MATH 30, 4/8-9/2020: L'HÔPITAL'S RULE

First of all, it used to be written "L'Hospital's Rule" (named after Guillaume de l'Hôpital because it's in his book, even though he learned it from Johann Bernoulli), but French spelling conventions changed over the years.

Remember that $\frac{0}{0}$ and $\frac{\infty}{\infty}$ are *meaningless* things to write.

Example. $\displaystyle\lim_{x\to 0}\frac{mx}{x} = \text{"}\frac{0}{0}\text{,"}$ but instead we can cancels the x's and get

$$\lim_{x\to 0}\frac{mx}{x} = m \qquad \text{for any value of } m.$$

Are you telling me that "$\dfrac{0}{0}$" is *every number*? Actually, I'm telling you that it makes no sense.

We call $\frac{0}{0}$ and $\frac{\infty}{\infty}$ "indeterminate forms." It just means "you need to do more work."

We've already seen examples of this:

Example. $\displaystyle\lim_{x\to 1}\frac{x^2-1}{x^2-x}$ "$= \dfrac{0}{0}$," but really you can factorize and cancel and get 2. [Check this.]

But how about more complicated examples like:

Example. $\displaystyle\lim_{x\to 0}\frac{\tan x - x}{x^3}$? Again it looks like $\frac{0}{0}$.

Here is the idea: Suppose we want $\displaystyle\lim_{x\to a}\frac{f(x)}{g(x)}$ where $f(a)=0$ and $g(a)=0$ and $g'(a)\neq 0$.

Then we can write:

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{\left(\frac{f(x)-f(a)}{x-a}\right)}{\left(\frac{g(x)-g(a)}{x-a}\right)} = \frac{f'(a)}{g'(a)}.$$

That is,

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

That's the gist of L'Hôpital's Rule.

Here it is, written out formally:

L'Hôpital's Rule. Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x\to a} f(x) = 0 \qquad \text{and} \qquad \lim_{x\to a} g(x) = 0$$

or that

$$\lim_{x\to a} f(x) = \pm\infty \qquad \text{and} \qquad \lim_{x\to a} g(x) = \pm\infty.$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.) Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

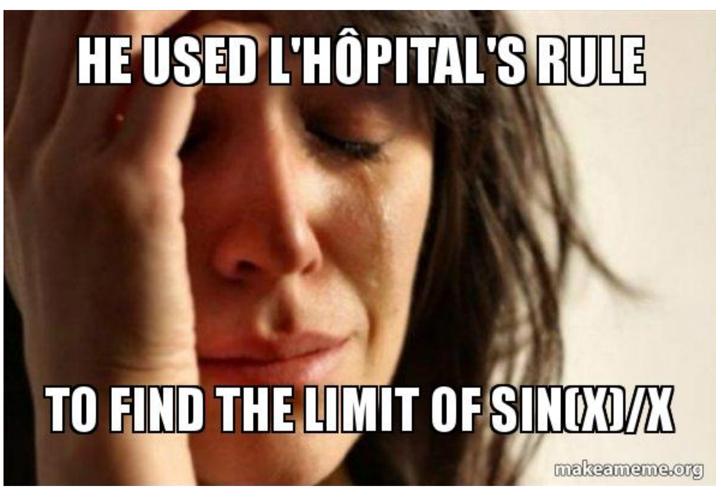
if the limit on the right-hand side exists (or is ∞ or $-\infty$).

Too-Basic Example. Find the limit: $\lim\limits_{x\to 0} \dfrac{\sin x}{x}$.

Note that all the conditions of L'Hôpital's Rule are satisfied, so $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = \lim\limits_{x\to 0} \dfrac{\cos x}{1} = 1$.

Even though this is technically correct, it is kind of ridiculous since *by the definition of the derivative* we have, with $h(x) = \sin x$,

$$(\cos(0) =)\ \ h'(0) = \lim_{x\to 0} \frac{h(x) - h(0)}{x - 0} \ \ \left(= \lim_{x\to 0} \frac{\sin x}{x}\right).$$



I could also say: You don't know the derivative of $\sin x$, but you're using L'Hôpital's Rule?!?!?! ☺

Here are some other examples:

Example. $\lim_{x\to 1} \frac{\ln x}{\sin(\pi x)} = ?$

Answer: $-\frac{1}{\pi}$.

Example. $\lim_{x\to 0} \frac{1-\cos x}{x^2} = ?$

Answer: $\frac{1}{2}$.

Example. $\lim_{x\to \infty} \frac{\ln x}{x^{1/3}} = ?$

Answer: 0.

Example. $\lim_{x\to 0} \frac{\tan x - x}{x^3} = ?$ (The example from earlier.)

Answer: In this one, you need to use L'Hôpital's Rule *twice*. Then you get $\frac{1}{3}$.

Trick Example. $\lim_{x\to \pi^-} \frac{\sin x}{1-\cos x} = ?$

It's tempting to use L'Hôpital's Rule and get $\lim_{x\to \pi^-} \frac{\cos x}{\sin x} = -\infty$. But this is *wrong* because the original limit is

$$\cdots = \frac{0}{1-(-1)} = 0.$$

That is, *we weren't allowed to use L'Hôpital's Rule because it wasn't an indeterminate form!*