HW 6

① $\begin{bmatrix} 2b+3c \\ -b \\ c \end{bmatrix} = \begin{bmatrix} 2b \\ -b \\ 0 \end{bmatrix} + \begin{bmatrix} 3c \\ 0 \\ c \end{bmatrix} = b\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \rightarrow$ let $\vec{u} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

thus $W = \text{Span}\{\vec{u}, \vec{v}\}$. By Theorem 1 of Vector Spaces W is a subspace of $\mathbb{R}^3$.

② $\begin{bmatrix} 2a-b \\ 3b-c \\ -a+3c \\ 3b \end{bmatrix} = \begin{bmatrix} 2a \\ 0 \\ -a \\ 0 \end{bmatrix} + \begin{bmatrix} -b \\ 3b \\ 0 \\ 3b \end{bmatrix} + \begin{bmatrix} 0 \\ -c \\ 3c \\ 0 \end{bmatrix} = a\underbrace{\begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \end{bmatrix}}_{\vec{u}} + b\underbrace{\begin{bmatrix} -1 \\ 3 \\ 0 \\ 3 \end{bmatrix}}_{\vec{v}} + c\underbrace{\begin{bmatrix} 0 \\ -1 \\ 3 \\ 0 \end{bmatrix}}_{\vec{w}}$ then $W = \text{Span}\{\vec{u}, \vec{v}, \vec{w}\}$

③ i) let $a=b=d=0$ then $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (the zero vector) is in H.

ii) let $\vec{u} = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} e & f \\ 0 & h \end{bmatrix}$ where each letter is a real number

then $\vec{u} + \vec{v} = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} + \begin{bmatrix} e & f \\ 0 & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ 0 & d+h \end{bmatrix}$ and notice that these sums are real number so $\vec{u} + \vec{v}$ is in H.

iii) let c be any real number then $c \cdot \vec{u} = c \cdot \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = \begin{bmatrix} ca & cb \\ 0 & cd \end{bmatrix}$ and each entry is a real number

thus $c \cdot \vec{u}$ is in H.

Therefore, H is a subspace of $M_{2\times 2}$.

④ a) We need $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ to be in W if it's a subspace, but note that $3r-2 = 3s+6$
$\rightarrow 3r-3s-t=2$

so if $r=s=t=0$ then $3\cdot 0 - 3\cdot 0 - 0 = 0 \neq 2$.

So, W does not contain the zero vector. so it cannot be a vector space.

④ cont'd

b) Note: $3a+b=c$ and $a+b+2c=2d$

$\rightarrow 3a+b-c=0$ $\rightarrow a+b+2c-2d=0$

If we put these equations into a matrix equation then we have:

$$\begin{bmatrix} 3 & 1 & -1 & 0 \\ 1 & 1 & 2 & -2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \text{this shows that } \vec{x} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \text{ is the null space of the given matrix.}$$

So, by Theorem 2 of this section, W is a vector space since it is a null space.

⑤ a)

$$\left[\begin{array}{cccc|c} 1 & 2 & 4 & 0 & 0 \\ 0 & 1 & 3 & -2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -2 & 4 & 0 \\ 0 & 1 & 3 & -2 & 0 \end{array}\right] \rightarrow \begin{array}{l} x_1 - 2x_3 + 4x_4 = 0 \\ x_2 + 3x_3 - 2x_4 = 0 \end{array} \rightarrow \begin{array}{l} x_1 = 2x_3 - 4x_4 \\ x_2 = -3x_3 + 2x_4 \end{array} \text{ So,}$$

$$\vec{x} = \begin{bmatrix} 2x_3 - 4x_4 \\ -3x_3 + 2x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3\begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} -4 \\ 2 \\ 0 \\ 1 \end{bmatrix} \quad \text{Nul}A = \text{Span}\left\{\begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ 0 \\ 1 \end{bmatrix}\right\}$$

b)

$$\begin{bmatrix} -1 & 3 & -4 & -3 & 1 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \left[\begin{array}{ccccc|c} -1 & 0 & 5 & -6 & 1 & 0 \\ 0 & 1 & -3 & 1 & 0 & 0 \\ 0 & \cdots & & & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & -5 & 6 & -1 & 0 \\ 0 & 1 & -3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \begin{array}{l} x_1 = 5x_3 - 6x_4 + x_5 \\ x_2 = 3x_3 - 4x_4 \end{array}$$

$$\vec{x} = \begin{bmatrix} 5x_3 - 6x_4 + x_5 \\ 3x_3 - 4x_4 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3\underbrace{\begin{bmatrix} 5 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\vec{u}} + x_4\underbrace{\begin{bmatrix} -6 \\ -4 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\vec{v}} + x_5\underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\vec{w}} \rightarrow \text{Nul}A = \text{Span}\{\vec{u}, \vec{v}, \vec{w}\}$$

⑥ This set looks like:

$$r\begin{bmatrix} 0 \\ 1 \\ 3 \\ 2 \end{bmatrix} + s\begin{bmatrix} 2 \\ -1 \\ 1 \\ -1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} \rightarrow A = \begin{bmatrix} 0 & 2 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 0 \\ 2 & -1 & -1 \end{bmatrix}$$

(7) Let $\vec{a}_4$ be the 4th column of A so $\vec{a}_4 = \begin{bmatrix} -2 \\ -2 \\ 0 \\ -2 \end{bmatrix}$ then $-1 \cdot \vec{a}_4 = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 2 \end{bmatrix} = \vec{w}$

Col A ↳ Let $A = [\vec{a}_1\ \vec{a}_2\ \vec{a}_3\ \vec{a}_4]$

therefore, $\vec{w} = 0 \cdot \vec{a}_1 + 0 \cdot \vec{a}_2 + 0 \cdot \vec{a}_3 - 1\vec{a}_4$ → $\underline{\vec{w} \text{ is in Col} A.}$

Nul A → Does $A \cdot \vec{w} = \vec{0}$?

$$A\vec{w} = \begin{bmatrix} (10\cdot 2)+(-8\cdot 2)+(-2\cdot 0)+(-2\cdot 2) \\ (0\cdot 2)+(2\cdot 2)+(2\cdot 0)+(-2\cdot 2) \\ (1\cdot 2)+(-1\cdot 2)+(6\cdot 0)+(0\cdot 2) \\ (1\cdot 2)+(1\cdot 2)+(0\cdot 0)+(-2\cdot 2) \end{bmatrix} = \begin{bmatrix} 20-16-4 \\ 4-4 \\ 2-2 \\ 2+2-4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \underline{\vec{w} \text{ is in Nul} A.}$$

(8) a) Consider $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, so $\det A = 1 \cdot 1 \cdot 1 = 1 \neq 0$ → A is invertible, thus columns of A are $\underline{L.I.}$

b) Consider $A = \begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -1 \\ -3 & -4 & 1 \end{bmatrix}$ then $\det A = 0 + 1 \cdot \begin{vmatrix} 1 & -2 \\ -3 & 1 \end{vmatrix} + (-1)(-1) \cdot \begin{vmatrix} 1 & 3 \\ -3 & -4 \end{vmatrix}$

(along 2nd row)

$= 1 \cdot (1-6) + 1 \cdot (-4-(-9))$

$= -5 + 5 = 0$

det A = 0 → so A is not invertible

thus, the columns of A are $\underline{\text{not } L.I.}$

[Also not $\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} - 1\begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$ → last vector is a linear combo of 1st two]

(9) Let $A = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3\ \vec{v}_4\ \vec{v}_5]$ → we are looking for a basis for Col A.

$$A \cdot \begin{bmatrix} 1 & 0 & 2 & 2 & 3 \\ 0 & 1 & -2 & -1 & -1 \\ -2 & 2 & -8 & 10 & -6 \\ 3 & 3 & 0 & 3 & 9 \end{bmatrix} \xrightarrow[-3R_1+R_4 \to R_4]{2R_1+R_3 \to R_3} \begin{bmatrix} 1 & 0 & 2 & 2 & 3 \\ 0 & 1 & -2 & -1 & -1 \\ 0 & 2 & -4 & 14 & 0 \\ 0 & 3 & -6 & -3 & 0 \end{bmatrix} \xrightarrow[-\frac{1}{3}R_4 \to R_4]{-\frac{1}{2}R_3 \to R_3} \begin{bmatrix} & & 〃 & & \\ 0 & 1 & -2 & -1 & -1 \\ 0 & -1 & 2 & -7 & 0 \\ 0 & -1 & 2 & 1 & 0 \end{bmatrix} \xrightarrow[R_2+R_4 \to R_4]{R_2+R_3 \to R_3} \begin{bmatrix} \underline{1} & 0 & 2 & 2 & 3 \\ 0 & \underline{1} & -2 & -1 & -1 \\ 0 & 0 & 0 & \underline{-8} & -1 \\ 0 & 0 & 0 & 0 & \underline{-1} \end{bmatrix}$$

The underlines are the pivots. By Theorem 6 of this section the pivot columns form a basis for Col A and the pivot columns are $\boxed{S = \{\vec{v}_1, \vec{v}_2, \vec{v}_4, \vec{v}_5\}}$

10 a) False, vectors in $\mathbb{R}^2$ have 2 entries vs 3 entries in $\mathbb{R}^3$.

b) True

c) True

d) False, only true for the zero vector

e) False, also need linear independence of the set of vectors,

f) True

g) False, it would be true if $f(t)=0$ for any t.

h) True