Math 30, Monday, May 4, 2020

1pm class

Your Final Exam is on Monday, May 11.
It will be available at 6am
and is due by 11:59pm

→ should take you $\approx$ 2 hours.

Questions?

Final Exam Review Probs are on Canvas now
(I sent an email yesterday)

Today: The last new topic: "Substitution".

It's just applying the Chain Rule to integrals...

Two types of integrals:

1. indefinite integrals:

$$\underbrace{\int x^2 dx}_{\text{stands for "all antideriv.s of } x^2\text{"}} = \frac{1}{3}x^3 + C$$

2. definite integrals: for these we have The F.T. of Calc, Part II:

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

Combine with the Chain Rule:

$$\frac{d}{dx}F(g(x)) = F'(g(x))g'(x)$$

① For indefinite integrals:

$$\int F'(g(x))g'(x)\,dx = \underbrace{F(g(x))}_{\text{The derivative of this}} + C$$

is this

② for definite integrals:

$$\int_a^b F'(g(x))g'(x)\,dx = F(g(b)) - F(g(a))$$

This is called "substitution"

That is,

$$\int_a^b F'(g(x))g'(x)\,dx$$

Chain Rule

$$= \int_a^b \frac{d}{dx} F(g(x))\,dx$$

F.T. of Calc. Part II

$$= F(g(b)) - F(g(a))$$

Here's why we call it "substitution"

("some say "u-substitution")

for an indefinite integral:

$$\int F'(g(x))\,g'(x)\,dx = ?$$

let $u = g(x)$. Think of it as a new variable.

$$\frac{du}{dx} = g'(x)$$

rewrite it as: $du = g'(x)\,dx$

"substitute"

$$= \int F'(u)\,du = F(u) + C$$

$$= F(g(x)) + C$$

Write it like this: (you'll do this a lot in Calc. II).

Example. $\int x\cos(x^2)\,dx = ?$

Try to choose u so that it looks like a composition / Chain Rule.

Try $u = x^2$) a good guess comes from experience
usually let u be the "inside function"

"move dx over"

$\frac{du}{dx} = 2x$. rewrite as $du = 2x\,dx$

"divide by 2"

rewrite as: $\frac{1}{2}du = x\,dx$

Now "substitute": write in terms of u, not x:

$$\int x\cos(x^2)\,dx = \int \cos u \cdot \frac{1}{2}du$$

$$= \frac{1}{2}\int \cos u\,du$$) easier integral!

Same example but faster:

Ex. $\int x\cos(x^2)\,dx = ?$

like in Chain Rule

Good guess: let $u = x^2$ (The "inside function")

$$\frac{du}{dx} = 2x, \quad du = 2x\,dx$$

$$\frac{1}{2}du = x\,dx$$

So

$$\int x\cos(x^2)\,dx = \int \cos u \cdot \frac{1}{2}du$$

$$= \frac{1}{2}\int \cos u\,du \quad \text{easy!}$$

$$= \frac{1}{2}\sin u + C$$

last step:
write in terms of the original variable

$$= \boxed{\frac{1}{2}\sin(x^2) + C}$$

Final answer:

$$\int x\cos(x^2)\,dx = \frac{1}{2}\sin(x^2) + C.$$

The deriv. of this is that

Important: we can check our answer!

$$\frac{d}{dx}\left(\frac{1}{2}\sin(x^2) + C\right)$$

Chain Rule (duh!)

$$= \frac{1}{2}\cos(x^2)\cdot 2x + 0$$

$$= x\cos(x^2)$$

we must be right!

$$\frac{d}{dx}\sin(x^2) = \frac{d}{dx}F(g(x))$$ where $F(u)=\sin u$, $g(x)=x^2$

& use Chain Rule

Ex. $\int \sin^4 x \cos x \, dx = ?$

Try $u = \sin x$ (so $u^4 = \sin^4 x$)

Then $\frac{du}{dx} = \cos x$ so $du = \cos x \, dx$

Now substitute:

write the integral in terms of u only (no x's)

$$\int \sin^4 x \cos x \, dx = \int u^4 \, du$$

easy integral!!

$$= \frac{1}{5} u^5 + C$$

$$\boxed{= \frac{1}{5} \sin^5 x + C}$$

That is,

$$\underbrace{\int \sin^4 x \cos x \, dx}_{\text{indefinite integral: "all antiderivatives of } \sin^4 x \cos x\text{"}} = \boxed{\frac{1}{5}\sin^5 x + C}$$

Check your answer by differentiating:

$$\frac{d}{dx}\left(\frac{1}{5}\sin^5 x + C\right)$$

$$= \frac{d}{dx}\left(\frac{1}{5}(\sin x)^5 + C\right)$$

Chain Rule (duh!)

$$= \sin^4 x \cdot \cos x + 0$$

Ex. $\displaystyle\int \frac{x+4}{x^2+1}\,dx = ?$

not so easy to see what u should be...

Trick: split into two integrals.

$$\cdots = \int \frac{x}{x^2+1}\,dx + \int \frac{4}{x^2+1}\,dx$$

Try: let $u = x^2+1$

$$du = 2x\,dx$$

$$\frac{1}{2}du = x\,dx$$

$$= \int \frac{1}{u}\cdot\frac{1}{2}du + 4\int \frac{1}{x^2+1}\,dx$$

$$\ldots = \int \frac{1}{u}\cdot\frac{1}{2}\,du + 4\int \frac{1}{x^2+1}\,dx$$

$$= \frac{1}{2}\int \frac{1}{u}\,du + 4\int \frac{1}{x^2+1}\,dx$$

easier!

recall: $\frac{d}{dx}\tan^{-1}x = \frac{1}{x^2+1}$

$$= \frac{1}{2}\ln|u| + 4\tan^{-1}x + C$$

last step: write in terms of the original variable

$u = x^2+1$

$$\boxed{= \frac{1}{2}\ln(x^2+1) + 4\tan^{-1}x + C}$$

Check by differentiating!

__Ex.__ $\int \tan x \, dx = ?$

not easy to find u...

rewrite it:

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx$$

Now try $u = \cos x$

$du = -\sin x \, dx$

$(-1)\,du = \sin x \, dx$

$$= \int \frac{1}{u} (-1)\, du$$ easy integral!

check by differentiating...

$$= -\ln|u| + C = \boxed{-\ln|\cos x| + C}$$

How not to do it:

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = \int \sin x \cdot \frac{1}{\cos x} dx$$

Try $u = \sin x$

$du = \cos x \, dx$

not the same thing!

You might not choose the right u on the first try...

Next time: substitution w/ definite integrals

See you on Wednesday!