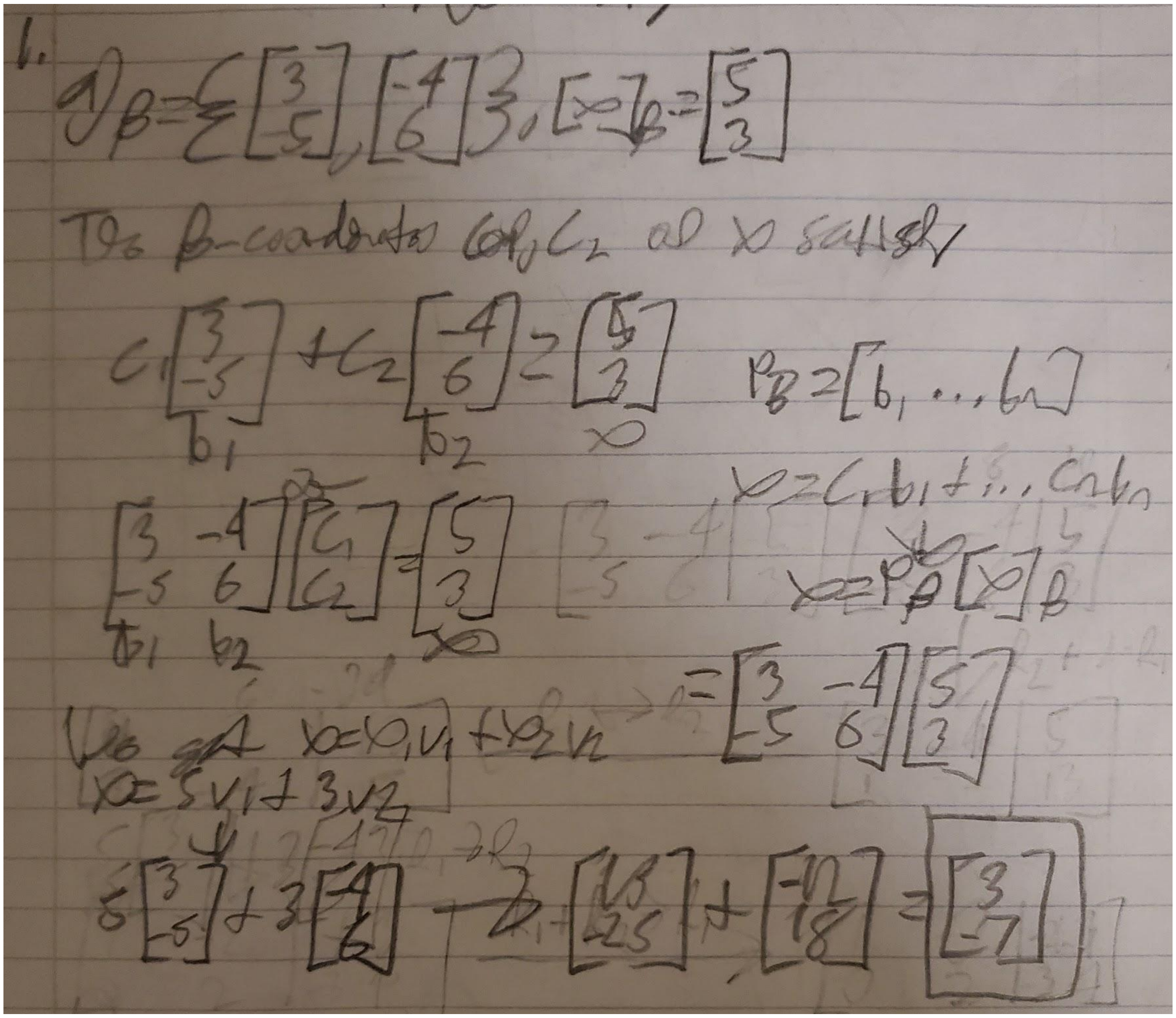
1. Find the vector $\mathbf{x}$ determined by the given coordinate vector $[x]_{\mathcal{B}}$ and the given basis $\mathcal{B}$.

(a) $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ -5 \end{bmatrix}, \begin{bmatrix} -4 \\ 6 \end{bmatrix} \right\}$, $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$



1.

a) $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ -5 \end{bmatrix}, \begin{bmatrix} -4 \\ 6 \end{bmatrix} \right\}, [x]_{\mathcal{B}} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

The $\mathcal{B}$-coordinates c_1, c_2 of x satisfy

$$\underbrace{c_1\begin{bmatrix} 3 \\ -5 \end{bmatrix}}_{b_1} + \underbrace{c_2\begin{bmatrix} -4 \\ 6 \end{bmatrix}}_{b_2} = \underbrace{\begin{bmatrix} 5 \\ 3 \end{bmatrix}}_{x} \qquad P_{\mathcal{B}} = [b_1, \dots, b_n]$$

$$x = c_1 b_1 + \dots + c_n b_n$$

$$\begin{bmatrix} 3 & -4 \\ -5 & 6 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 5 \\ 3 \end{bmatrix}}_{x} \qquad x = P_{\mathcal{B}}[x]_{\mathcal{B}}$$

$$= \begin{bmatrix} 3 & -4 \\ -5 & 6 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$x = x_1 v_1 + x_2 v_2$

$x = 5v_1 + 3v_2$

$$5\begin{bmatrix} 3 \\ -5 \end{bmatrix} + 3\begin{bmatrix} -4 \\ 6 \end{bmatrix} \to \begin{bmatrix} 15 \\ -25 \end{bmatrix} + \begin{bmatrix} -12 \\ 18 \end{bmatrix} = \boxed{\begin{bmatrix} 3 \\ -7 \end{bmatrix}}$$

(b) $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} \right\}$, $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$

b) $\beta = \left\{ \begin{bmatrix}1\\-2\\3\end{bmatrix}, \begin{bmatrix}5\\0\\-2\end{bmatrix}, \begin{bmatrix}4\\-3\\0\end{bmatrix} \right\}, [x]_\beta = \begin{bmatrix}1\\0\\-2\end{bmatrix}$

~~$x = x_1 V_1 + x_2 V_2 + x_3 V_3$~~

$x = 1 V_1 + \cancel{0 V_2} + (-2) V_3$

$= 1 \cdot \begin{bmatrix}1\\-2\\3\end{bmatrix} + \cancel{0 \begin{bmatrix}5\\0\\-2\end{bmatrix}} - 2 \cdot \begin{bmatrix}4\\-3\\0\end{bmatrix} \rightarrow \begin{bmatrix}1\\-2\\3\end{bmatrix} + \begin{bmatrix}8\\6\\0\end{bmatrix} = \boxed{\begin{bmatrix}-7\\4\\3\end{bmatrix}}$

2. Find the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ of $\mathbf{x}$ relative to the given basis $\mathcal{B} = \{\mathbf{b}_1, ..., \mathbf{b}_n\}$.

(a) $\mathbf{b}_1 = \begin{bmatrix}1\\-4\end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix}2\\-3\end{bmatrix}, \mathbf{x} = \begin{bmatrix}-1\\-6\end{bmatrix}$

2. a) $b_1 = \begin{bmatrix}1\\-4\end{bmatrix}, b_2 = \begin{bmatrix}2\\-3\end{bmatrix}, x = \begin{bmatrix}-1\\-6\end{bmatrix}$

$c_1 \begin{bmatrix}1\\-4\end{bmatrix} + c_2 \begin{bmatrix}2\\-3\end{bmatrix} = \begin{bmatrix}-1\\-6\end{bmatrix}$

$[b_1 \; b_2 \mid x]$

$\left[\begin{array}{cc|c}1 & 2 & -1\\-4 & -3 & -6\end{array}\right] \xrightarrow{R_2 + 4R_1 \to R_2} \left[\begin{array}{cc|c}1 & 2 & -1\\0 & 5 & -10\end{array}\right] \xrightarrow{\frac{1}{5}R_2} \left[\begin{array}{cc|c}1 & 2 & -1\\0 & 1 & -2\end{array}\right]$

$\xrightarrow{R_1 - 2 \cdot R_2 \to R_1} \left[\begin{array}{cc|c}1 & 0 & 3\\0 & 1 & -2\end{array}\right]$ $\quad [x]_\beta = \boxed{\begin{bmatrix}3\\-2\end{bmatrix}}$

(b) $\mathbf{b}_1 = \begin{bmatrix}1\\1\\3\end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix}2\\0\\8\end{bmatrix}, \mathbf{b}_3 = \begin{bmatrix}1\\-1\\3\end{bmatrix}, \mathbf{x} = \begin{bmatrix}0\\0\\-2\end{bmatrix}$

b) $b_1 = \begin{bmatrix}1\\1\\3\end{bmatrix}, b_2 = \begin{bmatrix}2\\0\\8\end{bmatrix}, b_3 = \begin{bmatrix}1\\-1\\3\end{bmatrix}, x = \begin{bmatrix}0\\0\\-2\end{bmatrix}$

$$c_1\begin{bmatrix}1\\1\\3\end{bmatrix} + c_2\begin{bmatrix}2\\0\\8\end{bmatrix} + c_3\begin{bmatrix}1\\-1\\3\end{bmatrix} = \begin{bmatrix}0\\0\\-2\end{bmatrix}$$

$[b_1 \ b_2 \ b_3 \mid x]$

$-6 \quad -3$

$$\left[\begin{array}{ccc|c}1&2&1&0\\1&0&-1&0\\3&8&3&-2\end{array}\right] \xrightarrow{R_2 - R_1 \to R_2} \left[\begin{array}{ccc|c}1&2&1&0\\0&-2&-2&0\\3&8&3&-2\end{array}\right]$$

$$\xrightarrow{R_3 - 3\cdot R_1 \to R_3} \left[\begin{array}{ccc|c}1&2&1&0\\0&-2&-2&0\\0&2&0&-2\end{array}\right] \xrightarrow{-\frac{1}{2}R_2} \left[\begin{array}{ccc|c}1&2&1&0\\0&1&1&0\\0&2&0&-2\end{array}\right]$$

$$\xrightarrow{R_3 - 2\cdot R_2 \to R_3} \left[\begin{array}{ccc|c}1&2&1&0\\0&1&1&0\\0&0&-2&-2\end{array}\right] \xrightarrow{R_1 - 2\cdot R_2 \to R_1} \left[\begin{array}{ccc|c}1&0&-1&0\\0&1&1&0\\0&0&-2&-2\end{array}\right]$$

$$\xrightarrow{-\frac{1}{2}R_3} \left[\begin{array}{ccc|c}1&0&-1&0\\0&1&1&0\\0&0&1&1\end{array}\right] \xrightarrow{R_2 - R_3 \to R_2} \left[\begin{array}{ccc|c}1&0&-1&0\\0&1&0&-1\\0&0&1&1\end{array}\right]$$

cont'd

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{R_1 + R_3 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$[x]_B = \boxed{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}$$

3. Use an inverse matrix to find $[\mathbf{x}]_{\mathcal{B}}$ for the given $\mathbf{x}$ and $\mathcal{B}$.

(a) $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \end{bmatrix} \right\}$, $\mathbf{x} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$

3. a) $B = \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \end{bmatrix} \right\}, x = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$ $\quad P_B^{-1} x = [x]_B$

$$\left[\begin{array}{cc|cc} 1 & -3 & 1 & 0 \\ -2 & 5 & 0 & 1 \end{array}\right] \xrightarrow{R_2 + 2 \cdot R_1 \to R_2} \left[\begin{array}{cc|cc} 1 & -3 & 1 & 0 \\ 0 & -1 & 2 & 1 \end{array}\right] \xrightarrow{-1 \cdot R_2} \left[\begin{array}{cc|cc} 1 & -3 & 1 & 0 \\ 0 & 1 & -2 & -1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & -3 & 1 & 0 \\ 0 & 1 & -2 & -1 \end{array}\right] \xrightarrow{R_1 + 3 \cdot R_2 \to R_1} \left[\begin{array}{cc|cc} 1 & 0 & -5 & -3 \\ 0 & 1 & -2 & -1 \end{array}\right]$$

$$[x]_B = P_B^{-1} x = \begin{bmatrix} 1 & -3 \\ -2 & 5 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 2 \\ -5 \end{bmatrix} = \begin{bmatrix} -5 & -3 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

$$2 \begin{bmatrix} -5 \\ -2 \end{bmatrix} - 5 \begin{bmatrix} -3 \\ -1 \end{bmatrix} \Rightarrow \begin{bmatrix} -10 \\ -4 \end{bmatrix} + \begin{bmatrix} 15 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

$$[x]_B = \boxed{\begin{bmatrix} 5 \\ 1 \end{bmatrix}}$$

(b) $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$, $\mathbf{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

b) $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}, x = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ Conf'd

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_2 + R_1 \to R_2} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right]$$

$$\downarrow R_1 - 2R_2 \to R_1$$

$$\left[\begin{array}{cc|cc} 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 1 \end{array}\right]$$

$$[x]_\mathcal{B} = P_\mathcal{B}^{-1} x = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$2\begin{bmatrix} -1 \\ 1 \end{bmatrix} + 3\begin{bmatrix} -2 \\ 1 \end{bmatrix} \to \begin{bmatrix} -2 \\ 2 \end{bmatrix} + \begin{bmatrix} -6 \\ 3 \end{bmatrix} = \begin{bmatrix} -8 \\ 5 \end{bmatrix}$$

$$[x]_\mathcal{B} = \boxed{\begin{bmatrix} -8 \\ 5 \end{bmatrix}}$$

4. The set $\mathcal{B} = \{1 - t^2, t - t^2, 2 - t + t^2\}$ is a basis for $\mathbb{P}_2$. Find the coordinate vector of $\mathbf{p}(t) = 1 + 3t - 6t^2$ relative to $\mathcal{B}$.

4. $\beta = \{1-t^2, t-t^2, 2-t+t^2\}$ basis for P_2

Find coord. vect. of $p(t) = 1+3t-6t^2$ relative to β.

Find $c_1, c_2,$ and c_3 such that

$c_1(1-t^2) + c_2(t-t^2) + c_3(2-t+t^2) = p(t) = 1+3t-6t^2$

$c_1 - c_1t^2 + c_2t - c_2t^2 + 2c_3 - c_3t + c_3t^2 = 1 + 3t - 6t^2$

Equating the coefficients gives us

$c_1 + \cancel{0 \cdot c_2} + 2c_3 = 1$

$\cancel{0 \cdot c_1} + c_2 - c_3 = 3$

$-c_1 - c_2 + c_3 = -6$

$$\begin{bmatrix} 1 & 0 & 2 & | & 1 \\ 0 & 1 & -1 & | & 3 \\ -1 & -1 & 1 & | & -6 \end{bmatrix} \xrightarrow{R_3 + R_1 \to R_3} \begin{bmatrix} 1 & 0 & 2 & | & 1 \\ 0 & 1 & -1 & | & 3 \\ 0 & -1 & 3 & | & -5 \end{bmatrix}$$

$$\xrightarrow{R_3 + R_2 \to R_3} \begin{bmatrix} 1 & 0 & 2 & | & 1 \\ 0 & 1 & -1 & | & 3 \\ 0 & 0 & 2 & | & -2 \end{bmatrix}$$

$-2/4 = -1/2$

$6/2 + 1/2 = 7/2$

$$\xrightarrow{\frac{1}{2} \cdot R_3} \begin{bmatrix} 1 & 0 & 2 & | & 1 \\ 0 & 1 & -1 & | & 3 \\ 0 & 0 & 1 & | & -1 \end{bmatrix} \xrightarrow{R_2 + R_3 \to R_2} \begin{bmatrix} 1 & 0 & 2 & | & 1 \\ 0 & 1 & 0 & | & 2 \\ 0 & 0 & 1 & | & -1 \end{bmatrix} 2$$

$$\xrightarrow{R_1 - 2R_3 \to R_1} \begin{bmatrix} 1 & 0 & 0 & | & 3 \\ 0 & 1 & 0 & | & 2 \\ 0 & 0 & 1 & | & -1 \end{bmatrix}$$

$$[p]_\beta = \boxed{\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}}$$

5. For the given subspace, H, (i) find a basis for the subspace, and (ii) state the dimension.

(a) $H = \left\{ \begin{bmatrix} 2c \\ a-b \\ b-3c \\ a+2b \end{bmatrix} : a, b, c \text{ in } \mathbb{R} \right\}$

5.

(a) $H = \left\{ \begin{bmatrix} 2c \\ a-b \\ b-3c \\ a+2b \end{bmatrix} ; a,b,c \text{ in } \mathbb{R} \right\}$

(i)

$0a + 0b + 2c$
$1a - 1b + 0c$
$0a + 1b - 3c$
$1a + 2b + 0c$

$\rightarrow V_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, V_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 2 \end{bmatrix}, V_3 = \begin{bmatrix} 2 \\ 0 \\ -3 \\ 0 \end{bmatrix}$

Section 4.3 Theorem 4!

$V_1 \neq 0$, V_2 not a multiple of V_1, and V_3 is not a linear combination of V_1 and V_2 $\rightarrow$ linearly independent!

↳ Thus, $\{V_1, V_2, V_3\}$ is a basis for H!

$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -3 \\ 0 \end{bmatrix}$

(ii) From above we know that there are 3 vectors in the basis, so the dimension of the subspace is 3!

$\boxed{\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -3 \\ 0 \end{bmatrix} \right\}, \text{ Dimension is } 3!}$

(b) $H = \{(a, b, c) : a - 3b + c = 0, b - 2c = 0, 2b - c = 0\}$

b) $H = \{(a,b,c) : a-3b+c=0,\ b-2c=0,\ 2b-c=0\}$

① $a-3b+c=0$

$b-2c=0$

$2b-c=0$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 2 & -1 & 0 \end{array}\right] \xrightarrow{R_1+3\cdot R_2 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & -5 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 2 & -1 & 0 \end{array}\right]$$

4

$$\xrightarrow{R_3-2\cdot R_2 \to R_3} \left[\begin{array}{ccc|c} 1 & 0 & -5 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right] \xrightarrow{\frac{1}{3}R_3} \left[\begin{array}{ccc|c} 1 & 0 & -5 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_2+2\cdot R_3 \to R_2} \left[\begin{array}{ccc|c} 1 & 0 & -5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_1+5\cdot R_3 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Homogeneous →

Only sol is trivial

↓

Only element of sol is zero vector

↓

$\boxed{\text{No basis. Dimension is } 0!}$

6. Find the dimension of the subspace spanned by the given vectors.

$$\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -6 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \\ 5 \end{bmatrix}$$

6. $\begin{bmatrix}1\\-2\\0\end{bmatrix}, \begin{bmatrix}-3\\-6\\0\end{bmatrix}, \begin{bmatrix}-2\\3\\5\end{bmatrix}, \begin{bmatrix}-3\\5\\5\end{bmatrix}$

$$\longrightarrow \begin{array}{cccc} & -6 & -4 & -6 \end{array}$$
$$\begin{bmatrix}1 & -3 & -2 & -3\\ -2 & -6 & 3 & 5\\ 0 & 0 & 5 & 5\end{bmatrix}$$

$$\xrightarrow{R_2+2\cdot R_1 \to R_2} \begin{bmatrix}1 & -3 & -2 & -3\\ 0 & -12 & -1 & -1\\ 0 & 0 & 5 & 5\end{bmatrix} \xrightarrow{\frac{1}{5}R_3} \begin{bmatrix}1 & -3 & -2 & -3\\ 0 & -12 & -1 & -1\\ 0 & 0 & 1 & 1\end{bmatrix}$$

$$\xrightarrow{R_2+R_3 \to R_2} \begin{bmatrix}1 & -3 & -2 & -3\\ 0 & -12 & 0 & 0\\ 0 & 0 & 1 & 1\end{bmatrix} \xrightarrow{-\frac{1}{12}R_2} \begin{bmatrix}1 & -3 & -2 & -3\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 1\end{bmatrix}$$

$$\xrightarrow{R_1+3\cdot R_2 \to R_1} \begin{bmatrix}1 & 0 & -2 & -3\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 1\end{bmatrix} \xrightarrow{R_1+2\cdot R_3 \to R_1} \begin{bmatrix}1 & 0 & 0 & -1\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 1\end{bmatrix}$$

3 Pivot cols, so dimension of subspace spanned by vectors is 3

$\boxed{\text{Dimension is 3!}}$

7. Determine the dimensions of Nul A and Col A for the given matrix: $A = \begin{bmatrix}1 & -6 & 9 & 0 & -2\\ 0 & 1 & 2 & -4 & 5\\ 0 & 0 & 0 & 5 & 1\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$

DEFINITION

A rectangular matrix is in **echelon form** (or **row echelon form**) if it has the following three properties:

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or **reduced row echelon form**):

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

EXAMPLE 1 The following matrices are in echelon form. The leading entries ($\blacksquare$) may have any nonzero value; the starred entries ($*$) may have any value (including zero).

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

The following matrices are in reduced echelon form because the leading entries are 1's, and there are 0's below *and above* each leading 1.

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

7. $A = \begin{bmatrix} 1 & -6 & 9 & 0 & -2 \\ 0 & 1 & 2 & -4 & 5 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

In echelon form?
1. ✓
2. ✓
3. ✓

Seeing how matrix A is already in vector form, we can just look at the pivot columns to figure out the rest.

3 pivot columns, so dimension of Col A is 3

2 columns without pivots, so $Ax = 0$ would have 2 free variables, thus dimension of Nul A is 2.

Free vars:
x_3, x_5

$\boxed{\begin{array}{l}\text{Dimension of Nul A is 2} \\ \text{Dimension of Col A is 3}\end{array}}$

8. The first four Hermite polynomials are $1, 2t, -2+4t^2$, and $-12t+8t^3$. These polynomials arise naturally in the study of differential equations in the fields of signal processing, probability, physics, and systems theory. Show that the first four Hermite polynomials form a basis of $\mathbb{P}_3$.

EXAMPLE 1 The standard basis for $\mathbb{R}^n$ contains n vectors, so $\dim \mathbb{R}^n = n$. The standard polynomial basis $\{1, t, t^2\}$ shows that $\dim \mathbb{P}_2 = 3$. In general, $\dim \mathbb{P}_n = n+1$. The space $\mathbb{P}$ of all polynomials is infinite-dimensional (Exercise 27).

THEOREM 12

The Basis Theorem

Let V be a p-dimensional vector space, $p \geq 1$. Any linearly independent set of exactly p elements in V is automatically a basis for V. Any set of exactly p elements that spans V is automatically a basis for V.

8. $1, 2t, -2+4t^2$, and $-12t+8t^3$

We can start by writing these polynomials as vectors

$$1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad 2t = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \quad -2+4t^2 = \begin{bmatrix} -2 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \quad -12t+8t^3 = \begin{bmatrix} 0 \\ -12 \\ 0 \\ 8 \end{bmatrix}$$

In echelon form?
1. ✓
2. ✓
3. ✓

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & 0 & -12 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix} \xrightarrow{\frac{1}{2} \cdot R_1} \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

4 pivot cols

Let's check in reduced echelon form!

Contd

(cont'd) 1*

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix} \xrightarrow{\frac{1}{4}\cdot R_3} \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

$$\downarrow \frac{1}{8}\cdot R_4$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xleftarrow{R_1+2\cdot R_3 \to R_1} \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\downarrow R_2+6\cdot R_4 \to R_2$$

$\dim(P_3) = 4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to 4 \text{ pivot columns}$$

↓ dimension of Col A is 4 → vectors span P^3

↓

identity matrix, so vectors independent!

Since these vectors are L.I., the polynomials are L.I. in P_3

The standard basis for $P_3 = \{1, t, t^2, t^3\}$

$\dim P_3 = 4$

$\boxed{\text{Hence, the polynomials are L.I. and hence they form a basis for } P_3}$

we got this from Basis Theorem!

9. Let $\mathcal{B}$ be the basis of $\mathbb{P}_3$ consisting of the Hermite polynomials in Question 7, and let $\mathbf{p}(t) = -1 + 8t^2 + 8t^3$. Find the coordinate vector of $\mathbf{p}$ relative to $\mathcal{B}$.

a. B: $1, 2t, -2+4t^2, -12t+8t^3$ | $p(t) = -1+8t^2+8t^3$

The coordinates of $p(t) = -1+8t^2+8t^3$ with respect to B satisfy

$$c_1(1) + c_2(2t) + c_3(-2+4t^2) + c_4(-12t+8t^3) = -1+8t^2+8t^3$$

Equating coefficients

$$c_1 - 2c_3 = -1$$

$$2c_2 - 12c_4 = 0$$

$$4c_3 = 8$$

$$8c_4 = 8$$

$$\frac{4c_3}{4} = \frac{8}{4} \quad c_3 = 2$$

$$\frac{8c_4}{8} = \frac{8}{8} \quad c_4 = 1$$

$$c_1 - 2(2) = -1$$

$$c_1 - 4 = -1 \quad c_1 = 3$$

$$+4 \quad +4$$

$$2c_2 - 12(1) = 0$$

$$2c_2 - 12 = 0 \qquad \frac{2c_2}{2} = \frac{12}{2} \quad c_2 = 6$$

$$+12 \quad +12$$

So, we get

$$\boxed{[p]_B = \begin{bmatrix} 3 \\ 6 \\ 2 \\ 1 \end{bmatrix}}$$

10. Assume that matrix A is row equivalent to B. Without calculations, list rank A and dim Nul A. Then find bases for Col A, Row A, and Nul A.

$$A = \begin{bmatrix} 1 & 3 & 4 & -1 & 2 \\ 2 & 6 & 6 & 0 & -3 \\ 3 & 9 & 3 & 6 & -3 \\ 3 & 9 & 0 & 9 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 & 4 & -1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

10. Assume matrix A is row equivalent to B.

$$A = \begin{bmatrix} 1 & 3 & 4 & -1 & 2 \\ 2 & 6 & 6 & 0 & -3 \\ 3 & 9 & 3 & 6 & -3 \\ 3 & 9 & 0 & 9 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 & 4 & -1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank of A is dimension of column space of A

↓

Assuming A is row equivalent to B, we see there are 3 pivot cols, so dim. of Col A is 3

↓

Rank of 3!

dim Nul A → 5 cols, but 3 are pivot cols so 2 cols have no pivots, giving us that dim. of Nul A is 2!

↓

dim. Nul A is 2!

Basis for A?

Looking at B, the pivots are in cols 1, 3, and 5, so cols 1, 3, and 5 would form a basis for Col A!

Basis for Col A: $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -3 \\ 0 \end{bmatrix} \right\}$

Cont'd

Cont'd

Row A? → Theorem 13!
According to Theorem 13, the nonzero rows of B form a basis for the row space of A as well as that for B!

Basis for Row A: $\{(1,3,4,-1,2),(0,0,1,-1,1),(0,0,0,0,-5)\}$

Nul A?
We need RREF! We can further reduce B

$$\begin{bmatrix} 1 & 3 & 4 & -1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 - 4\cdot R_2 \to R_1} \begin{bmatrix} 1 & 3 & 0 & 3 & -2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$+4 \quad -4$

$$\downarrow -\frac{1}{5}\cdot R_3$$

$$\begin{bmatrix} 1 & 3 & 0 & 3 & -2 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xleftarrow{R_2 - R_3 \to R_2} \begin{bmatrix} 1 & 3 & 0 & 3 & -2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\downarrow R_1 + 2R_3 \to R_1$$

$$\begin{bmatrix} 1 & 3 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{aligned} 1x_1 + 3x_2 + 3x_4 &= 0 \\ 1x_3 - 1x_4 &= 0 \\ 1x_5 &= 0 \end{aligned}$$

$\downarrow$

$x_1 = -3x_2 - 3x_4$
$x_3 = x_4$
$x_5 = 0$
x_2 is free
x_4 is free

Cont'd ★★

$x_1 = -3x_2 - 3x_4$

$x_3 = x_4$

$x_5 = 0$

x_2 is free

x_4 is free

$$\Rightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -3x_2 - 3x_4 \\ x_2 \\ x_4 \\ x_4 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

Basis for Null A: $\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$

$$\boxed{\begin{array}{l} \text{Rank } A = 3,\ \dim \text{Nul } A = 2 \\[1em] \text{Basis for col } A: \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -3 \\ 0 \end{bmatrix} \right\} \\[1em] \text{Basis for row } A: \{(1,3,4,-1,2), (0,0,1,-1,1), (0,0,0,0,-5)\} \\[1em] \text{Basis for Nul } A: \left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\} \end{array}}$$

11. If a 7×5 matrix A has rank 2, find dim Nul A, dim Row A, and rank A^T and briefly explain your reasoning for your answers.

THEOREM 14 **The Rank Theorem**

The dimensions of the column space and the row space of an $m \times n$ matrix A are equal. This common dimension, the rank of A, also equals the number of pivot positions in A and satisfies the equation

$$\text{rank } A + \dim \text{Nul } A = n$$

DEFINITION The **rank** of A is the dimension of the column space of A.

Since Row A is the same as Col A^T, the dimension of the row space of A is the rank of A^T. The dimension of the null space is sometimes called the **nullity** of A, though we will not use this term.

An alert reader may have already discovered part or all of the next theorem while working the exercises in Section 4.5 or reading Example 2 above.

PROOF By Theorem 6 in Section 4.3, rank A is the number of pivot columns in A. Equivalently, rank A is the number of pivot positions in an echelon form B of A. Furthermore, since B has a nonzero row for each pivot, and since these rows form a basis for the row space of A, the rank of A is also the dimension of the row space.

From Section 4.5, the dimension of Nul A equals the number of free variables in the equation $A\mathbf{x} = \mathbf{0}$. Expressed another way, the dimension of Nul A is the number of columns of A that are *not* pivot columns. (It is the number of these columns, not the columns themselves, that is related to Nul A.) Obviously,

$$\begin{Bmatrix} \text{number of} \\ \text{pivot columns} \end{Bmatrix} + \begin{Bmatrix} \text{number of} \\ \text{nonpivot columns} \end{Bmatrix} = \begin{Bmatrix} \text{number of} \\ \text{columns} \end{Bmatrix}$$

11. $2\{\left[\quad\right]$ ← Matrix A   Rank 2
$\downarrow$
dim of Col A is 2
5   dim. Nul A = 5 − 2 = 3

dim. Row A is the same as rank A as rank A is dim. of col. space of A and according to the Rank Theorem (Thm. 14), rank A equals the number of pivot positions in A.

dim. Row A = 2

rank A^T = dim Col A^T = dim. Row A = 2

dim Col A = dim Row A = Rank A

rank A + dim Nul A = n
2   $\downarrow$ 5
so,
dim. Nul A = 3

Dim. Nul A = 3 as rank A is 2, and there are 5 columns, so there would be 3 nonpivot positions. Dim. Row A = 2 as the dimension of row A is the same as rank A. Rank A^T is 2 as rank A^T is the same as dim. Row A and dim. Row A is the same as rank A!

*My handwriting is messy, so I am going to type the answer here for your convenience.
The dimension of Nul A is 3 as rank A is 2, and there are 5 columns, so after subtracting the number of pivot columns (*obtained from rank A) from 5, there would be 3 non-pivot positions, giving us that the dimension of Nul A is 3. The dimension of Row A is 2 as the dimension of Row A is the same as rank A (*see excerpt from book above). Finally, we have that rank A^T is 2 as it is the same as the dimension of Row A and the dimension of Row A is the same as rank A (*again, see the excerpts from the book posted above).

12. If the null space of a 4×6 matrix A is 3−dimensional, what is the dimension of the column space of A? Is Col $A = \mathbb{R}^3$? Why or why not?

THEOREM 14 **The Rank Theorem**

The dimensions of the column space and the row space of an $m \times n$ matrix A are equal. This common dimension, the rank of A, also equals the number of pivot positions in A and satisfies the equation

$$\text{rank } A + \dim \text{Nul } A = n$$

The Row Space

If A is an $m \times n$ matrix, each row of A has n entries and thus can be identified with a vector in $\mathbb{R}^n$. The set of all linear combinations of the row vectors is called the **row space** of A and is denoted by Row A. Each row has n entries, so Row A is a subspace of $\mathbb{R}^n$. Since the rows of A are identified with the columns of A^T, we could also write Col A^T in place of Row A.

12. $4\left\{\underbrace{\begin{bmatrix} \quad & \quad \end{bmatrix}}_{6}\right.$ dim. of Nul $A = 3$

dim of Col A?

$\Downarrow$

dim of Col $A = 6 - 3 = 3$

Is Col $A = \mathbb{R}^3$?

dim of Col $A = 3$, so Rank $A = 3$

Col A is not in $\mathbb{R}^3$. In this case col A is a 3 dimensional subspace of $\mathbb{R}^4$ because A has 4 rows.

dim Col A = dim Row A = Rank A

$\boxed{\text{The dim. Col } A = 3 \text{ as the dim. Nul } A = 3 \text{ and the total \# of cols. (6) minus the \# of non-pivot cols gives us 3 pivot cols. No, Col } A \neq \mathbb{R}^3 \text{ as Col } A \text{ is a 3-dimensional subspace of } \mathbb{R}^4 \text{ because } A \text{ has 4 rows.}}$

*For the answer, I wrote:

The dimension of Col A is 3 as the dimension of Nul A is 3 and the total number of columns (*6) minus the number of non-pivot columns gives us 3 pivot columns. No, Col A ≠ R^3 as Col A is a 3-dimensional subspace of R^4 because A has 4 rows.