1. Use augmented matrices and elementary row operations to solve the following systems. Please use row notation to show your steps.

For example: $-3R_1 + R_2 \rightarrow R_2$.

(a)

$$\begin{aligned} x_2 + 5x_3 &= -4 \\ x_1 + 4x_2 + 3x_3 &= -2 \\ 2x_1 + 7x_2 + x_3 &= -2 \end{aligned}$$

Week 1 HW

1.

a)

$$\begin{aligned} x_2 + 5x_3 &= -4 \\ x_1 + 4x_2 + 3x_3 &= -2 \\ 2x_1 + 7x_2 + x_3 &= -2 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 5 & -4 \\ 1 & 4 & 3 & -2 \\ 2 & 7 & 1 & -2 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 4 & 3 & -4 \\ 0 & 1 & 5 & -2 \\ 2 & 7 & 1 & -2 \end{array}\right]$$

$-8 \quad -6 \quad 8$

$$\xrightarrow{-2\cdot R_1 + R_3 \rightarrow R_3} \left[\begin{array}{ccc|c} 1 & 4 & 3 & -4 \\ 0 & 1 & 5 & -2 \\ 0 & -1 & -5 & 6 \end{array}\right] \xrightarrow{-4\cdot R_2 + R_1 \rightarrow R_1} \left[\begin{array}{ccc|c} 1 & 0 & -17 & 4 \\ 0 & 1 & 5 & -2 \\ 0 & -1 & -5 & 6 \end{array}\right]$$

$$\xrightarrow{R_2 + R_3 \rightarrow R_3} \left[\begin{array}{ccc|c} 1 & 0 & -17 & 4 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 0 & 4 \end{array}\right]$$

$\boxed{\text{No solution!}}$

This is False, so no

(b)

$$\begin{aligned} x_1 - 3x_3 &= 8 \\ 2x_1 + 2x_2 + 9x_3 &= 7 \\ x_2 + 5x_3 &= -2 \end{aligned}$$

b)

$$\begin{aligned} x_1 + 0x_2 - 3x_3 &= 8 \\ 2x_1 + 2x_2 + 9x_3 &= 7 \\ x_2 + 5x_3 &= -2 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 2 & 2 & 9 & 7 \\ 0 & 1 & 5 & -2 \end{array}\right]$$

$$\boxed{\begin{aligned} X &= 5 \\ Y &= 3 \\ Z &= -1 \end{aligned}}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 2 & 2 & 9 & 7 \end{array}\right] \xrightarrow{-2 \cdot R_1 + R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 0 & 2 & 15 & -9 \end{array}\right]$$

$$\xrightarrow{-2 \cdot R_2 + R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 5 & -5 \end{array}\right] \xrightarrow{\frac{1}{5} \cdot R_3} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\xrightarrow{-5 \cdot R_3 + R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow{3 \cdot R_3 + R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

(c)

$$\begin{aligned} 2x_1 - 6x_3 &= -8 \\ x_2 + 2x_3 &= 3 \\ 3x_1 + 6x_2 - 2x_3 &= -4 \end{aligned}$$

c) $$\begin{aligned} 2x_1 + 0x_2 - 6x_3 &= -8 \\ x_2 + 2x_3 &= 3 \\ 3x_1 + 6x_2 - 2x_3 &= -4 \end{aligned} \rightarrow \left[\begin{array}{ccc|c} 2 & 0 & -6 & -8 \\ 0 & 1 & 2 & 3 \\ 3 & 6 & -2 & -4 \end{array}\right]$$

$$\downarrow \tfrac{1}{2}\cdot R_1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -4 \\ 0 & 1 & 2 & 3 \\ 0 & 6 & 7 & 8 \end{array}\right] \xleftarrow{-3\cdot R_1 + R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -4 \\ 0 & 1 & 2 & 3 \\ 3 & 6 & -2 & -4 \end{array}\right]$$

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$$\downarrow -6\cdot R_2 + R_3 \to R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -5 & -10 \end{array}\right] \xrightarrow{-\frac{1}{5}\cdot R_3} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\downarrow 3\cdot R_3 + R_1 \to R_1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] \xleftarrow{-2\cdot R_3 + R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\boxed{\begin{aligned} x &= 2 \\ y &= -1 \\ z &= 2 \end{aligned}}$$

2. Construct an augmented matrix for a linear system whose solution set is $x_1 = 3$, $x_2 = -2$, and $x_3 = -1$.

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 1 & -1 & -1 \end{array}\right]$$

$2(3) + 1(-2) + 1(-1) = 3$
$0(3) + 1(-2) + 0(-1) = -2$
$0(3) + 1(-2) - 1(-1) = (-1)$

3. Determine which matrices are in reduced echelon form and which are only in echelon form (you don't need to care if the system is consistent or not):

(a) $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$

a) $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$ 1? ✓ 2? ✓ 3? ✓ In reduced echelon form

$\left[\begin{array}{ccc|c} \bullet & \star & \star & \star \\ 0 & \bullet & \star & \star \\ 0 & 0 & 0 & \bullet \end{array}\right]$ 4? ✓ 5? ✓

b) $\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 4 \end{array}\right]$ 1? ✓ 2? ✓ 3? ✓ 4? ✗ 5?

Only in echelon form

*Note that for part A, it is actually in reduced echelon form. The extra 1's in column 3 don't matter because there isn't a leading entry in that column.

DEFINITION

A rectangular matrix is in **echelon form** (or **row echelon form**) if it has the following three properties:

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or **reduced row echelon form**):

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

An **echelon matrix** (respectively, **reduced echelon matrix**) is one that is in echelon form (respectively, reduced echelon form). Property 2 says that the leading entries form an *echelon* ("steplike") pattern that moves down and to the right through the matrix. Property 3 is a simple consequence of property 2, but we include it for emphasis.

The "triangular" matrices of Section 1.1, such as

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

are in echelon form. In fact, the second matrix is in reduced echelon form. Here are additional examples.

EXAMPLE 1 The following matrices are in echelon form. The leading entries ($\blacksquare$) may have any nonzero value; the starred entries ($*$) may have any value (including zero).

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

The following matrices are in reduced echelon form because the leading entries are 1's, and there are 0's below *and above* each leading 1.

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

Any nonzero matrix may be **row reduced** (that is, transformed by elementary row operations) into more than one matrix in echelon form, using different sequences of row operations. However, the reduced echelon form one obtains from a matrix is unique. The following theorem is proved in Appendix A at the end of the text.

THEOREM 1

Uniqueness of the Reduced Echelon Form

Each matrix is row equivalent to one and only one reduced echelon matrix.

4. Row reduce the following matrix into reduced echelon form:
(Use row notation to show your steps)

$$\begin{bmatrix} 1 & 2 & 4 & 5 \\ 2 & 4 & 5 & 4 \\ 4 & 5 & 4 & 2 \end{bmatrix}$$

4. Row reduce the matrix into reduced echelon form:

$$\begin{bmatrix} 1 & 2 & 4 & 5 \\ 2 & 4 & 5 & 4 \\ 4 & 5 & 4 & 2 \end{bmatrix} \xrightarrow{-2\cdot R_1 + R_2 \to R_2} \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 0 & -3 & -6 \\ 4 & 5 & 4 & 2 \end{bmatrix}$$

$$\xrightarrow{-4\cdot R_1 + R_3 \to R_3} \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 0 & -3 & -6 \\ 0 & -3 & -12 & -18 \end{bmatrix} \xrightarrow{-\frac{1}{3}\cdot R_3} \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 0 & -3 & -6 \\ 0 & 1 & 4 & 6 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & -3 & -6 \end{bmatrix} \xrightarrow{-\frac{1}{3}\cdot R_3} \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Δ cont'd

Δ cont'd

$$\begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{-2\cdot R_2 + R_1 \to R_1} \begin{bmatrix} 1 & 0 & -4 & -7 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{-4\cdot R_3 + R_2 \to R_2} \begin{bmatrix} 1 & 0 & -4 & -7 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{4\cdot R_3 + R_1 \to R_1} \boxed{\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}}$$

5. Find the general solution for the augmented matrices:
(Use row notation to show your steps)

(a) $\left[\begin{array}{cccc} 1 & -1 & -1 & 4 \\ -2 & 4 & -5 & 6 \end{array}\right]$

(b) $\left[\begin{array}{cccc} 3 & -2 & 4 & 0 \\ 9 & -6 & 12 & 0 \\ 6 & -4 & 8 & 0 \end{array}\right]$

a) $\left[\begin{array}{ccc|c} 1 & -1 & -1 & 4 \\ -2 & 4 & -5 & 6 \end{array}\right] \xrightarrow{2\cdot R_1 + R_2 \to} \left[\begin{array}{ccc|c} 1 & -1 & -1 & 4 \\ 0 & 2 & -7 & 14 \end{array}\right]$

1st: Convert to row reduced echelon form!

$\downarrow \frac{1}{2}\cdot R_2$

$\left[\begin{array}{ccc|c} 1 & 0 & -4.5 & 11 \\ 0 & 1 & -3.5 & 7 \end{array}\right] \xleftarrow{R_1 + R_2 \to R_1} \left[\begin{array}{ccc|c} 1 & -1 & -1 & 4 \\ 0 & 1 & -3.5 & 7 \end{array}\right]$

Pivot columns! 1 and 2

$x_1 + 0x_2 - 4.5x_3 = 11$
$0x_1 + x_2 - 3.5x_3 = 7$

so free variable: x_3

basic variables are! x_1 and x_2

$$\boxed{\begin{cases} x_1 = 11 + 4.5x_3 \\ x_2 = 7 + 3.5x_3 \\ x_3 = \text{is free} \end{cases}}$$

b) $\left[\begin{array}{ccc|c} 3 & -2 & 4 & 0 \\ 9 & -6 & 12 & 0 \\ 6 & -4 & 8 & 0 \end{array}\right] \xrightarrow{\frac{1}{9}R_2} \left[\begin{array}{ccc|c} 3 & -2 & 4 & 0 \\ 1 & -0.\overline{66} & 1.\overline{33} & 0 \\ 6 & -4 & 8 & 0 \end{array}\right]$

$\downarrow -2R_1+R_3 \to R_3$

$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & -0.\overline{66} & 1.\overline{33} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xleftarrow{-3R_2+R_1 \to R_1} \left[\begin{array}{ccc|c} 3 & -2 & 4 & 0 \\ 1 & -0.\overline{66} & 1.\overline{33} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\downarrow R_1 \leftrightarrow R_2$

$\left[\begin{array}{ccc|c} 1 & -0.\overline{66} & 1.\overline{33} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Pivot column: 1 → Basic variables: x_1

Free variables: x_2 and x_3

$x_1 - 0.\overline{66}x_2 + 1.\overline{33}x_3 = 0$

$$\boxed{\begin{cases} x_1 = 0.\overline{66}x_2 - 1.\overline{33}x_3 \\ x_2 \text{ is free} \\ x_3 \text{ is free} \end{cases}}$$

6. For the following statements determine whether they are true or false. If false, then explain why it is false.

(a) Every elementary row operation is reversible.

True.

(b) Two matrices are row equivalent if they have the same number of rows.

False. Two matrices are considered row equivalent if there is a sequence of elementary row operations that transforms one matrix into the other.

(c) Elementary row operations on an augmented matrix never change the solution set of the associated linear system.

True.

(d) Two equivalent linear systems can have different solution sets.

False. They are called equivalent because they have the same solution set. That is, each solution of the first system is a solution of the second system, and vice versa.

(e) A consistent system of linear equations has one or more solutions.

True.

(f) In some cases, a matrix may be row reduced to more than one reduced echelon matrix if using different sequences of row operations.

False. Any nonzero matrix may be row reduced into more than one matrix in echelon form, using different sequences of row operations. However, the reduced echelon form one obtains from a matrix is unique. Each matrix is row equivalent to one and only one reduced echelon matrix.

(g) If one row in an echelon form of an augmented matrix reads

$$\begin{bmatrix} 0 & 0 & 0 & 5 & 0 \end{bmatrix}$$

then the associated matrix is inconsistent.

False. This row says that $5 * x_4 = 0$, so $x_4 = 0$. A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column. That is, if and only if an echelon form of the augmented matrix has no row of the form:

$$[0 \quad \cdots \quad 0 \quad b] \qquad \text{with } b \text{ nonzero}$$

(h) If a system has free variables, then there is only one unique solution.

False. Free variables could mean that there are infinitely many solutions. To state that "x_3 is free" means that you are free to choose any value for x_3.

7. A system of linear equations with fewer equations than unknowns (variables) is sometimes called an underdetermined system. Can such a system have a unique solution? Explain why or why not.

No. It cannot have a unique solution. Because there are more variables then there are equations, there must be at least one free variable. If the linear system is consistent with at least one free variable, then there are infinitely many solutions. If the linear system is inconsistent, there is no solution.

8. Suppose the coefficient matrix of a linear system of four equations in four variables has a pivot in each column. Explain why the system has a unique solution.

What would have to be true of a linear system to have a unique solution would be that it must be consistent and have no free variables. This is the case when there are four equations in four variables, with a pivot in each column.

9. Give an example of an inconsistent underdetermined system of two equations in three variables.

$$\boxed{\begin{aligned} x + 2y + 4z &= 0 \\ 0x + 0y + 0z &= 10 \end{aligned}}$$