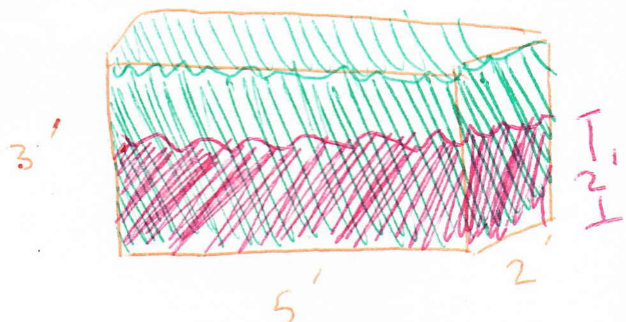
# 1 Applications to Physics and Engineering

1. An aquarium 5 ft long, 2 ft wide, and 3 ft deep is filled up to 2 ft of water. Find each of the following. Hint: the weight density of water is $\delta = \rho g = 62.5$ lb/ft$^3$.

(a) The hydrostatic pressure on the bottom of the aquarium.

(b) The hydrostatic force on the bottom.

(c) The hydrostatic force on one end of the aquarium.




(a) $P = \frac{F}{A} = \frac{\delta A d}{A} = \delta d$

$P = 62.5(2) = \boxed{125 \text{ lb/ft}^2}$

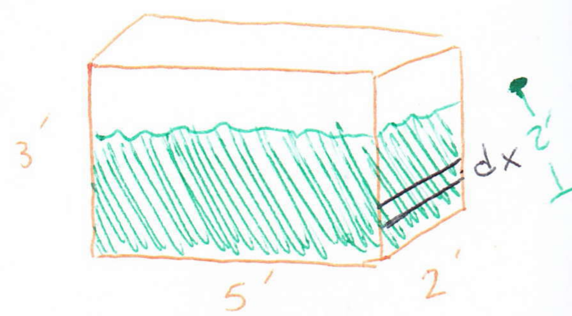
(b) $F = \delta A d = \delta d(l \cdot w)$

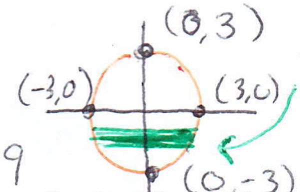
$F = 62.5(2)(5)(2) = \boxed{1,250 \text{ lb}}$

(c) $F = \delta A d = \int_0^2 62.5(2x)\,dx$

$= 125 \int_0^2 x\,dx$

$= 62.5 x^2 \Big|_0^2 = \boxed{250 \text{ lb}}$





$x^2+y^2=9$

$A = \underbrace{2\sqrt{9-x^2}}\,\underbrace{dx}$

$A = l \cdot w$

2. A milk truck carries milk with a density of 64.6 lb/ft$^3$ in a horizontal cylindrical tank with diameter 6 ft. Find the force exerted by the milk on one end of the tank when the tank is full. What if the tank is only half full?

$$F = \delta A d = \int_{-3}^{3} \overset{\delta}{64.6}\,\overset{d}{(3-x)}\,\overset{A}{2\sqrt{9-x^2}}\,dx$$

$$= 129.2\int_{-3}^{3}(3-x)\sqrt{9-x^2}\,dx$$

$$= 129.2\left(3\int_{-3}^{3}(9-x^2)^{1/2}dx - \cancel{\int_{-3}^{3}x(9-x^2)^{1/2}dx}\rightarrow 0\right)$$

$u = 9-x^2$

$du = -2x\,dx$

$\int \frac{1}{2}u^{1/2}du = \frac{1}{3}u^{3/2}$

$= \frac{1}{3}u^{3/2}\Big|_{-3}^{3}$

$= 0$

$$= 129.2\left(3\cdot\pi(3)^2 - 0\right) \approx \boxed{10{,}959\text{ lb}}$$

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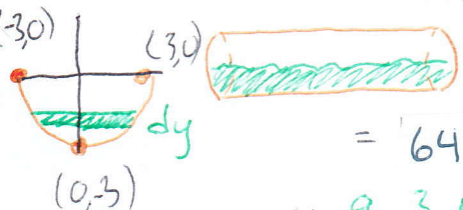


$$F = \delta A d = \int_{-3}^{0} \overset{\delta}{64.6}\,\overset{d}{(0-y)}\,\overset{A}{2\sqrt{9-y^2}}\,dy$$

$$= 64.6\int_{-3}^{0}-2y\sqrt{9-y^2}\,dy = 64.6\int_{0}^{9}u^{1/2}du = 64.6\left(\frac{2}{3}u^{3/2}\Big|_0^9\right)$$

$$= \boxed{1{,}162.8\text{ lb}}$$

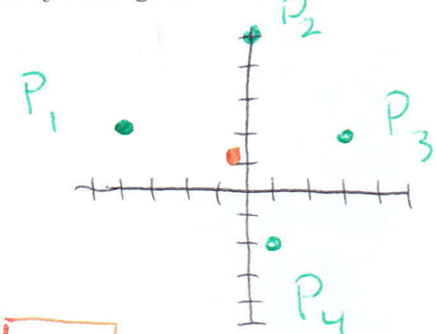
$u = 9-y^2$, $du = -2y\,dy$

$w = 2\sqrt{9-y^2}$

3. Find the moments $M_x$ and $M_y$ along with the center of mass of the system given where:

$$m_1 = 5, m_2 = 4, m_3 = 3, m_4 = 6$$

$$P_1(-4, 2), P_2(0, 5), P_3(3, 2), P_4(1, -2)$$



$$m = \sum_{i=1}^{4} m_i = 5+4+3+6 = 18$$

$$M_y = \sum_{i=1}^{4} m_i x_i = 5(-4)+4(0)+3(3)+6(1) = \boxed{-5}$$

$$M_x = \sum_{i=1}^{4} m_i y_i = 5(2)+4(5)+3(2)+6(-2) = \boxed{24}$$

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right) = \boxed{\left(-\frac{5}{18}, \frac{4}{3}\right)}$$

## 2 Applications to Economics and Biology

1. The marginal cost function $C'(x)$ is defined to be the derivative of the cost function. The marginal cost of producing $x$ gallons of orange juice is

$$C'(x) = 0.82 - 0.00003x + 0.000000003x^2$$

The fixed start up cost is $C(0) = \$18,000$. Use the net change theorem to find the cost of producing the first 4000 gallons of juice.

$$C(x) = 18000 + 0.82x - 0.000015x^2 + 0.000000001x^3$$

$$* \ C(4000) = 18000 + 0.82(4000) - 0.000015(4000)^2 + 0.000000001(4000)^3$$

$$= \boxed{\$21,104}$$

$$C(4000) - C(0) = \int_0^{4000} C'(x)\,dx$$

$$C(4000) = C(0) + \int_0^{4000} C'(x)\,dx$$

$$= 18000 + \left(0.82x\Big|_0^{4000} - 0.000015x^2\Big|_0^{4000} + 0.000000001x\Big|_0^{4000}\right)$$

$$= \boxed{\$21,104}$$

2. A demand curve is given by $p = 450/(x+8)$. Find the consumer surplus when the selling price is \$10.

$$P(x) = \frac{450}{x+8} \rightarrow 10 = \frac{450}{x+8} \rightarrow x+8 = 45 \rightarrow x = 37$$

$$\text{consumer surplus} = \int_0^X [p(x) - P]\,dx = \int_0^{37} \left(\frac{450}{x+8} - 10\right)dx$$

$$= 10\left[45\int_0^{37} (x+8)^{-1}\,dx - \int_0^{37} dx\right]$$

$$= 10\left[45 \cdot \ln(x+8)\Big|_0^{37} - x\Big|_0^{37}\right]$$

$$= 10\left[45(\ln 45 - \ln 8) - 37\right]$$

$$= \boxed{\$407.25}$$

3. If income is continuously collected at a rate of $f(t)$ dollars per year and will be invested at a constant rate $r$ (compounded continuously) for a period of $T$ years, then the future value of the income is given by $\int_0^T f(t)e^{r(T-t)}dt$. Compute the future value after 6 years for income received at a rate of $f(t) = 8000e^{0.04t}$ dollars per year and invested at 6.2% interest.

$$\text{Future Value} = \int_0^6 8000 e^{0.04t} \cdot e^{0.062(6-t)} dt$$

$$= 8000\int_0^6 e^{0.04t} \cdot e^{0.372} \cdot e^{-0.062t} dt$$

$$= 8000 e^{0.372} \int_0^6 e^{(0.04-0.062)t} dt$$

$$= 8000 e^{0.372} \int_0^6 e^{-0.022t} dt$$

$$= -\frac{4000000}{11} e^{0.372} \left( e^{-0.022t} \Big|_0^6 \right) = \boxed{\$65{,}230.48}$$

4. Pareto's Law of Income states that the number of people with incomes betwween $x = a$ and $x = b$ is $N = \int_a^b Ax^{-k}dx$, where $A$ and $k$ are constants with $A > 0$ and $k > 1$. The average income of these people is

$$\bar{x} = \frac{1}{N} \int_a^b Ax^{1-k}dx$$

Calculate the average income $\bar{x}$.

$$\bar{X} = \frac{1}{\int_a^b Ax^{-k}dx} \cdot \int_a^b Ax^{1-k} dx$$

$$= \frac{1}{\int_a^b x^{-k} dx} \cdot \int_a^b x^{1-k} dx$$

$$= \frac{1}{\frac{1}{-k+1} x^{-k+1}} \cdot \frac{1}{1-k+1} x^{1-k+1} \Big|_a^b$$

$$= \frac{1-k}{x^{1-k}} \cdot \frac{x^{-k}}{-k} \Big|_a^b = \frac{k-1}{k} \cdot x^{-1} \Big|_a^b = \boxed{\frac{k-1}{k(b-a)}}$$

## 3 Probability

1. Let $f(x) = 30x^2(1-x)^2$ for $0 \le x \le 1$ and $f(x) = 0$ for all other values of $x$.

(a) Verify that $f$ is a probability density function.

(b) Find $P(X \le \frac{1}{3})$.

(a) $f(x) \ge 0$ for all $x$

$$\int_{-\infty}^{\infty} f(x)dx = \int_0^1 30x^2(1-x)^2 dx = 30\int_0^1 (x^2 - 2x^3 + x^4)dx$$

$$= 30\left(\frac{1}{3}x^3\Big|_0^1 - \frac{2}{4}x^4 + \frac{1}{5}x^5\Big|_0^1\right) = 1$$

(b) $$P(X \le \tfrac{1}{3}) = \int_{-\infty}^{1/3} f(x)dx = \int_0^{1/3} 30x^2(1-x^2)dx$$

$$= 30\left(\frac{1}{3}x^3\Big|_0^{1/3} - \frac{1}{2}x^4\Big|_0^{1/3} + \frac{1}{5}x^5\Big|_0^1\right)$$

$$\boxed{= 0.2099}$$

2. The following density function is an example of a logistic distribution

$$f(x) = \frac{e^{3-x}}{(1+e^{3-x})^2}$$

Verify that $f$ is a probability density function and find $P(3 \le X \le 4)$.

* $f(x) \ge 0$ for all $x$

Let $u = 1 + e^{3-x}$

$du = -e^{3-x}dx$

* $$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{3} \frac{e^{3-x}}{(1+e^{3-x})^2}dx + \int_3^{\infty} \frac{e^{3-x}}{(1+e^{3-x})^2}dx$$

$$= \int_{\infty}^{2} -u^{-2}du + \int_2^1 -u^{-2}du = \int_2^{\infty} u^{-2}du + \int_1^2 u^{-2}du$$

$$= \lim_{t\to\infty}\int_2^t u^{-2}du + \int_1^2 u^{-2}du = \lim_{t\to\infty}\left(-\frac{1}{u}\Big|_2^t\right) + \left(-\frac{1}{u}\Big|_1^2\right)$$

$$= \frac{1}{2} + \frac{1}{2} = \boxed{1}$$

$$P(3 \le X \le 4) = \int_3^4 \frac{e^{3-x}}{(1+e^{3-x})^2}dx = \int_2^{1+\frac{1}{e}} -u^{-2}du = \frac{1}{u}\Big|_2^{1+\frac{1}{e}} = \boxed{0.2311}$$

3. Let $f(x) = k(3x - x^2)$ if $0 \le x \le 3$ and $f(x) = 0$ for all other values of $x$. For what value of $k$ is $f$ a probability density function? Find the mean of the distribution.

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \rightarrow \int_0^3 k(3x-x^2)\,dx = 1 \rightarrow k\int_0^3 (3x-x^2)\,dx = 1$$

$$\rightarrow k\left[\frac{3}{2}x^2\Big|_0^3 - \frac{1}{3}x^3\Big|_0^3\right] = 1 \rightarrow k\left(\frac{27}{2} - \frac{27}{3}\right) = 1 \rightarrow \boxed{k = \frac{2}{9}}$$

$$\mu = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \int_0^3 x \cdot \frac{2}{9}(3x-x^2)\,dx = \frac{2}{9}\int_0^3 (3x^2 - x^3)\,dx$$

$$= \frac{2}{9}\left[x^3\Big|_0^3 - \frac{1}{4}x^4\Big|_0^3\right] = \frac{2}{9}\left(27 - \frac{81}{4}\right) = \frac{54}{36} = \boxed{\frac{3}{2}}$$

4. The time between infection and the display of symptoms for streptococcal sore throat is a random variable whose probability density function can be approximated by $f(t) = \frac{1}{15676}t^2e^{-0.05t}$ if $0 \le t \le 150$ and $f(t) = 0$ otherwise.

(a) What is the probability that an infected patient will display symptoms within 48 hours?

(b) What is the probability that an infected patient will not display symptoms until after 36 hours?

(a) $P(0 \le T \le 48) = \int_0^{48} \frac{1}{15676} t^2 e^{-0.05t}\,dt = \frac{1}{15676}\int_0^{48} t^2 e^{-0.05t}\,dt$

Let $u = t^2$, $du = 2t\,dt$

$v = -20e^{-0.05t}$, $dv = e^{-0.05t}\,dt$

Let $u = t$, $du = dt$

$v = -20e^{-0.05t}$, $dv = e^{-0.05t}$

$$= \frac{1}{15676}\left[-20t^2e^{-0.05t}\Big|_0^{48} + 40\int_0^{48} te^{-0.05t}\,dt\right]$$

$$= \frac{1}{15676}\left[-20t^2e^{-0.05t}\Big|_0^{48} + 40\left(-20te^{-0.05t}\Big|_0^{48} + 20\int_0^{48} e^{-0.05t}\,dt\right)\right]$$

$$= \frac{1}{15676}\left[-20t^2e^{-0.05t}\Big|_0^{48} - 800te^{-0.05t}\Big|_0^{48} - 16000e^{-0.05t}\Big|_0^{48}\right]$$

$= \boxed{0.4392}$

(b) $P(36 \le T) = P(36 \le T \le 150)$

$$= \frac{1}{15676}\left[-20t^2e^{-0.05t}\Big|_{36}^{150} - 800te^{-0.05t}\Big|_{36}^{150} - 16000e^{-0.05t}\Big|_{36}^{150}\right]$$

$= \boxed{0.725}$