MATH 30, 3/27/2020: INTRO TO INCREASING/DECREASING FUNCTIONS

Last time we looked at:

Mean Value Theorem. Let f be a function such that:
(1) f is continuous on $[a, b]$
(2) f is differentiable on (a, b).
Then there is a number c in (a, b) where

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

That is,

$$f(b) = f(a) + (b - a)f'(c).$$

[Picture.]

It might look complicated, but it's a lot easier to understand if you draw the picture [see Zoom notes].

Example. Use the Mean Value Theorem to prove:

$$|\sin x - \sin y| \leq |x - y| \qquad \text{for all } x, y.$$

Proof. We may assume $x < y$. Let $f(t) = \sin t$. By the MVT there is some c with $x < c < y$ such that

$$(\cos(c) = \quad) \qquad f'(c) = \frac{f(y) - f(x)}{y - x}.$$

But also $|\cos(c)| \leq 1$ is always true, which shows that

$$\left| \frac{f(y) - f(x)}{y - x} \right| \leq 1.$$

This is what we needed to prove.

New Section: What do f' and f'' reveal about the shape of the curve $y = f(x)$?

Example. Where is $f(x) = 3x^4 - 8x^3 + 6x^2$ increasing? decreasing? Can you answer this question without drawing the graph?

Because of Coronavirus, I will be brief:

If $f'(c) > 0$, then the slope of the tangent line is positive at that point, so the function is increasing near that point.

If $f'(c) < 0$, then the slope of the tangent line is negative at that point, so the function is decreasing near that point.

You can give a precise mathematical proof of this using the Mean Value Theorem, but we will skip that (you can see the book).

This gives another way to approach Min/Max problems.

Example. Find the global min and max of the function $f(x) = x^5 + x + 1$ on $[-1, 1]$.

Solution. We have $f'(x) = 5x^4 + 1 \geq 1$, so f is always increasing. Thus the global min is $f(-1) = -1$ and the global max is $f(1) = 3$.

Example. Find the global min and max of the function $f(x) = 3x^4 - 8x^3 + 6x^2$ on $[-\frac{1}{2}, \frac{1}{2}]$.

Solution. We have $f'(x) = 12x^3 - 24x^2 + 12x = 12x(x-1)^2$. Since $f'(x) < 0$ for $x < 0$, f is *decreasing* on the interval $[-\frac{1}{2}, 0)$. Since $f'(x) > 0$ for $x > 0$, f is *increasing* on the interval $(0, \frac{1}{2}]$. There is a critical point at $x = 0$ ($x = 1$ is not in the domain).

[Sketch.]

So we see the global min is at $x = 0$, and then compare the endpoints to see that $f(-\frac{1}{2}) = \frac{43}{16}$ is the global max.

Here are some problems for you:

Problem 1. Consider the function $f(x) = x^3 - x^2 + x + 3$. Note that $f(-1) = 0$. At $x = -1$ does f change from positive to negative or from negative to positive? Make a rough sketch.

Problem 2. On what intervals is $f(x) = x^3 - 2x + 6$ increasing? On what intervals is it decreasing? Where are the local maxima and minima? Make a rough sketch.

After Spring Break we will see what the second derivative f'' tells us about the shape of the graph.