HW 8

(1) a) $\begin{bmatrix} -1-3 & 4 \\ 6 & 9-3 \end{bmatrix} = \begin{bmatrix} -4 & 4 \\ 6 & 6 \end{bmatrix} = A - \lambda I$ and $\det(A-\lambda I) = 0$, so yes.

b) $\begin{bmatrix} 3 & 6 & 7 \\ 3 & 2 & 7 \\ 5 & 6 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 3-12+14 \\ 3-4+14 \\ 5-12+8 \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \\ 1 \end{bmatrix} \neq \lambda \vec{x}$, so no.

c) $\begin{bmatrix} 4-1 & -2 & 3 \\ 0 & -1-1 & 3 \\ -1 & 2 & -2-1 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 3 \\ 0 & -2 & 3 \\ -1 & 2 & -3 \end{bmatrix} = A - 1I$

$\left[\det(A-1I) = 3 \cdot \begin{vmatrix} -2 & 3 \\ 2 & -3 \end{vmatrix} - 1 \cdot \begin{vmatrix} -2 & 3 \\ -2 & 3 \end{vmatrix} = 3 \cdot (6-6) - 1 \cdot (-6+6) = 0 \rightarrow \text{so } \lambda = 1 \text{ is an eigenvalue}\right]$

$\left[\begin{array}{ccc|c} 3 & -2 & 3 & 0 \\ 0 & -2 & 3 & 0 \\ -1 & 2 & -3 & 0 \end{array}\right] \xrightarrow{3R_3+R_1 \rightarrow R_1} \left[\begin{array}{ccc|c} 0 & 4 & -6 & 0 \\ 0 & -2 & 3 & 0 \\ -1 & 2 & -3 & 0 \end{array}\right] \xrightarrow[R_2+R_3 \rightarrow R_3]{2R_2+R_1 \rightarrow R_1} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & -2 & 3 & 0 \\ -1 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow$ $x_1 = 0$, $-2x_2 + 3x_3 = 0$

$\hookrightarrow x_2 = \frac{3}{2} x_3$

So, eigenvector: $\vec{x} = \begin{bmatrix} 0 \\ 3/2\, x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 3/2 \\ 1 \end{bmatrix}$ let $x_3 = 2 \rightarrow \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$.

(2) a) For $\lambda = -1 \rightarrow \begin{bmatrix} 1-(-1) & -3 \\ -4 & 5-(-1) \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix} \rightarrow \left[\begin{array}{cc|c} 2 & -3 & 0 \\ -4 & 6 & 0 \end{array}\right] \xrightarrow{2R_1+R_2 \rightarrow R_2} \left[\begin{array}{cc|c} 2 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow 2x_1 - 3x_2 = 0$

$\hookrightarrow x_1 = 3/2\, x_2$

eigenvector $\rightarrow \vec{x} = \begin{bmatrix} 3/2\, x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$, Basis for eigenspace ($\lambda = -1$) $\left\{\begin{bmatrix} 3/2 \\ 1 \end{bmatrix}\right\}$ or $\left\{\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right\}$

For $\lambda = 7 \rightarrow \begin{bmatrix} -6 & -3 \\ -4 & -2 \end{bmatrix} \xrightarrow{-\frac{1}{3}R_1 \rightarrow R_1} \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} \xrightarrow{-2R_1+R_2 \rightarrow R_2} \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \rightarrow 2x_1 + x_2 = 0 \rightarrow \vec{x} = \begin{bmatrix} -1/2\, x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$

$\hookrightarrow x_1 = -\frac{1}{2} x_2$

Basis for eigenspace ($\lambda = 7$) $\left\{\begin{bmatrix} -1/2 \\ 1 \end{bmatrix}\right\}$ or $\left\{\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right\}$

(2) cont'd

b) $\lambda=3 \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 3 & -3 & 3 \\ 2 & -2 & 2 \end{bmatrix} \xrightarrow[-2R_1+R_3\to R_3]{-3R_1+R_2\to R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & -3 & 6 \\ 0 & -2 & 4 \end{bmatrix} \xrightarrow{-\frac{2}{3}R_2\to R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & -4 \\ 0 & -2 & 4 \end{bmatrix} \xrightarrow[\frac{1}{2}R_2\to R_2]{R_2+R_3\to R_3} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow$

$x_1 - x_3 = 0 \rightarrow x_1 = x_3$

$x_2 - 2x_3 = 0 \quad x_2 = 2x_3$

So, $\vec{x} = \begin{bmatrix} x_3 \\ 2x_3 \\ x_3 \end{bmatrix} \rightarrow x_3\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and a basis for the eigenspace $(\lambda=3)$ $\left\{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right\}$.

(3) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(4) Assume λ is an eigenvalue of an invertible matrix A. Since λ is an eigenvalue of A then assume $\vec{x}$ satisfies $A\vec{x} = \lambda\vec{x}$. Since A is invertible, then A is square and A^{-1} exists, so...

linear transformation

$$A^{-1}(A\vec{x}) = A^{-1}(\lambda\vec{x}) \rightarrow I\vec{x} = A^{-1}(\lambda\vec{x}) \rightarrow \vec{x} = \lambda(A^{-1}\vec{x}) \rightarrow \frac{1}{\lambda}\vec{x} = A^{-1}\vec{x} \rightarrow \underline{\lambda^{-1}\vec{x} = A^{-1}\vec{x}}$$

Therefore, λ^{-1} is an eigenvalue of A^{-1}. (Note that λ is a scalar).

(5) a) $\begin{bmatrix} -4 & -1 \\ 6 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} -4-\lambda & -1 \\ 6 & 1-\lambda \end{bmatrix} \rightarrow \det(A-\lambda I) = (-4-\lambda)(1-\lambda) - (-1\cdot 6)$

$= (\lambda+4)(\lambda-1) + 6$

$= (\lambda^2+3\lambda-4)+6 = \lambda^2+3\lambda+2$ characteristic equation

Thus $\lambda^2+3\lambda+2=0$

$\rightarrow (\lambda+1)(\lambda+2)=0$ $\lambda=-1,-2$ eigenvalues

b) $\begin{bmatrix} 5 & 3 \\ -4 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 5-\lambda & 3 \\ -4 & 4-\lambda \end{bmatrix} \rightarrow \det(A-\lambda I) = (5-\lambda)(4-\lambda) - (3\cdot -4)$

$= (\lambda-5)(\lambda-4) + 12$

$= (\lambda^2-9\lambda+20)+12 = \lambda^2-9\lambda+32$ characteristic equation

Thus, $\lambda^2-9\lambda+32=0 \rightarrow \dfrac{-(-9)\pm\sqrt{(-9)^2-4(1)(32)}}{2(1)} = \dfrac{9\pm\sqrt{81-128}}{2}$

$= \dfrac{9}{2} \pm \dfrac{\sqrt{-47}}{2} \rightarrow$ no real eigenvalues only complex.

⑥ a) $\begin{bmatrix} 4-\lambda & 0 & -1 \\ 0 & 4-\lambda & -1 \\ 1 & 0 & 2-\lambda \end{bmatrix} \rightarrow \det(A-\lambda I) = 0 + (4-\lambda)\begin{vmatrix} 4-\lambda & -1 \\ 1 & 2-\lambda \end{vmatrix} + 0$

(along 2^{nd} column)

$\rightarrow (4-\lambda)\left[(4-\lambda)(2-\lambda)-(-1\cdot 1)\right] = (4-\lambda)\left[(\lambda-4)(\lambda-2)+1\right] = (4-\lambda)\left[(\lambda^2-6\lambda+8)+1\right]$

$\rightarrow (4-\lambda)(\lambda^2-6\lambda+9) = (4-\lambda)(\lambda-3)^2$ ← either one

or $4\lambda^2-24\lambda+36-\lambda^3+6\lambda^2-9\lambda = -\lambda^3+10\lambda^2-33\lambda+36$ ←

b) $\begin{bmatrix} -1-\lambda & 0 & 2 \\ 3 & 1-\lambda & 0 \\ 0 & 1 & 2-\lambda \end{bmatrix} \rightarrow \det(A-\lambda I) = 0 + (1-\lambda)\begin{vmatrix} -1-\lambda & 2 \\ 0 & (2-\lambda) \end{vmatrix} - 1\begin{vmatrix} -1-\lambda & 2 \\ 3 & 0 \end{vmatrix}$

(along 2^{nd} column)

$\rightarrow (1-\lambda)\left[(-1-\lambda)(2-\lambda)+0\right] - 1(0-6) = (1-\lambda)(\lambda+1)(\lambda-2)+6 = (1-\lambda)(\lambda^2-\lambda-2)+6$

$\rightarrow (\lambda^2-\lambda-2)+(-\lambda^3+\lambda^2+2\lambda)+6 = \underline{-\lambda^3+2\lambda^2+\lambda+4}$

⑦ Since A and B are similar, then A is similar to B, thus there exists an invertible matrix P such that $A = PBP^{-1}$ and

$$\det(A) = \det(PBP^{-1}) = \det(P)\det(B)\det(P^{-1}) = \det(P)\det(B)\frac{1}{\det(P)} = \det(B).$$

Therefore, $\det A = \det B$.