Math 30, Friday April 24, 2020

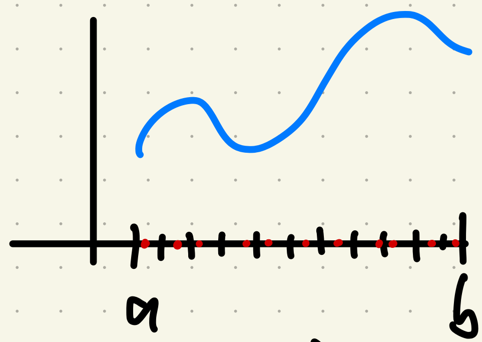
1pm class

Questions?

Review of yesterday:

Let f be defined for $a \le x \le b$

zoom in and see!



Divide the interval $[a,b]$ into n equal parts

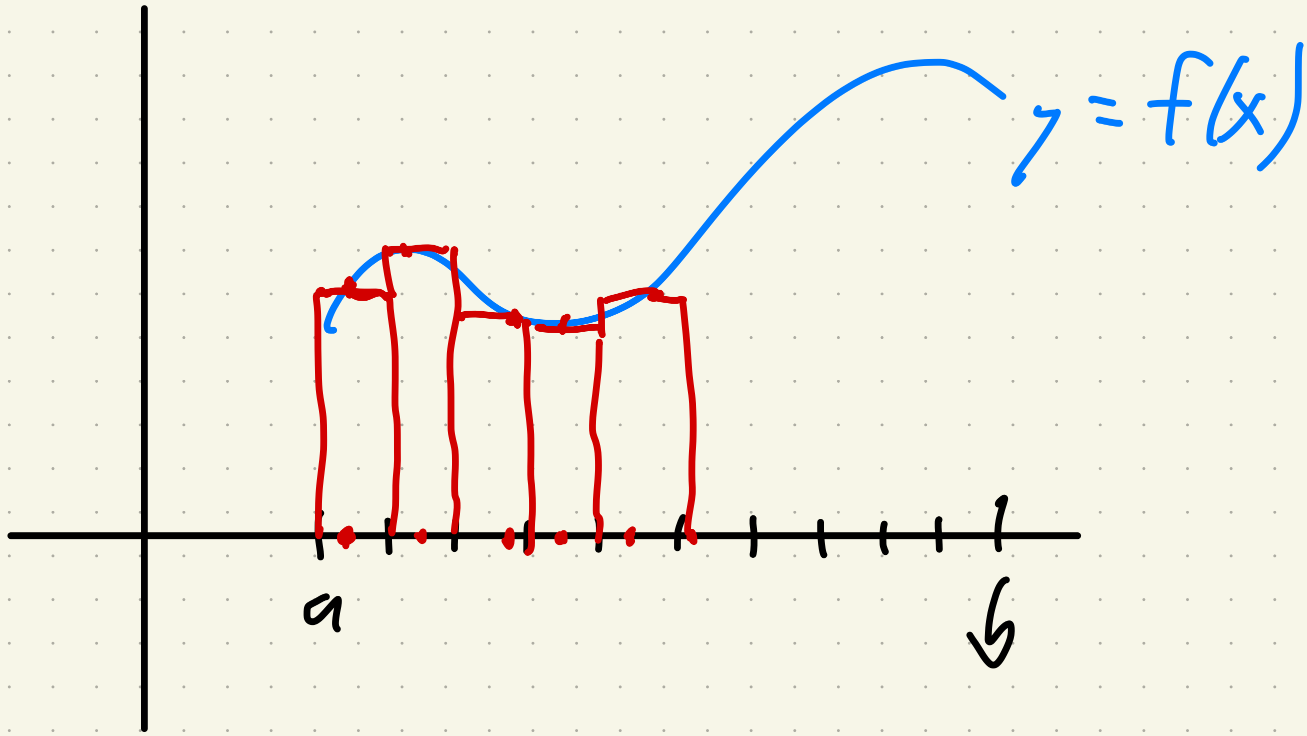
each has width $\underbrace{\Delta x}_{\text{"change in } x\text{"}} = \dfrac{b-a}{n}$

Let $x_1^*, x_2^*, \ldots, x_n^*$ be "sample points",

one in each subinterval

Then the height of the j^{th} rectangle

is $f(x_j^*)$



So the area under the blue curve is approximately the sum of the areas of rectangles:

$$\underbrace{\sum_{j=1}^{n}}_{\text{add them up.}} \underbrace{f(x_j^*)}_{\text{height of } j^{th} \text{ rectangle}} \underbrace{\Delta x}_{\text{width of } j^{th} \text{ rectangle}}$$

to get the <u>exact</u> area under the curve
take the limit as $n \to \infty$
(more & more rectangles)
(they get thinner.

exact area under the curve:

$$\lim_{n \to \infty} \sum_{j=1}^{n} f(x_j^*) \Delta x$$

$$\parallel \quad \text{new notation}$$

$$\int_a^b f(x)\, dx .$$

"fairy godmother"

like Cinderella, in the limit

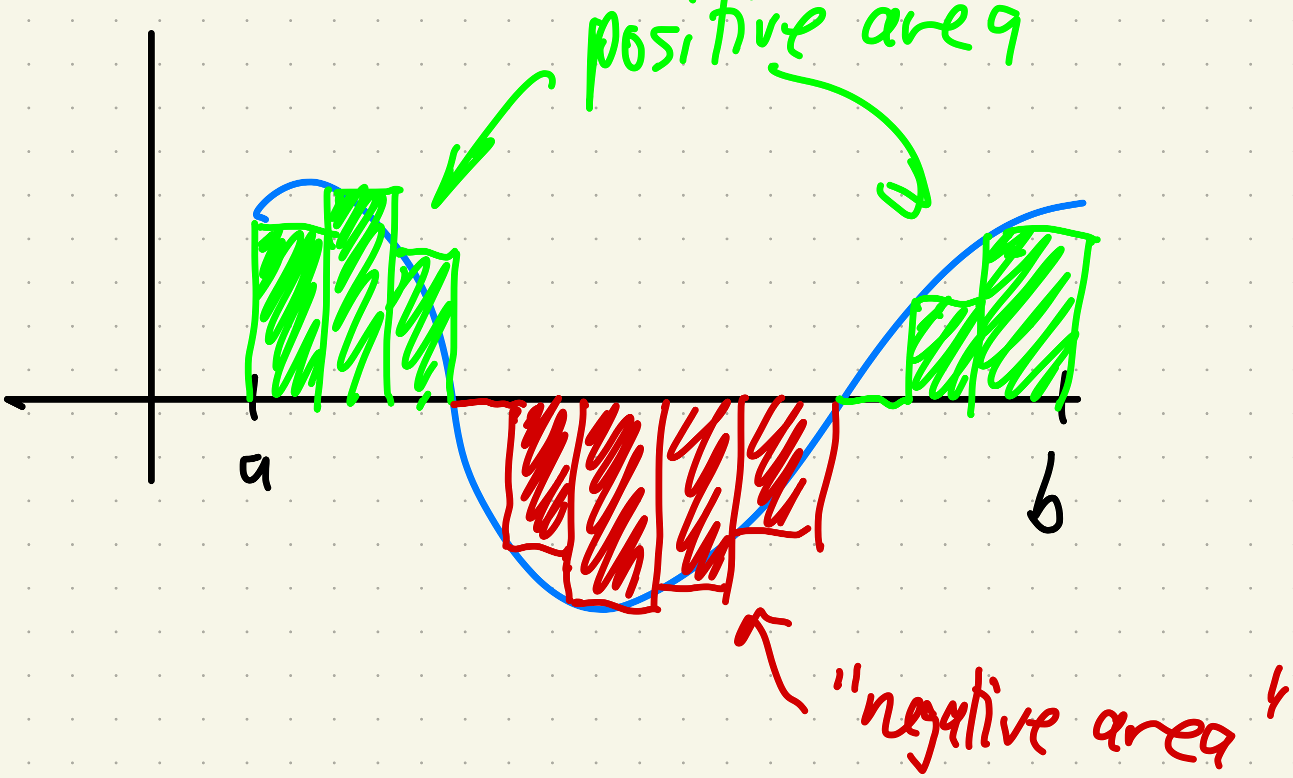
$\sum_{j=1}^{n}$ turns into $\int_a^b$ and Δx turns into dx

That is,

$$\int_a^b f(x)\,dx \overset{\text{def}}{=} \lim_{n\to\infty} \sum_{j=1}^{n} f(x_j^*)\Delta x$$

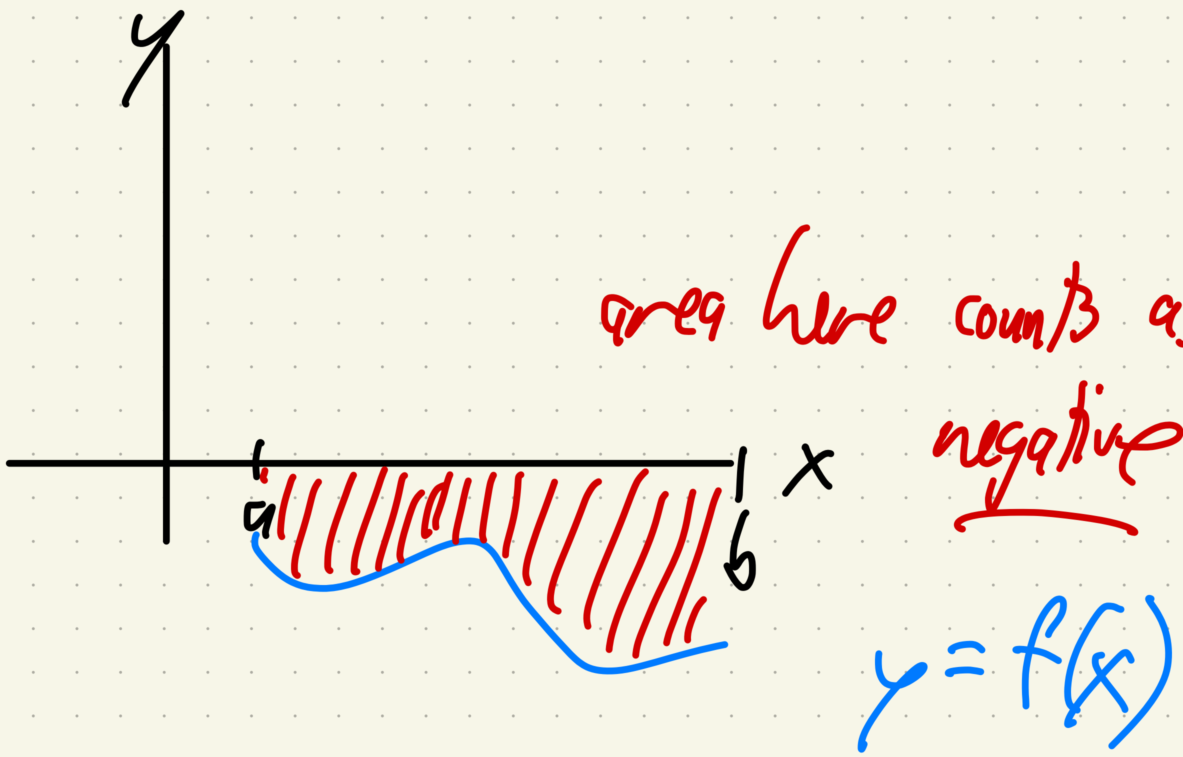
This is called "The definite integral of f from a to b."

<u>Note</u>: $\int_a^b f(x)\,dx$ is a "net area" — might have cancellation



For example, if $f(x) < 0$ everywhere,

then $\int_a^b f(x)\,dx < 0$:

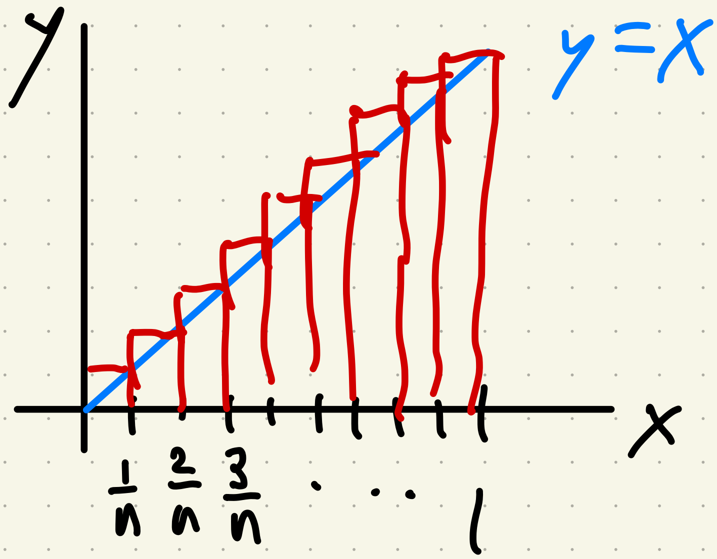


Fact: If f is continuous,

it actually doesn't matter which

sample points you take.

Ex. What is $\int_0^1 x\,dx$? Here $f(x) = x$.



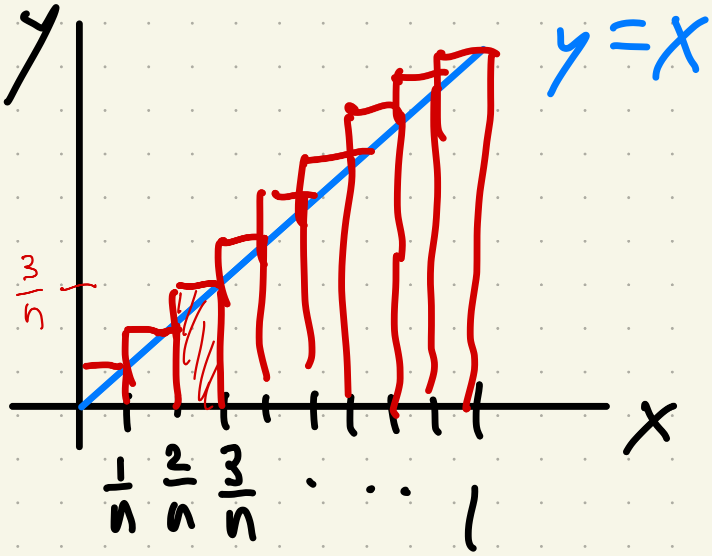
The long way: practice with this method...

Split $[0,1]$ into n equal parts:

each has width $\frac{1}{n} = \Delta x$

Let's use the right endpts for heights of rectangles.

height of j^{th} rectangle is $f\left(\frac{j}{n}\right) = \frac{j}{n}$



Add up areas of rectangles:

$$\sum_{j=1}^{n} \underbrace{\frac{j}{n}}_{\substack{\text{height}\\ \text{of } j^{th}\\ \text{rectangle}}} \cdot \underbrace{\frac{1}{n}}_{\substack{\text{width}\\ \text{of } j^{th}\\ \text{rectangle}}} = \sum_{j=1}^{n} \underbrace{\frac{j}{n^2}}_{\substack{\text{area of}\\ j^{th}\text{ rectangle}}}$$

n does not depend on j, so factor it out

$$= \frac{1}{n^2} \sum_{j=1}^{n} j$$

$$= \frac{1}{n^2}\left(\frac{n(n+1)}{2}\right)$$

formula from last time

$$\cdots = \frac{1}{n^2}\left(\frac{n(n+1)}{2}\right)$$

} The sum of the areas of the rectangles.

$$= \frac{n(n+1)}{n^2(2)}$$

$$= \frac{n+1}{2n}$$

an approximation to the area under the curve

$$= \frac{1}{2} + \frac{1}{2n}$$

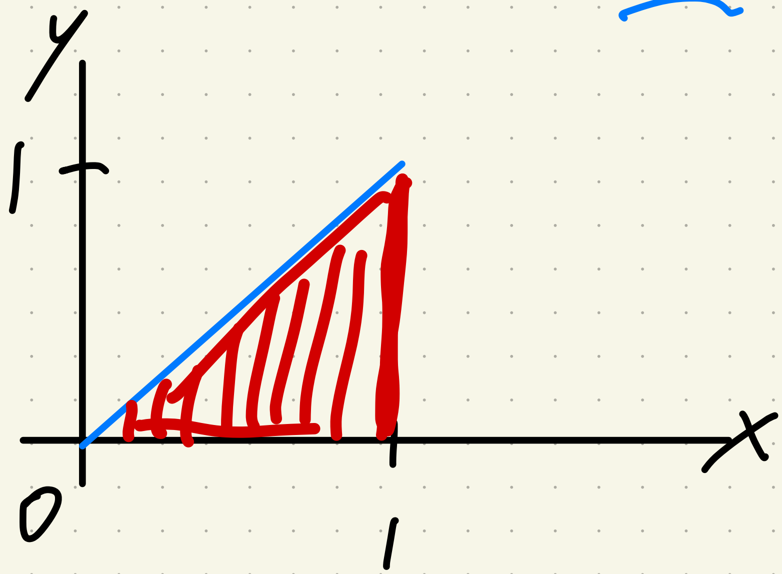
a little bigger... than the exact area

To get the exact area, take the limit as $n \to \infty$:

get $\lim_{n\to\infty}\left(\frac{1}{2} + \frac{1}{2n}\right) = \boxed{\frac{1}{2}}$

Summary: $\int_0^1 x\,dx = \boxed{\frac{1}{2}}$

This for practice — of course it's $\frac{1}{2}$!



area of triangle is $\frac{1}{2}$

doing $\lim_{n\to\infty} \sum_{j=1}^{n} f(x_j^*)\Delta x$

is doing it the long way ...

I'll show you tricks to make it easier.

just like with derivatives,

there are shortcuts...

Can prove things about integrals

using what we know about finite sums

and limits:

<u>Theorem</u>. Let f, g be continuous functions

and let c be a constant.

1. $\int_a^b c\,dx = ?$ We'll figure it out!

2. $\int_a^b [f(x) + g(x)]\,dx = ?$

3. $\int_a^b c f(x)\,dx = ?$

1. $\int_a^b c\,dx \overset{\text{def}}{=} \lim_{n\to\infty} \sum_{j=1}^{n} c\,\Delta x$ indp. of j

$f(x) = c$

constant f=

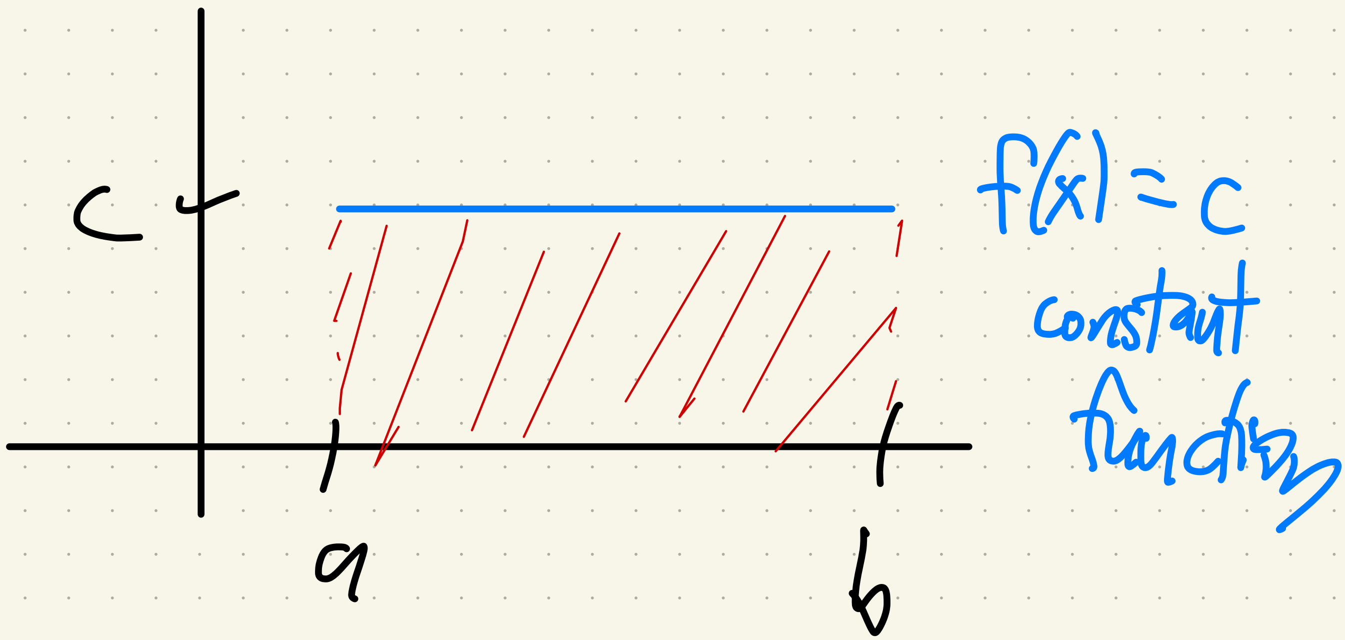
$$= \lim_{n\to\infty} (\underbrace{c\Delta x + c\Delta x + \cdots + c\Delta x}_{n \text{ times}})$$

$$= \lim_{n\to\infty} nc\Delta x$$

$\Delta x = \frac{b-a}{n}$

$$= \lim_{n\to\infty} \frac{nc(b-a)}{n} = \boxed{c(b-a)}$$

Oh yeah...



Of course

$$\int_a^b c\,dx = c(b-a)$$

area of rectangle

(negative if $c<0$).

Property 2.

again use known facts about sums and limits

$$\int_a^b [f(x)+g(x)]\,dx$$

def

$$= \lim_{n\to\infty} \sum_{j=1}^{n} \left(f(x_j^*) + g(x_j^*)\right)\Delta x$$

property of addition (last time)

$$= \lim_{n\to\infty} \left(\sum_{j=1}^{n} f(x_j^*)\Delta x + \sum_{j=1}^{n} g(x_j^*)\Delta x \right)$$

property of limits

$$= \lim_{n\to\infty} \left(\sum f(x_j^*)\Delta x \right) + \lim \sum g \,\Delta x$$

to be confirmed (pretty much done)

Other Q's?

see class webpage for final schedule