1) a)

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ -3 & 1 & 6 & 3 \\ 2 & -2 & -1 & -1 \end{array}\right] \xrightarrow[-2R_1+R_3\to R_3]{3R_1+R_2\to R_2} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & -3 & -3 \\ 0 & -2 & 5 & 3 \end{array}\right] \xrightarrow{2R_2+R_3\to R_3} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & -1 & -3 \end{array}\right] \to \left[\begin{array}{ccc|c} \sim & & & - \\ \sim & & & - \\ 0 & 0 & 1 & 3 \end{array}\right] \to$$

$3R_3+R_2\to R_2$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 3 \end{array}\right] \to \vec{x} = \begin{bmatrix} 7 \\ 6 \\ 3 \end{bmatrix} \to \text{unique, all entries accounted for.}$$

$3R_3+R_1\to R_1$

b)

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & 1 \\ 3 & -8 & 8 & 6 \\ 0 & 1 & 2 & 3 \\ 1 & 0 & 8 & 10 \end{array}\right] \xrightarrow{-3R_1+R_2\to R_2} \left[\begin{array}{ccc|c} 1 & -3 & 2 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \\ 1 & 0 & 8 & 10 \end{array}\right] \xrightarrow{-1R_2+R_3\to R_3} \left[\begin{array}{ccc|c} 1 & -3 & 2 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 8 & 10 \end{array}\right] \xrightarrow[R_3\leftrightarrow R_4]{3R_2+R_1\to R_4} \left[\begin{array}{ccc|c} 1 & 0 & 8 & 10 \\ 0 & 1 & 2 & 3 \\ 1 & 0 & 8 & 10 \\ 0 & 0 & 0 & 0 \end{array}\right] \to$$

$-R_1+R_3\to R_3$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 8 & 10 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \begin{array}{l} x_1 + 8x_3 = 10 \\ x_2 + 2x_3 = 3 \\ x_3 = \text{free variable} \end{array} \to \begin{array}{l} x_1 = 10 - 8x_3 \\ x_2 = 3 - 2x_3 \\ x_3 = x_3 \end{array} \to \vec{x} = \begin{bmatrix} 10-8x_3 \\ 3-2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -8 \\ -2 \\ 1 \end{bmatrix}$$

there is no x_4

↳ let $x_3 = 1$ →

$$\vec{x} = \begin{bmatrix} 10 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} -8 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

⇓

a solution.

Not unique, there is a free variable.

② Want $T(\vec{x}) = \vec{0} \rightarrow A\vec{x} = \vec{0}$

$$\left[\begin{array}{cccc|c} 3 & 2 & 10 & -6 & 0 \\ 1 & 0 & 2 & -4 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 1 & 4 & 10 & 8 & 0 \end{array}\right] \xrightarrow[-1R_2+R_4\to R_4]{-3R_2+R_1\to R_1} \left[\begin{array}{cccc|c} 0 & 2 & 4 & 6 & 0 \\ 1 & 0 & 2 & -4 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 4 & 8 & 12 & 0 \end{array}\right] \xrightarrow{\frac{1}{2}R_1\to R_1,\ \frac{1}{4}R_4\to R_4} \left[\begin{array}{cccc|c} 0 & 1 & 2 & 3 & 0 \\ 1 & 0 & 2 & -4 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 3 & 0 \end{array}\right] \xrightarrow{-R_1+R_3\to R_3,\ -R_1+R_4\to R_4} \left[\begin{array}{cccc|c} 0 & 1 & 2 & 3 & 0 \\ 1 & 0 & 2 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightarrow$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cccc|c} 1 & 0 & 2 & -4 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \begin{array}{l} x_1 + 2x_3 - 4x_4 = 0 \\ x_2 + 2x_3 + 3x_4 = 0 \\ x_3, x_4 \text{ are free} \end{array} \rightarrow \begin{array}{l} x_1 = -2x_3 + 4x_4 \\ x_2 = -2x_3 - 3x_4 \\ x_3 = x_3 \\ x_4 = x_4 \end{array}$$

$$\rightarrow \vec{x} = \begin{bmatrix} -2x_3 + 4x_4 \\ -2x_3 - 3x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ -2x_3 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 4x_4 \\ -3x_4 \\ 0 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 4 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

③ Based on the information given we have that $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$

a) We want $T\left(\begin{bmatrix} 5 \\ -3 \end{bmatrix}\right)$: Note that $\begin{bmatrix} 5 \\ -3 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -3 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 0 \end{bmatrix} + (-3)\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Therefore,

by properties of Linear transformations $T\left(\begin{bmatrix} 5 \\ -3 \end{bmatrix}\right) = T\left(5\begin{bmatrix} 1 \\ 0 \end{bmatrix} + (-3)\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$

$$= 5\,T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + (-3)\,T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$
$$= 5\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right) + (-3)\begin{bmatrix} -1 \\ 6 \end{bmatrix}$$
$$= \begin{bmatrix} 10 + 3 \\ 25 - 18 \end{bmatrix} = \begin{bmatrix} 13 \\ 7 \end{bmatrix},$$

b) Likewise, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, thus

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = x_1 \cdot T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + x_2 \cdot T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$
$$= x_1 \cdot \begin{bmatrix} 2 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 6 \end{bmatrix}$$
$$= \begin{bmatrix} 2x_1 - x_2 \\ 5x_1 + 6x_2 \end{bmatrix}$$

④ By the assumptions $Span\{\vec{v}_1, \dots \vec{v}_p\} = \mathbb{R}^n$, so for any $\vec{x}$ in $\mathbb{R}^n$

$\vec{x} = c_1\vec{a}_1 + c_2\vec{a}_2 + \cdots c_p\vec{a}_p$, thus $T(\vec{x}) = T(c_1\vec{a}_1 + \cdots + c_p\vec{a}_p)$

(T is linear transform) $= c_1 T(\vec{a}_1) + c_2 T(\vec{a}_2) + \cdots c_p T(\vec{a}_p)$

$= c_1 \cdot \vec{0} + c_2 \cdot \vec{0} + \cdots c_p \cdot \vec{0}$

(since $T(\vec{v}_i) = \vec{0}$, for all $i = 1, 2, \dots p$) $= \vec{0}$.

So, for any $\vec{x}$ in $\mathbb{R}^n$, $T(\vec{x}) = \vec{0}$.

⑤ Note $5I_3 = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$. So, $A - 5I_3 = \begin{bmatrix} (5-5) & -1-0 & 3-0 \\ -4-0 & (3-5) & -6-0 \\ -3-0 & 1-0 & 2-5 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 3 \\ -4 & -2 & -6 \\ -3 & 1 & -3 \end{bmatrix}$ ✓

and $(5I_3) \cdot A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \cdot \begin{bmatrix} 5 & -1 & 3 \\ -4 & 3 & -6 \\ -3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 \cdot 5 + 0 & -1 \cdot 5 + 0 & 5 \cdot 3 + 0 \\ -4 \cdot 5 + 0 & 3 \cdot 5 + 0 & -6 \cdot 5 + 0 \\ -3 \cdot 5 + 0 & 1 \cdot 5 + 0 & 2 \cdot 5 + 0 \end{bmatrix} = \begin{bmatrix} 25 & -5 & 15 \\ -20 & 15 & -30 \\ -15 & 5 & 10 \end{bmatrix}$ ✓

⑥ By the assumption, A is invertible, and it also shows that A is row equivalent to I_n so $A \sim I_n$ (not equal!); Thus, the system

$\left[A \,\middle|\, \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix}\right] \sim \left[I_n \,\middle|\, \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix}\right]$ meaning that $\vec{x} = \vec{0}$ since there are no free variables,

Since A is invertible and row equivalent to I_n, then A must have the same number of rows and columns.

HW-4 (2, 4, 7, 8, 9, 10, 11)

7) Consider $\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 5 & 12 & 0 & 1 \end{array}\right] \xrightarrow{-5R_1 + R_2 \to R_2} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 2 & -5 & 1 \end{array}\right] \xrightarrow{-R_2 + R_1 \to R_1} \left[\begin{array}{cc|cc} 1 & 0 & 6 & -1 \\ 0 & 2 & -5 & 1 \end{array}\right] \xrightarrow{\frac{1}{2}R_2 \to R_2} \left[\begin{array}{cc|cc} 1 & 0 & 6 & -1 \\ 0 & 1 & -\frac{5}{2} & \frac{1}{2} \end{array}\right]$

Therefore, $A^{-1} = \begin{bmatrix} 6 & -1 \\ -\frac{5}{2} & \frac{1}{2} \end{bmatrix}$.

For $A\vec{x} = \vec{b}_1 \rightarrow \vec{x} = A^{-1}\vec{b}_1 \rightarrow \begin{bmatrix} 6 & -1 \\ -\frac{5}{2} & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 6\cdot(-1) + (-1)\cdot 3 \\ (-\frac{5}{2})\cdot(-1) + \frac{1}{2}\cdot 3 \end{bmatrix} = \begin{bmatrix} -6 - 3 \\ \frac{5}{2} + \frac{3}{2} \end{bmatrix} = \begin{bmatrix} -9 \\ 4 \end{bmatrix}$

For $A\vec{x} = \vec{b}_2 \rightarrow \vec{x} = A^{-1}\vec{b}_2 \rightarrow \begin{bmatrix} 6 & -1 \\ -5/2 & 1/2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -5 \end{bmatrix} = \begin{bmatrix} 6\cdot 1 + (-1)\cdot(-5) \\ (-\frac{5}{2})\cdot 1 + (\frac{1}{2})(-5) \end{bmatrix} = \begin{bmatrix} 6 + 5 \\ -\frac{5}{2} - \frac{5}{2} \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \end{bmatrix}$

For $A\vec{x} = \vec{b}_3 \rightarrow \vec{x} = A^{-1}\vec{b}_3 \rightarrow \begin{bmatrix} 6 & -1 \\ -\frac{5}{2} & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 6\cdot 2 + (-1)\cdot 6 \\ (-\frac{5}{2})\cdot 2 + (\frac{1}{2})\cdot 6 \end{bmatrix} = \begin{bmatrix} 12 - 6 \\ -5 + 3 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$

For $A\vec{x} = \vec{b}_4 \rightarrow \vec{x} = A^{-1}\vec{b}_4 = \begin{bmatrix} 6 & -1 \\ -\frac{5}{2} & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 6\cdot 3 + (-1)\cdot 5 \\ (-\frac{5}{2})\cdot 3 + (\frac{1}{2})\cdot 5 \end{bmatrix} = \begin{bmatrix} 18 - 5 \\ -\frac{15}{2} + \frac{5}{2} \end{bmatrix} = \begin{bmatrix} 13 \\ -5 \end{bmatrix}$

b) $\left[\begin{array}{cc|cccc} 1 & 2 & -1 & 1 & 2 & 3 \\ 5 & 12 & 3 & -5 & 6 & 5 \end{array}\right] \xrightarrow{\text{same as top}} \left[\begin{array}{cc|cccc} 1 & 2 & -1 & 1 & 2 & 3 \\ 0 & 2 & 8 & -10 & -4 & -10 \end{array}\right]$

$\xrightarrow{\text{Same as top}} \left[\begin{array}{cc|cccc} 1 & 0 & -9 & 11 & 6 & 13 \\ 0 & 2 & 8 & -10 & -4 & -10 \end{array}\right] \xrightarrow{\text{Same as top}} \left[\begin{array}{cc|cccc} 1 & 0 & -9 & 11 & 6 & 13 \\ 0 & 1 & 4 & -5 & -2 & -5 \end{array}\right]$.

(8) P is invertible, thus P^{-1} exists and $P \cdot P^{-1} = I = P^{-1} \cdot P$

So, $A = PBP^{-1} \rightarrow A \cdot P = PB(P^{-1} \cdot P) \rightarrow (A \cdot P) = (P \cdot B) \rightarrow P^{-1}(A \cdot P) = P^{-1}(PB)$

$= PBI$

$= PB$

$= (P^{-1} \cdot P) B$

$= IB$

$= B$

thus $B = P^{-1} \cdot A \cdot P$.

(9) A is invertible, thus A^{-1} exists and $A \cdot A^{-1} = I = A^{-1} \cdot A$

So, $AD = I \rightarrow A^{-1} \cdot (AD) = A^{-1} \cdot I \longrightarrow (A^{-1} \cdot A) \cdot D = A^{-1} \rightarrow I \cdot D = A^{-1} \rightarrow D = A^{-1}$

(10) A is invertible, thus $\exists A^{-1}$ such that $A \cdot A^{-1} = I = A^{-1} \cdot A$

So, $AB = AC \rightarrow A^{-1} \cdot (AB) = A^{-1} \cdot (AC) \rightarrow (A^{-1} \cdot A) \cdot B = (A^{-1} \cdot A) \cdot C$

$\rightarrow I \cdot B = I \cdot C \rightarrow B = C$.

(11) A few different answers work, but...

Since matrix A is invertible, then A^T is invertible, by the Invertible Matrix Theorem (IMT).

Since A^T is invertible, then by the IMT the columns of A are linearly independent.

(12) No. Consider a $n \times n$ matrix A and the matrix equation $A\vec{x} = \vec{0}$.

If A has two identical rows, then A will have a free variable, thus there will be more than just the trivial solution meaning the columns of A are not linearly independent. By the IVM, A cannot be invertible.

(13) AB is invertible, thus there exists a matrix W such that $(AB) \cdot W = I$.

Since $(A \cdot B)W = I$ then $A(BW) = I$. Let $BW = C$, then there exists a matrix C such that $AC = I$, thus A is invertible.