HW 10

① Note that $\vec{u}_1 \cdot \vec{u}_2 = 0$ $\quad \vec{y}\cdot\vec{u}_1 = (-1\cdot1)+(4\cdot1)+(3\cdot1) = 6$ $\quad \vec{y}\cdot\vec{u}_2 = (-1\cdot-1)+(4\cdot3)+(3\cdot-2) = 7$

$\vec{u}_1\cdot\vec{u}_1 = (1\cdot1)+(1\cdot1)+(1\cdot1) = 3$ $\quad \vec{u}_2\cdot\vec{u}_2 = (-1\cdot-1)+(3\cdot3)+((-2)(-2)) = 14$

$$\hat{y} = \text{proj}_W \vec{y} = \frac{6}{3}\vec{u}_1 + \frac{7}{14}\vec{u}_2 = 2\begin{bmatrix}1\\1\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}-1\\3\\-2\end{bmatrix} = \begin{bmatrix}2-1/2\\2+3/2\\2-1\end{bmatrix} = \begin{bmatrix}3/2\\7/2\\1\end{bmatrix}$$

$$\vec{z} = \vec{y}-\hat{y} = \begin{bmatrix}-1\\4\\3\end{bmatrix} - \begin{bmatrix}3/2\\7/2\\1\end{bmatrix} = \begin{bmatrix}-5/2\\1/2\\2\end{bmatrix} \rightarrow \vec{y} = \hat{y} + \vec{z} = \begin{bmatrix}3/2\\7/2\\1\end{bmatrix} + \begin{bmatrix}-5/2\\1/2\\2\end{bmatrix}$$

② Note: $\vec{v}_1\cdot\vec{v}_2 = 0$ $\quad y\cdot v_1 = (3\cdot1)+(-1\cdot-2)+(1\cdot-1)+(13\cdot2) = 30$ $\quad y\cdot v_2 = (3\cdot-4)+(-1\cdot1)+(1\cdot0)+(13\cdot3) = 26$

$v_1\cdot v_1 = (1\cdot1)+(-2\cdot-2)+(-1\cdot-1)+(2\cdot2) = 10$ $\quad v_2\cdot v_2 = (-4\cdot-4)+(1\cdot1)+(0\cdot0)\cdot(3\cdot3) = 26$

$$\hat{y} = \text{proj}_W \vec{y} = \frac{30}{10}v_1 + \frac{26}{26}v_2 = 3\begin{bmatrix}1\\-2\\-1\\2\end{bmatrix} + \begin{bmatrix}-4\\1\\0\\3\end{bmatrix} = \begin{bmatrix}-1\\-5\\-3\\9\end{bmatrix}$$

③ Note: $\vec{v}_1\cdot\vec{v}_2 = 0$ $\quad z\cdot v_1 = (3\cdot2)+(-7\cdot-1)+(2\cdot-3)+(3\cdot1) = 10$ $\quad z\cdot v_2 = (3\cdot1)+(-7\cdot1)+(2\cdot0)+(3\cdot-1) = -7$

$v_1\cdot v_1 = (2\cdot2)+(-1\cdot-1)+(-3\cdot-3)+(1\cdot1) = 15$ $\quad \vec{v}_2\cdot v_2 = (1\cdot1)+(1\cdot1)+(0\cdot0)+(-1\cdot-1) = 3$

$$\hat{y} = \text{proj}_{\text{Span}(v_1,v_2)}\hat{y} = \underline{\frac{10}{15}v_1 + \frac{-7}{3}v_2} = \underline{\frac{2}{3}\begin{bmatrix}2\\-1\\-3\\1\end{bmatrix} - \frac{7}{3}\begin{bmatrix}1\\1\\0\\-1\end{bmatrix}} = \begin{bmatrix}4/3-7/3\\-2/3-7/3\\-2+0\\2/3+7/3\end{bmatrix} = \begin{bmatrix}-3/3\\-9/3\\-2\\9/3\end{bmatrix} = \begin{bmatrix}-1\\-3\\-2\\3\end{bmatrix}$$

* Any underlined answer is enough.

④ a) $$UU^T = \begin{bmatrix}2/3 & -2/3\\1/3 & 2/3\\2/3 & 1/3\end{bmatrix}\cdot\begin{bmatrix}2/3 & 1/3 & 2/3\\-2/3 & 2/3 & 1/3\end{bmatrix} = \begin{bmatrix}(4/9+4/9) & (2/9-4/9) & (4/9-2/9)\\(2/9-4/9) & (1/9+4/9) & (2/9+2/9)\\(4/9-2/9) & (2/9+2/9) & (4/9+1/9)\end{bmatrix} = \begin{bmatrix}8/9 & -2/9 & 2/9\\-2/9 & 5/9 & 4/9\\2/9 & 4/9 & 5/9\end{bmatrix}$$

$$U^TU = \begin{bmatrix}2/3 & 1/3 & 2/3\\-2/3 & 2/3 & 1/3\end{bmatrix}\cdot\begin{bmatrix}2/3 & -2/3\\1/3 & 2/3\\2/3 & 1/3\end{bmatrix} = \begin{bmatrix}4/9+1/9+4/9 & -4/9+2/9+2/9\\-4/9+2/9+2/9 & 4/9+4/9+1/9\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

b) $y\cdot u_1 = 8/3+8/3+2/3 = \frac{18}{3} = 6$. $\quad y\cdot u_2 = -8/3 + 16/3 + 1/3 = 9/3 = 3$ $\quad .\ \hat{y} = \text{proj}_W \vec{y} = \frac{6}{1}u_1 + \frac{3}{1}u_2 = 6\begin{bmatrix}2/3\\1/3\\2/3\end{bmatrix} + 3\begin{bmatrix}-2/3\\2/3\\1/3\end{bmatrix}$

$u_1\cdot u_1 = (2/3)^2+(1/3)^2+(2/3)^2 = \frac{9}{9} = 1$, $\quad u_2\cdot u_2 = (-2/3)^2+(2/3)^2+(1/3)^2 = 9/9 = 1$

$$= \begin{bmatrix}4\\2\\4\end{bmatrix} + \begin{bmatrix}-2\\2\\1\end{bmatrix} = \begin{bmatrix}2\\4\\5\end{bmatrix}$$

$$\begin{bmatrix}8/9 & -2/9 & 2/9\\-2/9 & 5/9 & 4/9\\2/9 & 4/9 & 5/9\end{bmatrix}\begin{bmatrix}4\\8\\1\end{bmatrix} = \begin{bmatrix}32/9 - 16/9 + 2/9\\-8/9 + 40/9 + 4/9\\8/9 + 32/9 + 5/9\end{bmatrix} = \begin{bmatrix}18/9\\36/9\\45/9\end{bmatrix} = \begin{bmatrix}2\\4\\5\end{bmatrix}$$

(5) (1)(a) $A^TA = \begin{bmatrix}1&1&1\\3&-1&1\end{bmatrix}\begin{bmatrix}1&3\\1&-1\\1&1\end{bmatrix} = \begin{bmatrix}1+1+1 & 3-1+1\\3-1+1 & 9+1+1\end{bmatrix} = \begin{bmatrix}3&3\\3&11\end{bmatrix}$, $A^T\vec{b} = \begin{bmatrix}1&1&1\\3&-1&1\end{bmatrix}\begin{bmatrix}5\\1\\0\end{bmatrix} = \begin{bmatrix}5+1+0\\15-1+0\end{bmatrix} = \begin{bmatrix}6\\14\end{bmatrix}$

(b) $\underline{\begin{bmatrix}3&3\\3&11\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}6\\4\end{bmatrix}}$ $\rightarrow \left[\begin{array}{cc|c}3&3&6\\3&11&14\end{array}\right] \xrightarrow[\frac{1}{3}R_1\to R_1]{-R_1+R_2\to R_2} \left[\begin{array}{cc|c}1&1&2\\0&8&8\end{array}\right] \xrightarrow{\frac{1}{8}R_2\to R_2} \left[\begin{array}{cc|c}1&1&2\\0&1&1\end{array}\right] \xrightarrow{-R_2+R_1\to R_1} \left[\begin{array}{cc|c}1&0&1\\0&1&1\end{array}\right] \rightarrow \underline{\hat{x} = \begin{bmatrix}1\\1\end{bmatrix}}$

(c) $\|\vec{b}-A\hat{x}\| \Rightarrow \begin{bmatrix}5\\1\\0\end{bmatrix} - \begin{bmatrix}1&3\\1&-1\\1&1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}5\\1\\0\end{bmatrix} - \begin{bmatrix}1+3\\1-1\\1+1\end{bmatrix} = \begin{bmatrix}5\\1\\0\end{bmatrix} - \begin{bmatrix}4\\0\\2\end{bmatrix} = \begin{bmatrix}1\\1\\-2\end{bmatrix} \rightarrow \left\|\begin{bmatrix}1\\1\\-2\end{bmatrix}\right\| = \sqrt{(1)^2+(1)^2+(-2)^2} = \underline{\sqrt{6}}$

(2)(a) $A^TA = \begin{bmatrix}1&-1&0&2\\-2&2&3&5\end{bmatrix}\begin{bmatrix}1&-2\\-1&2\\0&3\\2&5\end{bmatrix} = \begin{bmatrix}1+1+0+4 & -2-2+0+10\\-2-2+0+10 & 4+4+9+25\end{bmatrix} = \begin{bmatrix}6&6\\6&42\end{bmatrix}$, $A^T\vec{b} = \begin{bmatrix}1&-1&0&2\\-2&2&3&5\end{bmatrix}\begin{bmatrix}3\\1\\-4\\2\end{bmatrix} = \begin{bmatrix}3-1+0+4\\-6+2-12+10\end{bmatrix} = \begin{bmatrix}6\\-6\end{bmatrix}$

(b) $\underline{\begin{bmatrix}6&6\\6&42\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}6\\-6\end{bmatrix}}$ $\rightarrow \left[\begin{array}{cc|c}6&6&6\\6&42&-6\end{array}\right] \xrightarrow[\frac{1}{6}R_1\to R_1]{-R_1+R_2\to R_2} \left[\begin{array}{cc|c}1&1&1\\0&36&-12\end{array}\right] \xrightarrow{\frac{1}{36}R_2\to R_2} \left[\begin{array}{cc|c}1&1&1\\0&1&-1/3\end{array}\right] \xrightarrow{-R_2+R_1\to R_1} \left[\begin{array}{cc|c}1&0&4/3\\0&1&-1/3\end{array}\right] \rightarrow \underline{\hat{x} = \begin{bmatrix}4/3\\-1/3\end{bmatrix}}$

(c) $\|\vec{b}-A\hat{x}\| \Rightarrow \begin{bmatrix}3\\1\\-4\\2\end{bmatrix} - \begin{bmatrix}1&-2\\-1&2\\0&3\\2&5\end{bmatrix}\begin{bmatrix}4/3\\-1/3\end{bmatrix} = \begin{bmatrix}3\\1\\-4\\2\end{bmatrix} - \begin{bmatrix}4/3+2/3\\-4/3-2/3\\0-1\\8/3-5/3\end{bmatrix} = \begin{bmatrix}3\\1\\-4\\2\end{bmatrix} - \begin{bmatrix}2\\-2\\-1\\1\end{bmatrix} = \begin{bmatrix}1\\3\\-3\\1\end{bmatrix} \rightarrow \left\|\begin{bmatrix}1\\3\\-3\\1\end{bmatrix}\right\| = \sqrt{(1)^2+3^2+(-3)^2+(1)^2}$

$= \underline{\sqrt{20}} = 2\sqrt{5}$

Let $A = [\vec{a}_1, \vec{a}_2]$

(6) (1) Note: $\vec{a}_1 \cdot \vec{a}_2 = 0$ (a) $b \cdot a_1 = (3\cdot 1)+(-1\cdot -1)+(5\cdot 1) = 9$ $b\cdot a_2 = (3\cdot 2)+(-1\cdot 4)+(5\cdot 2) = 12$

$a_1\cdot a_1 = (1)^2+(-1)^2+(1)^2 = 3$ $a_2\cdot a_2 = 2^2+4^2+2^2 = 24$

$\hat{y} = \text{proj}_{\text{Col}A}\vec{b} = \frac{9}{3}\vec{a}_1 + \frac{12}{24}\vec{a}_2 = 3\begin{bmatrix}1\\-1\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}2\\4\\2\end{bmatrix} = \begin{bmatrix}3+1\\-3+2\\3+1\end{bmatrix} = \underline{\begin{bmatrix}4\\-1\\4\end{bmatrix}}$

(b) $\hat{x} = \begin{bmatrix}9/3\\12/24\end{bmatrix} = \underline{\begin{bmatrix}3\\1/2\end{bmatrix}}$

(2) Let $A = [\vec{a}_1\ \vec{a}_2\ \vec{a}_3]$ (a) $b\cdot a_1 = (2\cdot 1)+(5\cdot 1)+(6\cdot 0)+(6\cdot -1) = 1$ $b\cdot a_2 = (2\cdot 1)+(5\cdot 0)+(6\cdot 1)+(6\cdot 1) = 14$

$a_2\cdot a_2 = (1)^2+(1)^2+(1)^2 = 3$

Note: $a_1\cdot a_2 = 0$, $a_1\cdot a_3 = 0$, $a_2\cdot a_3 = 0$

$a_1\cdot a_1 = 1^2+1^2+0^2+(-1)^2 = 3$

$b\cdot a_3 = (2\cdot 0)+(5\cdot -1)+(6\cdot 1)+(6\cdot -1) = -5$ $a_3\cdot a_3 = 0^2+(-1)^2+(1)^2+(-1)^2 = 3$

$\hat{y} = \text{proj}_{\text{Col}A}\vec{b} = \frac{1}{3}\vec{a}_1 + \frac{14}{3}\vec{a}_2 + \frac{-5}{3}\vec{a}_3 = \frac{1}{3}\begin{bmatrix}1\\1\\0\\-1\end{bmatrix} + \frac{14}{3}\begin{bmatrix}1\\0\\1\\1\end{bmatrix} - \frac{5}{3}\begin{bmatrix}0\\-1\\1\\-1\end{bmatrix} = \begin{bmatrix}1/3+14/3+0\\1/3+0+5/3\\0+14/3-5/3\\-1/3+14/3+5/3\end{bmatrix} = \begin{bmatrix}15/3\\6/3\\9/3\\18/3\end{bmatrix} = \underline{\begin{bmatrix}5\\2\\3\\6\end{bmatrix}}$

(b) $\hat{x} = \begin{bmatrix}1/3\\14/3\\-5/3\end{bmatrix}$

(7) Assume $A\vec{x} = \vec{0}$, then since A^TA is invertible by IMT (invertible matrix Thm) if $A^TA\vec{x} = \vec{0}$, then $\vec{x} = \vec{0}$. (only solution, columns of A^TA are L.I.)

And since $\vec{x} = \vec{0}$ is the only solution for $A\vec{x} = \vec{0}$, then columns of A are L.I. by definition of Linear Independence.