STAT 50 HW #12 Sections 4.7 and 4.8

Section 4.7 #'s 1, 3, 4, 7, 8, 12, 15

1.

Let T ~ Exp(0.45). Find

a. μ_T

$E(x) = \int_0^\infty x \lambda e^{-\lambda x} = \lambda \int_0^\infty x e^{-\lambda x}\,dx$

$u = x \qquad v = e^{-\lambda x}\,dx \qquad \int u\,dv = uv - \int v\,du$

$du = dx \qquad dv = -\frac{1}{\lambda} e^{-\lambda x}$

$\lambda \int_0^\infty x e^{-\lambda x}\,dx = \lambda\left[\frac{-x e^{-\lambda x}}{\lambda}\Big|_0^\infty + \frac{1}{\lambda}\int_0^\infty e^{-\lambda x}\,dx\right]$

$= \lambda\left[\frac{-x e^{-\lambda x}}{\lambda} - \frac{1}{\lambda^2} e^{-\lambda x}\right]\Big|_0^\infty$

$= \lambda\left[0 - 0 + 0 + \frac{1}{\lambda^2}\cdot 1\right] = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$

So, $E(x) = \frac{1}{0.45}$ or $\boxed{2.22}$

b. σ_T^2

b) $\sigma_T^2 = V(x) \quad V(x) = E(x^2) - (E(x))^2$

$$E(x^2) = \int_0^\infty x^2 \lambda e^{-\lambda x}\,dx = \lambda \int_0^\infty x^2 e^{-\lambda x}\,dx$$

$u = x^2 \qquad v = -\frac{1}{\lambda} e^{-\lambda x} \qquad \int u\,dv = uv - \int v\,du$

$du = 2x\,dx \qquad dv = e^{-\lambda x}\,dx$

$$\lambda \int_0^\infty x^2 e^{-\lambda x}\,dx = \lambda\left[\frac{-x^2 e^{-\lambda x}}{\lambda}\Big|_0^\infty + \frac{2}{\lambda}\int_0^\infty x e^{-\lambda x}\,dx\right]$$

$$= \lambda\left[-\frac{x^2}{\lambda} e^{-\lambda x}\Big|_0^\infty + \frac{2}{\lambda^2}\underbrace{\int_0^\infty x \lambda e^{-\lambda x}\,dx}_{E(x) = \frac{1}{\lambda}}\right]$$

$$= \lambda\left[-0 + 0 + \frac{2}{\lambda^2}\cdot\frac{1}{\lambda}\right]$$

$$= \frac{2}{\lambda^2} = E(x^2) \qquad \text{so, } \rightarrow \frac{1}{0.45^2} \text{ or } \boxed{4.938}$$

$$V(x) = E(x^2) - (E(x))^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

c. P(T > 3)

c) $P(T > 3)$

$T \sim Exp(0.45)$

$= 1 - P(T \leq 3) = 1 - F(3)$

$= 1 - [1 - e^{-0.45 \cdot 3}] = 1 - [1 - e^{-1.35}]$

$= e^{-1.35} = \boxed{0.2592}$

d. The median of T

d) Median of T

$F(x_m) = 0.5$

$1 - e^{-0.45 x_m} = 0.5$

$e^{-0.45 x_m} = 0.5$

$\frac{-0.45 x_m}{-0.45} = \frac{\ln(0.5)}{-0.45} = \boxed{1.5403}$

3.

A catalyst researcher states that the diameters, in microns, of the pores in a new product she has made have the exponential distribution with parameter λ = 0.25.

a. **What is the mean pore diameter?**

3. $T \sim \text{Exp}(0.25)$

a) $\mu_T = \frac{1}{0.25} = \boxed{4 \text{ microns}}$

b. **What is the standard deviation of the pore diameters?**

b) $\sigma_T = \sqrt{\sigma_T^2} = \sqrt{\frac{1}{\lambda^2}} = \sqrt{\frac{1}{0.25^2}} = \sqrt{16} = \boxed{4 \text{ microns}}$

c. **What proportion of the pores are less than 3 microns in diameter?**

c) $P(T<3) = F(3) = 1 - e^{-0.25 \cdot 3} = \boxed{0.5276}$

d. **What proportion of the pores are greater than 11 microns in diameter?**

d) $P(T > 11) = 1 - P(T \leq 11) = 1 - F(11)$

$= 1 - [1 - e^{-0.25 \cdot 11}] = e^{-2.75} = \boxed{0.0639}$

e. What is the median pore diameter?

e) $F(m) = 0.5$

$1 - e^{-0.25 \cdot m} = 0.5$

$e^{-0.25m} = 0.5 \rightarrow \frac{-0.25m}{-0.25} = \frac{\ln(0.5)}{-0.25}$

$\boxed{m = 2.7726 \text{ microns}}$

f. What is the third quartile of the pore diameters?

f) 3rd quartile? $\rightarrow 0.75$

$F(a) = 0.75$

$1 - e^{-0.25a} = 0.75$

$-1 \qquad\qquad -1$

$-e^{-0.25a} = -0.25 \rightarrow \frac{-0.25a}{-0.25} = \frac{\ln(0.25)}{-0.25}$

$\boxed{a = 5.5452 \text{ microns}}$

g. What is the 99th percentile of the pore diameters?

a) 99th percentile?

$F(p) = 0.99$

$1 - e^{-0.25p} = 0.99$

$-1 \qquad -1$

$-e^{-0.25p} = -0.01 \rightarrow \frac{-0.25p}{-0.25} = \frac{\ln(0.01)}{-0.25}$

$\boxed{p = 18.4207 \text{ meters}}$

4.

The distance between flaws on a long cable is exponentially distributed with mean 12 m.

a. **What is the value of the parameter λ?**

4. a) What is λ? $T \sim Exp(\lambda)$

mean = 12 m

$\frac{1}{\lambda} = 12\lambda$

$\frac{1}{12} = \frac{12\lambda}{12}$

$\lambda = 0.0833$

$\boxed{\lambda = 0.0833}$

b. **Find the median distance.**

b) Median?

$F(m) = 0.5$

$1 - e^{-0.0833m} = 0.5$

$e^{-0.0833m} = 0.5$

$\frac{-0.0833m}{-0.0833} = \frac{\ln(0.5)}{-0.0833} = \boxed{8.3177 \text{ m}}$

c. **Find the standard deviation of the distances.**

c) Std dev?

$$\sigma_T = \sqrt{\sigma_T^2} = \sqrt{\frac{1}{\lambda^2}} = \sqrt{\frac{1}{0.0833^2}} \approx \boxed{12\,m}$$

d. **Find the 65th percentile of the distances.**

d) 65th percentile?

$F(m) = 0.65$

$1-e^{-0.0833p} = 0.65$

$e^{-0.0833p} = 0.35$

$$\frac{-0.0833p}{-0.0833} = \frac{\ln(0.35)}{-0.0833} \qquad p = \boxed{12.5979\,m}$$

7.

Refer to Exercise 4.

a. **Find the probability that there are exactly 5 flaws in a 50 m length of cable.**

7. Prob that there are 5 flaws in 50m of cable

Poisson Distribution types of question

Mean = 1 flaw per 12m

0.166 per 2m → 4.166 per 50m

$$e^{-4.166}\frac{(4.166)^5}{5!} = \boxed{0.1623}$$

b. **Find the probability that there are more than two flaws in a 20 m length of cable.**

b) Prob of more than 2 flaws in 20 m of cable

0.166 per $2 \Rightarrow 1.66$ per 20m

$$P(X>2) = 1 - P(X \le 2)$$
$$= 1 - (P(X=0) + P(X=1) + P(X=2))$$
$$= 1 - 0.7659 = \boxed{0.2340}$$

c. Find the probability that there are no flaws in a 15 m length of cable.

c) No flaws in 15m

1 per 12

0.25 per $3 \Rightarrow 1.25$ per 15

$$P(X=0) = \frac{e^{-1.25}(1.25)^0}{0!} = \boxed{0.2865}$$

d. Find the probability that the distance between two flaws is greater than 15 m.

d) Prob that dist between 2 flaws is greater than 15m.

CDF

$$P(X \ge 15) = \int_{15}^{\infty} 0.0833\, e^{-0.0833x}\, dx$$
$$= 0.0833 \int_{15}^{\infty} e^{-0.0833x}\, dx$$
$$= \cancel{0.0833}\left(\frac{e^{-0.0833x}}{\cancel{-0.0833}}\right)\Bigg|_{15}^{\infty}$$
$$- \left(-e^{-0.0833 \cdot 15}\right)$$
$$e^{-0.0833 \cdot 15} = \boxed{0.2865}$$

e. Find the probability that the distance between two flaws is between 8 and 20 m.

8) Prob that dist. between 2 flaws is between 8 and 20m

$P(8 \le X \le 20) = \int_{8}^{20} 0.0833 e^{-0.0833x} dx$

$\downarrow$

$\cancel{0.0833} \dfrac{e^{-0.0833x}}{-\cancel{0.0833}} \Big|_{8}^{20}$

$= -e^{-0.0833 \cdot 20} - (-e^{-0.0833 \cdot 8})$

$= -0.1889 + 0.5134 = \boxed{0.3245}$

8.

Someone claims that the waiting time, in minutes, between hits at a certain website has the exponential distribution with parameter λ = 1.

a. **Let X be the waiting time until the next hit. If the claim is true, what is P(X ≥ 5)?**

8. $T \sim Exp(1)$

a) $P(X \ge 5)$?

$1 - P(X < 5) = 1 - P(X \le 5)$

$= 1 - (1 - e^{-1(5)}) = e^{-5} = \boxed{0.0067}$

b. **Based on the answer to part (a), if the claim is true, is five minutes an unusually long time to wait?**

Yes. Five minutes is an unusually long time to wait.

c. **If you waited five minutes until the next hit occurred, would you still believe the claim? Explain.**

If I waited five minutes until the next hit occured, I would still believe the claim because I have only observed one trial in this case. I am sure that if I measured the time many times, the waiting time would be closer to 1 on average.

12.

The distance between consecutive flaws on a roll of sheet aluminum is exponentially distributed with mean distance 3 m. Let X be the distance, in meters, between flaws.

a. **What is the mean number of flaws per meter?**

12. $T \sim Exp(\lambda)$

a) mean = 1 flaw per 3m → 0.33 flaws per 1m

so $\boxed{\text{mean} = 0.\overline{33}}$

b. **What is the probability that a 5 m length of aluminum contains exactly two flaws?**

b) $5 * 0.\overline{33} = 1.\overline{66}$

Prob that 5m has 2 flaws?

$P(X=2) = \frac{e^{-1.66}(1.66)^2}{2!} = \boxed{0.2623}$

15.

A light fixture contains five lightbulbs. The lifetime of each bulb is exponentially distributed with mean 200 hours. Whenever a bulb burns out, it is replaced. Let T be the time of the first bulb replacement. Let Xi , i = 1, …, 5, be the lifetimes of the five bulbs. Assume the lifetimes of the bulbs are independent.

a. **Find P(X1 > 100).**

a) mean = 200 hrs

$\lambda \frac{1}{\lambda} = 200\lambda$

$\frac{1}{200} = \frac{200\lambda}{200}$ $\lambda = \frac{1}{200}$ or 0.005

$P(X_1 > 100) = 1 - P(X_1 \leq 100) = 1 - F(100)$

$= 1 - [1 - e^{-0.005 \cdot 100}] = e^{-0.5} = \boxed{0.6065}$

b. **Find P(X1 > 100 and X2 > 100 and ⋯ and X5 > 100).**

b) $P(X_1 > 100 \ldots X_5 > 100)$, ∵ independent

$0.6065^5 = \boxed{0.0821}$

c. **Explain why the event T > 100 is the same as {X1 > 100 and X2 > 100 and ⋯ and X5 > 100}.**

T (*the time of the first replacement) will be greater than 100 hours if and only if each of the bulbs lasts longer than 100 hours. We don't know what bulb X1 will be from the five bulbs, but we can only guarantee that the first bulb will last longer than 100 hours if all bulbs selected last longer than 100 hours.

d. **Find P(T ≤ 100).**

d) $P(T \leq 100) \rightarrow \int_0^{\infty} \lambda e^{-\lambda x}\,dx$

5 bulbs, so $5 * 0.005$ $= -e^{-\lambda x}\Big|_0^{n} = -e^{-\lambda n} + 1$

0.025 $= 1 - e^{-\lambda n} \rightarrow 1 - e^{-0.025 \cdot 100}$

$P(T \leq 100) = \boxed{0.9179}$

e. **Let t be any positive number. Find P(T ≤ t), which is the cumulative distribution function of T.**

e) Find $P(T \leq t)$, what is the CDF of T?

$$\boxed{P(T \leq t) = 1 - e^{-0.025t}}$$

f. **Does T have an exponential distribution?**

Yes. It would be $T \sim EXP(0.025)$ in this case.

g. **Find the mean of T.**

$$\mu_T = \frac{1}{0.025} = 40 \text{ hours}$$

h. **If there were n lightbulbs, and the lifetime of each was exponentially distributed with parameter λ, what would be the distribution of T?**

$T \sim Exp(n\lambda)$

Section 4.8 #'s 1, 3, 6

1.

The distance advanced in a day by a tunnel boring machine, in meters, is uniformly distributed on the interval (30, 50).

a. **Find the mean distance.**

1. Uniformly distributed on the interval (30, 50)

a) Mean distance? $E(X) = \int_a^b x \cdot \frac{1}{b-a}\,dx$

$X \sim U(a,b)$

$X \sim U(30,50)$

$$\frac{x^2}{2(b-a)}\Big|_a^b \rightarrow \frac{b^2-a^2}{2(b-a)} = \frac{(b+a)(b-a)}{2(b-a)}$$

$$\frac{a+b}{2} \rightarrow \frac{30+50}{2} = \boxed{40}$$

b. **Find the standard deviation of the distances.**

b) Std dev?

$\sigma^2 = \frac{(b-a)^2}{12} = \frac{(50-30)^2}{12} = \sqrt{33.33} = \boxed{5.7735}$

c. **Find the probability that the distance is between 35 and 42 meters.**

c) Prob between 35 and 42 m?

$P(35 < x < 42) = \int_{35}^{42} \frac{1}{20}\,dx = \frac{x}{20}\Big|_{x=35}^{x=42} = \frac{42}{20} - \frac{35}{20} = \frac{7}{20}$

$\boxed{0.35}$

$f(x) = \begin{cases} \frac{1}{20} & 30 < x < 50 \\ 0 & \text{otherwise} \end{cases}$

d. **Suppose the distances advanced on different days are independent. What is the probability the machine advances more than 45 meters on exactly four out of 10 days?**

☆From independence, a Bin variable may be defined!

d) $P(X > 45)$ on exactly 4 of 10 days?

$X \sim Bin(10, 0.25)$

$P(X > 45) = \int_{45}^{50} \frac{1}{20}\,dx = \frac{x}{20}\Big|_{x=45}^{x=50} = \frac{50}{20} - \frac{45}{20} = 0.25$

$P(X_{Bin} = 4) = \binom{10}{4}(0.25)^4(0.75)^6$

$\to 210(0.0039)(0.1779) = \boxed{0.1460}$

$\frac{10!}{6!\,4!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot \cancel{6!}}{\cancel{6!} \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 210$

3.

Let T ~ Γ(4, 0.5).

a. **Find µT.**

3. $T \sim \Gamma(4, 0.5)$

$r \quad \lambda$

a) $\mu_T = \frac{r}{\lambda} = \frac{4}{0.5} = \boxed{8}$

b. Find σT.

b) $\sigma_T \qquad \sigma_x^2 = \frac{r}{\lambda^2} = \frac{4}{0.5^2} = \sqrt{16} = \boxed{4}$

c. Find P(T ≤ 1).

c) $P(T \leq 1)$

$= P(T \leq 1) = P(X \geq 4)$

$= 1 - P(X \leq 3)$

$= 1 - [P(X=0) + P(X=1) + P(X=2) + P(X=3)]$

$\mu = 1 \times 0.5 = 0.5$

$= 1 - \left(\frac{e^{-0.5} 0.5^0}{0!} + \frac{e^{-0.5} 0.5^1}{1!} + \frac{e^{-0.5} 0.5^2}{2!} + \frac{e^{-0.5} 0.5^3}{3!} \right)$

$X \sim Poisson(0.5)$

$= \boxed{0.00175}$

1×0.5

d. Find P(T ≥ 4).

A) $P(T \geq 4) \rightarrow \int_4^{\infty} \frac{0.5^4}{\Gamma(4)} x^{4-1} e^{-0.5x}\,dx$

$X \sim \Gamma(4, 2)$

$X \sim \Gamma(4, 0.5) \rightarrow \int_0^{\infty} \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x}\,dx$

$\Gamma(4) = (4-1)! = 3! = 6 \quad \frac{0.0625}{6} \int_4^{\infty} x^3 e^{-0.5x}\,dx$

$\boxed{0.8571}$

u-sub $\rightarrow u = -0.5x$

$\int \frac{8}{0.5} e^u u^3\,du$

$0.0104 \left(e^{-0.5x}(-2x^3 - 12x^2 - 48x - 96)\right)\Big|_4^{\infty}$

$16 \int e^u u^3\,du \rightarrow$ integration by parts

$16\left(u^3 e^u - \int 3u^2 e^u\,du\right) \leftarrow u = u^3 \quad du = 3u^2 \quad v = e^u \quad dv = e^u$

$16\left(u^3 e^u - 3\left(u^2 e^u - 2(e^u u - e^u)\right)\right)$

6.

The lifetime, in years, of a type of small electric motor operating under adverse conditions is exponentially distributed with λ = 3.6. Whenever a motor fails, it is replaced with another of the same type. Find the probability that fewer than six motors fail within one year.

6. $\lambda = 3.6 \quad t = 6 \quad X \sim \text{Poisson}(3.6)$

$P(X < 6)$? $\quad T \sim \Gamma(6, 3.6)$

$1 \cdot 3.6 = 3.6$

$P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=5)$

$\frac{e^{-3.6} 3.6^0}{0!} + \frac{e^{-3.6} 3.6^1}{1!} + \frac{e^{-3.6} 3.6^2}{2!} + \frac{e^{-3.6} 3.6^3}{3!} + \frac{e^{-3.6} 3.6^4}{4!} + \frac{e^{-3.6} 3.6^5}{5!}$

$0.0273 + 0.0984 + 0.1771 + 0.2125 + 0.1912 + 0.1377$

$= \boxed{0.8441}$