Math 30, Monday April 20, 2020

1pm class

Intro to Antiderivatives and Integrals

Hope to grade exams soon...

Today: antiderivatives and integration.

Last major topic of the semester!

Questions?

Related Rates Question:

I'll type solutions soon...

So far: given a function $f(x)$, find its derivative $f'(x)$.

$$\underset{\text{start with } f}{f(x)} \xrightarrow[\text{take its derivative}]{} f'(x)$$

Now: Try to go the other direction.

Given a function $g(x)$, find a function $f(x)$ whose derivative is $g(x)$

$$\begin{tikzcd} f(x) \arrow[r, bend left, "\text{take derivative}"] & f'(x) = g(x) \arrow[l, bend left, "?"] \end{tikzcd}$$

If $f'(x) = g(x)$

Then g is called the <u>derivative</u> of f

and f is called the antiderivative of g,

$$\begin{tikzcd} f(x) \arrow[r, bend left, "\text{take derivative}"] & g(x) = f'(x) \arrow[l, bend left, "\text{find the antiderivative}"] \end{tikzcd}$$

"undo" the derivative.

Easiest examples come from experience:

Ex. $g(x) = x^2$.

Can you find a function $f(x)$ whose derivative is $g(x)$?

want $f'(x) = x^2$

$f(x) = \frac{1}{3}x^3$

check: $f'(x) = x^2$ ✓ ☺

Is that the only function with $f'(x) = x^2$?

$$f(x) = \frac{1}{3}x^3 + C$$

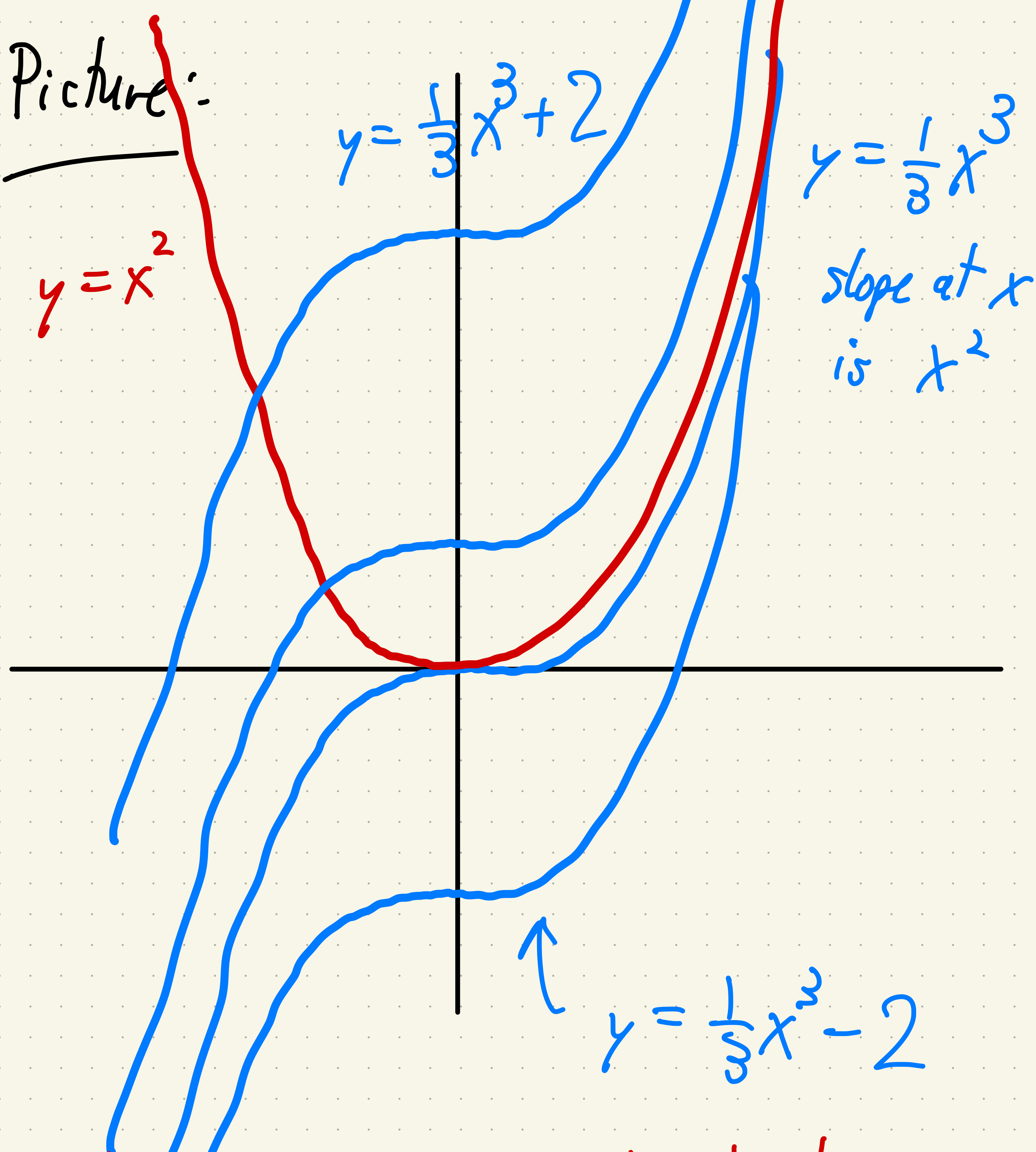
also has $f'(x) = x^2$

for <u>any</u> constant C.

<u>Fact</u>: All solutions of $f'(x) = x^2$ have the form $f(x) = \frac{1}{3}x^3 + C$ where C is a constant.

We proved this using the Mean Value Theorem!

Picture:



shifting up or down doesn't change the slope at x.

slope blue curve is the y-value of red curve.

More examples based on experience:

Find the most general antiderivative of:

(a) $g(x) = \sin x$

(b) $g(x) = \dfrac{1}{x}$

(c) $g(x) = x^n$ for $n \neq -1$.

(a) Want $f(x)$ so that $f'(x) = \sin x$

Based on experience:

$$f(x) = -\cos x + C$$

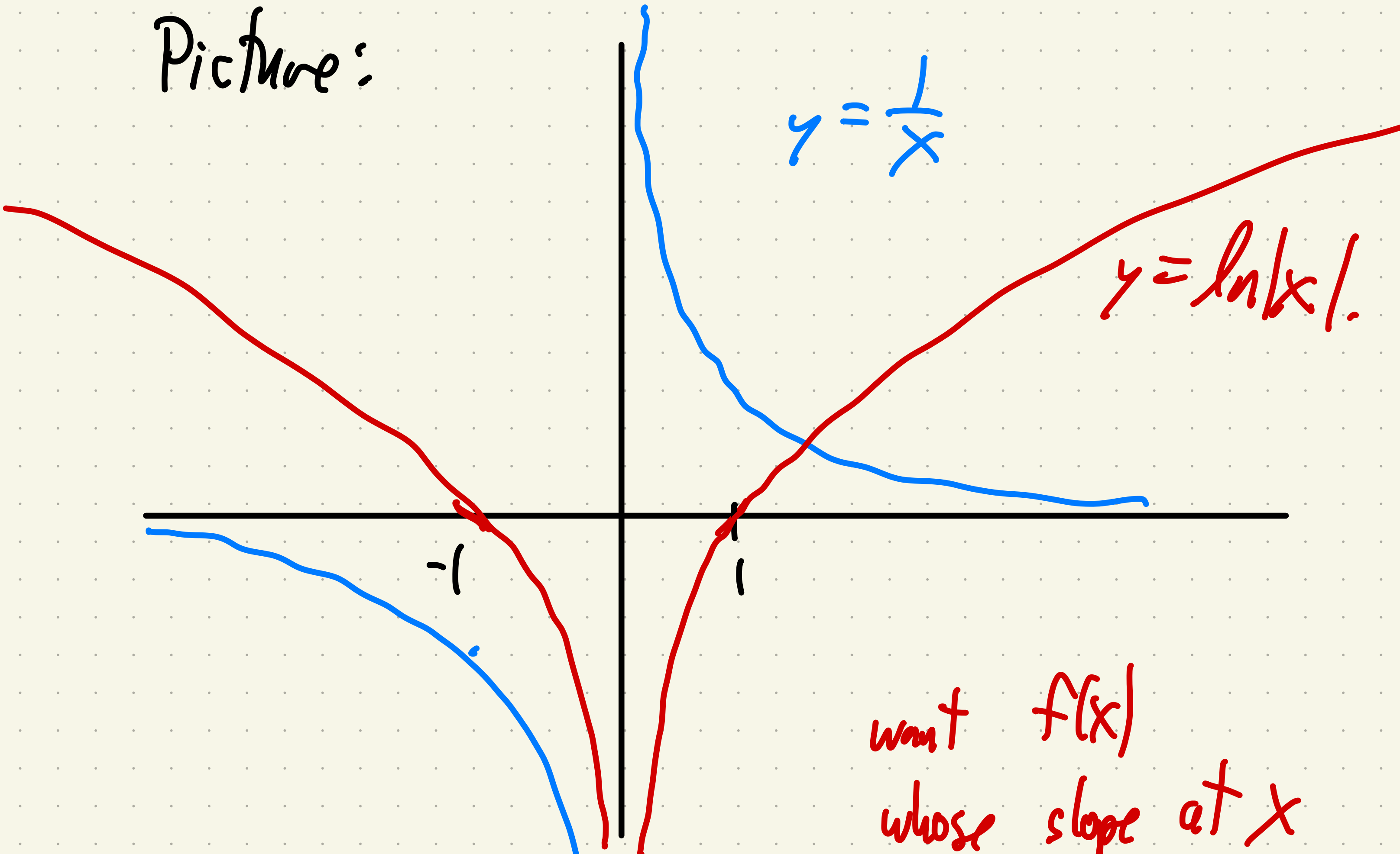
where C is any constant.

(b) Find $f(x)$ satisfying $f'(x) = \frac{1}{x}$.

$f(x) = \ln x + C$ works for $x > 0$.

What about for $x < 0$?

Picture:



want $f(x)$ whose slope at x is $\frac{1}{x}$

$\boxed{f(x) = \ln|x| + C}$ for any constant C.

(c) Find $f(x)$ such that $f'(x) = x^n$

(here $n \neq -1$).

$$f(x) = \frac{1}{n+1} x^{n+1} + C$$

for any constant C.

see why it's important that $n \neq -1$?

Power Rule

Check: $f'(x) = x^n + 0$

ok!

Remember: Power Rule says:

For any n,

$$\frac{d}{dx} x^n = n x^{n-1}$$

To "undo" This ...

Want $f(x)$ so that

$$\frac{d}{dx} f(x) = x^n$$

(where $n \neq -1$)

$$f(x) = \frac{1}{n+1} x^{n+1} + C$$

works.

Can do other examples, but it gets trickier:

Ex. $g(x) = x^4 + e^{2x} + \cos x + 1$

Find the most general antideriv. of g.

Remember the Addition Rule for Derivatives:
to find $f(x)$ such that

$$f'(x) = x^4 + e^{2x} + \cos x + 1.$$

$$f(x) = \frac{1}{5}x^5 + \frac{1}{2}e^{2x} + \sin x + x + C$$

for any constant C.

Check:

good! ☺

$$f'(x) = x^4 + e^{2x} + \cos x + 1 + 0$$

remember: $\frac{d}{dx} e^{2x}$

looks like $h(r(x))$

where $r(x) = 2x$ $\quad r'(x) = 2$

$h(y) = e^y$

$h'(y) = e^y$

So Chain Rule says:

$$\frac{d}{dx} h(r(x)) = h'(r(x)) r'(x)$$

$$= e^{2x} \cdot 2$$

So $$\frac{d}{dx}\left(\frac{1}{2} e^{2x}\right) = e^{2x}$$

Ex. $g(x) = xe^x$.

Find the most general antiderivative of g.

That is, find all f with

$$f'(x) = xe^x.$$

Method for now: experiment & use experience.

Guess: (remember: $\frac{d}{dx}e^x \neq xe^{x-1}$)

$$\frac{d}{dx}x^n = nx^{n-1}$$

Again, $g(x) = xe^x$.

Find $f(x)$ so that $f'(x) = xe^x$.

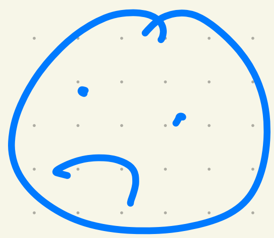
<u>Hint</u>: Think in terms of the Product Rule.

Try $f(x) = \frac{1}{2}x^2e^x$.

Then $f'(x) = xe^x + \frac{1}{2}x^2e^x$

by the Product Rule

$\neq xe^x$



$g(x) = xe^x$.

Solve $f'(x) = xe^x$

Try $f(x) = xe^x$.

By Product Rule,

$$f'(x) = e^x + xe^x$$

not quite...

want to remove this...

Next try

$$\boxed{f(x) = xe^x - e^x + C}$$

Check by differentiating...

Super Amazing Fact:

(The Main Result in Calculus)

Problems about tangents (The derivative is the slope of the tangent line) are related to Problems about Areas.

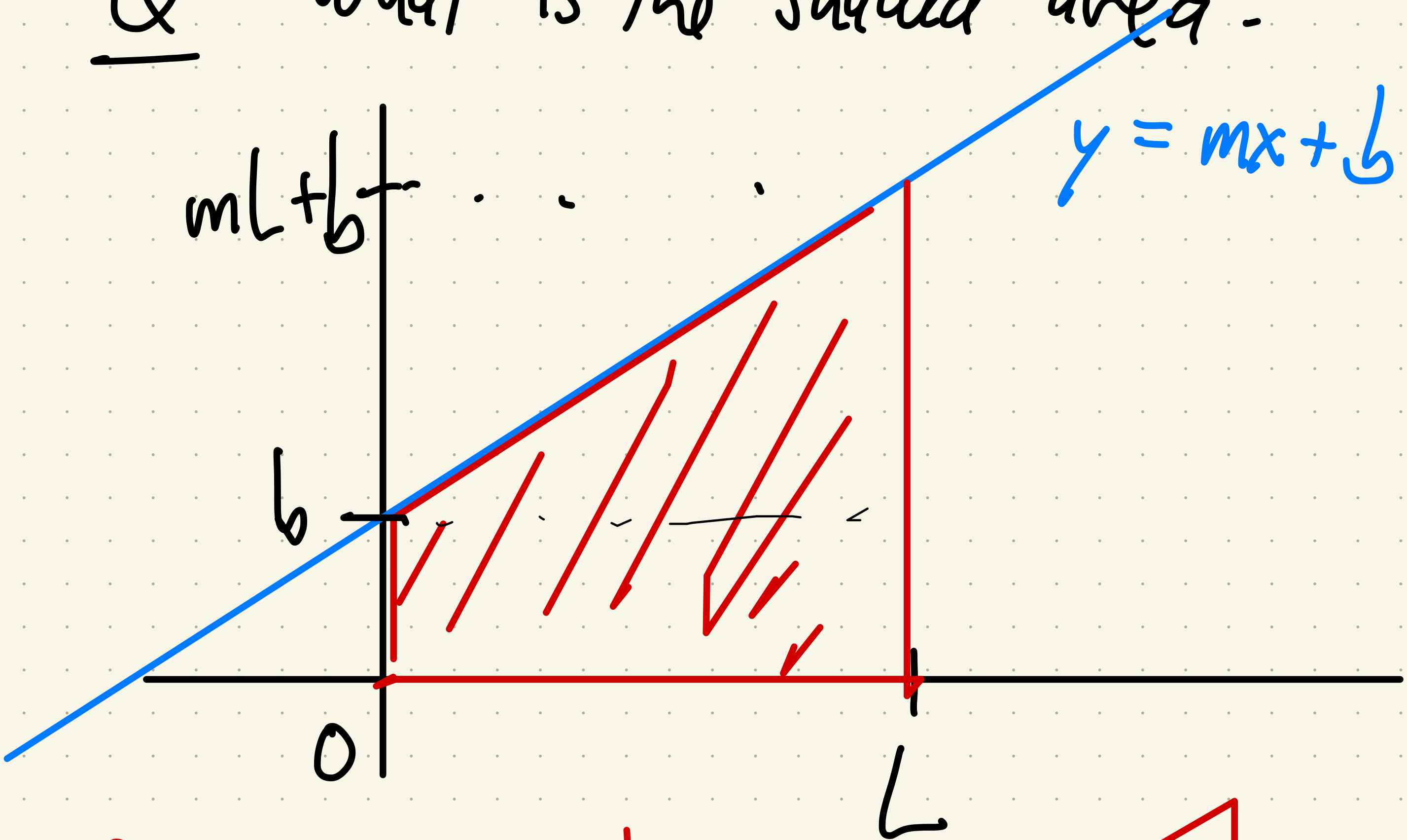
Wow!

__Example.__ Let $g(x) = mx + b$

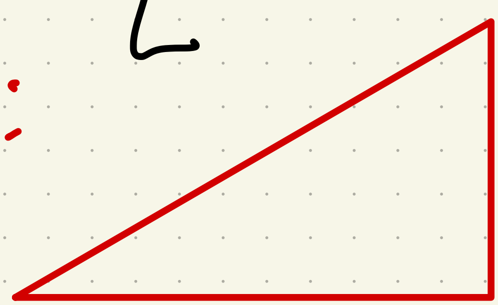
where $m > 0$, $b > 0$

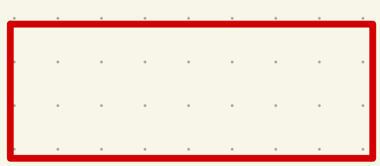
are fixed #s.

__Q__: What is the shaded area?



Can use geometry:





$$bL \quad + \quad \frac{1}{2}L(mL)$$

To be continued...

See you on Wednesday!