MATH 30, SPRING 2020: LECTURE ON DERIVS OF INVERSE FUNCTIONS

Say that g is the inverse function of f. By definition, this means $g(f(x)) = x$ for **every** x in the domain of f. This means the two sides are equal *as functions*, so they have the same derivatives. The Chain Rule then says:

$$g'(f(x))f'(x) = 1.$$

I recommend *not* memorizing the formula. Just use the Chain Rule from scratch every time!

Example. By definition, $f(x) = \ln x$ ("the natural log") is the inverse of $g(y) = e^y$, and we know $g'(y) = e^y$. By differentiating both sides of $g(f(x)) = x$ (which is valid for all $x > 0$), we get $g'(f(x))f'(x) = 1$. That is,

$$e^{\ln x} f'(x) = 1.$$

This shows that the derivative of $f(x) = \ln x$ is $f'(x) = \frac{1}{x}$.

Example. We already know the derivative of $f(x) = \sqrt{x}$ (we found it from the definition of the derivative, using limits). Here is another way:

Of course $g(y) = y^2$ is the inverse of $f(x) = \sqrt{x}$. ("Take the square root of x, then square it, and you get x back.") So $g(f(x)) = x$ is true for all $x > 0$. Taking the derivative of both sides, we get $g'(f(x))f'(x) = 1$. But $g'(y) = 2y$, so we get $2f(x) \cdot f'(x) = 1$. That is,

$$f'(x) = \frac{1}{2\sqrt{x}}.$$

Notice that this can be rewritten as $\frac{d}{dx}x^{1/2} = \frac{1}{2}x^{-1/2}$, which looks like the Power Rule!

The Generalized Power Rule. We only proved the Power Rule $\frac{d}{dx}x^n = nx^{n-1}$ when $n = 1, 2, 3, 4, \ldots$. In fact, the Power Rule is true when the power is *any* real number (fractions, negative numbers, etc.)! Here is why:

For any a, and $x > 0$, rewrite x^a as

$$x^a = (e^{\ln x})^a = e^{a \ln x}.$$

Now differentiate this using the Chain Rule (and the fact that $\frac{d}{dx}\ln x = \frac{1}{x}$) to get:

$$\frac{d}{dx}x^a = e^{a\ln x} \cdot \frac{a}{x} = x^a \cdot \frac{a}{x} = \frac{ax^a}{x} = ax^{a-1}.$$

We can similarly find the derivatives of a^x and its inverse, $\log_a x$:

Example. First, rewrite a^x as

$$a^x = (e^{\ln a})^x = e^{x \ln a}.$$

Now differentiate, using the Chain Rule:

$$\frac{d}{dx}a^x = (\ln a)e^{x\ln a} = (\ln a)a^x.$$

Example. The inverse of $g(y) = a^y$ is $f(x) = \log_a x$, and now we know

$$g'(y) = (\ln a)a^y.$$

Since f and g are inverses, we have $g(f(x)) = x$ for all $x > 0$. Differentiating both sides, we get

$$g'(f(x)) \cdot f'(x) = 1 \qquad \text{for all } x > 0.$$

That is,

$$(\ln a)a^{\log_a x} \cdot \frac{d}{dx} \log_a x = 1.$$

That is,

$$(\ln a)x \cdot \frac{d}{dx} \log_a x = 1,$$

which we can rewrite as

$$\frac{d}{dx} \log_a x = \frac{1}{x \ln a}.$$

Example. The inverse of $g(y) = \tan y$ is $f(x) = \tan^{-1} x$ (sometimes written as $\arctan x$). We know $g'(y) = \sec^2 y$, so the Chain Rule gives $g'(f(x)) \cdot f'(x) = 1$ for all x. That is,

$$\sec^2(\tan^{-1} x) \cdot f'(x) = 1.$$

Here's the interesting part: Now use the trig identity $1 + \tan^2\theta = \sec^2\theta$. (It's the same as $\cos^2\theta + \sin^2\theta = 1$, after dividing by $\cos^2\theta$.) Plug in $\theta = \tan^{-1} x$. Then

$$\sec^2(\tan^{-1} x) = 1 + (\tan(\tan^{-1} x))^2 = 1 + x^2.$$

So we finally get

$$\frac{d}{dx} \tan^{-1} x = \frac{1}{1 + x^2}.$$

You can do something similar to find:

$$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1 - x^2}}.$$