Question 1

$$\Lambda = \frac{p_1}{p_0} = \frac{\frac{1}{\sqrt{(2\pi\sigma^2)^N}} e^{\frac{-1}{2\sigma^2}((x-\theta)(x-\theta)^T)}}{\frac{1}{\sqrt{(2\pi\sigma^2)^N}} e^{\frac{-1}{2\sigma^2}(xx^T)}} = e^{\frac{-1}{2\sigma^2}(xx^T - x\theta^T - \theta x^T + \theta\theta^T - xx^T)}$$

$$\boxed{\Lambda = e^{\frac{-1}{2\sigma^2}(-x\theta^T - \theta x^T + \theta\theta^T)}} = e^{\frac{-1}{2\sigma^2}(-2x^T\theta + \|\theta\|^2)}$$

Question 2:

$$\forall x \in \mathbb{R}^d,\ g_{MPE}(x) = \begin{cases} 1 & \text{if } \pi_1 p_1(x) > \pi_0 p_0(x) \\ 0 & \text{otherwise} \end{cases}$$

$$\pi_1 p_1(x) > \pi_0 p_0(x) \Leftrightarrow \frac{p_1(x)}{p_0(x)} > \frac{\pi_0}{\pi_1} \Leftrightarrow e^{\frac{-1}{2\sigma^2}(-2x^T\theta + \|\theta\|^2)} > \frac{\pi_0}{\pi_1}$$

$$\Leftrightarrow \frac{1}{2\sigma^2}(-2x^T\theta + \|\theta\|^2) > \log\left(\frac{\pi_0}{\pi_1}\right) \Leftrightarrow x^T\theta > \sigma^2 \log\left(\frac{\pi_0}{\pi_1}\right) + \frac{\|\theta\|^2}{2}$$

$$\Rightarrow \forall x \in \mathbb{R}^d,\ g_{MPE}(x) = \begin{cases} 1 & \text{if } x^T\theta > \sigma^2 \log\left(\frac{\pi_0}{\pi_1}\right) + \frac{\|\theta\|^2}{2} \\ 0 & \text{otherwise} \end{cases}$$

Question 3:

$$\mathbb{P}e\{g_{MPE}\} = \mathbb{P}[g(x) \neq \varepsilon] = \pi_0 \mathbb{P}[g(X_0)=1] + \pi_1 \mathbb{P}[g(X_1)=0]$$

$$\mathbb{P}[g(X_0)=1] = \mathbb{P}\left[x_0^T\theta > \sigma^2 \lg(\pi_0/\pi_1) + \frac{\|\theta\|^2}{2}\right] = \mathbb{P}\Bigg[\overbrace{\frac{x_0^T\theta}{\sigma\|\theta\|}}^{\mathcal{N}(0,1)\ (\mathrm{I})} > \frac{\sigma}{\|\theta\|}\log(\pi_0/\pi_1) + \frac{\|\theta\|}{2\sigma}\Bigg]$$

$$= 1 - \phi\left(\frac{1}{\rho}\log\left(\frac{\pi_0}{\pi_1}\right) + \frac{\rho}{2}\right) \quad \text{with } \rho = \frac{\|\theta\|}{\sigma}$$

$$\mathbb{P}[g(X_1)=0] = \mathbb{P}\left[x_1^T\theta < \sigma^2 \log(\pi_0/\pi_1) + \frac{\|\theta\|^2}{2}\right] = \mathbb{P}\Bigg[\overbrace{\frac{x_1^T\theta - \|\theta\|^2}{\sigma\|\theta\|}}^{\mathcal{N}(0,1)\ (\mathrm{II})} < \frac{\sigma}{\|\theta\|}\log\left(\frac{\pi_0}{\pi_1}\right) - \frac{\|\theta\|^2}{2}\Bigg]$$

$$= \phi\left(\frac{1}{\rho}\log\left(\frac{\pi_0}{\pi_1}\right) - \frac{\rho}{2}\right)$$

$$\Rightarrow \mathbb{P}e\{g_{MPE}\} = \pi_0\left(1 - \phi\left(\frac{1}{\rho}\log\left(\frac{\pi_0}{\pi_1}\right) + \frac{\rho}{2}\right)\right) + \pi_1\left(\frac{1}{\rho}\log\left(\frac{\pi_0}{\pi_1}\right) - \frac{\rho}{2}\right)$$

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* if $Z \sim \mathcal{N}(a\theta, \sigma^2 I_n) \Rightarrow Z^T\theta \sim \mathcal{N}(a\|\theta\|^2, \|\theta\|^2\sigma^2)$

because the moments $E[(Z^T\theta)^r] = \sum_{x_i \in \varepsilon} (Z^T\theta)^r \mathbb{P}(Z^T = z) = \theta^r \underbrace{\sum_{x_i \in \varepsilon} z^{T^r} \mathbb{P}(Z^T = z) z}_{E[(z^T)^r]}$

so $E[(z^T\theta)] = \theta \underbrace{E[(z^T)]}_{a\theta^T} = a\|\theta\|^2$

$E[(z^T\theta)^2] - E[z^T\theta]^2 = \|\theta\|^2(\sigma^2 I_n + a^2\|\theta\|^2) - a^2\|\theta\|^4 = \|\theta\|^2\sigma^2$

Thus $z^T\theta \sim \mathcal{N}(a\|\theta\|^2, \|\theta\|^2\sigma^2)$

I) So, as $X_0 \sim \mathcal{N}(0, \sigma^2 I_n) \Rightarrow X_0^T\theta \sim \mathcal{N}(0, \|\theta\|^2\sigma^2) \Rightarrow \boxed{\frac{X_0^T\theta}{\|\theta\|^2\sigma^2} \sim \mathcal{N}(0,1)}$

II) $X_1 \sim \mathcal{N}(\theta, \sigma^2 I_n) \Rightarrow X_1^T\theta \sim \mathcal{N}(\|\theta\|^2, \|\theta\|^2\sigma^2) \Rightarrow \boxed{\frac{X_1^T\theta - \|\theta\|^2}{\sigma\|\theta\|} \sim \mathcal{N}(0,1)}$

Question 4

The error probabilities of both tests are:

$$P_e(g_{MPE}) = \pi_0\left(1 - \Phi\left(\frac{1}{\rho}\log\left(\frac{\pi_0}{\pi_1}\right) + \frac{\rho}{2}\right)\right) + \pi_1\,\Phi\left(\frac{1}{\rho}\log\left(\frac{\pi_0}{\pi_1}\right) - \frac{\rho}{2}\right)$$

$$P_e(g_{NP}^{\gamma}) = \pi_0\gamma + \pi_1\,\Phi\left(\Phi^{-1}(1-\gamma) - \rho\right)$$

Considering that the probability of error of the MPE test is lower than any other test by definition:

$$P_e(g_{MPE}) \leq P_e(g_{NP}^{\gamma})$$

Now, if we replace the expressions with $\pi_0 = \pi_1 = \frac{1}{2}$ and $\rho = A$:

$$\frac{1}{2}\left(1 - \Phi\left(\frac{A}{2}\right)\right) + \frac{1}{2}\Phi\left(-\frac{A}{2}\right) \leq \frac{1}{2}\gamma + \frac{1}{2}\Phi\left(\Phi^{-1}(1-\gamma) - A\right)$$

, as $\frac{1}{2}\Phi\left(-\frac{A}{2}\right) = \frac{1}{2}\left(1 - \Phi\left(\frac{A}{2}\right)\right)$

we obtain that $\forall \gamma \in (0,1)$ and $\forall A > 0$

$$1 - \Phi\left(\frac{A}{2}\right) \leq \frac{1}{2}\left(\gamma + \Phi\left(\Phi^{-1}(1-\gamma) - A\right)\right)$$

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