Math 30, Monday April 27, 2020

1pm class

Questions?

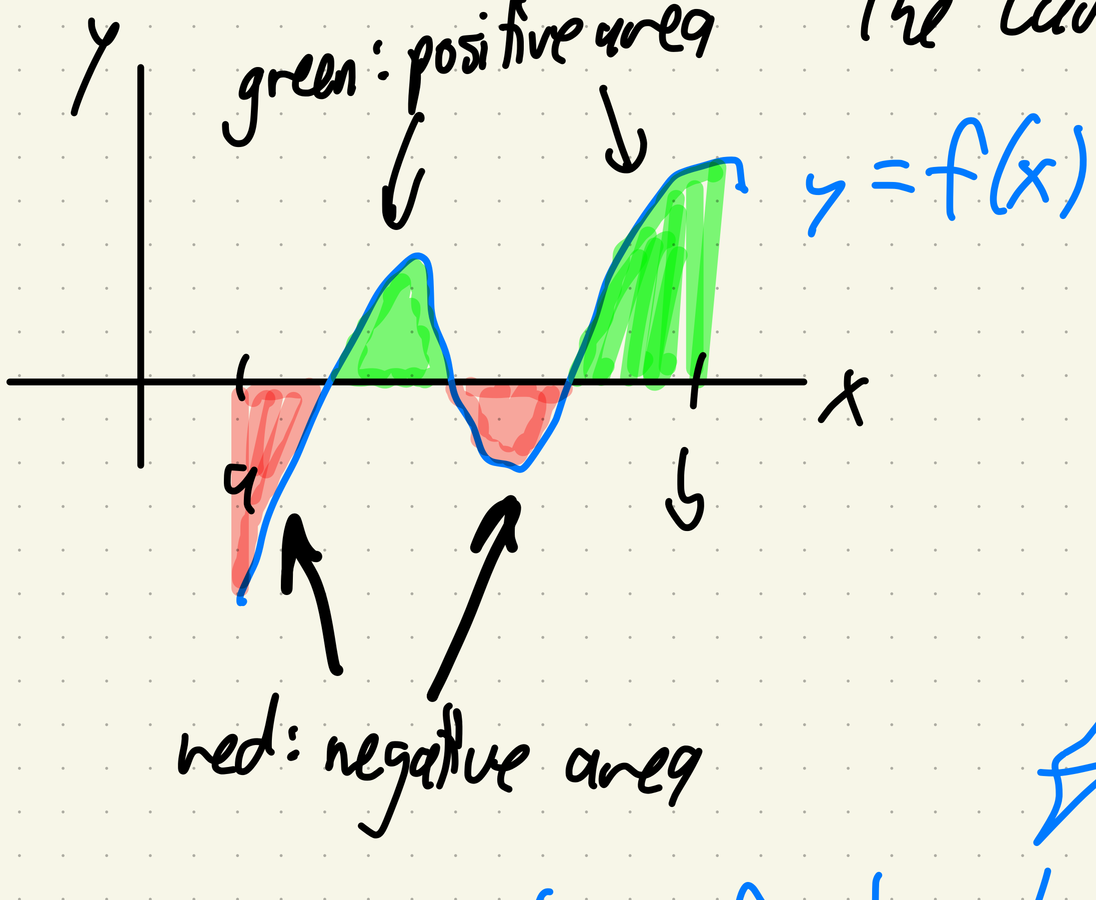
$\frac{b-a}{n}$

Recall:

$$\int_a^b f(x)\,dx \stackrel{\text{def}}{=} \lim_{n\to\infty} \sum_{j=1}^{n} f(x_j^*)\,\underbrace{\Delta x}_{h}$$

sum followed by a limit

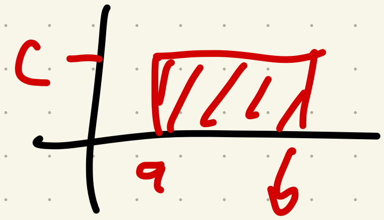
geometric meaning: "net area under the curve"



So can prove properties of integrals using known properties of sums and limits.

<u>Theorem.</u> (from Friday). Let f and g be continuous functions and c be a constant. Then:

1. $$\underbrace{\int_a^b c\,dx}_{\text{area of a rectangle}} = c(b-a)$$



2. $$\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

3. $$\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx$$

similar to addition: "factor out the c."

sketch of proof of #3:

$$\int_a^b c f(x)\,dx \overset{def}{=} \lim_{n\to\infty} \sum_{j=1}^{n} c f(x_j^*)\Delta x$$

factor out c from each term

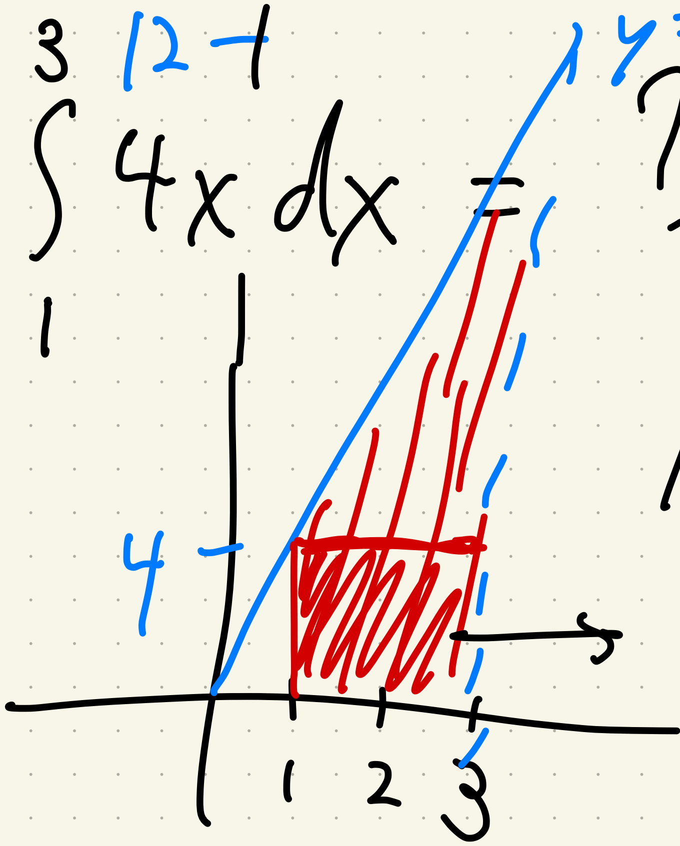
then pull c from out of limit. (limit law).

$$= c \lim_{n\to\infty} \sum_{j=1}^{n} f(x_j^*)\Delta x$$

$$\overset{def}{=} c \int_a^b f(x)\,dx$$

Ex. $\int_1^3 4x\,dx = ?$

y = 4x (not to scale)



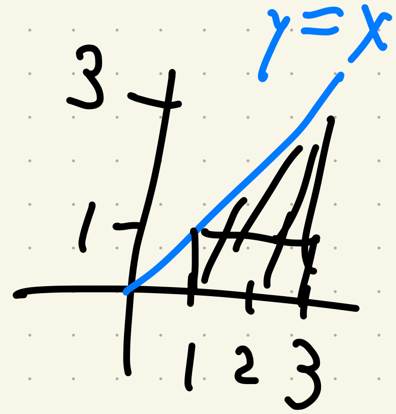
triangle area: $\frac{1}{2}(2)(8) = 8$

rectangle area $= 8$

geometrically: area under curve is

$8 + 8 = 16.$

Same as: $4\int_1^3 x\,dx = 16$ check



rectangle area $= 2$

triangle area $= \frac{1}{2} 2 \cdot 2 = 2$

Honestly, sometimes in math it's easier to think abstractly:

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx.$$

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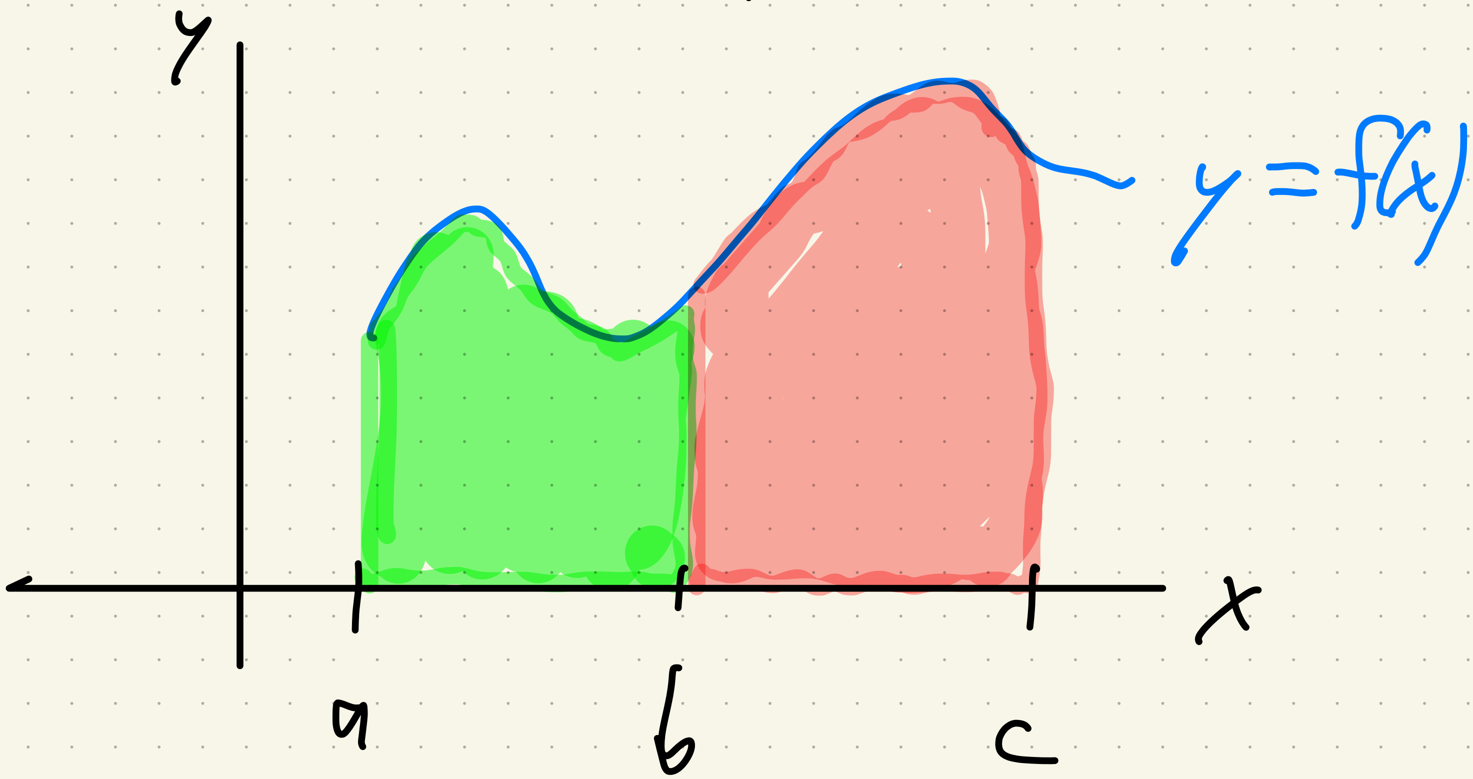
seems almost paradoxical

but true!

a 4^{th} property:

4. $$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

easy to understand the picture:



Another way to write it: write $c = b + h$.

$$\int_a^{b+h} f(x)\,dx - \int_a^b f(x)\,dx = \int_b^{b+h} f(x)\,dx$$

we'll use this soon!

Next <u>Section</u>: The Fundamental Theorem of Calculus.

1. it shows how areas/integrals are related to slopes of tangent lines/derivatives. It connects the two halves of calculus.

2. it gives an easy and fast way to do certain integrals.

Remember antiderivatives:

say $f'(x) = g(x)$

we say: g is the derivative of f

and f is an antiderivative of g.

Picture:

$$f(x) \underset{\text{antiderivative}}{\overset{\text{deriv.}}{\rightleftarrows}} g(x) = f'(x)$$

Q: How do you find antideriv.s?

Q: How to find antiderivatives

one way: experience & guess & check

Ex. $g(x) = x\cos x$

Find antideriv.s of g.

Guess: $x\sin x$?

Check: $\frac{d}{dx}(x\sin x) \overset{\text{Product Rule}}{=} \sin x + x\cos x$

not quite...

modify it:

let $f(x) = x\sin x + \cos x + C$

Check: $f'(x) = \underbrace{\sin x + x\cos x}_{\text{Product Rule}} - \sin x + 0$

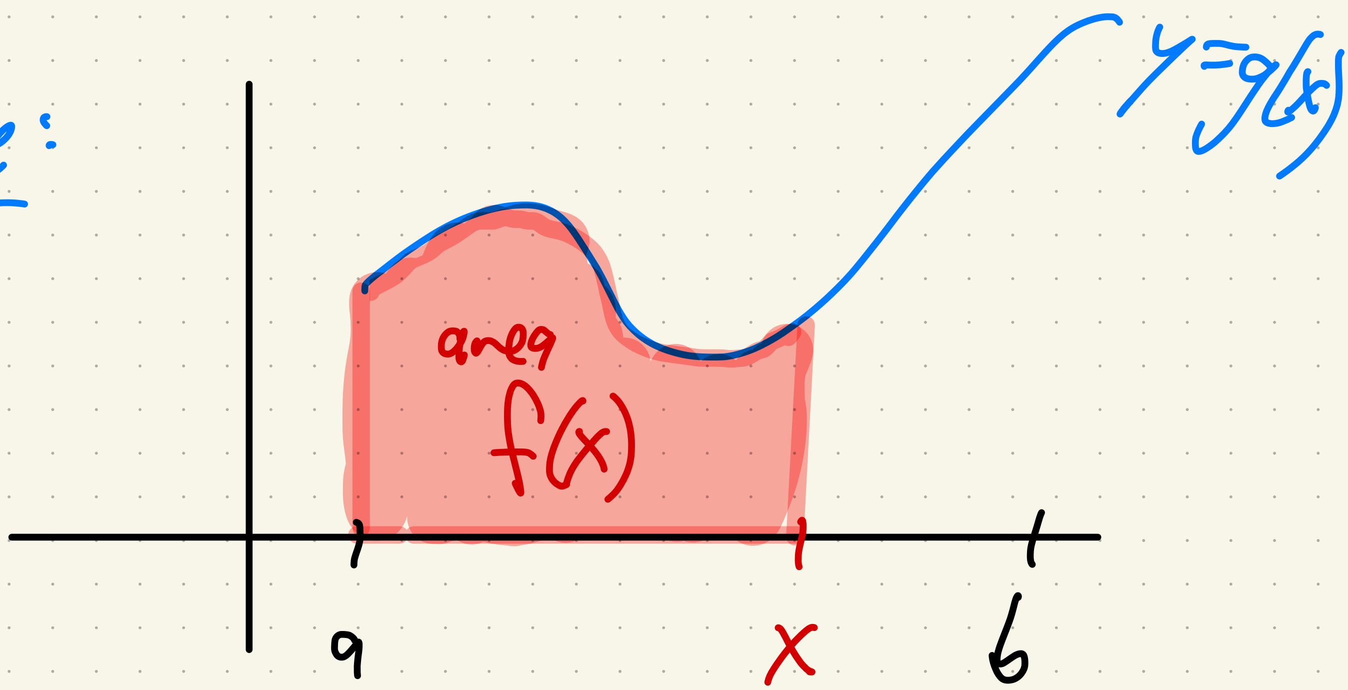
Another way to find antiderivatives: use integrals!

Let $g(x)$ be continuous on $[a,b]$.

For $a \le x \le b$ define a function by

$$f(x) \overset{\text{def}}{=} \int_a^x g(t)\,dt$$

Picture:



as x changes, the red area changes

In particular, $f(a) = 0$, $f(b) = \int_a^b g(t)\,dt$

It turns out, f is an antiderivative of g.

The Fund. Theorem of Calc. Part I:

If g is a continuous function on $[a,b]$, then the function

$$f(x) = \int_a^x g(t)\,dt, \quad a \le x \le b,$$

is an antiderivative of g:

$$f'(x) = g(x).$$

Wow!

integrals & derivatives are related!

$\downarrow$ areas

$\downarrow$ slopes of tangent lines

<u>Reason</u>: Again, f is defined to be

$$f(x) = \int_a^x g(t)\,dt.$$

Two Parts: (Want to show $f'(x) = g(x)$).

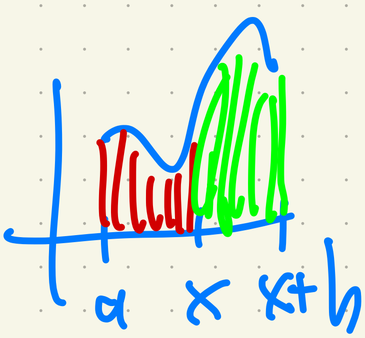
<u>I.</u>

$$f'(x) \overset{\text{def}^n}{=} \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

(def^n of derivative)

$$= \lim_{h \to 0} \frac{1}{h} \left[\int_a^{x+h} g(t)\,dt - \underbrace{\int_a^{x} g(t)\,dt}_{\text{red}} \right]$$

(def^n of f)



$$= \lim_{h \to 0} \frac{1}{h} \underbrace{\int_x^{x+h} g(t)\,dt}_{\text{green}}$$

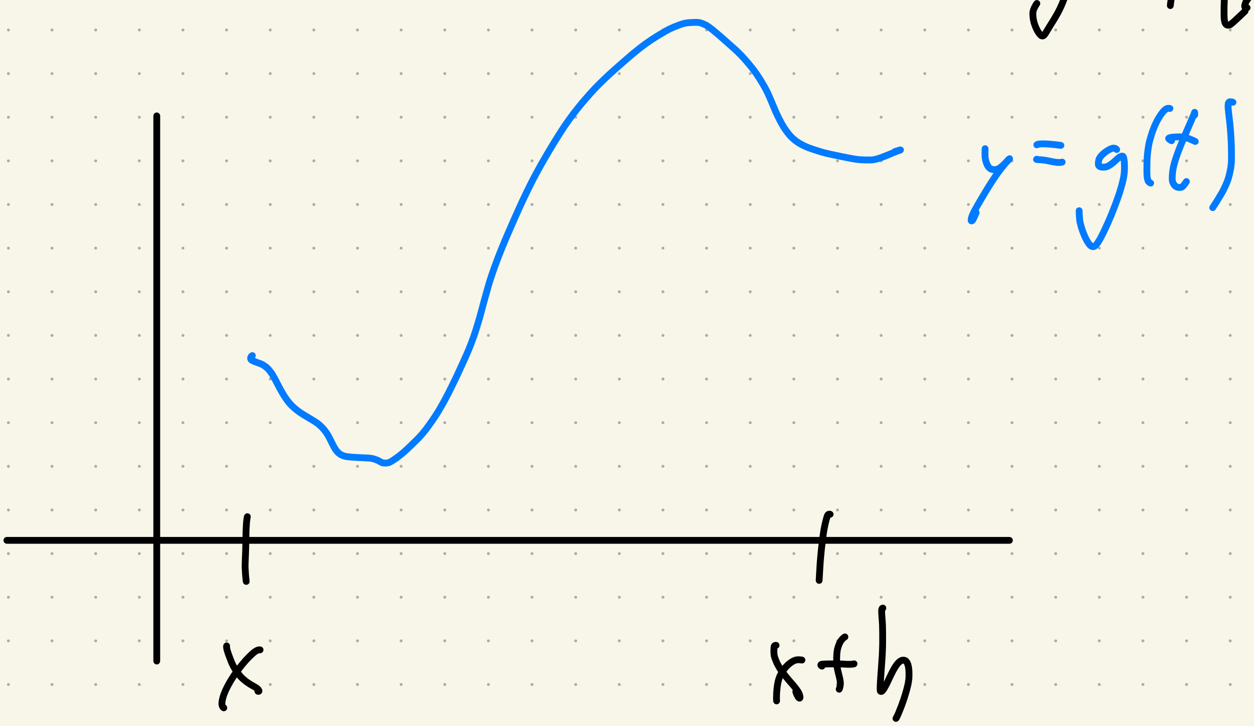
(Property 4)

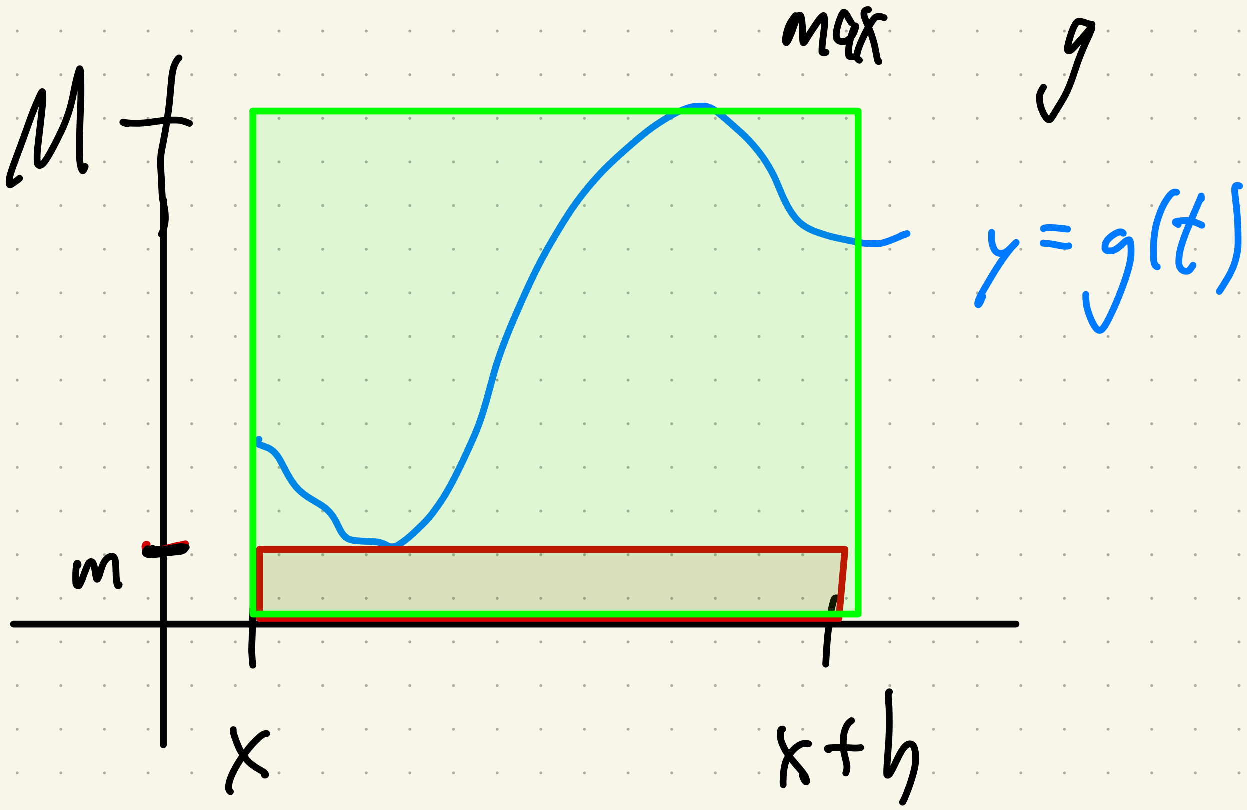
Summary: I. $f'(x) = \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} g(t)\,dt$

Second part:

II. Let m be the minimum of g on $[x, x+h]$

Let M be the max. of g on $[x, x+h]$.





Then compare areas:

$$\underbrace{mh}_{\text{area of red rectangle}} \le \underbrace{\int_x^{x+h} g(t)\,dt}_{\text{area under blue curve}} \le \underbrace{Mh}_{\text{area of green rectangle}}$$

Divide by h and let $h \to 0$.

out of time