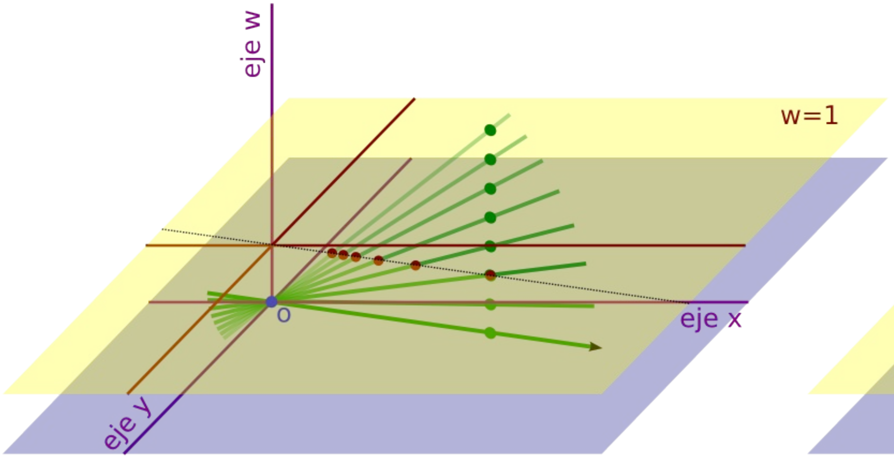
Espacio Proyectivo

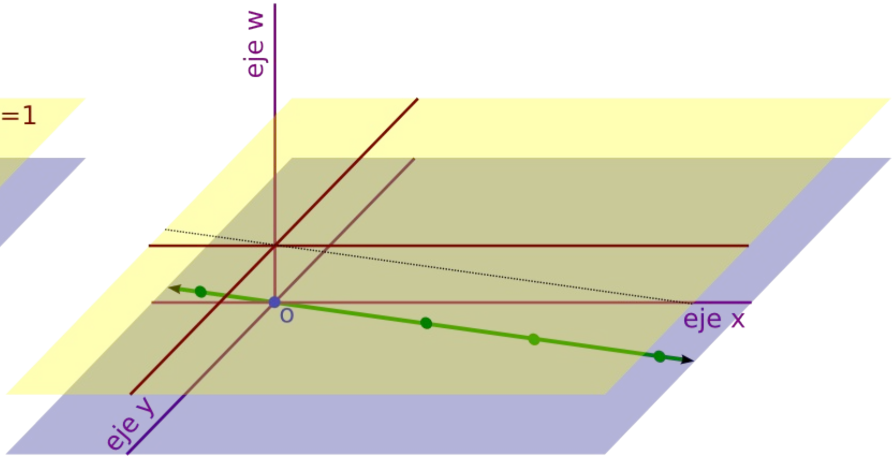


Espacio Proyectivo y Plano Ideal

Punto ideal:

- Si w≠0: la recta corta al plano en (x/w,y/w,1)
- Si w=0: la recta es horizontal, y se corresponde con un punto ideal, en el infinito, definido por un "vector" dirección.





El efecto de una transformación lineal en $\mathbb{R}^3$ sobre los puntos de $\mathbb{P}^2$ se denomina transformación proyectiva.

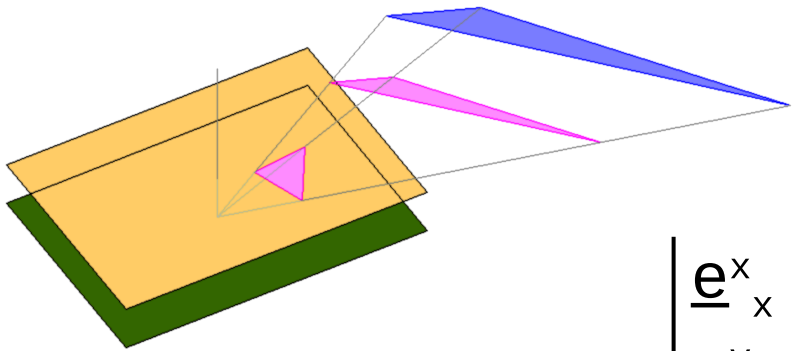
No proyecta

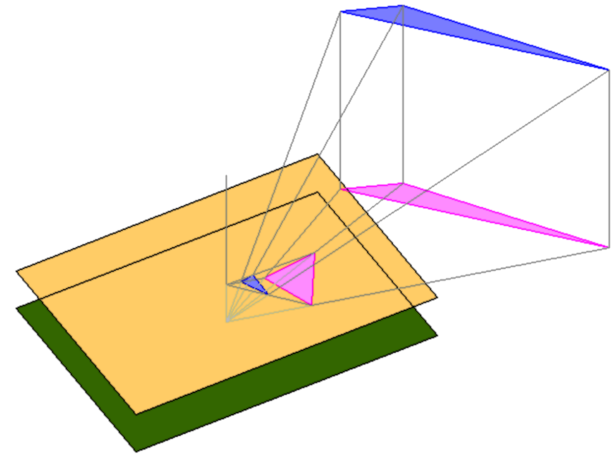
$$\begin{vmatrix} p^x \\ p^y \\ p^w \end{vmatrix}$$

$$\begin{vmatrix} \underline{e}^x{}_x & \underline{e}^x{}_y & \underline{e}^x{}_w \\ \underline{e}^y{}_x & \underline{e}^y{}_y & \underline{e}^y{}_w \\ \underline{e}^w{}_x & \underline{e}^w{}_y & \underline{e}^w{}_w \end{vmatrix} \begin{vmatrix} \underline{p}^x \\ \underline{p}^y \\ \underline{p}^w \end{vmatrix}$$

$$\begin{matrix} \Uparrow & \Uparrow & \Uparrow \\ \underline{\mathbf{e}}_x & \underline{\mathbf{e}}_y & \underline{\mathbf{e}}_w \end{matrix}$$

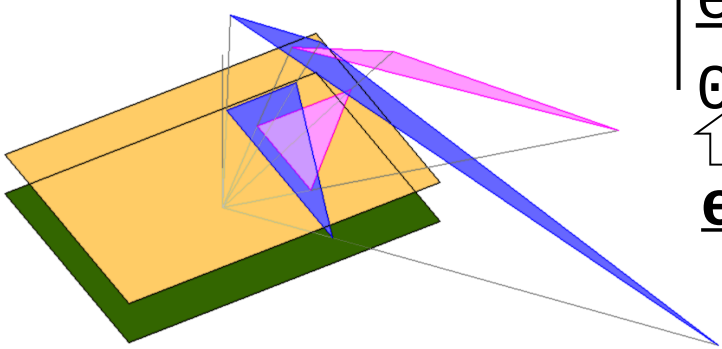
Transformaciones Afines en P^2

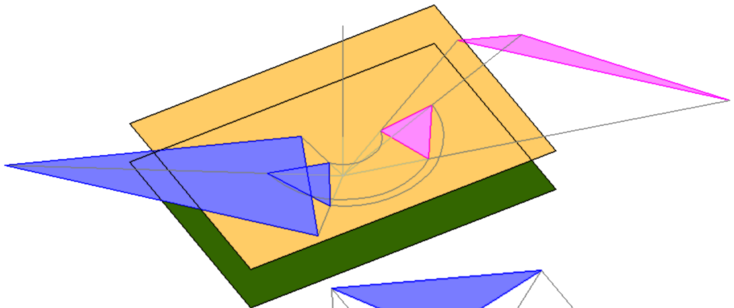


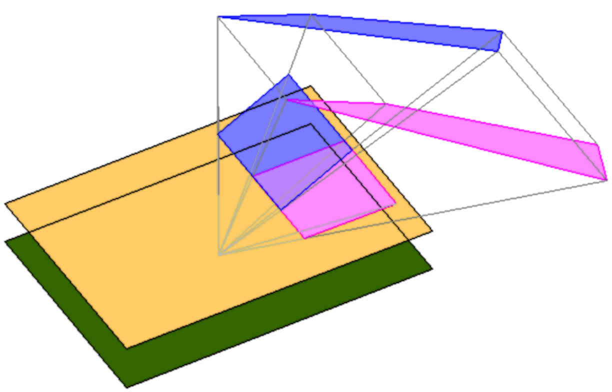


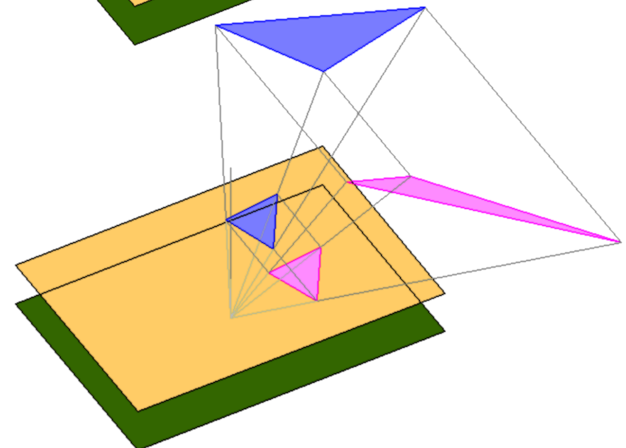
$$\begin{vmatrix} \underline{e}^x{}_x & \underline{e}^x{}_y & \underline{O}^x \\ \underline{e}^y{}_x & \underline{e}^y{}_y & \underline{O}^y \\ 0 & 0 & 1 \end{vmatrix}$$

$$\begin{array}{ccc} \Uparrow & \Uparrow & \Uparrow \\ \underline{\mathbf{e}}_x & \underline{\mathbf{e}}_y & \underline{\mathbf{e}}_w \end{array}$$









Transformaciones Afines en P^3

Parte lineal 3D

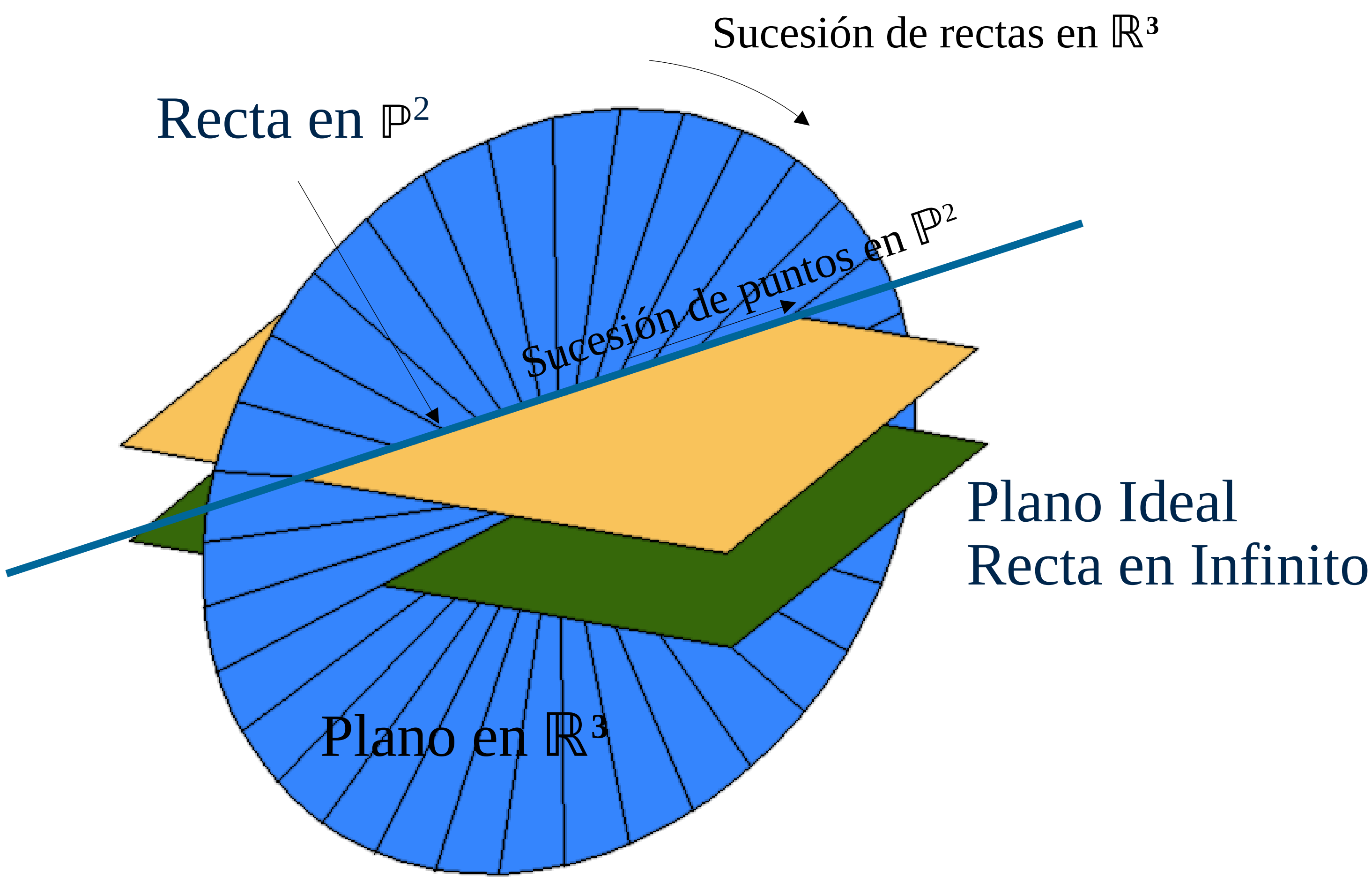
Traslación

$$
\begin{array}{cccc}
\multicolumn{4}{c}{\left|\begin{array}{cccc}
\underline{e}_x^{\,x} & \underline{e}_y^{\,x} & \underline{e}_z^{\,x} & \underline{O}^x \\
\underline{e}_x^{\,y} & \underline{e}_y^{\,y} & \underline{e}_z^{\,y} & \underline{O}^y \\
\underline{e}_x^{\,z} & \underline{e}_y^{\,z} & \underline{e}_z^{\,z} & \underline{O}^z \\
0 & 0 & 0 & 1
\end{array}\right|} \\
\Uparrow & \Uparrow & \Uparrow & \Uparrow \\
\underline{\mathbf{e}}_x & \underline{\mathbf{e}}_y & \underline{\mathbf{e}}_z & \underline{\mathbf{e}}_w\texttt{(origen)}
\end{array}
$$

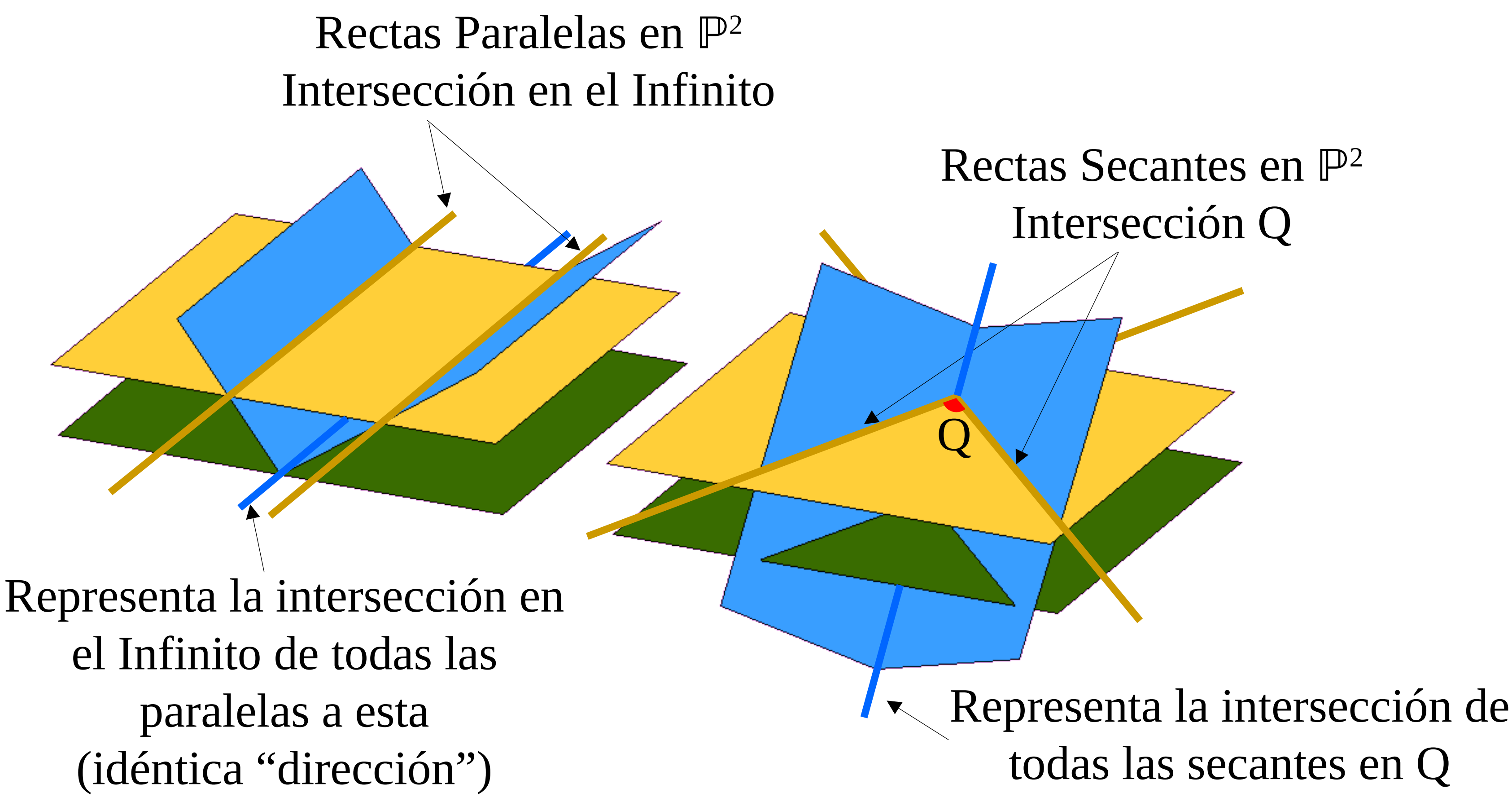
El plano ideal se mapea sobre si mismo

(puede girar o deformarse, pero no puede inclinarse)

Planos por el Origen en P^2 = Rectas en R^2



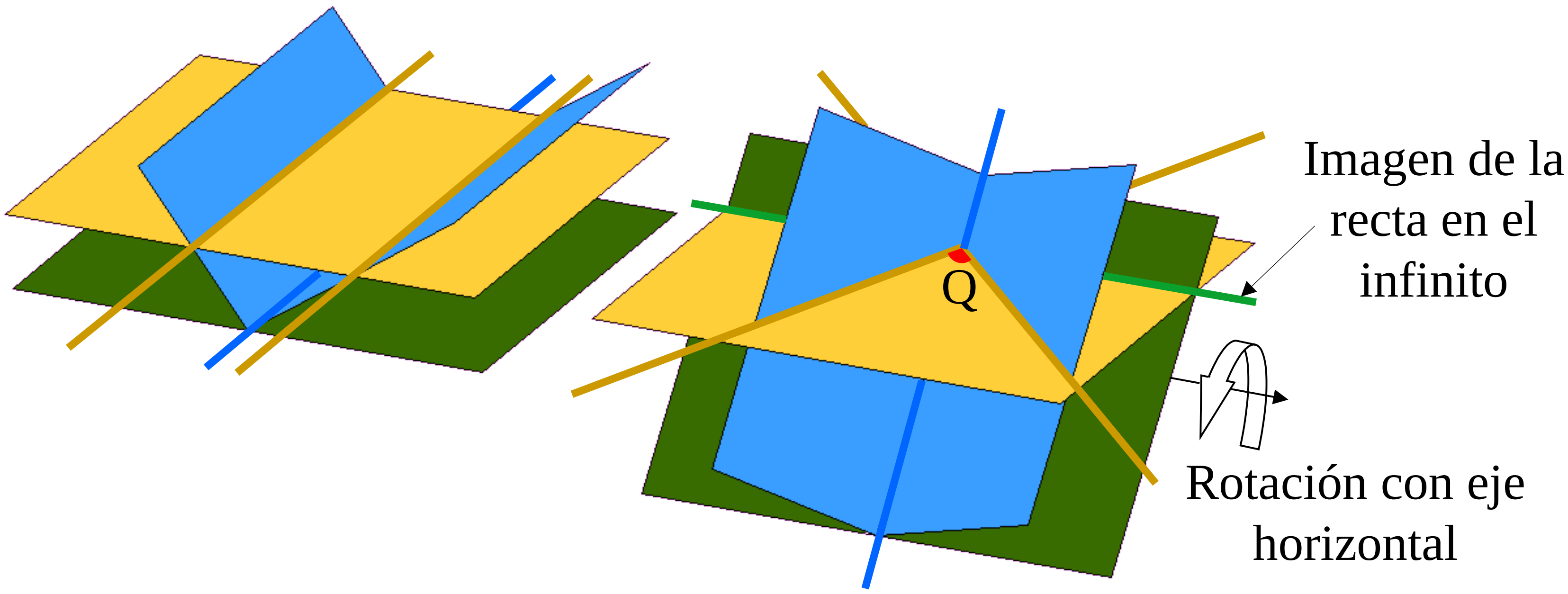
Planos por el Origen en P^2 = Rectas en R^2



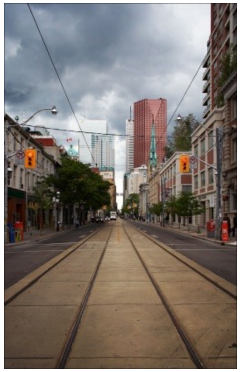
En el espacio proyectivo todo par de rectas tiene un punto común.

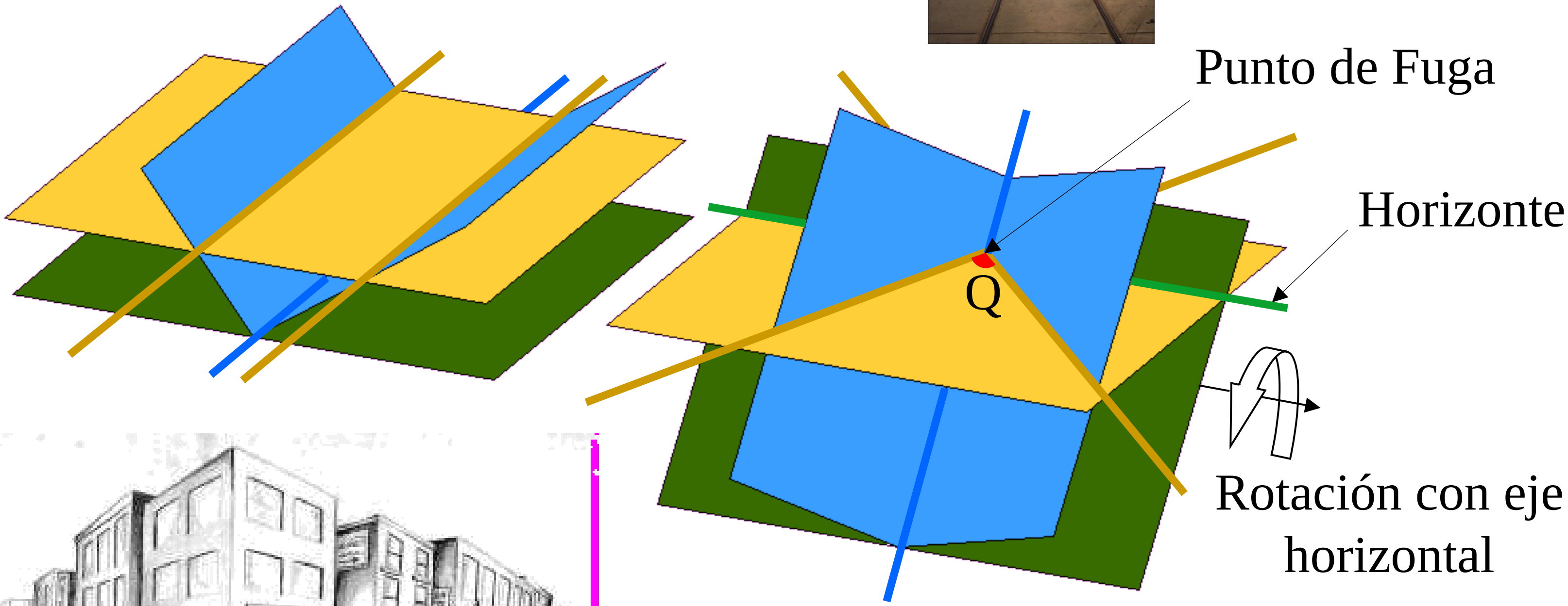
Transformación Proyectiva General

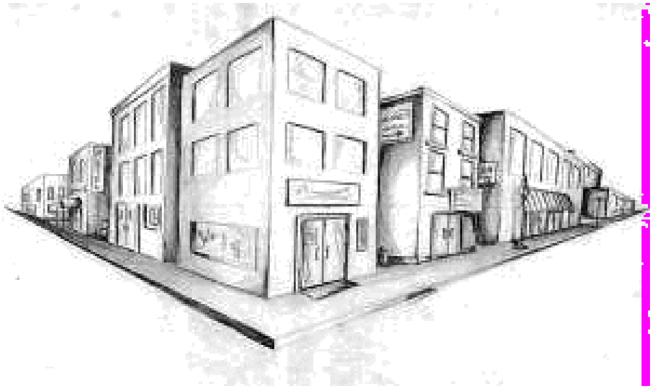
El plano ideal puede inclinarse



Transformación Proyectiva General

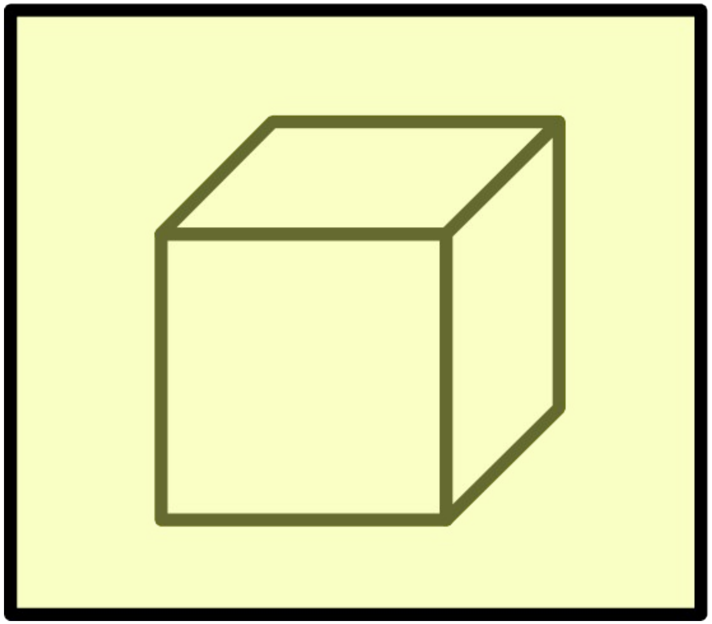


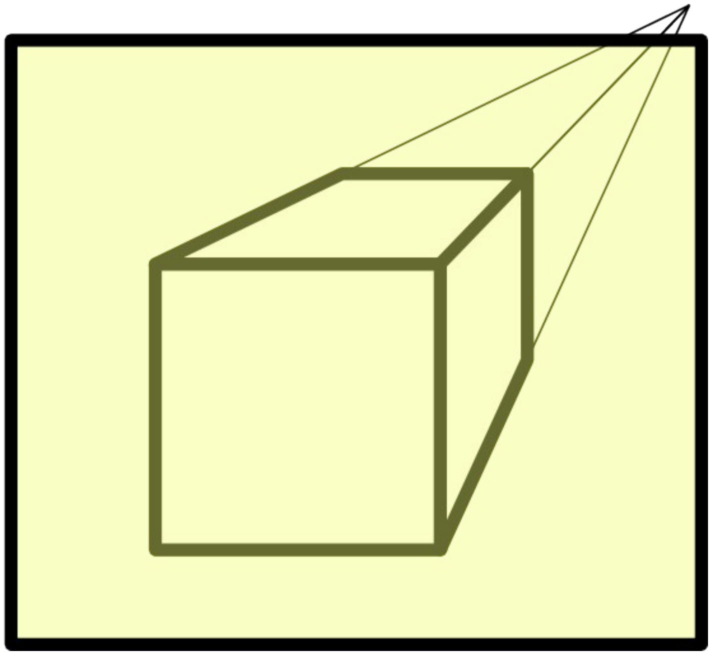




Proyecciones

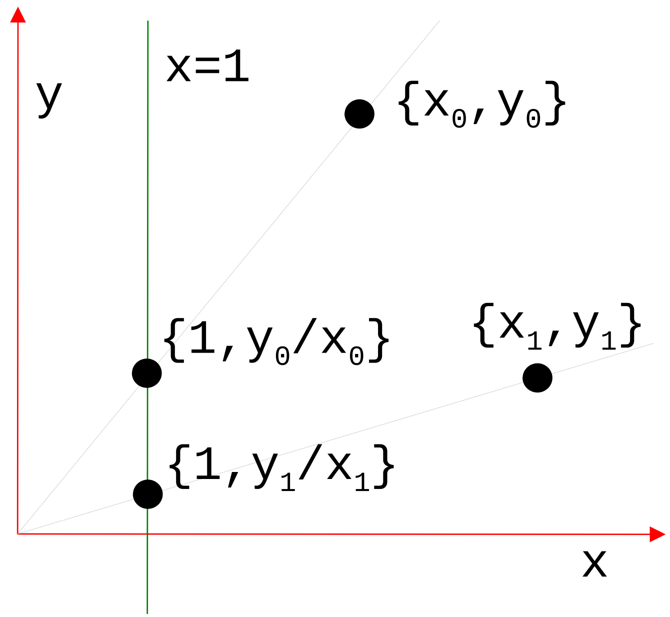
Transformación no invertibles, de rango incompleto.

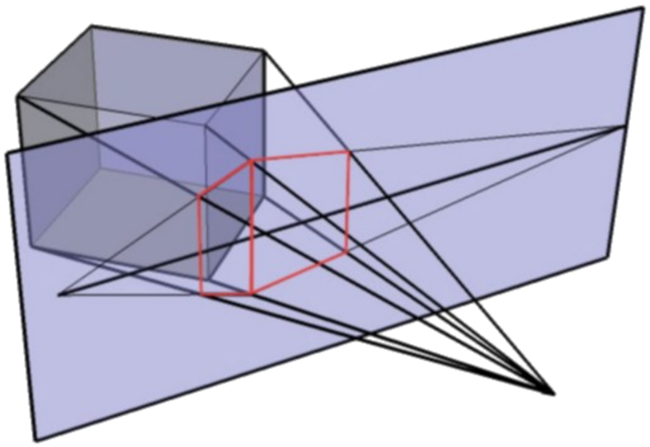




Ortogonal **Perspectiva**

Ejemplo: Perspectiva Central





$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} 1/x & 0 & 0 \\ 0 & 1/x & 0 \\ 0 & 0 & 1/x \end{bmatrix} \begin{bmatrix} 1 \\ y/x \\ z/x \end{bmatrix}$$

$$\begin{bmatrix} wx \\ wy \\ wz \\ w \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} wx \\ wy \\ wz \\ wx \end{bmatrix}$$

Transformación Proyectiva General 3D

$$\begin{array}{cc} \mathbf{P} & \mathbf{v} \\ \Downarrow & \Downarrow \\ \begin{vmatrix} wx \\ wy \\ wz \\ w \end{vmatrix} & \begin{vmatrix} x \\ y \\ z \\ 0 \end{vmatrix} \end{array}$$

$$\begin{array}{cccccc} \left| \begin{matrix} \underline{e}_x{}^x & \underline{e}_y{}^x & \underline{e}_z{}^x & \underline{e}_w{}^x \\ \underline{e}_x{}^y & \underline{e}_y{}^y & \underline{e}_z{}^y & \underline{e}_w{}^y \\ \underline{e}_x{}^z & \underline{e}_y{}^z & \underline{e}_z{}^z & \underline{e}_w{}^z \\ \underline{e}_x{}^w & \underline{e}_y{}^w & \underline{e}_z{}^w & \underline{e}_w{}^w \end{matrix} \right| & & \begin{vmatrix} \underline{wx} \\ \underline{wy} \\ \underline{wz} \\ \underline{w} \end{vmatrix} & \begin{vmatrix} \underline{x} \\ \underline{y} \\ \underline{z} \\ 0 \end{vmatrix} \\ \Uparrow \quad \Uparrow \quad \Uparrow \quad \Uparrow & & \Uparrow & \Uparrow \\ \underline{\mathbf{e}}_x \quad \underline{\mathbf{e}}_y \quad \underline{\mathbf{e}}_z \quad \underline{\mathbf{e}}_w & & \underline{\mathbf{P}} & \underline{\mathbf{v}} \end{array}$$

¿Preguntas?