Given $f(x,y) = \begin{cases} \frac{2}{3}(x+2y), & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$

PDF

a) Find $f_X(x)$ and $f_Y(y)$

marginal density

$$f_Y(y) = \int_0^1 \left(\frac{2}{3}(x+2y)\right) dx = \frac{2}{3}\int_0^1 (x+2y)\,dx \rightarrow \frac{2}{3}\left[\frac{x^2}{2} + 2yx\right]\Big|_0^1$$

$$\frac{2}{3}\left[\frac{1}{2} + 2y(1)\right] = \frac{2}{3}\left[\frac{1}{2} + 2y\right] = \frac{2}{6} + \frac{4y}{3},\quad 0 \le y \le 1$$

$$f_X(x) = \int_0^1 \frac{2}{3}(x+2y) = \frac{2}{3}\int_0^1 (x+2y)\,dy = \frac{2}{3}\left(xy + \frac{2y^2}{2}\right)\Big|_0^1$$

$$\frac{2}{3}\left(x(1) + (1)^2\right) = \frac{2}{3}(x+1) = \frac{2}{3}x + \frac{2}{3},\quad 0 \le x \le 1$$

$$\boxed{\begin{aligned} f_Y(y) &= \frac{2}{6} + \frac{4y}{3},\quad 0 \le y \le 1 \\ f_X(x) &= \frac{2}{3}x + \frac{2}{3},\quad 0 \le x \le 1 \end{aligned}}$$

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b) Are X & Y independent? Explain

$P(x,y) = P_X(x) P_Y(y)$ ← discrete

or

$f(x,y) = f_X(x) f_Y(y)$ ← continuous

$$\frac{2}{3}(x+2y) = \left(\frac{2}{3}x + \frac{2}{3}\right)\left(\frac{1}{6} + \frac{4y}{3}\right) ?$$

$$\frac{2}{3}(x+2y) \neq \frac{4}{18}x + \frac{8}{9}xy + \frac{4}{18} + \frac{8}{9}y$$

$f(x,y) \qquad f_X(x) f_Y(y)$

No, They are not independent because: $f(x,y) \neq f_X(x) f_Y(y)$. In other words, the joint PDF function is not equal to the multiplication of the marginal densities of X and Y.

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c) Find $\sigma_x^2 \leftarrow$ Var

$\sigma_x^2 = E(x^2) - E(x)^2$

$E(x) = \int_0^1 x\left(\frac{2}{3}x + \frac{2}{3}\right) \rightarrow \int_0^1 \left(\frac{2}{3}x^2 + \frac{2}{3}x\right)dx$

$\left(\frac{2x^3}{9} + \frac{x^2}{3}\right)\Big|_0^1 \leftarrow \left(\frac{2}{3}\frac{x^3}{3} + \frac{2}{3}\frac{x^2}{2}\right)\Big|_0^1$

$\frac{2}{9} + \frac{1}{3} = \frac{2}{9} + \frac{3}{9} = \frac{5}{9}$

$E(x^2) = \int_0^1 \cancel{x^2}\left(\frac{2}{3}x + \frac{2}{3}\right)dx = \int_0^1 \left(\frac{2}{3}x^3 + \frac{2}{3}x^2\right)dx$

$\frac{2x^4}{12} + \frac{2x^3}{9}\Big|_0^1 \leftarrow \frac{2}{3}\frac{x^4}{4} + \frac{2}{3}\frac{x^3}{3}\Big|_0^1$

$\frac{x^4}{6} + \frac{2x^3}{9}\Big|_0^1 \rightarrow \frac{1}{6} + \frac{2}{9} \rightarrow \frac{3}{18} + \frac{4}{18} = \frac{7}{18}$

$\sigma_x^2 = \frac{7}{18} - \left(\frac{5}{9}\right)^2 = \frac{7}{18} - \frac{25}{81} = \boxed{\frac{117}{1458}}$

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d) Find $E(Y)$

$$E(Y) = \int_0^1 y\left(\frac{2}{6} + \frac{4y}{3}\right) dy = \int_0^1 \left(\frac{2}{6}y + \frac{4y^2}{3}\right) dy$$

$$\left(\frac{y^2}{6} + \frac{4y^3}{9}\right)\Bigg|_0^1 \leftarrow \left(\frac{2}{6}\frac{y^2}{\cancel{2}} + \frac{4}{3}\frac{y^3}{3}\right)\Bigg|_0^1$$

$$\left(\frac{1}{6} + \frac{4}{9}\right) = \frac{3}{18} + \frac{8}{18} = \frac{11}{18}$$

$$\boxed{E(Y) = \frac{11}{18}}$$

e) Find $Cov(X,Y)$

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

$$E(XY) = \int_0^1\int_0^1 xy\, f(x,y)\, dx\, dy = \int_0^1\int_0^1 xy\left[\frac{2}{3}(x+2y)\right] dx\, dy$$

$$= \int_0^1\int_0^1 y\left(\frac{2}{3}(x^2+2xy)\right) dx\, dy = \int_0^1 \left(\int_0^1 \frac{2}{3}(x^2y + 2xy^2)\, dx\right) dy$$

$$= \int_0^1\int_0^1 y\left(\frac{2}{3}(x^2+2xy)\right) dx\, dy = \int_0^1 \left(\int_0^1 \frac{2}{3}(x^2y + 2xy^2)\, dx\right) dy$$

$$\left(\frac{x^3}{3}y + x^2y^2\right)\Bigg|_0^1 \leftarrow \int_0^1 \left(\left(\frac{x^3}{3}y + \frac{2x^2}{2}y^2\right)\Bigg|_0^1\right) dy$$

$$\int_0^1 \left(\frac{1}{3}y + y^2\right) dy \rightarrow \left(\frac{1}{3}\frac{y^2}{2} + \frac{y^3}{3}\right)\Bigg|_0^1 = \frac{1}{6} + \frac{1}{3} = \frac{1}{2}$$

$$\frac{1}{2} - \left(\frac{5}{9}\right)\left(\frac{11}{18}\right) = \frac{1}{2} - \frac{55}{162} = 0.16 \qquad \boxed{Cov(X,Y) = 0.16}$$