1 Integration by Parts

Evaluate each of the following using integration by parts.

1. $\int x \csc^2 x dx$

$u = x$, $du = dx$

$v = -\cot x$, $dv = \csc^2 x dx$

$\int x \csc^2 x dx = -x\cot x + \int \cot x dx$

$= -x\cot x + \int \frac{\cos x}{\sin x} dx$ $u = \sin x$, $du = \cos x dx$

$= -x\cot x + \int \frac{1}{u} du$

$= -x\cot x + \ln|u| + C = \boxed{-x\cot x + \ln|\sin x| + C}$

2. $\int_0^{1/2} x\cos(\pi x)dx$

$u = x$, $du = dx$

$v = \frac{1}{\pi}\sin(\pi x)$, $dv = \cos(\pi x)dx$

$\int_0^{1/2} x\cos(\pi x)dx = \frac{1}{\pi}x\sin(\pi x)\Big|_0^{1/2} - \frac{1}{\pi}\int_0^{1/2}\sin(\pi x)dx$

$= \frac{1}{2\pi} + \frac{1}{\pi^2}\cos(\pi x)\Big|_0^{1/2}$

$= \boxed{\frac{1}{2\pi} - \frac{1}{\pi^2} = \frac{\pi - 2}{2\pi^2} \approx 0.0578}$

3. $\int t^2 \sin(2t)dt$

$u = t^2$, $du = 2t dt$

$v = -\frac{1}{2}\cos(2t)$, $dv = \sin(2t)dt$

$\int t^2\sin(2t)dt = -\frac{1}{2}t^2\cos(2t) + \cancel{\frac{1}{2}}\int \cancel{2}t\cos(2t)dt$ $u = t$, $du = dt$, $v = \frac{1}{2}\sin(2t)$, $dv = \cos(2t)dt$

$= -\frac{1}{2}t^2\cos(2t) + \frac{1}{2}t\sin(2t) - \frac{1}{2}\int\sin(2t)dt$

$= \boxed{-\frac{1}{2}t^2\cos(2t) + \frac{1}{2}t\sin(2t) + \frac{1}{4}\cos(2t) + C}$

4. $\int_0^\pi \theta\sin\theta\cos\theta d\theta$

$u = \theta\sin\theta$, $du = (\sin\theta + \theta\cos\theta)d\theta$

$v = \sin\theta$, $dv = \cancel{\sin}\cos\theta d\theta$

$\int_0^\pi \theta\sin\theta\cos\theta d\theta = \cancel{\theta\sin^2\theta}\Big|_0^\pi - \int_0^\pi \sin^2\theta d\theta - \int\theta\sin\theta\cos\theta d\theta$

$2\int_0^\pi \theta\sin\theta\cos\theta d\theta = -\int_0^\pi\left(\frac{1-\cos 2\theta}{2}\right)d\theta = -\int_0^\pi \frac{1}{2}d\theta + \frac{1}{2}\cancel{\int_0^\pi \cos 2\theta d\theta}$

$\int_0^\pi \theta\sin\theta\cos\theta d\theta = \boxed{-\frac{\pi}{4}}$

2 Trigonometric Substitutions

Evaluate each of the following using trigonometric substitutions.

1. $\int \sin^3\theta \cos^4\theta d\theta$ $= \int \sin^2\theta \cos^4\theta \sin\theta d\theta = \int (1-\cos^2\theta)\cos^4\theta \sin\theta d\theta$

$= -\int (1-u^2)u^4 du = -\int (u^4-u^6)du$

Let $u = \cos\theta$
$du = -\sin\theta d\theta$

$= -\left(\frac{1}{5}u^5 - \frac{1}{7}u^7\right) = \boxed{\frac{1}{7}\cos^7\theta - \frac{1}{5}\cos^5\theta} + C$

2. $\int_0^{\pi/2} (2-\sin\theta)^2 d\theta$ $= \int_0^{\pi/2} (4 - 4\sin\theta + \sin^2\theta)d\theta = \int_0^{\pi/2} \left[4 - 4\sin\theta + \left(\frac{1-\cos 2\theta}{2}\right)\right]d\theta$

$= \int_0^{\pi/2} \left(\frac{9}{2} - 4\sin\theta - \frac{1}{2}\cos 2\theta\right)d\theta = \frac{9}{2}\theta\Big|_0^{\frac{\pi}{2}} + 4\cos\theta\Big|_0^{\frac{\pi}{2}} - \frac{1}{4}\sin(2\theta)\Big|_0^{\frac{\pi}{2}}$

$= \boxed{\frac{9\pi}{4} - 4 = \frac{9\pi - 16}{4} \approx 3.069}$

3. $\int \tan^3 t \sec t dt$ $= \int \tan^2 t \cdot \sec t \cdot \tan t dt = \int (\sec^2 t - 1)\sec t \tan t dt$

$= \int (u^2-1)du = \frac{1}{3}u^3 - u = \boxed{\frac{1}{3}\sec^3 t - \sec t + C}$

Let $u = \sec t$
$du = \sec t \tan t dt$

4. $\int (\tan^2 x + \tan^4 x)dx$ $= \int \tan^2 x(1 + \tan^2 x)dx = \int \tan^2 x \sec^2 x dx$

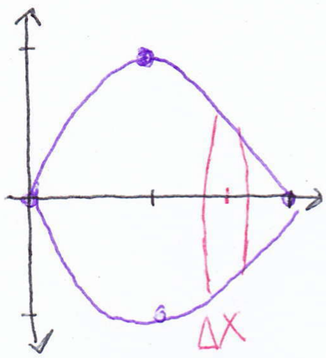
$= \int u^2 du = \frac{1}{3}u^3 + C = \boxed{\frac{1}{3}\tan^3 x + C}$

Let $u = \tan x$
$du = \sec^2 x dx$

5. $\int \sqrt{1-\cos(4\theta)}d\theta = \int \frac{\sqrt{2}}{\sqrt{2}} \cdot \sqrt{1-\cos(4\theta)}\, d\theta = \sqrt{2} \int \sqrt{\frac{1-\cos(4\theta)}{2}}\, d\theta$

$= \sqrt{2} \int \sqrt{\sin^2(2\theta)}\, d\theta = \sqrt{2} \int \sin(2\theta) d\theta = \boxed{-\frac{\sqrt{2}}{2} \cos(2\theta) + C}$

6. Find the volume obtained by rotating the region bounded by the curves $y = \sin^2 x$ and $y = 0$, $0 \leq x \leq \pi$, about the x-axis.



$A(x) = \pi(\sin^2 x)^2 = \pi \sin^4 x$

$V = \int_a^b A(x)dx = \pi \int_0^\pi \sin^4 x\, dx$

$V = \pi \int_0^\pi \left(\frac{1-\cos 2x}{2} \right)^2 dx$

$V = \frac{\pi}{4} \int_0^\pi (1 - 2\cos 2x + \cos^2 2x) dx$

$V = \frac{\pi}{4} \int_0^\pi \left[1 - 2\cos 2x + \left(\frac{1+\cos 4x}{2}\right)\right] dx$

$V = \frac{\pi}{4} \int_0^\pi \left(\frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x\right) dx$

$V = \frac{\pi}{4} \left(\frac{3}{2}x \Big|_0^\pi - \overset{0}{\cancel{\sin 2x \Big|_0^\pi}} + \overset{0}{\cancel{\frac{1}{8}\sin 4x \Big|_0^\pi}} \right)$

$V = \frac{\pi}{4}\left(\frac{3\pi}{2}\right)$

$\boxed{V = \frac{3\pi^2}{8} \text{ un.}^3}$