Math 30, Thursday April 30, 2020

1pm class

The Fund. Thm. of Calc., Part II.

Quiz tomorrow (see instructions on Canvas)

Last time:

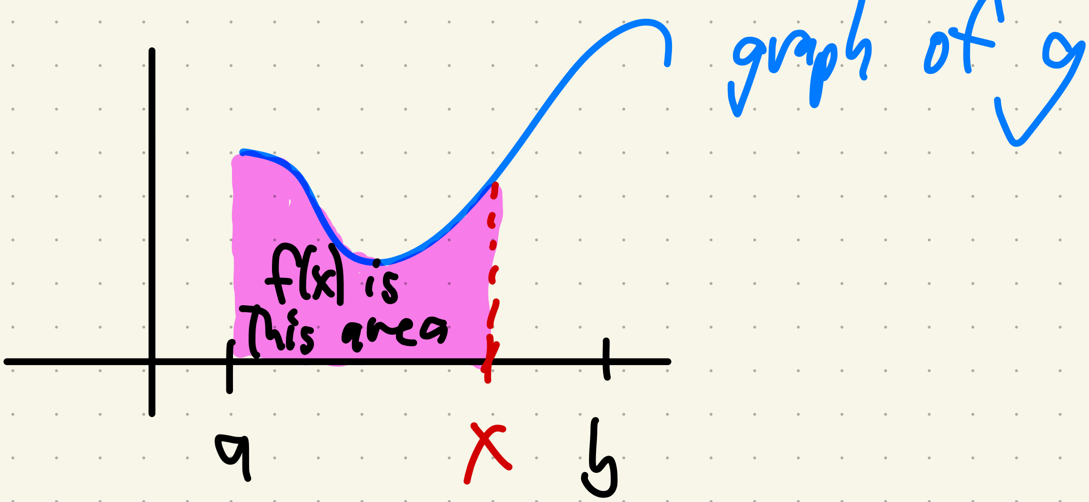
The Fundamental Theorem of Calculus, Part I → F.T. of C.

If g is continuous on $[a,b]$, define a function $f(x) = \int_a^x g(t)\,dt$.

Then f is an antiderivative of g:

$$f'(x) = g(x).$$

Picture:



$f(x)$ is the area under the graph of g from a to x.

Note: We could have also defined a function

$$F(x) \overset{\text{def}}{=} \int_A^x g(t)\,dt.$$

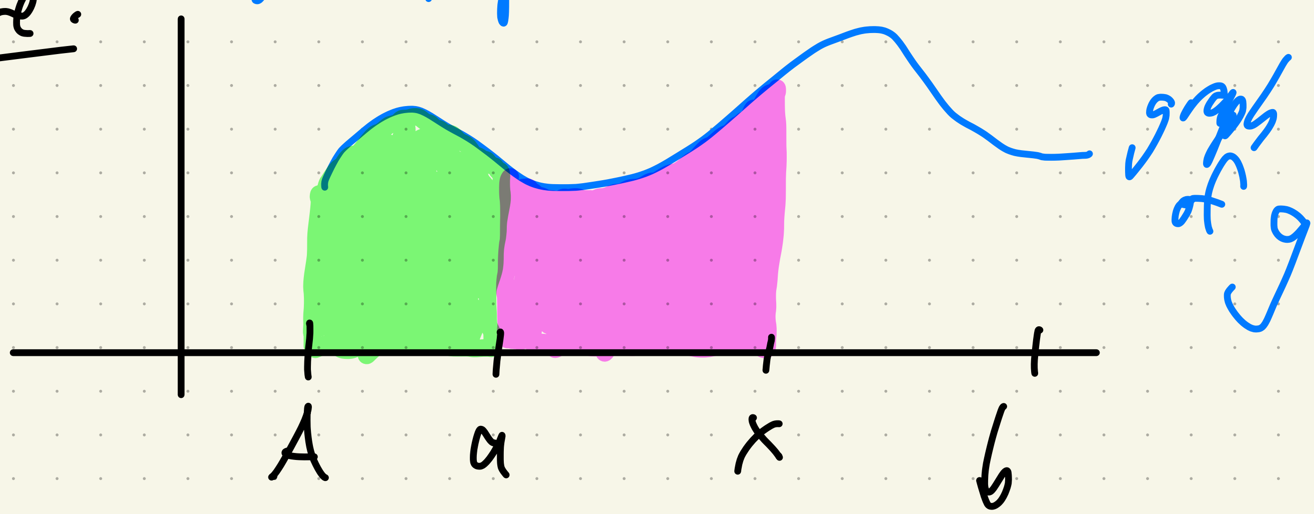
$\downarrow$ by "Property 4"

"+C"

$$= \int_A^a g(t)\,dt + \int_a^x g(t)\,dt$$

doesn't dep. on x

Picture:



So

$$F'(x) = 0 + g(x)$$

So $F(x)$ is another antiderivative of g

Summary: every antiderivative of g can be obtained by integrating!

antiderivs of g are all of the form

$$f(x) = \int_a^x g(t)\,dt + C$$

For that reason, the family of all antiderivatives of g is denoted by:

$$\underbrace{\int g(x)\,dx}_{}$$

called an "indefinite integral"

Example. $\int x^6 \, dx = ?$

$\underbrace{\int}$ all antiderivatives of $g(x) = x^6$

"whose derivative is $g(x) = x^6$?"

$= \frac{1}{7} x^7 + C$

$$\boxed{\int x^6 \, dx = \frac{1}{7} x^7 + C}$$

For Emphasis: a definite integral is a number

$$\int_0^1 x^3\,dx \text{ is a number}$$

(actually it's $\frac{1}{4}$)

and an indefinite integral is a family of functions

$$\int x^3\,dx = \frac{1}{4}x^4 + C$$

Last time: integrals & averages.

The average of a function g over the interval $[a,b]$ is

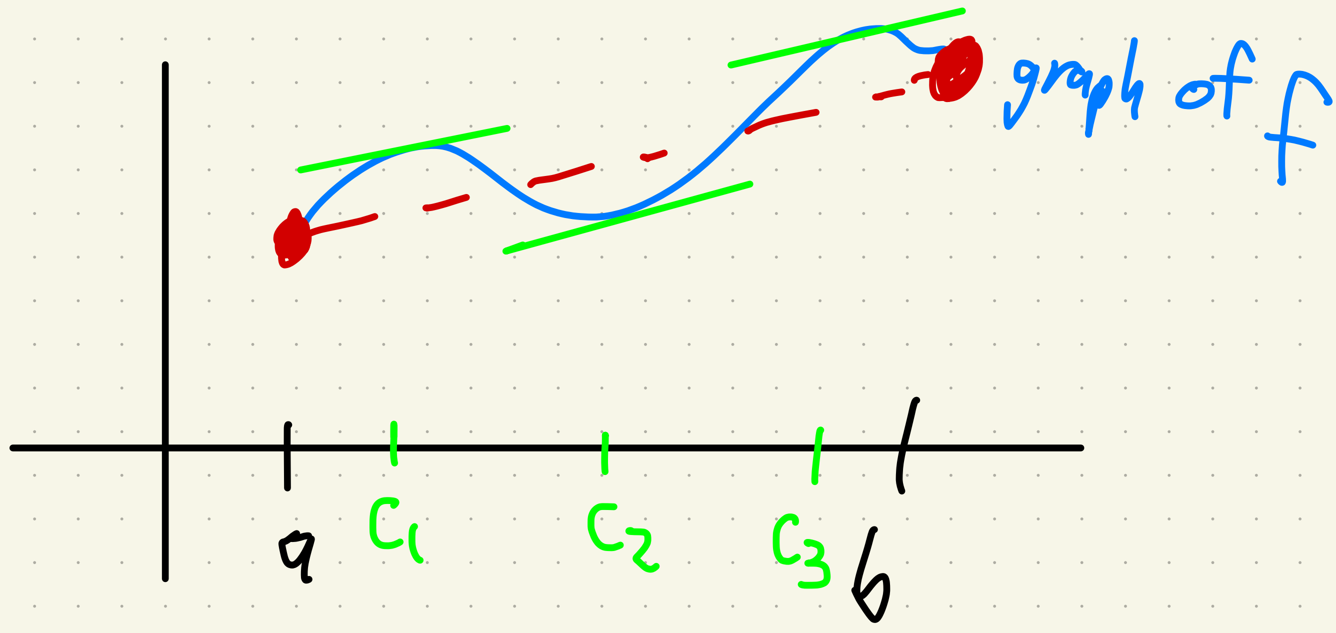
$$\mathrm{Avg}(g) = \underbrace{\frac{1}{b-a}}_{\text{divided by length of interval.}} \underbrace{\int_a^b g(t)\,dt}_{\text{area under curve}}$$

This generalizes the "usual" average of numbers.

Q: Is this related to the $\underbrace{\text{Mean}}_{\text{another word for "average."}}$ Value Theorem?

A: Yes!

Recall: The MVT says: if f looks like



$$\underbrace{\frac{f(b)-f(a)}{b-a}}_{\text{slope of line between endpts}} = \underbrace{f'(c)}_{\text{slope of tangent line at } c}$$

for some $a < c < b$.

Apply the MVT to the function

$$f(x) = \int_a^x g(t)\,dt$$

where g is continuous.

$f(a) = 0$

$f(b) = \int_a^b g(t)\,dt$

The MVT says:

for some c between a and b we have

$$f'(c) = \frac{f(b) - f(a)}{b-a} = \frac{1}{b-a}\int_a^b g(t)\,dt$$

by F.T. of Calc. II

$g(c)$

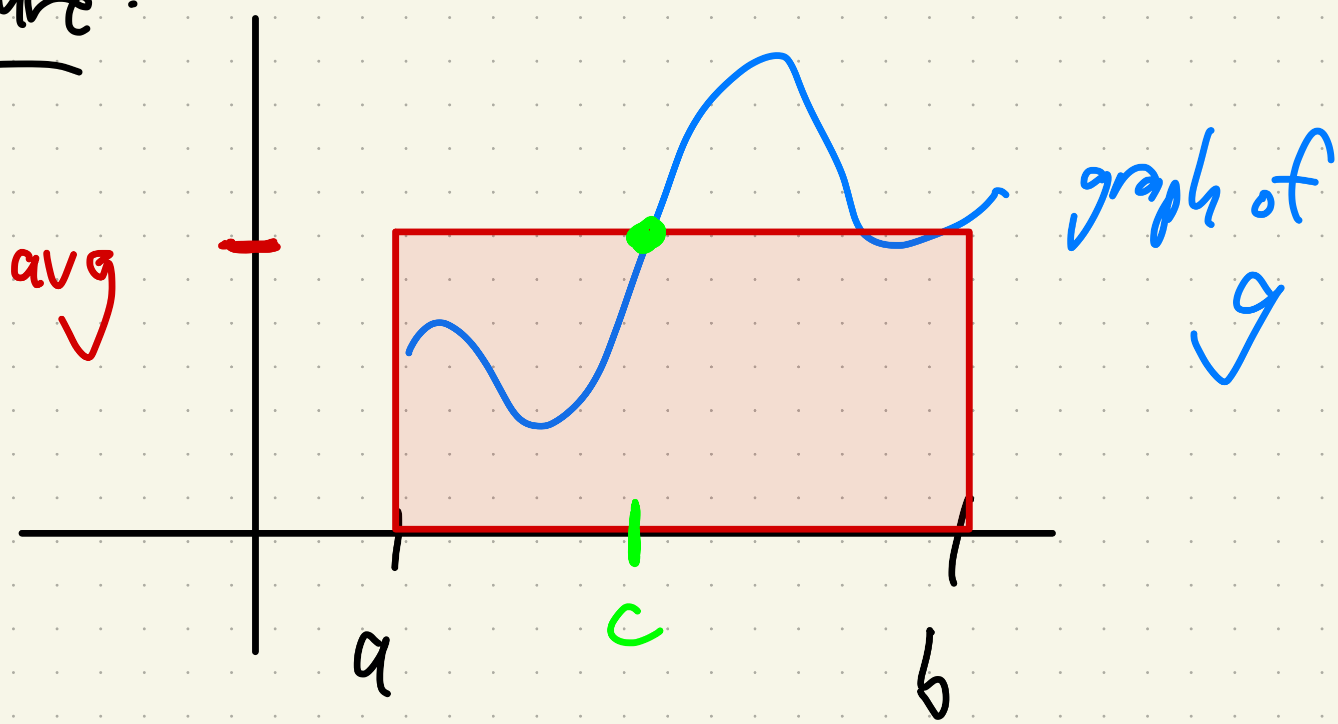
Summary: if g is continuous,

Then at some point c between a and b

it equals its average:

$$g(c) = \underbrace{\frac{1}{b-a} \int_a^b g(t)\,dt}_{\text{The avg. of } g \text{ over } [a,b].}$$

Picture:



recall: the avg of g is the height of the rectangle with the same area

MVT says: at some point g attains its average.

If g is not continuous, this isn't true in general.

Ex. "number of children in family x" is not a continuous function.

On Average, American family has 2.3 children.

The average is not attained

The F. T. of Calc. Part I is about

"differentiating an integral":

$$\frac{d}{dx}\int_a^x g(t)\,dt = g(x).$$

Q: What about

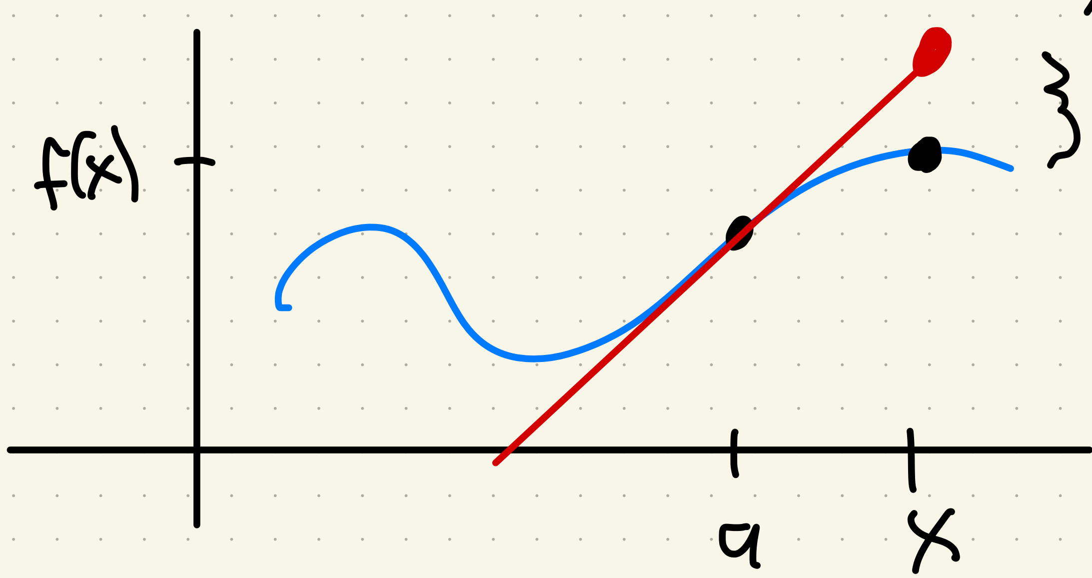
"integrating a derivative"?

$$\int_a^b f'(x)\,dx = ?$$

That's the F. T. of Calc. Part II!

Warm Up: linear approximation:

$$f(x) \approx \underbrace{f(a) + f'(a)(x-a)}_{\text{value on tangent line}}$$



pretty close when x is close to a

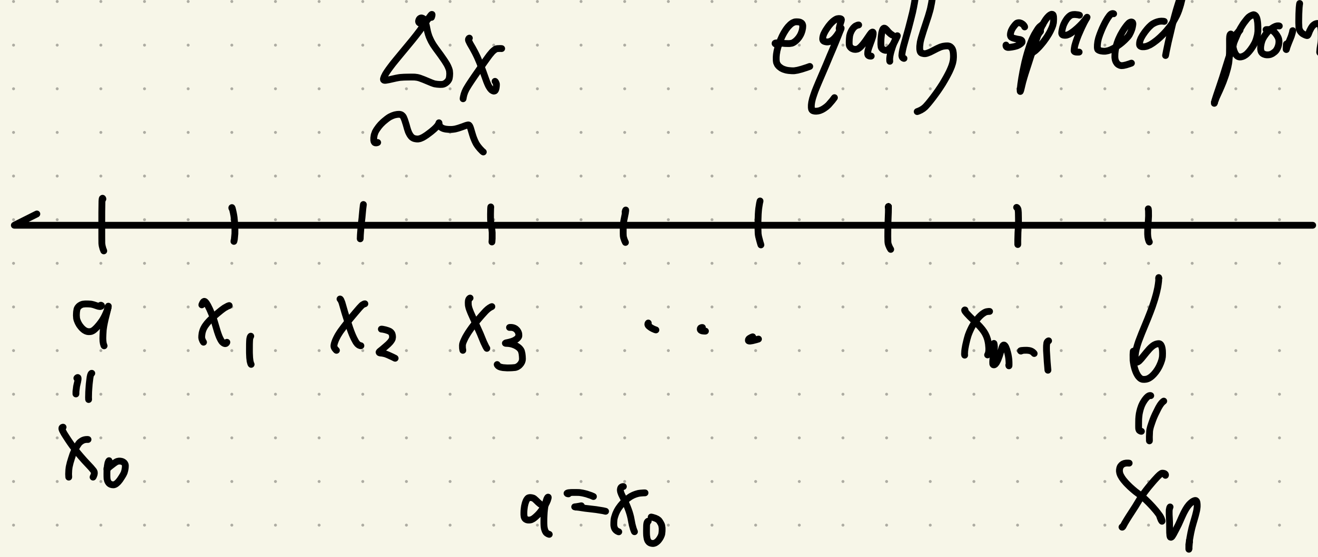
Rewrite it:

$$\underbrace{f(x) - f(a)}_{\text{vertical change}} \approx \underbrace{f'(a)}_{\text{slope}}\underbrace{(x-a)}_{\text{horiz. change}}$$

$$f(x) - f(a) \approx f'(a)(x-a)$$

Do this over & over:

equally spaced points



$$f(x_1) - f(a) \approx f'(x_0)\Delta x$$

$$f(x_2) - f(x_1) \approx f'(x_1)\Delta x$$

$$f(x_3) - f(x_2) \approx f'(x_2)\Delta x$$

$\vdots$ etc. once for each segment

$$f(b) - f(x_{n-1}) \approx f'(x_{n-1})\Delta x$$

$$f(x_1) - f(a) \approx f'(x_0)\Delta x$$

$$f(x_2) - f(x_1) \approx f'(x_1)\Delta x$$

$$f(x_3) - f(x_2) \approx f'(x_2)\Delta x$$

$\vdots$ etc. once for each segment

$$f(b) - f(x_{n-1}) \approx f'(x_{n-1})\Delta x$$

Add them up:

$$(f(x_1) - f(a)) + (f(x_2) - f(x_1)) + (f(x_3) - f(x_2)) + \cdots + (f(b) - f(x_{n-1}))$$

$$\approx \sum_{j=0}^{n-1} f'(x_j)\Delta x$$

I'll finish tomorrow...

Questions?

See you tomorrow!