1. Determine if b is a linear combination of a1, a2, and a3. (Use row notation to show your steps)

$$\mathbf{a}_1 = \begin{bmatrix}1\\0\\1\end{bmatrix} \quad \mathbf{a}_2 = \begin{bmatrix}-2\\3\\-2\end{bmatrix} \quad \mathbf{a}_3 = \begin{bmatrix}-6\\7\\5\end{bmatrix} \quad \mathbf{b} = \begin{bmatrix}11\\-5\\9\end{bmatrix}$$

$$a_1 = \begin{bmatrix}1\\0\\1\end{bmatrix} \quad a_2 = \begin{bmatrix}-2\\3\\-2\end{bmatrix} \quad a_3 = \begin{bmatrix}-6\\7\\5\end{bmatrix} \quad b = \begin{bmatrix}11\\-5\\9\end{bmatrix}$$

$x_1 a_1 + x_2 a_2 + x_3 a_3 = b$?

$$x_1\begin{bmatrix}1\\0\\1\end{bmatrix} + x_2\begin{bmatrix}-2\\3\\-2\end{bmatrix} + x_3\begin{bmatrix}-6\\7\\5\end{bmatrix} = \begin{bmatrix}11\\-5\\9\end{bmatrix}$$

$\downarrow$ same as

$$\begin{bmatrix}x_1\\0x_1\\x_1\end{bmatrix} + \begin{bmatrix}-2x_2\\3x_2\\-2x_2\end{bmatrix} + \begin{bmatrix}-6x_3\\7x_3\\5x_3\end{bmatrix} = \begin{bmatrix}11\\-5\\9\end{bmatrix} \text{ and } \begin{bmatrix}x_1 - 2x_2 - 6x_3\\0x_1 + 3x_2 + 7x_3\\x_1 - 2x_2 + 5x_3\end{bmatrix} = \begin{bmatrix}11\\-5\\9\end{bmatrix}$$

$$\begin{bmatrix}1 & -2 & -6 & 11\\0 & 3 & 7 & -5\\1 & -2 & 5 & 9\end{bmatrix} \xrightarrow{-1\cdot R_1 + R_3 \to R_3} \begin{bmatrix}1 & -2 & -6 & 11\\0 & 3 & 7 & -5\\0 & 0 & 11 & -2\end{bmatrix}$$

$\downarrow \frac{1}{3}\cdot R_2$

$$\begin{bmatrix}1 & 0 & -1.33 & 7.66\\0 & 1 & 7/3 & -5/3\\0 & 0 & 11 & -2\end{bmatrix} \xleftarrow{2R_2 + R_1 \to R_1} \begin{bmatrix}1 & -2 & -6 & 11\\0 & 1 & 7/3 & -5/3\\0 & 0 & 11 & -2\end{bmatrix}$$

$\downarrow \frac{1}{11}\cdot R_3$

$$\begin{bmatrix}1 & 0 & -1.33 & 7.66\\0 & 1 & 7/3 & -5/3\\0 & 0 & 1 & -2/11\end{bmatrix} \xrightarrow{\frac{4}{3}\cdot R_3 + R_1 \to R_1} \begin{bmatrix}1 & 0 & 0 & 7.42\\0 & 1 & 7/3 & -5/3\\0 & 0 & 1 & -2/11\end{bmatrix}$$

$\downarrow -\frac{7}{3}\cdot R_3 + R_2 \to R_2$

$$\begin{bmatrix}1 & 0 & 0 & 7.42\\0 & 1 & 0 & -1.24\\0 & 0 & 1 & -2/11\end{bmatrix}$$

cont'd

(cont'd)

$x_1 = 7.\overline{42}$ $x_2 = -1.\overline{24}$ $x_3 = -2/11$

$$7.\overline{42}\begin{bmatrix}1\\0\\1\end{bmatrix} - 1.\overline{24}\begin{bmatrix}-2\\3\\-2\end{bmatrix} - 2/11\begin{bmatrix}-6\\7\\5\end{bmatrix} = \begin{bmatrix}11\\-5\\9\end{bmatrix}$$

R_1? ✓
R_2? ✓
R_3? ✓

$\boxed{\text{b is a linear combination of } a_1, a_2, \text{ and } a_3}$

2. Determine if b is a linear combination of the vectors formed from the columns of the matrix A.

$$A = \begin{bmatrix} 1 & -4 & 2 \\ 0 & 3 & 5 \\ -2 & 8 & -4 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 3 \\ -7 \\ -3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -4 & 2 \\ 0 & 3 & 5 \\ -2 & 8 & -4 \end{bmatrix} \quad b = \begin{bmatrix} 3 \\ -7 \\ -3 \end{bmatrix} \quad \begin{bmatrix} 1 & -4 & 2 & 3 \\ 0 & 3 & 5 & -7 \\ -2 & 8 & -4 & -3 \end{bmatrix}$$

$$\downarrow 2 \cdot R_1 + R_3 \rightarrow R_3$$

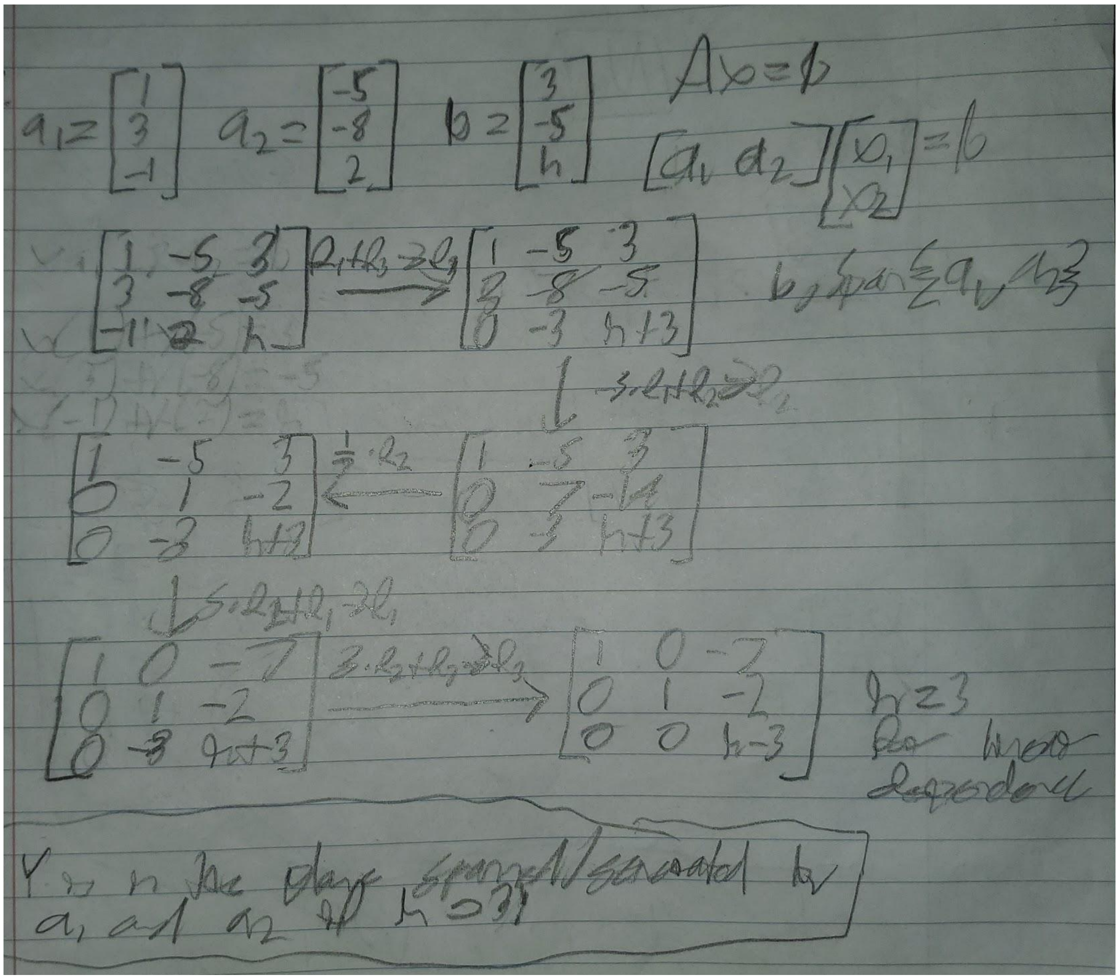
An augmented matrix is consistent if and only if it does not contain a row of the form $[0 \ldots 0 \; b]$

$$\begin{bmatrix} 1 & -4 & 2 & 3 \\ 0 & 3 & 5 & -7 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

No b is not a linear combination of the 3 vectors formed from the columns of A.

3. For what value(s) of h is y in the plane generated by v1 and v2?

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \qquad \mathbf{a}_2 = \begin{bmatrix} -5 \\ -8 \\ 2 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 3 \\ -5 \\ h \end{bmatrix}$$



$a_1 = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$ $a_2 = \begin{bmatrix} -5 \\ -8 \\ 2 \end{bmatrix}$ $b = \begin{bmatrix} 3 \\ -5 \\ h \end{bmatrix}$ $Ax = b$

$[a_1\ a_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b$

$\begin{bmatrix} 1 & -5 & 3 \\ 3 & -8 & -5 \\ -1 & 2 & h \end{bmatrix} \xrightarrow{R_1+R_3 \to R_3} \begin{bmatrix} 1 & -5 & 3 \\ 3 & -8 & -5 \\ 0 & -3 & h+3 \end{bmatrix}$ $b, \mathrm{Span}\{a_1, a_2\}$

$\downarrow -3R_1+R_2 \to R_2$

$\begin{bmatrix} 1 & -5 & 3 \\ 0 & 1 & -2 \\ 0 & -3 & h+3 \end{bmatrix} \xleftarrow{\frac{1}{7}R_2} \begin{bmatrix} 1 & -5 & 3 \\ 0 & 7 & -14 \\ 0 & -3 & h+3 \end{bmatrix}$

$\downarrow 5R_2+R_1 \to R_1$

$\begin{bmatrix} 1 & 0 & -7 \\ 0 & 1 & -2 \\ 0 & -3 & h+3 \end{bmatrix} \xrightarrow{3R_2+R_3 \to R_3} \begin{bmatrix} 1 & 0 & -7 \\ 0 & 1 & -2 \\ 0 & 0 & h-3 \end{bmatrix}$ $h=3$ for linear dependence

Y is in the plane spanned/generated by a_1 and a_2 if $h=3$

4. Let $A = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 3 & -2 \\ -2 & 6 & 3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ -4 \end{bmatrix}$.

Denote the columns of A by $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, and let $W = \mathrm{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3.\}$

1. $A = \begin{array}{c} \begin{matrix} a_1 & a_2 & a_3 \end{matrix} \\ \begin{bmatrix} 1 & 0 & -4 \\ 0 & 3 & -2 \\ -2 & 6 & 3 \end{bmatrix} \end{array} \quad b = \begin{bmatrix} 4 \\ 1 \\ -4 \end{bmatrix} \quad \left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}, \begin{bmatrix} -4 \\ -2 \\ 3 \end{bmatrix} \right\}$

Denote the columns of A by a_1, a_2, a_3, and let $W = \text{Span}\{a_1, a_2, a_3\}$

a) Is b in $\{a_1, a_2, a_3\}$?

There are only 3 vectors in the set and b would not be one of them, so no.

$\boxed{\text{No}}$

b) How many vectors are in $\{a_1, a_2, a_3\}$?

$\boxed{3}$

columns of the matrix A

? linear comb of Span{...}

(a) Is b in {a1, a2, a3}?

No. There are only 3 vectors in the set and b is not one of them.

(b) How many vectors are in {a1, a2, a3}?

3.

(c) Is b in W? (Show your work)

$\boxed{3}$

c) is b a linear comb of Span {...} ← columns of matrix A

c) Is b in W, Find if consistent and has a unique solution

Get to REF Row Echelon Form!

$$\left[\begin{array}{ccc|c} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ -2 & 6 & 3 & -4 \end{array}\right] \xrightarrow{2\cdot R_1 + R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ 0 & 6 & -5 & 4 \end{array}\right]$$

$$\downarrow -2\cdot R_2 + R_3 \to R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 3 & -2 & 1 \\ 0 & 0 & -1 & 2 \end{array}\right] \xleftarrow{4 R_3 + R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ 0 & 0 & -1 & 2 \end{array}\right]$$

$$\downarrow \tfrac{1}{3} R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & -2/3 & 1/3 \\ 0 & 0 & -1 & 2 \end{array}\right] \xrightarrow{\frac{2}{3} R_3 + R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 2 \end{array}\right]$$

$-\frac{4}{3} + \frac{1}{3} = -1$

cont'd

cont'd

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 2 \end{array}\right] \xrightarrow{-1\cdot R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Thus, we get b is in W!

If A was written as a system of equations b would be a possible solution!

$\boxed{\text{b is in W!}}$

(d) How many vectors are in W?

Infinitely many.

(e) Show that a1 is in W. [Hint Row operations are unnecessary.]

e) $\boxed{a_1 = a_1 + 0a_2 + 0a_3}$

5. Use the definition of Ax to write the following matrix equation as a vector equation.

$$\begin{bmatrix} 1 & 2 & -3 & 1 \\ -2 & -3 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -3 & 1 \\ -2 & -3 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

Vector eqn:
$v_1 x_1 + v_2 x_2 = b$

$$\boxed{2\begin{bmatrix} 1 \\ -2 \end{bmatrix} + (-1)\begin{bmatrix} 2 \\ -3 \end{bmatrix} + (1)\begin{bmatrix} -3 \\ 1 \end{bmatrix} + (-1)\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}}$$

6. Use the definition of Ax to write the following vector equation as a matrix equation. Then write this vector equation as a system of equations.

Note: A matrix equation is not the same as an augmented matrix.

$$z_1 \begin{bmatrix} 2 \\ -4 \end{bmatrix} + z_2 \begin{bmatrix} -1 \\ 5 \end{bmatrix} + z_3 \begin{bmatrix} -4 \\ 3 \end{bmatrix} + z_4 \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$$

6. $z_1\begin{bmatrix}2\\-4\end{bmatrix}+z_2\begin{bmatrix}-1\\5\end{bmatrix}+z_3\begin{bmatrix}-4\\3\end{bmatrix}+z_4\begin{bmatrix}0\\2\end{bmatrix}=\begin{bmatrix}5\\12\end{bmatrix}$

$$\boxed{\begin{array}{c}\begin{bmatrix}2 & -1 & -4 & 0\\-4 & 5 & 3 & 2\end{bmatrix}\begin{bmatrix}z_1\\z_2\\z_3\\z_4\end{bmatrix}=\begin{bmatrix}5\\12\end{bmatrix}\\ \text{and}\\ 2z_1+(1)z_2-4z_3+0z_4=5\\ -4z_1+5z_2+3z_3+2z_4=12\end{array}}$$

7. Given A and b, write the augmented matrix for the linear system that corresponds to the matrix equation Ax = b. Then solve the system and write the solution as a vector.

$$A=\begin{bmatrix}1 & 3 & -4\\1 & 5 & 2\\-3 & -7 & 6\end{bmatrix}\qquad \mathbf{b}=\begin{bmatrix}-2\\4\\12\end{bmatrix}$$

7. $A = \begin{bmatrix} 1 & 3 & -4 \\ 1 & 5 & 2 \\ -3 & -7 & 6 \end{bmatrix} \quad b = \begin{bmatrix} -2 \\ 4 \\ 12 \end{bmatrix}$

RREF

$$\left[\begin{array}{ccc|c} 1 & 3 & -4 & -2 \\ 1 & 5 & 2 & 4 \\ -3 & -7 & 6 & 12 \end{array}\right] \xrightarrow{-1 \cdot R_1 + R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 3 & -4 & -2 \\ 0 & 2 & 6 & 6 \\ -3 & -7 & 6 & 12 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2} \cdot R_2} \left[\begin{array}{ccc|c} 1 & 3 & -4 & -2 \\ 0 & 1 & 3 & 3 \\ -3 & -7 & 6 & 12 \end{array}\right] \xrightarrow{3 \cdot R_1 + R_3 \to R_3} \begin{array}{cccc} & & -12 & -6 \end{array} \left[\begin{array}{ccc|c} 1 & 3 & -4 & -2 \\ 0 & 1 & 3 & 3 \\ 0 & 2 & -6 & 6 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2} \cdot R_3} \left[\begin{array}{ccc|c} 1 & 3 & -4 & -2 \\ 0 & 1 & 3 & 3 \\ 0 & 1 & -3 & 3 \end{array}\right] \xrightarrow{-1 \cdot R_2 + R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 3 & -4 & -2 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & -6 & 0 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{6} \cdot R_3} \begin{bmatrix} 1 & 3 & -4 & -2 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{-3 \cdot R_2 + R_1 \to R_1} \begin{bmatrix} 1 & 0 & -13 & -11 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow{13 \cdot R_3 + R_1 \to R_1} \begin{bmatrix} 1 & 0 & 0 & -11 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{-3 \cdot R_3 + R_2 \to R_2} \begin{bmatrix} 1 & 0 & 0 & -11 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\boxed{x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -11 \\ 3 \\ 0 \end{bmatrix}}$$

8. Let $A = \begin{bmatrix} 1 & -2 & -1 \\ -2 & 2 & 0 \\ 4 & -1 & 3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

(a) Show that the equation Ax = b does not have a solution for all possible b. (Meaning there is not a solution for every possible combination of b1, b2, b3.)

$$A = \begin{bmatrix} 1 & -2 & -1 \\ -2 & 2 & 0 \\ 4 & -1 & 3 \end{bmatrix} \text{ and } b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Set up Ax=b as an augmented matrix

(a) $$\begin{bmatrix} 1 & -2 & -1 \\ -2 & 2 & 0 \\ 4 & -1 & 3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & -1 & b_1 \\ -2 & 2 & 0 & b_2 \\ 4 & -1 & 3 & b_3 \end{bmatrix}$$

$\downarrow -\frac{1}{2} R_2$ $\quad -\frac{1}{2}b_2 - b_1 \downarrow b_1 + \frac{1}{2}b_2$

$$\begin{bmatrix} 1 & -2 & -1 & b_1 \\ 1 & -1 & 0 & -\frac{1}{2}b_2 \\ 4 & -1 & 3 & b_3 \end{bmatrix}$$

$8 \quad 4 \qquad -(-2)+2$

$$\begin{bmatrix} 1 & -2 & -1 & b_1 \\ 0 & -1 & -1 & b_1 + \frac{b_2}{2} \\ 4 & -1 & 3 & b_3 \end{bmatrix} \xleftarrow{b_1 - (-\frac{1}{2}b_2) b_2}$$

$\downarrow b_3 - 4b_1 \rightarrow b_3$

$$\begin{bmatrix} 1 & -2 & -1 & b_1 \\ 0 & -1 & -1 & b_1 + \frac{b_2}{2} \\ 0 & 7 & 7 & b_3 - 4b_1 \end{bmatrix} \xrightarrow{\frac{b_3 - 4b_1}{7}} \begin{bmatrix} 1 & -2 & -1 & b_1 \\ 0 & -1 & -1 & b_1 + \frac{b_2}{2} \\ 0 & 1 & 1 & \frac{b_3 - 4b_1}{7} \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & -2 & -1 & b_1 \\ 0 & -1 & -1 & b_1 + \frac{b_2}{2} \\ 0 & 0 & 0 & \left(\frac{b_3 - 4b_1}{7}\right) + b_1 + \frac{b_2}{2} \end{bmatrix}$$

$\frac{b_3 - 4b_1}{7} + b_1 + \frac{b_2}{2}$

$\downarrow$

$\frac{b_3}{7} - \frac{4b_1}{7} + b_1 + \frac{b_2}{2}$

$\downarrow$

$b_3 - 4b_1 + 7b_1 + \frac{7b_2}{2}$

$\downarrow$

$2b_3 - 8b_1 + 14b_1 + 7b_2$

$\downarrow$

$2b_3 + 6b_1 + 7b_2 = 0$

System would only be consistent when $2b_3 + 6b_1 + 7b_2 = 0$, otherwise it is inconsistent. For example, there are no solutions for $b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

$-\frac{3}{7} + 1 + \frac{1}{2} \quad \frac{3}{2} + 1\frac{1}{2}$

(b) Based on (a), do the columns of A span R3? In other words, can every vector in R3 be written as a linear combination of the columns of A? Why or why not?

No. It does not span R3 as it does not have 3 pivots when in echelon form.

(c) Describe the set of all b for which Ax = b does have a solution.

If 2b3 + 6b1 + 7b2 = 0, then the system is consistent. For any b where the equation 2b3 + 6b1+ 7b2 = 0, there is a solution. You could also say that the set of all b such that b is a plane through the origin in R3. *I decided to do more research and the page from the book below helped. Sorry, I can be slow at times.

1.4 The Matrix Equation $A\mathbf{x} = \mathbf{b}$ 37

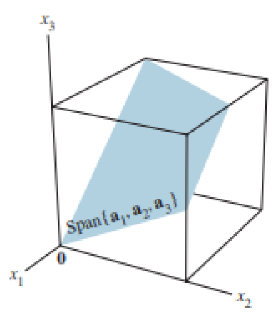


FIGURE 1
The columns of $A = [\mathbf{a}_1 \ \ \mathbf{a}_2 \ \ \mathbf{a}_3]$ span a plane through $\mathbf{0}$.

The reduced matrix in Example 3 provides a description of all $\mathbf{b}$ for which the equation $A\mathbf{x} = \mathbf{b}$ *is* consistent: The entries in $\mathbf{b}$ must satisfy

$$b_1 - \tfrac{1}{2}b_2 + b_3 = 0$$

This is the equation of a plane through the origin in $\mathbb{R}^3$. The plane is the set of all linear combinations of the three columns of A. See Fig. 1.

The equation $A\mathbf{x} = \mathbf{b}$ in Example 3 fails to be consistent for all $\mathbf{b}$ because the echelon form of A has a row of zeros. If A had a pivot in all three rows, we would not care about the calculations in the augmented column because in this case an echelon form of the augmented matrix could not have a row such as $[0 \ \ 0 \ \ 0 \ \ 1]$.

In the next theorem, the sentence "The columns of A span $\mathbb{R}^m$" means that *every* $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A. In general, a set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^m$ **spans** (or **generates**) $\mathbb{R}^m$ if every vector in $\mathbb{R}^m$ is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_p$—that is, if Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} = \mathbb{R}^m$.

9. For the following statements determine whether they are true or false. If false, then explain why it is false. If true, then point to a Theorem or definition that supports the statement.

(a) The solution set of the linear system whose augmented matrix is [a1 a2 a3 b] is the same as the solution set for x1a1 + x2a2 + x3a3 = b.

True. Refer to theorem 3 under section 1.4 on page 36. Also posted below.

THEOREM 3

If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and if $\mathbf{b}$ is in $\mathbb{R}^m$, the matrix equation

$$A\mathbf{x} = \mathbf{b} \tag{4}$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b} \tag{5}$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix} \tag{6}$$

(b) Asking whether the linear system corresponding to an augmented matrix [a1 a2 a3 b] has a solution amounts to asking whether b is in Span{a1, a2, a3}.

True. Refer to page 30. See definition of span and theorem 4. *I don't know what to call this definition, I just know this is my proof posted below (the explanation starts on the sentence with the highlighted part):

One of the key ideas in linear algebra is to study the set of all vectors that can be generated or written as a linear combination of a fixed set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of vectors.

DEFINITION

If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in $\mathbb{R}^n$, then the set of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ is denoted by Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ and is called the **subset of** $\mathbb{R}^n$ **spanned** (or **generated**) **by** $\mathbf{v}_1, \ldots, \mathbf{v}_p$. That is, Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is the collection of all vectors that can be written in the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

with $c_1, \ldots, c_p$ scalars.

Asking whether a vector $\mathbf{b}$ is in Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ amounts to asking whether the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{b}$$

has a solution, or, equivalently, asking whether the linear system with augmented matrix $[\,\mathbf{v}_1 \;\cdots\; \mathbf{v}_p \;\; \mathbf{b}\,]$ has a solution.

Note that Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ contains every scalar multiple of $\mathbf{v}_1$ (for example), since $c\mathbf{v}_1 = c\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_p$. In particular, the zero vector must be in Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

(c) The equation Ax = b is consistent if the augmented matrix [A b] has a pivot position in every row.

False. If the augmented matrix [A b] has a pivot position in “every” row, the equation Ax = b may or may not be consistent. If the pivot position is in the column representing b, then we say it is not consistent.

(d) If the columns of an m × n matrix A span R m, then the equation Ax = b is consistent for each b in R m.

True. If the columns span R^m, this says that every b in R^m is in the span of the columns, which is another way of saying that any b is a linear combination of the columns. Then the equation is consistent. You could also just look at the explanation and 4th theorem posted below:

13] span a plane

In the next theorem, the sentence “The columns of A span $\mathbb{R}^m$” means that *every* $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A. In general, a set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^m$ **spans** (or **generates**) $\mathbb{R}^m$ if every vector in $\mathbb{R}^m$ is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_p$—that is, if Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} = \mathbb{R}^m$.

THEOREM 4

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all true statements or they are all false.

a. For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.
b. Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.
c. The columns of A span $\mathbb{R}^m$.
d. A has a pivot position in every row.

(e) If the equation Ax = b is consistent, then b is in the set spanned by the columns of A.

True. The equation Ax = b has a non-empty solution set if and only if b is a linear combination of the columns of A. This is theorem 4 again, also posted below:

*You could also see existence of solutions in notes

13] span a plane

In the next theorem, the sentence "The columns of A span $\mathbb{R}^m$" means that *every* $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A. In general, a set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in $\mathbb{R}^m$ **spans** (or **generates**) $\mathbb{R}^m$ if every vector in $\mathbb{R}^m$ is a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_p$—that is, if Span $\{\mathbf{v}_1, \dots, \mathbf{v}_p\} = \mathbb{R}^m$.

THEOREM 4

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all true statements or they are all false.

a. For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.
b. Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.
c. The columns of A span $\mathbb{R}^m$.
d. A has a pivot position in every row.

(f) Any linear combination of vectors can always be written in the form Ax for a suitable matrix A and vector x.

True. I can’t properly put it into words, but the image below explains it for me:

*True by definition of Ax

SOLUTION From the definition,

$$\begin{bmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1\begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} + x_2\begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix} + x_3\begin{bmatrix} 4 \\ -3 \\ 8 \end{bmatrix}$$

$$= \begin{bmatrix} 2x_1 \\ -x_1 \\ 6x_1 \end{bmatrix} + \begin{bmatrix} 3x_2 \\ 5x_2 \\ -2x_2 \end{bmatrix} + \begin{bmatrix} 4x_3 \\ -3x_3 \\ 8x_3 \end{bmatrix} \tag{7}$$

$$= \begin{bmatrix} 2x_1 + 3x_2 + 4x_3 \\ -x_1 + 5x_2 - 3x_3 \\ 6x_1 - 2x_2 + 8x_3 \end{bmatrix}$$

The first entry in the product $A\mathbf{x}$ is a sum of products (sometimes called a *dot product*), using the first row of A and the entries in $\mathbf{x}$. That is,

$$\begin{bmatrix} 2 & 3 & 4 \\ & & \\ & & \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 + 3x_2 + 4x_3 \\ \\ \\ \end{bmatrix}$$

This matrix shows how to compute the first entry in $A\mathbf{x}$ directly, without writing down all the calculations shown in (7). Similarly, the second entry in $A\mathbf{x}$ can be calculated at once by multiplying the entries in the second row of A by the corresponding entries in $\mathbf{x}$ and then summing the resulting products:

$$\begin{bmatrix} & & \\ -1 & 5 & -3 \\ & & \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \\ -x_1 + 5x_2 - 3x_3 \\ \\ \end{bmatrix}$$

Likewise, the third entry in $A\mathbf{x}$ can be calculated from the third row of A and the entries in $\mathbf{x}$. ∎

Row–Vector Rule for Computing $A\mathbf{x}$

If the product $A\mathbf{x}$ is defined, then the ith entry in $A\mathbf{x}$ is the sum of the products of corresponding entries from row i of A and from the vector $\mathbf{x}$.