MATH 30, 3/25-26/2020: ROLLE'S THEOREM AND THE MVT

Thought experiment #1. A point moves along a straight line. [Picture.] It's position at time t is denoted by $x(t)$. Suppose it starts at $x = 0$ and later comes back to $x = 0$.

True or false: at some time its *speed* is 0.

Now plot time as a horizontal axis: [Picture.] Speed is zero at points where the derivative $x'(t)$ is zero. This is an example of *Rolle's Theorem.*

Rolle's Theorem. Let f be a function such that:

(1) f is continuous on $[a, b]$
(2) f is differentiable on (a, b)
(3) $f(a) = f(b)$.

Then there is some c in (a, b) such that $f'(c) = 0$.

[Picture.]

This theorem has a bunch of cool applications...

Example. Show that the equation

$$1 + 2x + x^3 + 4x^5 = 0$$

has exactly one solution. [We'll do the details in class.]

Note: Rolle's theorem doesn't tell you exactly *where* the solution is, just that it exists.

Thought experiment #2. Start driving on the freeway, and suppose that one hour later you are exactly 100 miles away.

True or false: at some time your speed was exactly 100 miles per hour.

Now plot time as a horizontal axis: [Picture.] Think of some special cases: (1) constant speed (2) starting slow, ending fast (3) starting fast, ending slow.

At some point, $x'(t) = 100$. This is an example of the Mean Value Theorem ("mean" is another word for "average").

Mean Value Theorem. Let f be a function such that:

(1) f is continuous on $[a, b]$
(2) f is differentiable on (a, b).

Then there is a number c in (a, b) where

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

That is,

$$f(b) = f(a) + (b - a)f'(c).$$

[Picture.]

Proof. Apply Rolle's theorem to the cleverly-designed function

$$h(x) = f(x) - f(a) = \left[\frac{f(b) - f(a)}{b - a}\right](x - a).$$

Corollary of the MVT. If $f'(x) = 0$ for all x in (a, b), then f is constant on (a, b).

Proof. Suppose f is *not* constant. Then $f'(c) \neq 0$ at some point c.

We solved our first differential equation!