Math 100 # Exam 2 Fall 21

Instructions: Please answer the following legibly, logically, and **show all work** on a separate sheet of paper. No credit will be given for unjustified or unclear work. Please clearly label every problem and work them in order. When you are finished, please scan your work (or take pictures) and submit via the Exams link on Canvas.

Note: $M_{2\times 2}$ is the vector space of all 2×2 matrices whose entries are real numbers.

1. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in a vector space V, and let H be any subspace of V that contains both $\mathbf{u}$ and $\mathbf{v}$. Explain why H also contains $\text{Span}\{\mathbf{u}, \mathbf{v}\}$.

2. Let H and K be subspaces of a vector space V. The **intersection** of H and K, denoted by $H \cap K$, is the set of $\mathbf{v}$ in V such that $\mathbf{v}$ belongs to both H and K. Show that $H \cap K$ is a subspace of V.

3. Let $T : V \to W$ be a linear transformation from a vector space V into a vector space W. Prove that the range of T is a subspace of W. [*Hint:* Vectors in the range of T have the form $T(\mathbf{v})$ for some $\mathbf{v}$ in V.]

4. Define $T : \mathbb{P}_2 \to \mathbb{R}^2$ by $T(\mathbf{p}) = \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix}$.

   For example if $\mathbf{p}(t) = 3 + 4t + 5t^2$ then $T(\mathbf{p}) = \begin{bmatrix} 3 \\ 12 \end{bmatrix}$.

   (a) Show that T is a linear transformation.

   (b) Find a polynomial in the kernel of T.

   (c) What is the range of T?

5. Is the following set of polynomials linearly independent in $\mathbb{P}_3$? Explain your work.

$$1 + 2t^3, 2 + t - 3t^2, -t + 2t^2 - t^3$$

6. Let H be a subspace of $M_{2\times 2}$ whose vectors are of the form $\begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$. Then, $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \right\}$ is a basis for H.

   Find the coordinate vector of $\mathbf{v} = \begin{bmatrix} 7 & -1 \\ 0 & 0 \end{bmatrix}$ according to the basis, $\mathcal{B}$.

7. Let H be the subspace of $M_{2\times 2}$ in Question 6 with the added restriction that $a + 2b + 3c = 0$. Find a basis for H, and state the dimension of H.

8. Let H be the set of all vectors of the forms given below and either find a basis for the vector space, or give an example to show that it is not a vector space.

(a) $\left\{ \begin{bmatrix} 3q-4p \\ 2p \\ q+1 \\ 2p+5q \end{bmatrix} : p, q \text{ are real} \right\}$ (b) $\left\{ \begin{bmatrix} 2c-b \\ 3a-2b \\ 0 \\ a+4b+3c \end{bmatrix} \quad a, b, c \text{ are real} \right\}$

9. Assume that matrix A is row equivalent to matrix B, and

$$A = \begin{bmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & -1 & 5 \\ 0 & -2 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) State dim Col A and dim Nul A.

(b) Give a basis for Col A, Row A, and Nul A.

10. If A is a 5×4 matrix, what is the largest possible dimension of Row A? If A is a 4×5 matrix, what is the largest possible dimension of Row A? Explain.