Math 30, Monday April 13, 2020

Applied Optimization Problems

today: last day of "new" material before the Exam on Friday

not totally new — just applications.

Exam: no class on Friday.

I'll post the exam at 6am on Friday —

submit your work by 11:59pm on Friday.

Questions?

today: word problems for "real life":

in each one:

find the appropriate function,

then minimize or maximize it.

#1. $f(x)$ = vert. dist. b/w parabolas

#2. $f(x)$ = time it takes to reach the island if you run for $6-x$ miles

#3. $f(x)$ = cost of making box for cat

etc.

This is on Canvas — please work on it now —

MATH 30, 4/13/2020: APPLIED OPTIMIZATION PROBLEMS

I'll help w/ 2 & 3 today (or answer other Q's..)

- Read and understand the problem.
- Draw a diagram.
- Introduce notation.
- Write the quantity Q to be optimized in terms of your notation.
- Write Q as a function of a single variable, $Q = f(x)$.
- Find the global maximum and/or minimum of f.

In this worksheet, you can use a calculator in the last step if you want.

(1) Draw the parabolas $y = x^2 + 1$ and $y = x - x^2$ on the same axes. What is the minimum vertical distance between these parabolas?

(2) An island is 2 miles due north of its closest point along a straight shoreline. A visitor is staying at a cabin on the shore that is 6 miles west of that point. The visitor is planning to go from the cabin to the island. Suppose the visitor runs at a rate of 8 mph and swims at a rate of 3 mph. How far should the visitor run before swimming to minimize the time it takes to reach the island?

(3) You are constructing a box for your cat to sleep in. The plush material for the square bottom of the box costs \$5 per square foot (ft^2) and the material for the sides costs \$2/ft^2. You need a box with volume 4 ft^3. Find the dimensions of the box that minimize cost. Use x to represent the length of the side of the box.

(4) A cylindrical can without a top is made to contain V cm^3 of liquid. Find the dimensions of the can that will minimize the cost of the metal to make the can.

(5) An object with weight W is dragged along a horizontal plane by a force acting along a rope attached to the object. If the rope makes an angle θ with the plane, then the magnitude of the force is

$$F = \frac{\mu W}{\mu \sin\theta + \cos\theta}$$

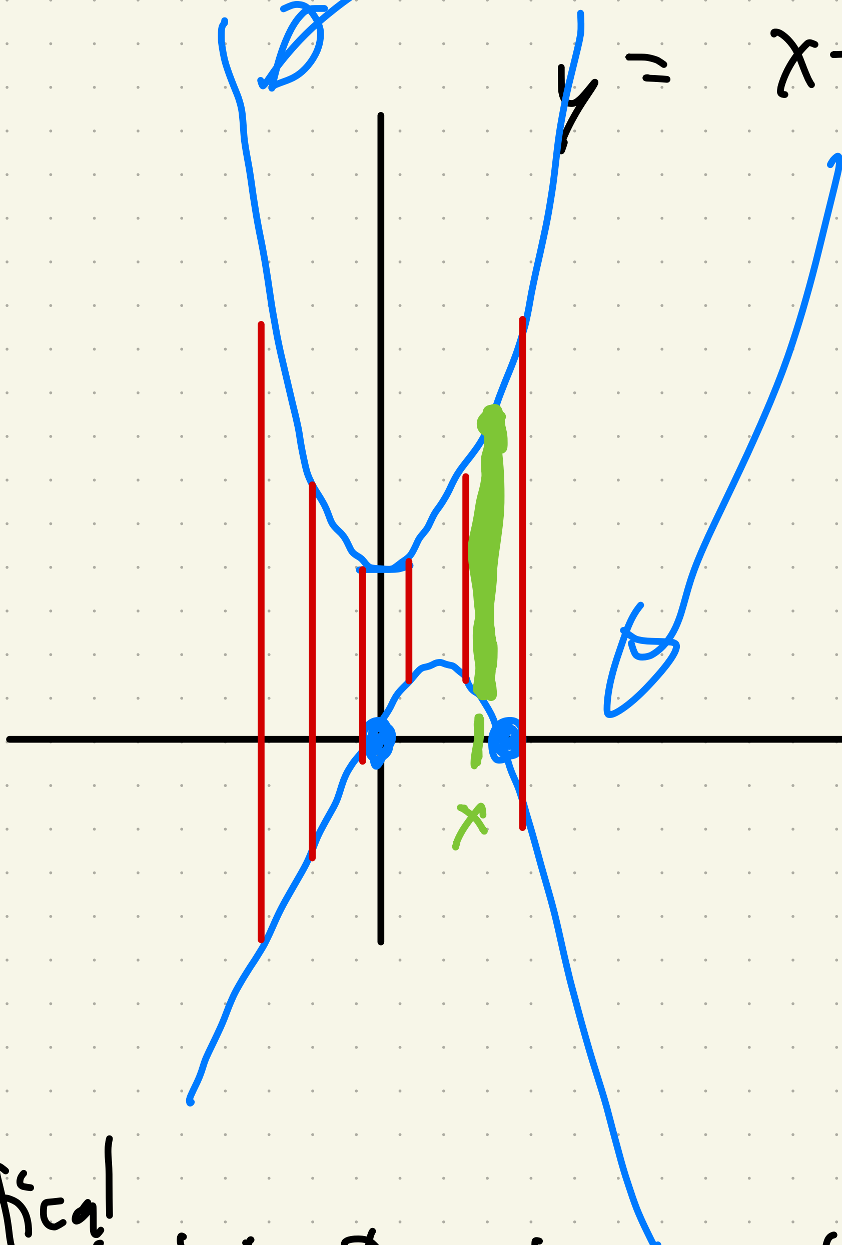
where μ is a constant called the coefficient of friction. For what value of θ is F smallest?

(6) Owners of a car rental company have determined that if they charge customers p dollars per day to rent a car, where $50 \leq p \leq 200$, then the number of cars n they rent per day can be modeled by the linear function $n(p) = 1000 - 5p$. If they charge \$50 per day or less, they will rent all their cars. If they charge \$200 per day or more, they will not rent any cars. Assuming the owners plan to charge customers between \$50 per day and \$200 per day to rent a car, how much should they charge to maximize their revenue?

\#1. parabolas

$y = x^2 + 1$

$y = x - x^2$



Q: which of the red segments is shortest?

vertical dist. b/w the two parabolas at x is

$$f(x) = \underbrace{(x^2+1)}_{\text{top y-value}} - \underbrace{(x-x^2)}_{\text{bottom y-value}} = 2x^2 - x + 1$$

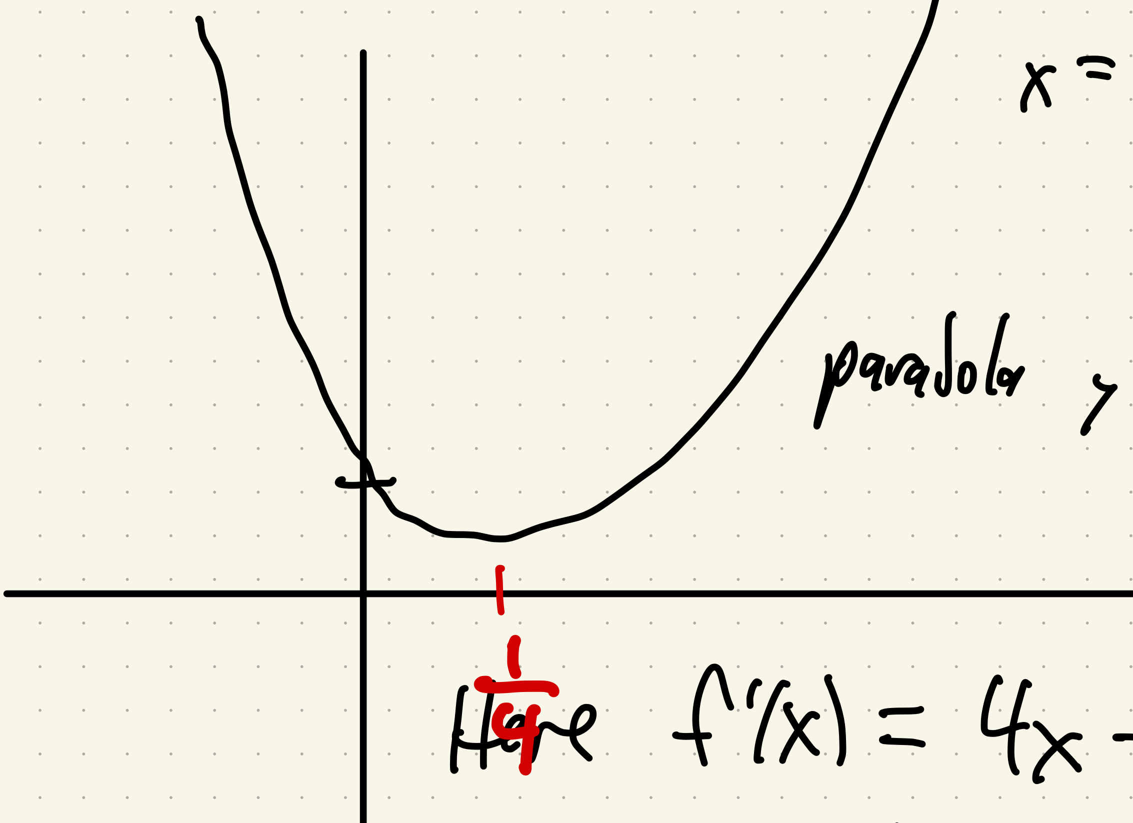
Math problem:

minimize $f(x) = 2x^2 - x + 1$

Find where f is incr/decr.

roots are $x = 1 \pm \sqrt{1 -}$

no roots



paradola $y = f(x)$

Here $f'(x) = 4x - 1$

crit. pt: $x = \frac{1}{4}$

$f' < 0$ when $x < \frac{1}{4} \longrightarrow f$ is decr

$f' > 0$ when $x > \frac{1}{4} \longrightarrow f$ is incr.

local min. at $x = \frac{1}{4}$

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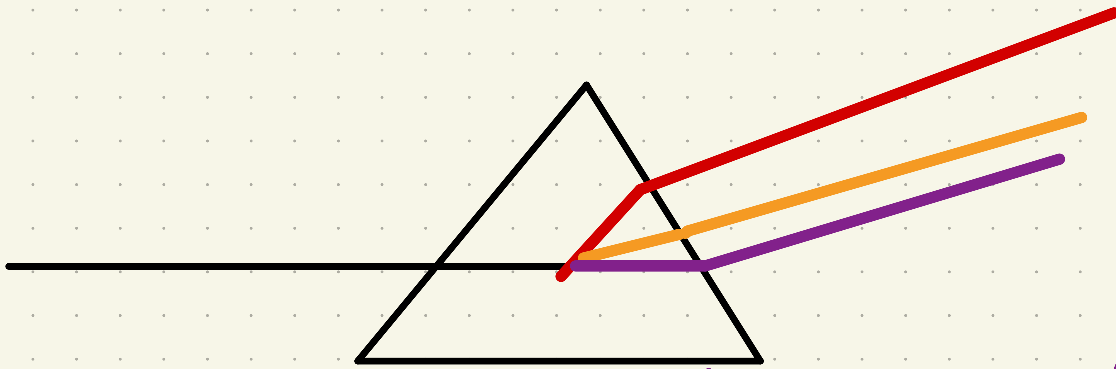
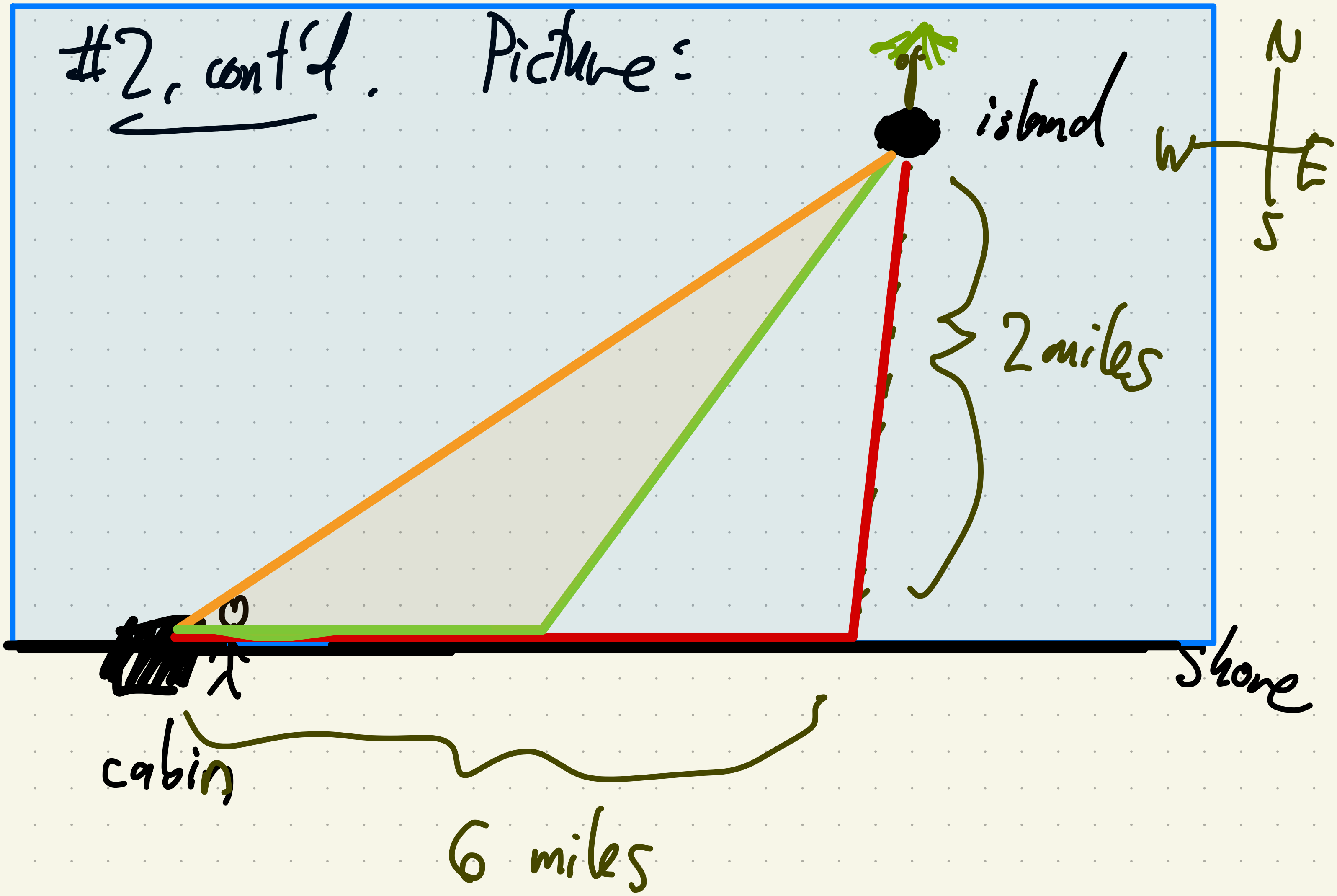
#2. Draw a diagram.

Cool problem — same method also explains the law of refraction ("Snell's Law") in optics



explains prisms & rainbows

#2, cont'd. Picture:



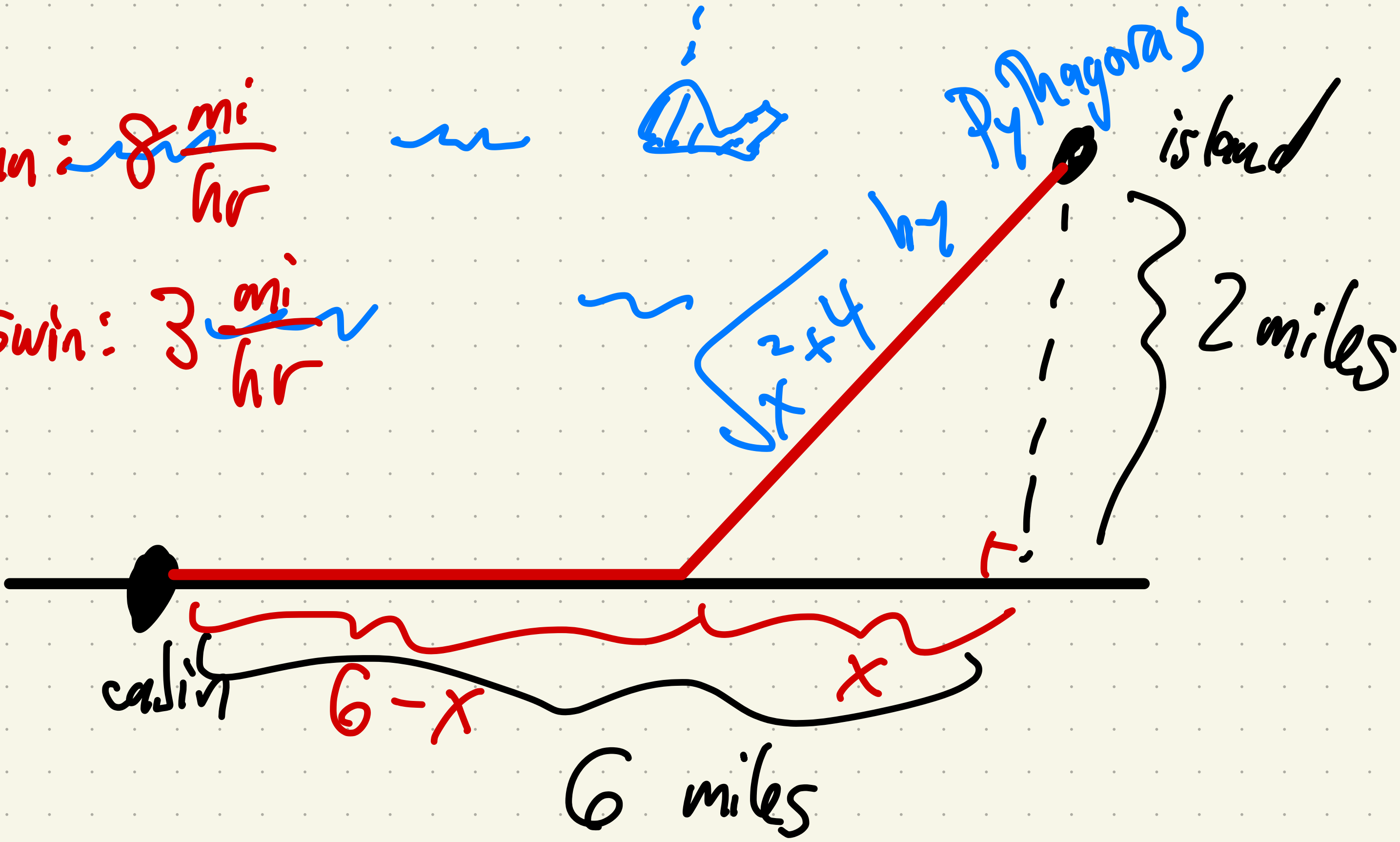
can run at $8 \frac{mi}{hr}$ and swim at $3 \frac{mi}{hr}$.

how far should he run before jumping in & swimming?

lots of options of "run then swim"

run: $8\ \frac{mi}{hr}$

swim: $3\ \frac{mi}{hr}$



to find "The total time" it takes to reach The island.

Key fact: speed × time = dist.

$\frac{\text{miles}}{\text{hour}} \times \text{hours} = \text{miles}$

sometimes stated as a "lifeguard problem"

$$\text{time} = \frac{\text{dist}}{\text{speed}}$$

time to swim: $t_2 = \frac{\sqrt{x^2+4}}{3}$

time to run: $t_1 = \frac{6-x}{8}$

total time to reach the island is $t_1 + t_2$.

So minimize

$$f(x) = \underbrace{\frac{6-x}{8}}_{\text{run time}} + \underbrace{\frac{\sqrt{x^2+4}}{3}}_{\text{swim time}}$$

Where is it incr./decr.?

$$f'(x) = -\frac{1}{8} + \frac{1}{3}\left(\frac{1}{2}(x^2+4)^{-1/2} \cdot \underbrace{2x}_{\text{Chain Rule}}\right)$$

$$= -\frac{1}{8} + \frac{x}{3\sqrt{x^2+4}}$$

$$f'(x) = -\frac{1}{8} + \frac{x}{3\sqrt{x^2+4}}$$

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critical pts: where $f' = 0$.

Same as: $$\frac{1}{8} = \frac{x}{3\sqrt{x^2+4}}$$

$$\Updownarrow$$

$$3\sqrt{x^2+4} = 8x$$

$$\Updownarrow$$

$$9(x^2+4) = 64x^2$$

$$\Updownarrow$$

$$36 = 55x^2$$

only care about $0 \le x \le 6$ in this example.

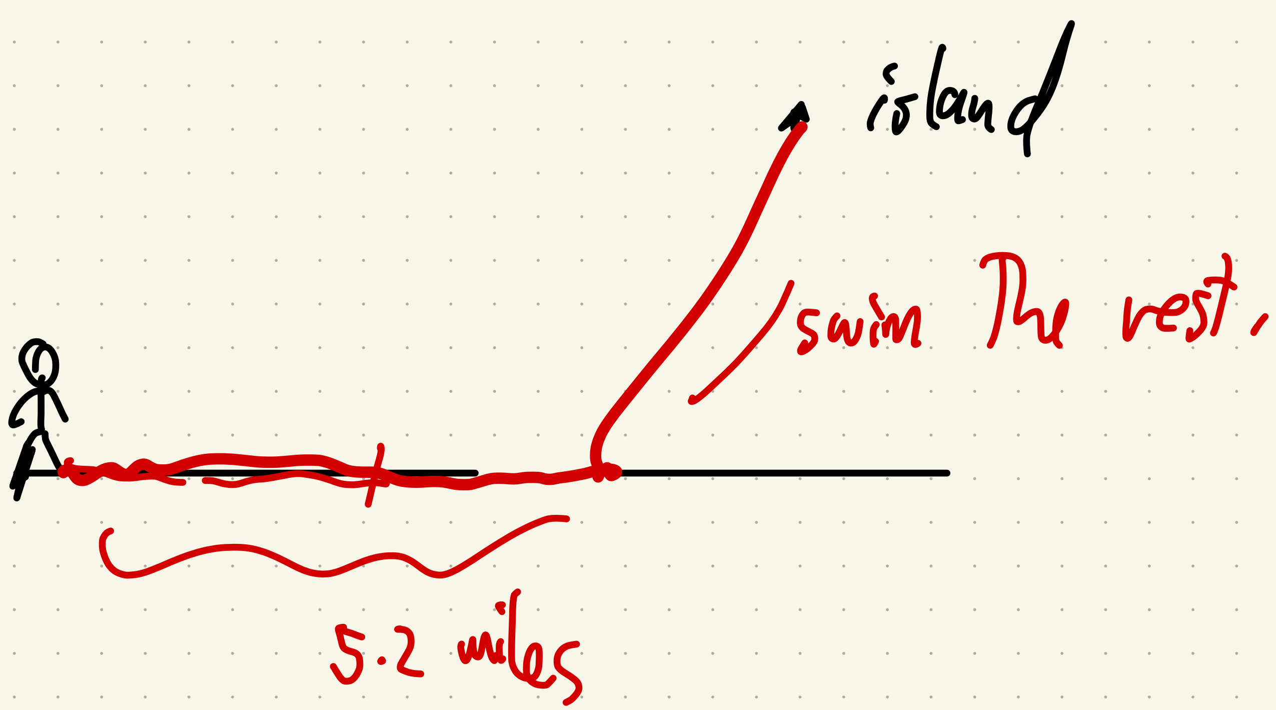
So $x = \sqrt{\frac{36}{55}}$ is the only critical pt

$$x = \frac{6}{\sqrt{55}} \approx 0.81 \text{ miles.}$$

Now use incr./decr. or concavity to see That it's a local min.

and global min.

so $6 - x \approx 5.2$



See you on Wednesday!