DIFERENCIAS FINITAS

Buscamos aproximar los operadores diferenciales presentes en las ecuaciones por **operadores en diferencias**, para transformar una ecuación diferencial en un sistema de ecuaciones algebraicas donde las incógnitas son nodos del dominio en los cuales obtendremos la solución**.**

$$\rho c_p \frac{\partial \emptyset}{\partial t} = k \frac{\partial^2 \emptyset}{\partial x^2} - c\emptyset + G(x) \quad \rightarrow \quad \mathbf{K}\phi = \mathbf{F}$$

Series de Taylor: $\emptyset_{\mathrm{i+k}} = \emptyset_{\mathrm{i}} + \sum_{\mathrm{m=1}}^{\infty} \frac{(\mathrm{k}\Delta \mathrm{x})^{\mathrm{m}}}{\mathrm{m}!} \frac{\partial^{\mathrm{m}} \emptyset}{\partial \mathrm{x}^{\mathrm{m}}}\Big|_{\mathrm{i}}$

Ejemplos:

$$\emptyset_{i+1} = \emptyset_i + \Delta x \frac{\partial \emptyset}{\partial x}\Big|_i + \frac{(\Delta x)^2}{2} \frac{\partial^2 \emptyset}{\partial x^2}\Big|_i + \frac{(\Delta x)^3}{3!} \frac{\partial^3 \emptyset}{\partial x^3}\Big|_i + \frac{(\Delta x)^4}{4!} \frac{\partial^4 \emptyset}{\partial x^4}\Big|_i + \dots$$

$$\emptyset_{i-1} = \emptyset_i - \Delta x \frac{\partial \emptyset}{\partial x}\Big|_i + \frac{(\Delta x)^2}{2} \frac{\partial^2 \emptyset}{\partial x^2}\Big|_i - \frac{(\Delta x)^3}{3!} \frac{\partial^3 \emptyset}{\partial x^3}\Big|_i + \frac{(\Delta x)^4}{4!} \frac{\partial^4 \emptyset}{\partial x^4}\Big|_i + \dots$$

$$\emptyset_{i+2} = \emptyset_i + 2\Delta x \frac{\partial \emptyset}{\partial x}\Big|_i + \frac{(2\Delta x)^2}{2} \frac{\partial^2 \emptyset}{\partial x^2}\Big|_i + \frac{(2\Delta x)^3}{3!} \frac{\partial^3 \emptyset}{\partial x^3}\Big|_i + \frac{(2\Delta x)^4}{4!} \frac{\partial^4 \emptyset}{\partial x^4}\Big|_i + \dots$$

importante para desarrollar taylor

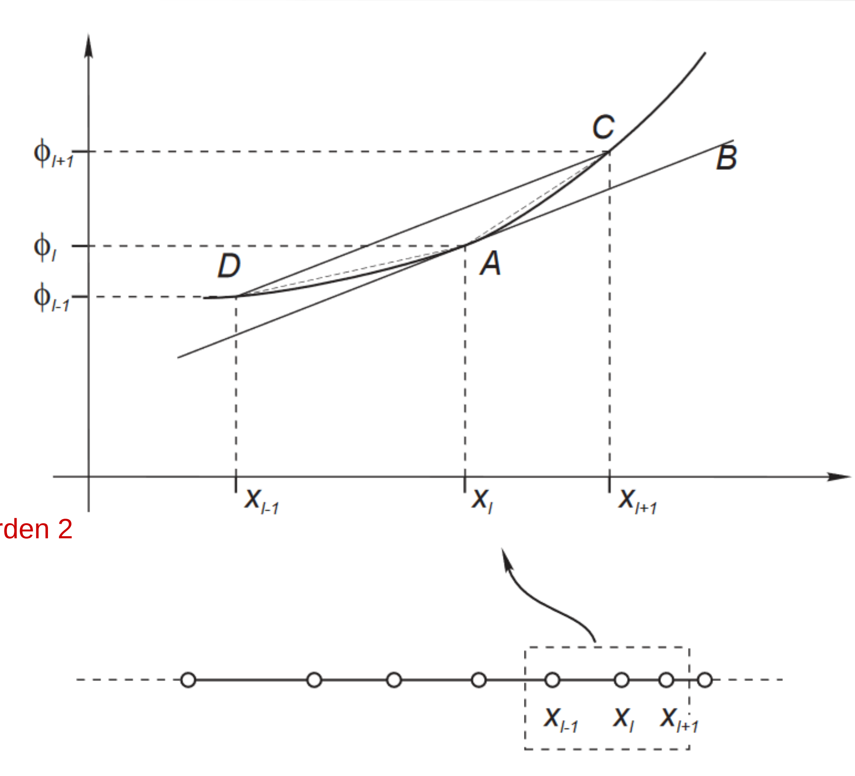
DIFERENCIAS FINITAS

Resumen de las aproximaciones más comunes a utilizar:

Operador	Orden del error
$\left.\frac{\partial \emptyset}{\partial x}\right\|_i \approx \frac{\emptyset_i - \emptyset_{i-1}}{\Delta x}$	$o(\Delta x)$
$\left.\frac{\partial \emptyset}{\partial x}\right\|_i \approx \frac{\emptyset_{i+1} - \emptyset_i}{\Delta x}$	$o(\Delta x)$
$\left.\frac{\partial \emptyset}{\partial x}\right\|_i \approx \frac{\emptyset_{i+1} - \emptyset_{i-1}}{2\Delta x}$	$o(\Delta x^2)$
$\left.\frac{\partial^2 \emptyset}{\partial x^2}\right\|_i \approx \frac{\emptyset_{i+1} - 2\emptyset_i + \emptyset_{i-1}}{\Delta x^2}$	$o(\Delta x^2)$

orden 1

orden 2



Relación entre puntos usados, orden de la derivada, precisión del error: **N ≥ k + p**

En aproximaciones centradas y mallas uniformes ganamos un orden de aproximación…

DIFERENCIAS FINITAS

Retomando la ecuación diferencial:

$$\rho c_p \frac{\partial \emptyset}{\partial t} = k\frac{\partial^2 \emptyset}{\partial x^2} - c\emptyset + G(x)$$

Y sus posibles condiciones de borde: (sumado a una condición inicial)

$$\emptyset = \overline{\emptyset}, \qquad -k\frac{\partial \emptyset}{\partial \eta} = q, \qquad -k\frac{\partial \emptyset}{\partial \eta} = h(\emptyset - \emptyset_\infty)$$

Discretización completa (mallas uniformes):

$$\rho c_p \frac{\emptyset_i^{n+1} - \emptyset_i^n}{\Delta t} = k\left(\frac{\emptyset_{i-1}^{n+\theta} - 2\emptyset_i^{n+\theta} + \emptyset_{i+1}^{n+\theta}}{\Delta x^2}\right) - c\emptyset_i^{n+\theta} + G_i$$

$$-k\left(\frac{\emptyset_{i+1}^{n+\theta} - \emptyset_{i-1}^{n+\theta}}{2\Delta x}\right)(\pm 1) = q, \qquad -k\left(\frac{\emptyset_{i+1}^{n+\theta} - \emptyset_{i-1}^{n+\theta}}{2\Delta x}\right)(\pm 1) = h\left(\emptyset_i^{n+\theta} - \emptyset_\infty\right)$$

DIFERENCIAS FINITAS

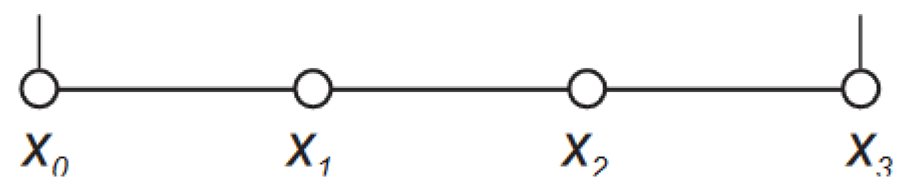
Ejercicio 1a GTP: $\rho c_p = 0; k = 2; c = 0; G(x) = 100$

$$2\frac{\partial^2 \emptyset}{\partial \mathrm{x}^2} + 100 = 0; \quad \forall \mathrm{x}[0,1]$$

$$\emptyset(0) = 10; \emptyset(1) = 50$$

Solución analítica: $\emptyset(x) = -25x^2 + 65x + 10$

Discretización propuesta: $\Delta x = {}^1/_3$



$$2\left(\frac{\emptyset_{i-1} - 2\emptyset_i + \emptyset_{i+1}}{\Delta x^2}\right) + 100 = 0$$

Incógnitas del problema? $\emptyset_1 \; y \; \emptyset_2$

Siempre buscaremos armar un sistema de N ecuaciones con N incógnitas…

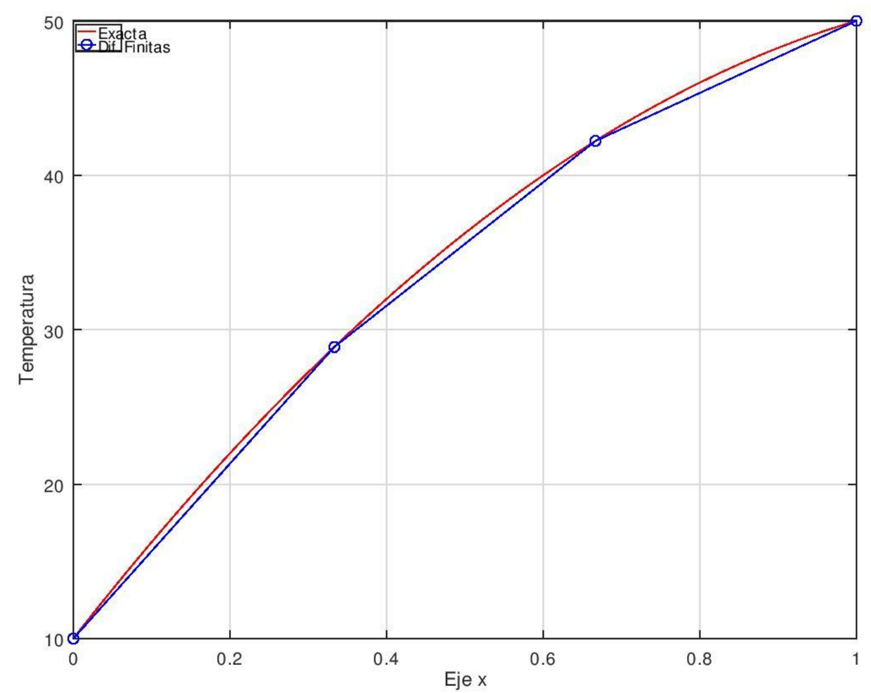
DIFERENCIAS FINITAS

Sistema de ecuaciones	Sistema matricial	Sistema matricial ampliado
$-2\emptyset_1 + \emptyset_2 = -50\Delta x^2 - \emptyset_0$ $\emptyset_1 - 2\emptyset_2 = -50\Delta x^2 - \emptyset_3$	$\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} \emptyset_1 \\ \emptyset_2 \end{pmatrix} = \begin{pmatrix} -140/9 \\ -500/9 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \emptyset_0 \\ \emptyset_1 \\ \emptyset_2 \\ \emptyset_3 \end{pmatrix} = \begin{pmatrix} 10 \\ -50/9 \\ -50/9 \\ 50 \end{pmatrix}$

$$\emptyset = K^{-1}F \quad \rightarrow \quad \emptyset = [10 \quad 28.89 \quad 42.22 \quad 50]$$

Generalizando **K** para n nodos interiores:

1	0	0	0	0	0
1	-2	1	0	0	0
0	1	-2	1	0	0
0	0	1	-2	1	0
0	0	0	1	-2	1
0	0	0	0	0	1



DIFERENCIAS FINITAS

Ejercicio 1c GTP: $\rho c_p = 0; k = 1; c = 0; G(x) = 100(x-3)^2$

$$\frac{\partial^2 \emptyset}{\partial \mathrm{x}^2} + 100(x-3)^2 = 0;\ \ \forall \mathrm{x}[1,5]$$

$$\mathrm{q}(1) = 2;\ \emptyset(5) = 0$$

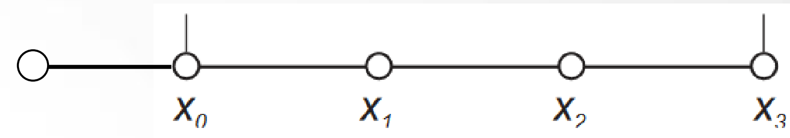
Solución analítica: $\emptyset(x) = \frac{-25x^4+300x^3-1350x^2+1906x+2345}{3}$

Discretización propuesta: $\Delta x = \frac{L}{N-1} = \frac{4}{3}$

$$(\emptyset_{i-1} - 2\emptyset_i + \emptyset_{i+1}) = -100\Delta x^2(x_i - 3)^2$$

Incógnitas del problema? $\emptyset_0,\ \emptyset_1\ y\ \emptyset_2$

DIFERENCIAS FINITAS



Planteo de condiciones de borde:

- Sin nodo ficticio o(Δx) (quitamos calidad a la solucion)

$$q(1) = 2 \;\rightarrow\; -k\frac{\partial \emptyset}{\partial x}(-1) = 2 \;\rightarrow\; k\left(\frac{\emptyset_1 - \emptyset_0}{\Delta x}\right) = 2$$ (pensar el extremo x=5)

- Con nodo ficticio o(Δx²)

nodo ficticio despejado

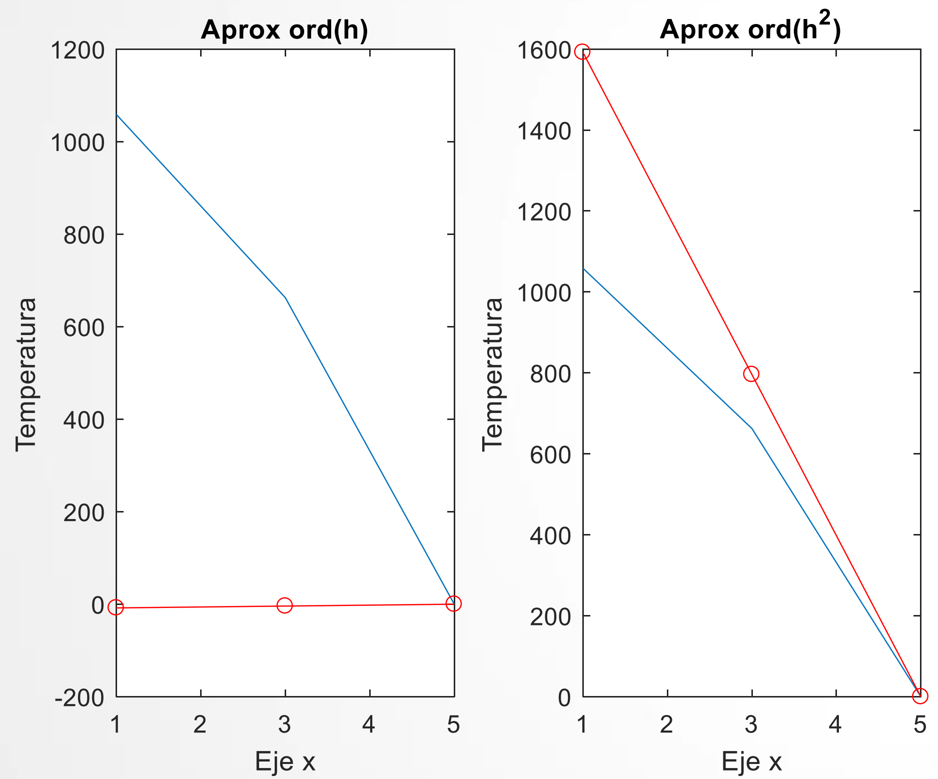
$$q(1) = 2 \;\rightarrow\; -k\frac{\partial \emptyset}{\partial x}(-1) = 2 \;\rightarrow\; k\left(\frac{\emptyset_1 - \emptyset_f}{2\Delta x}\right) = 2 \;\rightarrow\; \emptyset_f = \emptyset_1 - \frac{4\Delta x}{k}$$

Sistema matricial orden Δx	Sistema matricial orden Δx²
$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \emptyset_0 \\ \emptyset_1 \\ \emptyset_2 \\ \emptyset_3 \end{pmatrix} = \begin{pmatrix} 8/3 \\ -79.0123 \\ -79.0123 \\ 0 \end{pmatrix}$	$\begin{pmatrix} -2 & 2 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \emptyset_0 \\ \emptyset_1 \\ \emptyset_2 \\ \emptyset_3 \end{pmatrix} = \begin{pmatrix} -711.11 + 16/3 \\ -79.0123 \\ -79.0123 \\ 0 \end{pmatrix}$

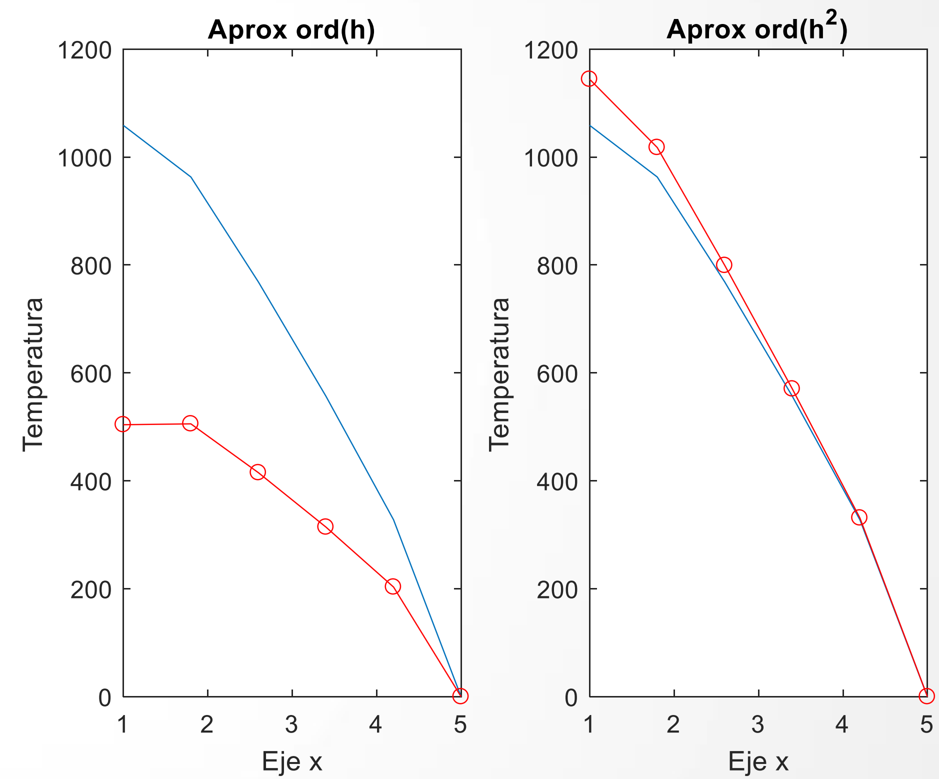
Fila 1: en orden Δx utilizo CB, mientras que en Δx² escribo la ecuación diferencial y despejo el nodo ficticio de la CB.

DIFERENCIAS FINITAS

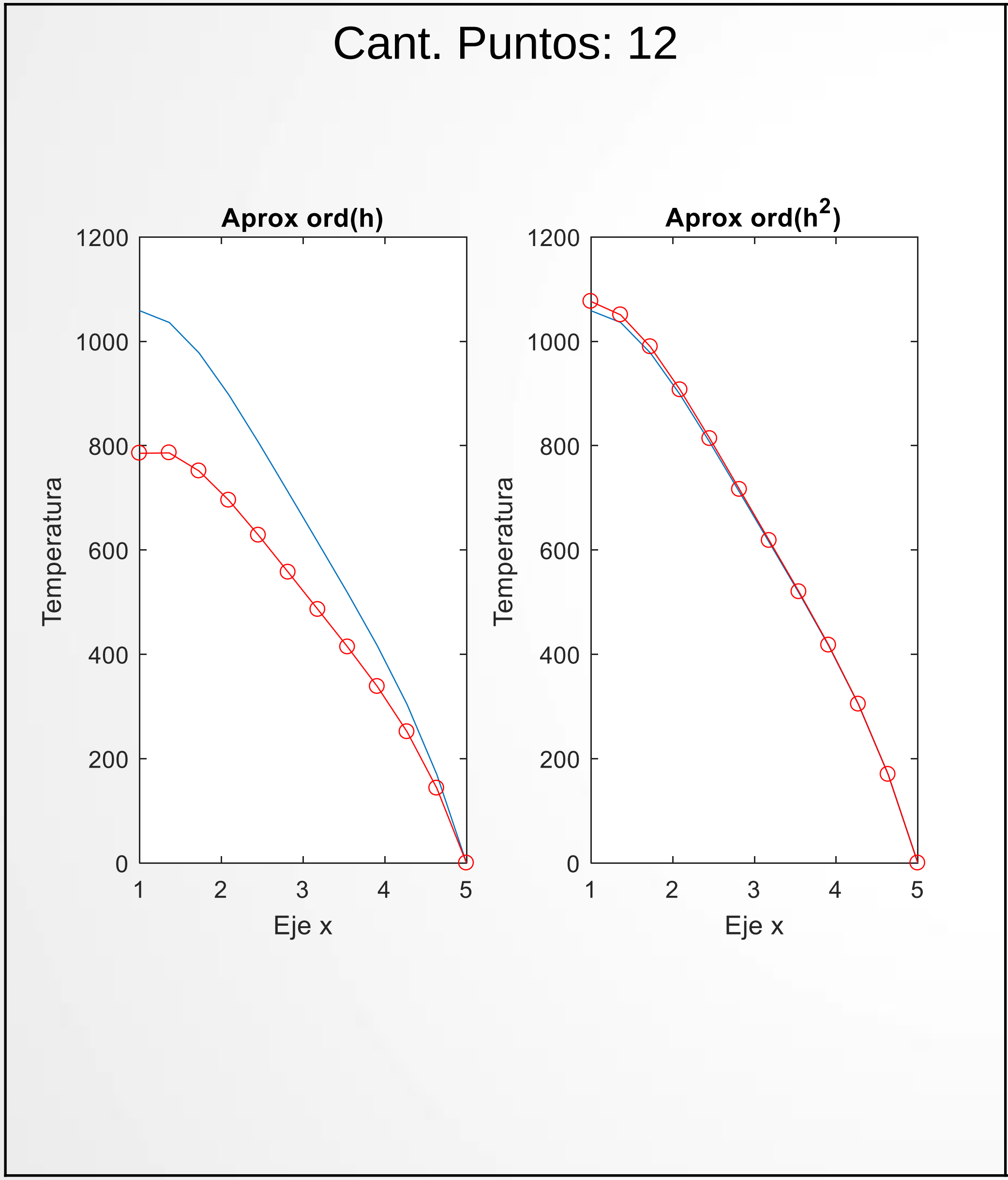
Cant. Puntos: 3

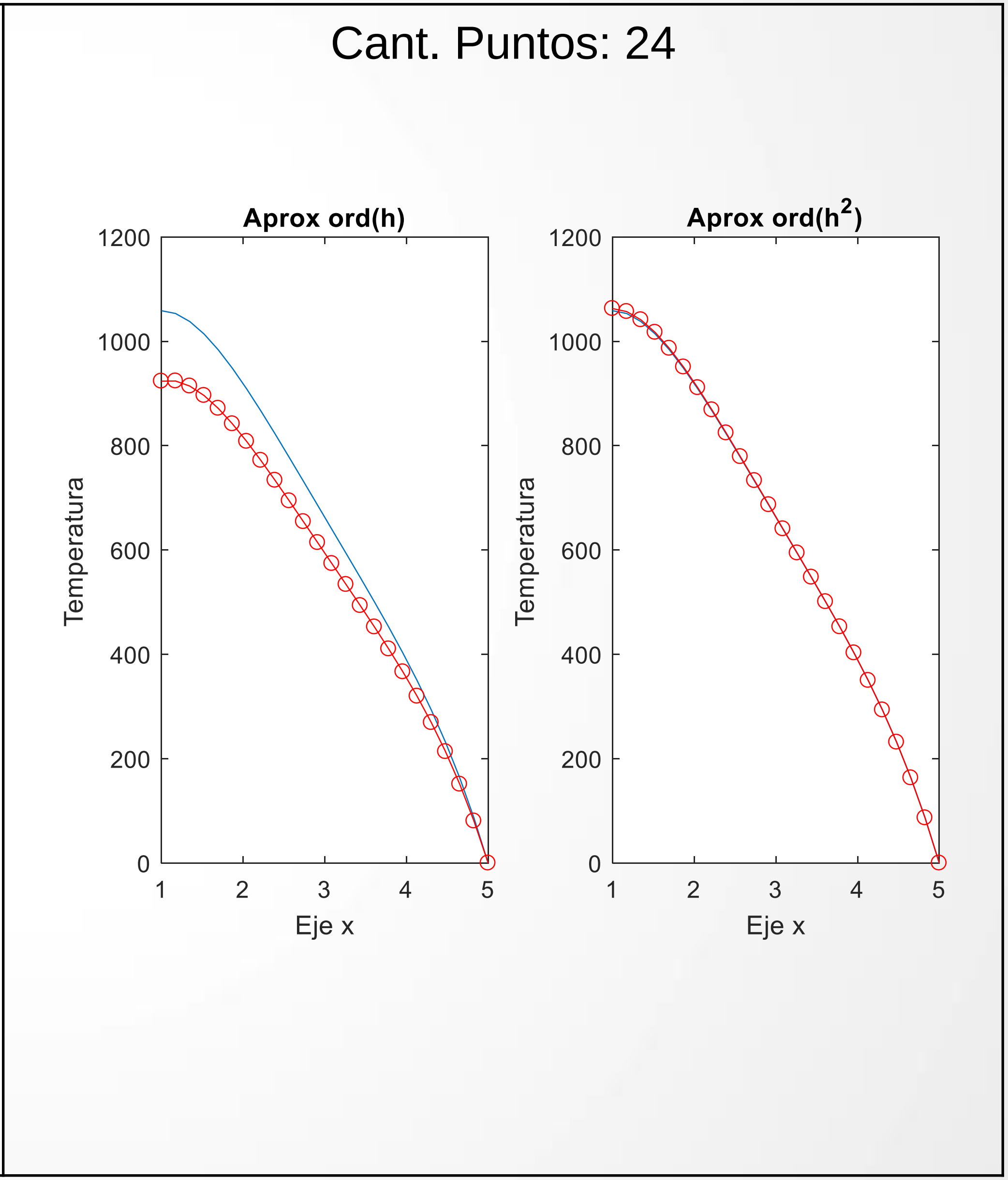


Cant. Puntos: 6

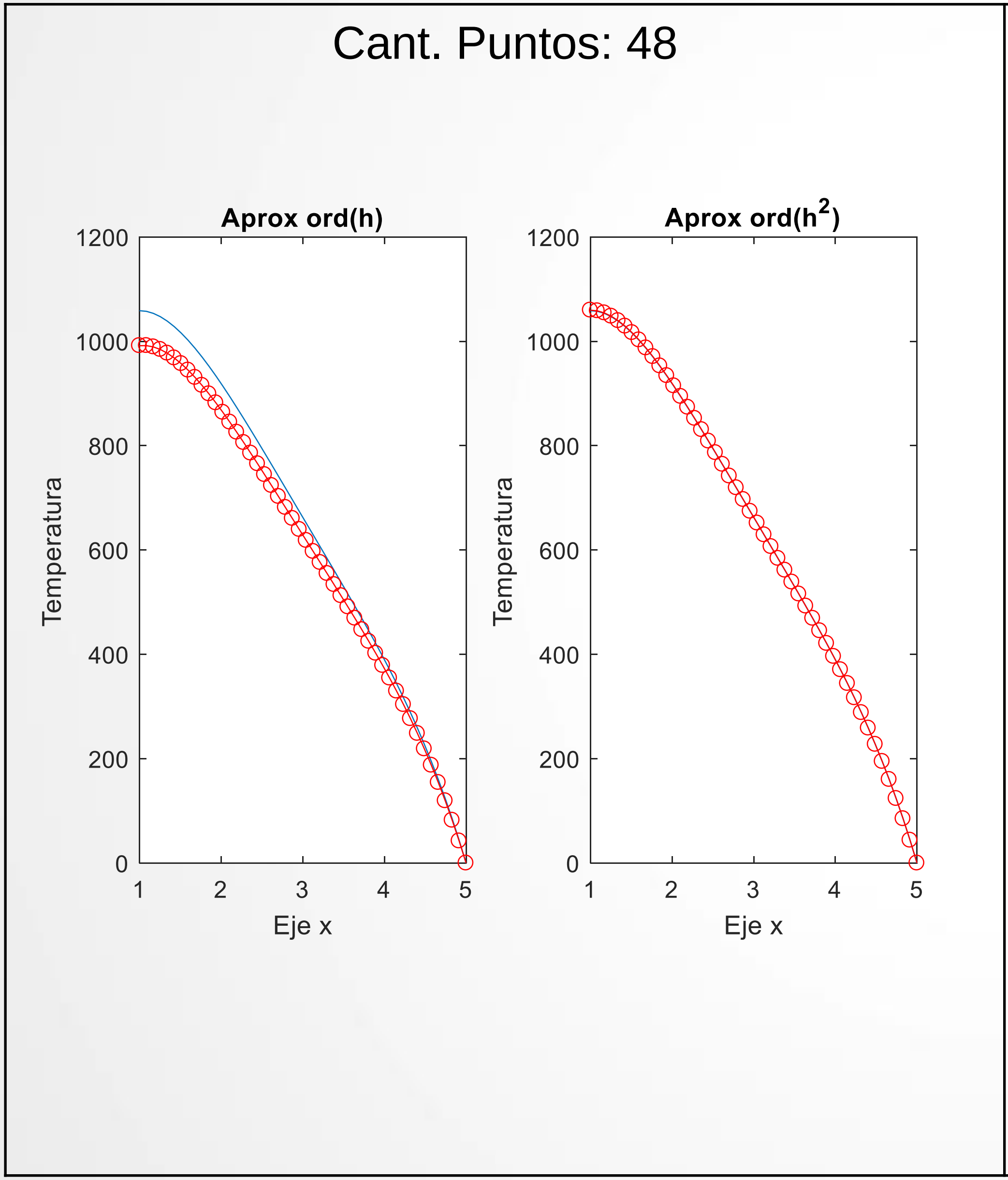


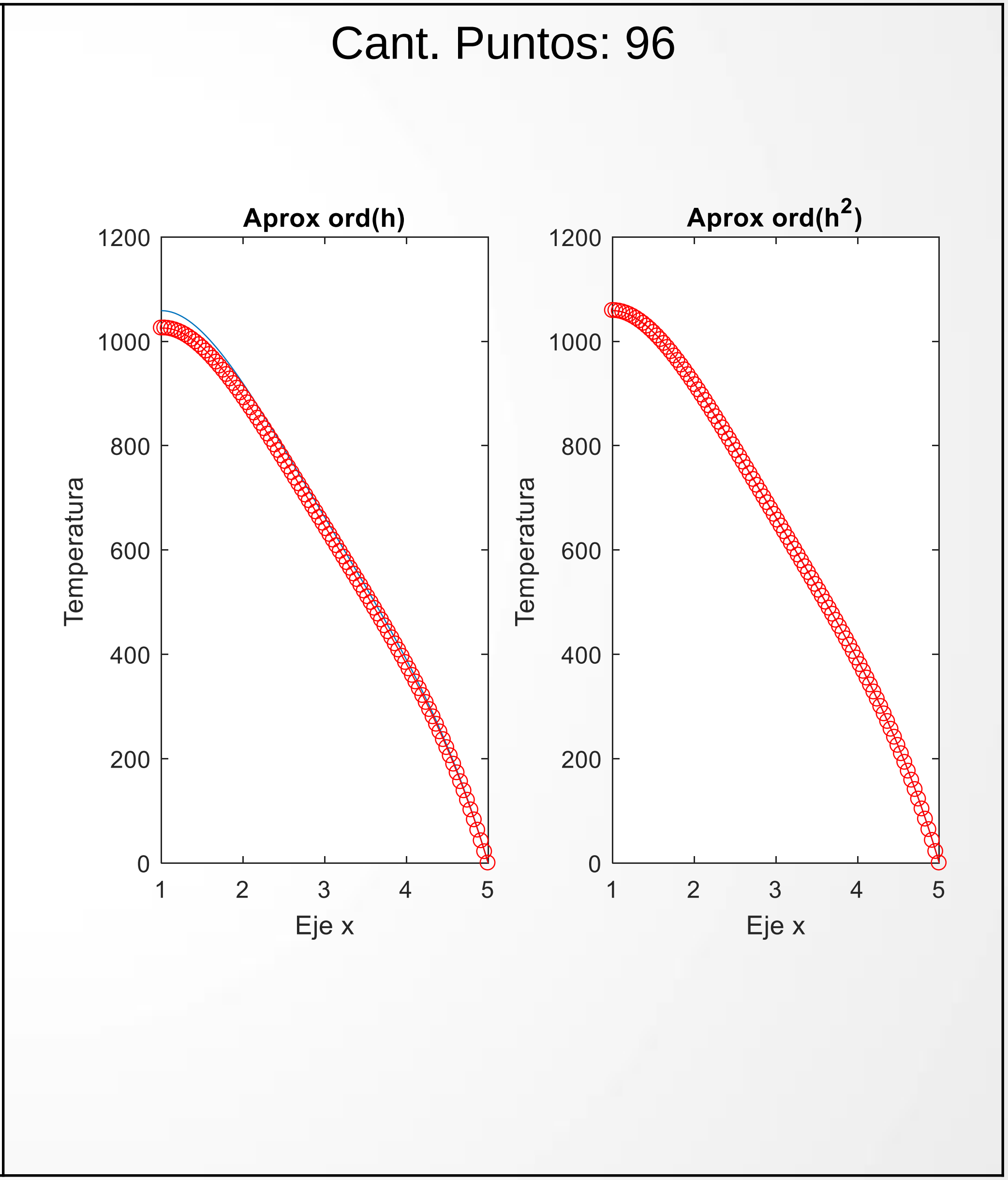
DIFERENCIAS FINITAS





DIFERENCIAS FINITAS



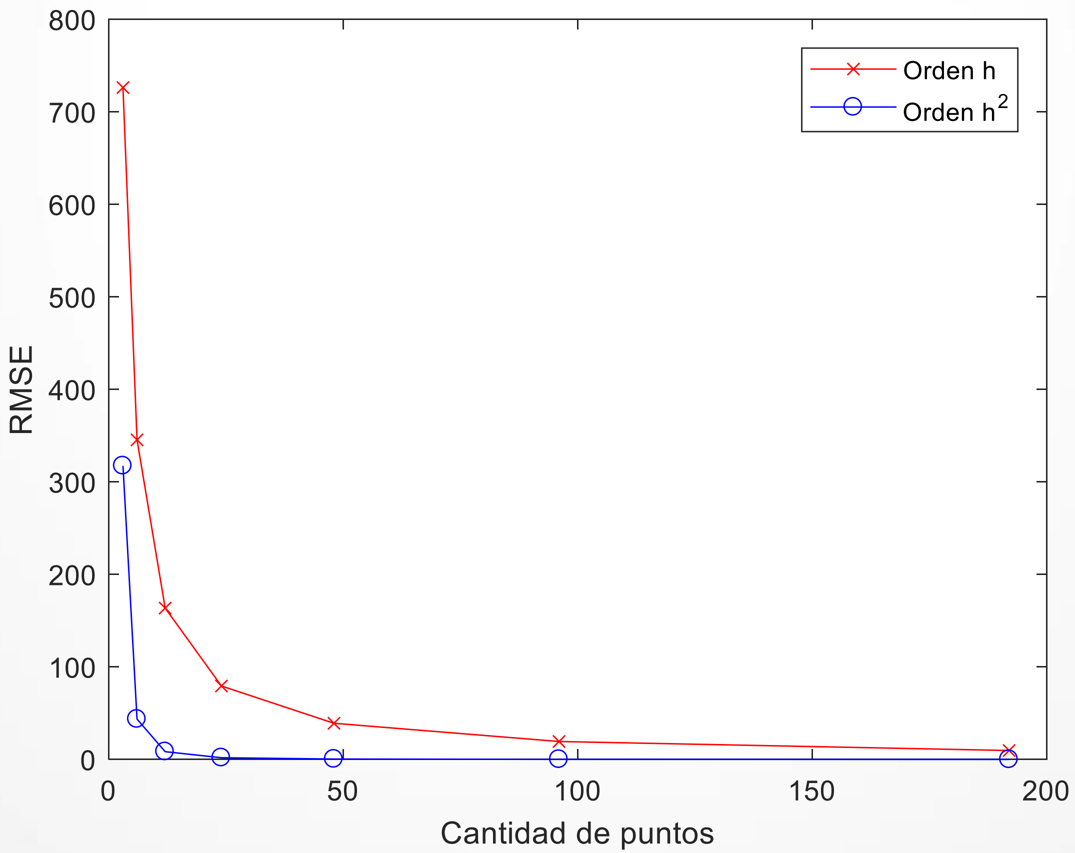


DIFERENCIAS FINITAS

Medimos el RMSE (raíz del error cuadrático medio) para cada aproximación:

$$Err_{oh} = \sqrt{\frac{\sum_{i=1}^{N}(\emptyset ex_i - Qap\_oh_i)^2}{N}}\,;\, Err_{oh2} = \sqrt{\frac{\sum_{i=1}^{N}(\emptyset ex_i - Qap\_oh2_i)^2}{N}}$$

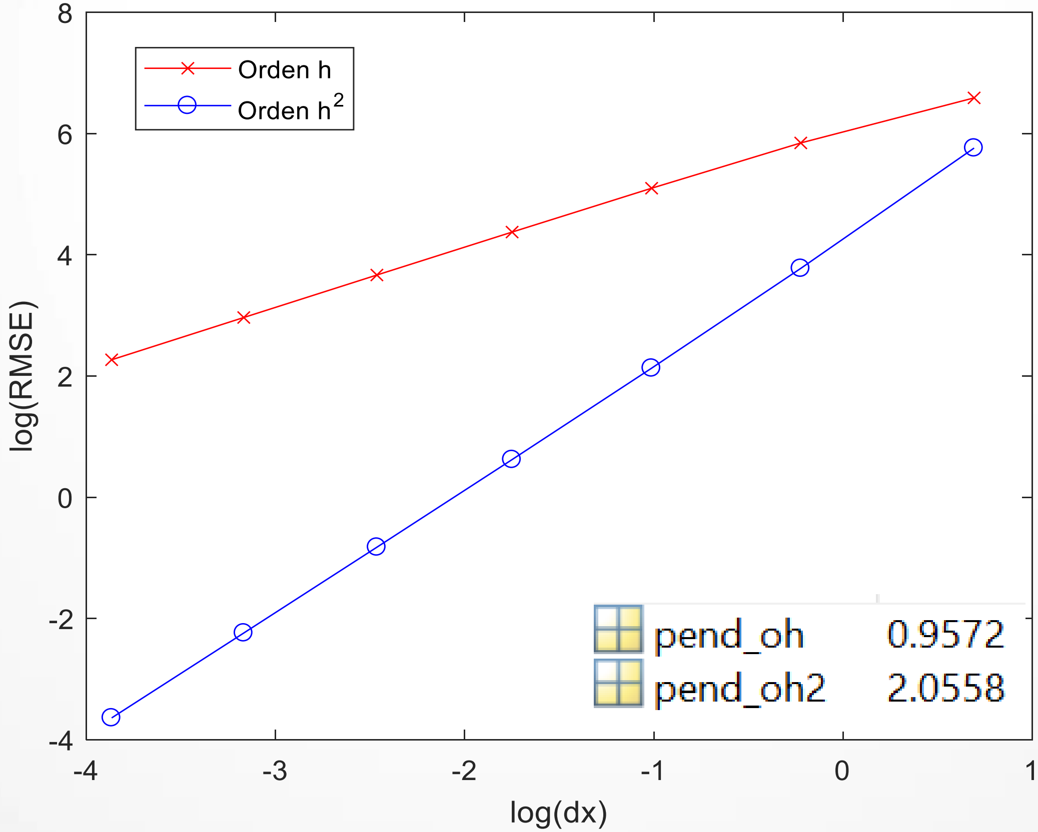
puede entrar en el parcial
chequear estos errores



DIFERENCIAS FINITAS

Sabemos que el error en l aproximación se comporta como: $E \approx C\Delta x^p$

Corroboramos la convergencia analizando: $\log E = \log C + p \log \Delta x$, cuya pendiente p determina si la solución converge de forma lineal o curadrática.



DIFERENCIAS FINITAS

```
function [T_ap] = solOrden1(h)
    k=1;
    x = 1:h:5;
    n = length(x);
    G = 100*((x(1:n-1)-3).^2);
    K = zeros(n);
    b = zeros(n,1);

    fila = [1 -2 1];
    for i=2:(n-1)
        K(i,i-1:i+1) = fila;
    end
    K(1,1:2) = [-1 1];
    K(n,n) = 1;

    b(1:n-1) = (-G./k)*(h^2);
    b(1) = 2*h;

    T_ap = K\b;
end
```

```
function [T_ap] = solOrden2(h)
    k=1;
    x = 1:h:5;
    n = length(x);
    G = 100*((x(1:n-1)-3).^2);
    K = zeros(n);
    b = zeros(n,1);

    fila = [1 -2 1];
    for i=2:(n-1)
        K(i,i-1:i+1) = fila;
    end
    K(1,1:2) = [-2 2];
    K(n,n) = 1;

    b(1:n-1) = (-G./k)*(h^2);
    b(1) = b(1) + 4*h;

    T_ap = K\b;
end
```

DIFERENCIAS FINITAS

Generalizando esquemas numéricos estacionarios (mallas uniformes):

$$k\frac{\partial^2 \emptyset}{\partial \mathrm{x}^2} - c\emptyset + G(x) = 0 \quad \rightarrow \quad -\emptyset_{i-1} + \left(2 + \frac{\Delta x^2 c}{k}\right)\emptyset_i - \emptyset_{i+1} = \frac{\Delta x^2 G_i}{k}$$

Condición de borde Neumann: $-k\frac{\partial \emptyset}{\partial \eta} = q$

- Extremo izquierdo: $-k\left(\frac{\phi_{int}-\phi_{fic}}{2\Delta x}\right)(-1) = q$
- Extremo derecho: $-k\left(\frac{\phi_{fic}-\phi_{int}}{2\Delta x}\right)(1) = q$

$$\longrightarrow \quad \boxed{\emptyset_{fic} = \emptyset_{int} - \frac{2\Delta x q}{k}}$$

$$\boxed{\emptyset_{fic} = \emptyset_{int} - \frac{2\Delta x h}{k}\emptyset_b + \frac{2\Delta x h}{k}\emptyset_{inf}}$$

Condición de borde Robin: $k\frac{\partial \emptyset}{\partial \eta} + h(\emptyset_b - \emptyset_{inf}) = 0$

- Extremo izquierdo: $k\left(\frac{\phi_{int}-\phi_{fic}}{2\Delta x}\right)(-1) + h(\emptyset_b - \emptyset_{inf}) = 0$
- Extremo derecho: $k\left(\frac{\phi_{fic}-\phi_{int}}{2\Delta x}\right)(1) + h(\emptyset_b - \emptyset_{inf}) = 0$

DIFERENCIAS FINITAS

Entonces para nodos interiores la ecuación a escribir es:

$$-\emptyset_{i-1}+\left(2+\frac{\Delta x^2 c}{k}\right)\emptyset_i-\emptyset_{i+1}=\frac{\Delta x^2 G_i}{k}$$

Para nodos en el extremo izquierdo: $-\emptyset_{fic}+\left(2+\frac{\Delta x^2 c}{k}\right)\emptyset_i-\emptyset_{i+1}=\frac{\Delta x^2 G_i}{k}$

Para nodos en el extremo derecho: $-\emptyset_{i-1}+\left(2+\frac{\Delta x^2 c}{k}\right)\emptyset_i-\emptyset_{fic}=\frac{\Delta x^2 G_i}{k}$

Condición de borde Neumann: $\emptyset_{fic}=\emptyset_{int}-\frac{2\Delta x q}{k}$

Condición de borde Robin: $\emptyset_{fic}=\emptyset_{int}-\frac{2\Delta x h}{k}\emptyset_b+\frac{2\Delta x h}{k}\emptyset_{inf}$

Despejar stencil final reemplazando nodo ficticio en la ecuación

DIFERENCIAS FINITAS

Generalizando esquemas numéricos no estacionarios (mallas uniformes):

$$\rho c_p \frac{\emptyset_i^{n+1} - \emptyset_i^n}{\Delta t} = k\left(\frac{\emptyset_{i-1}^{n+\theta} - 2\emptyset_i^{n+\theta} + \emptyset_{i+1}^{n+\theta}}{\Delta x^2}\right) - c\emptyset_i^{n+\theta} + G_i$$

Esquemas temporales:

- $\theta = 0$, Forward Euler, esquema explícito de orden Δt
- $\theta = 1$, Backward Euler, esquema implícito de orden Δt
- $\theta = {}^1/_2$, Crank-Nicholson, esquema implícito de orden Δt^2

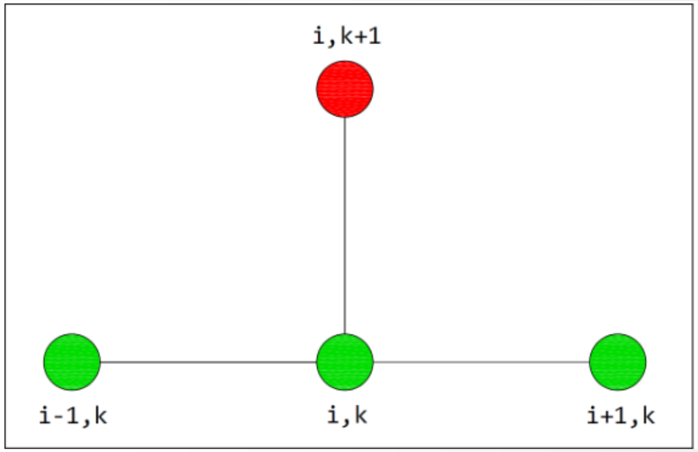
DIFERENCIAS FINITAS

Forward Euler: condicionalmente estable

podriamos hacer un doble for para los tiempos y para cada nodo

$$\rho c_p \frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} = k\left(\frac{\phi_{i-1}^n - 2\phi_i^n + \phi_{i+1}^n}{\Delta x^2}\right) - c\phi_i^n + G_i$$

$$\phi_i^{n+1} = \frac{\Delta t}{\rho c_p}\left(\frac{k}{\Delta x^2}(\phi_{i-1}^n - 2\phi_i^n + \phi_{i+1}^n) - c\phi_i^n + G_i\right) + \phi_i^n$$



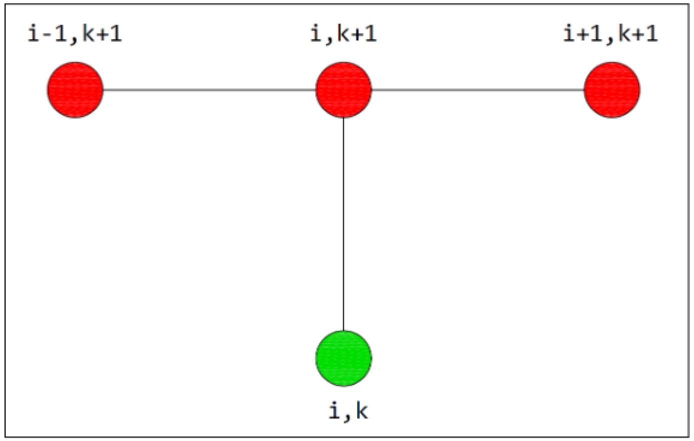
Condición de estabilidad: $Fo = \frac{\alpha \Delta t}{\Delta x^2} \leq 0.5 \;\rightarrow\; \Delta t \leq \frac{0.5\Delta x^2}{\alpha}$

si la cond de borde varia en el tiempo debemos elegir un deltaT de manera que detecte la longitud de onda y no pase de largo

Backward Euler: incondicionalmente estable

$$\rho c_p \frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} = k\left(\frac{\phi_{i-1}^{n+1} - 2\phi_i^{n+1} + \phi_{i+1}^{n+1}}{\Delta x^2}\right) - c\phi_i^{n+1} + G_i$$

$$-\frac{k}{\Delta x^2}\phi_{i-1}^{n+1} + \left(\frac{2k}{\Delta x^2} + c + \frac{\rho c_p}{\Delta t}\right)\phi_i^{n+1} - \frac{k}{\Delta x^2}\phi_{i+1}^{n+1} = G_i + \frac{\rho c_p}{\Delta t}\phi_i^n$$



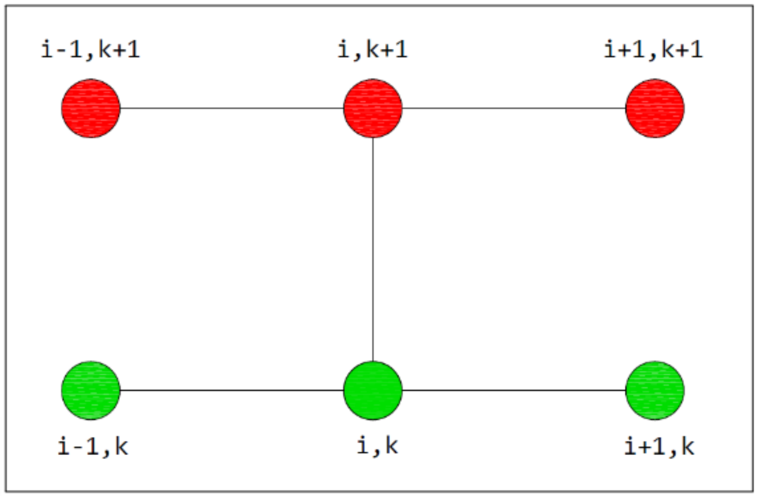
DIFERENCIAS FINITAS

Crank-Nicholson: incondicionalmente estable (oscilaciones espurias)

la ventaja de crank nicholson es que tengo orden 2 en el tiempo

$$\rho c_p \frac{\emptyset_i^{n+1} - \emptyset_i^n}{\Delta t} = \frac{1}{2}\left(k\left(\frac{\emptyset_{i-1}^n - 2\emptyset_i^n + \emptyset_{i+1}^n}{\Delta x^2}\right) - c\emptyset_i^n + G_i + k\left(\frac{\emptyset_{i-1}^{n+1} - 2\emptyset_i^{n+1} + \emptyset_{i+1}^{n+1}}{\Delta x^2}\right) - c\emptyset_i^{n+1} + G_i\right)$$

Desarrollo…



DIFERENCIAS FINITAS

Buenas noticias!

Discretización nodo a nodo:

$$\rho c_p \frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} = k\left(\frac{\phi_{i-1}^{n+\theta} - 2\phi_i^{n+\theta} + \phi_{i+1}^{n+\theta}}{\Delta x^2}\right) - c\phi_i^{n+\theta} + G_i$$

$$\downarrow$$

Planteo vectorial:

$$\frac{\rho c_p}{\Delta t}\left(\overline{\phi}^{n+1} - \overline{\phi}^n\right) = \overline{\overline{K}}\,\overline{\phi}^{n+\theta} + \overline{G}$$

Forward Euler

$$\overline{\phi}^{n+1} = \frac{\Delta t}{\rho c_p}\left(\overline{\overline{K}}\,\overline{\phi}^n + \overline{G}\right) + \overline{\phi}^n$$

Backward Euler

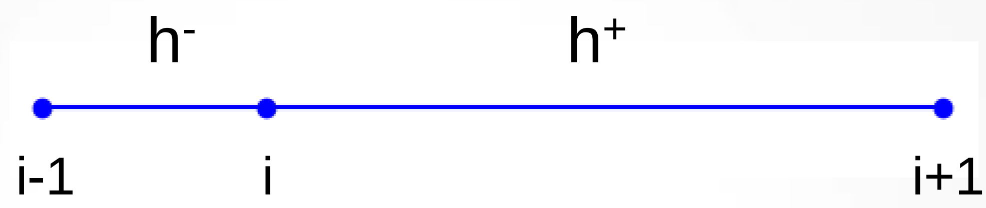
$$\left(\frac{\rho c_p}{\Delta t}\overline{\overline{I}} - \overline{\overline{K}}\right)\overline{\phi}^{n+1} = \overline{G} + \frac{\rho c_p}{\Delta t}\overline{\phi}^n$$

Crank-Nicholson

$$\left(\frac{\rho c_p}{\Delta t}\overline{\overline{I}} - \frac{1}{2}\overline{\overline{K}}\right)\overline{\phi}^{n+1} = \left(\frac{1}{2}\overline{\overline{K}} + \frac{\rho c_p}{\Delta t}\overline{\overline{I}}\right)\overline{\phi}^n + \overline{G}$$

DIFERENCIAS FINITAS

Mallas no uniformes:



$$\emptyset_{i+1} = \emptyset_i + h^+ \left.\frac{\partial \emptyset}{\partial x}\right|_i + \frac{(h^+)^2}{2}\left.\frac{\partial^2 \emptyset}{\partial x^2}\right|_i + \frac{(h^+)^3}{3!}\left.\frac{\partial^3 \emptyset}{\partial x^3}\right|_i + \frac{(h^+)^4}{4!}\left.\frac{\partial^4 \emptyset}{\partial x^4}\right|_i + \ldots$$

$$\emptyset_{i-1} = \emptyset_i - h^- \left.\frac{\partial \emptyset}{\partial x}\right|_i + \frac{(h^-)^2}{2}\left.\frac{\partial^2 \emptyset}{\partial x^2}\right|_i - \frac{(h^-)^3}{3!}\left.\frac{\partial^3 \emptyset}{\partial x^3}\right|_i + \frac{(h^-)^4}{4!}\left.\frac{\partial^4 \emptyset}{\partial x^4}\right|_i + \ldots$$

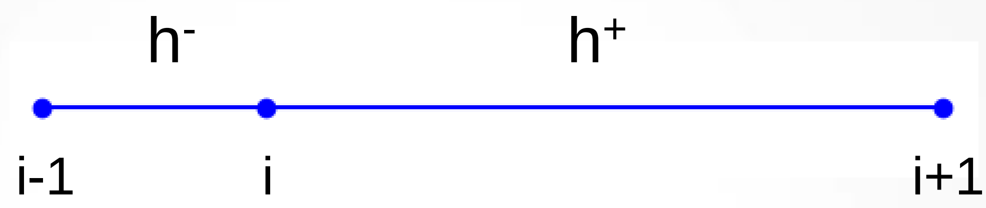
$$\left.\frac{\partial^2 \emptyset}{\partial x^2}\right|_i = a\emptyset_{i+1} + b\emptyset_i + c\emptyset_{i-1} = a\left(\emptyset_i + h^+ \left.\frac{\partial \emptyset}{\partial x}\right|_i + \frac{(h^+)^2}{2}\left.\frac{\partial^2 \emptyset}{\partial x^2}\right|_i\right) + b\emptyset_i + c\left(\emptyset_i - h^- \left.\frac{\partial \emptyset}{\partial x}\right|_i + \frac{(h^-)^2}{2}\left.\frac{\partial^2 \emptyset}{\partial x^2}\right|_i\right)$$

$$\left.\frac{\partial^2 \emptyset}{\partial x^2}\right|_i = \overbrace{(a+b+c)}^{=0}\emptyset_i + \overbrace{(ah^+ - ch^-)}^{=0}\left.\frac{\partial \emptyset}{\partial x}\right|_i + \underbrace{\left(a\frac{(h^+)^2}{2} + c\frac{(h^-)^2}{2}\right)}_{=1}\left.\frac{\partial^2 \emptyset}{\partial x^2}\right|_i$$

3 ecuaciones con 3 incógnitas…

DIFERENCIAS FINITAS

Mallas no uniformes:



$$\begin{pmatrix} 1 & 1 & 1 \\ h^{+} & 0 & -h^{-} \\ (h^{+})^{2}/2 & 0 & (h^{+})^{2}/2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \Longrightarrow \quad \begin{aligned} a &= \frac{2}{h^{+}(h^{+}+h^{-})} \\ b &= \frac{-2}{h^{+}h^{-}} \\ c &= \frac{2}{h^{-}(h^{+}+h^{-})} \end{aligned}$$

$$\left.\frac{\partial^{2}\emptyset}{\partial x^{2}}\right|_{i} = \frac{2\emptyset_{i+1}}{h^{+}(h^{+}+h^{-})} - \frac{2\emptyset_{i}}{h^{+}h^{-}} + \frac{2\emptyset_{i-1}}{h^{-}(h^{+}+h^{-})}$$

orden h

Si $h^{+}=h^{-}$ entonces recuperamos el stencil clásico

DIFERENCIAS FINITAS

Manipulación de las series de Taylor: podemos aproximar cualquier operador diferencial con cualquier combinación de puntos, equiespaciados o no…

$$\left.\frac{\partial^3 \emptyset}{\partial x^3}\right|_i = a\emptyset_{i+3} + b\emptyset_{i+1} + c\emptyset_i + d\emptyset_{i-2} + e\emptyset_{i-4} \quad \rightarrow \quad n \geq k + p$$

ponemos 1 la derivada que buscamos

$$\begin{bmatrix} 1, & 1, & 1, & 1, & 1 \\ 3*h, & h, & 0, & -2*h, & -4*h \\ (9*h^2)/2, & h^2/2, & 0, & 2*h^2, & 8*h^2 \\ (9*h^3)/2, & h^3/6, & 0, & -(4*h^3)/3, & -(32*h^3)/3 \\ (27*h^4)/8, & h^4/24, & 0, & (2*h^4)/3, & (32*h^4)/3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\left.\frac{\partial^3 \emptyset}{\partial x^3}\right|_i = \frac{1}{7h^3}\emptyset_{i+3} - \frac{3}{5h^3}\emptyset_{i+1} + \frac{1}{2h^3}\emptyset_i + 0\emptyset_{i-2} - \frac{3}{70h^3}\emptyset_{i-4}$$