Prove factorial of n is n!

$$f(n) = n! \quad \forall n \geq 0$$

by using recursive formula and Induction.

$$f(n+1) = n \times f(n).$$

Base case:

$$f(0) = 1$$

Inductive step:

$f(n) = n!$ is the Inductive hypothesis

to prove $f(n+1) = (n+1)!$

we know $f(n+1) = (n+1) \times f(n)$

$= (n+1) \times n!$

$= (n+1)!$ — QED

Prove a^n by Induction using recursion formula $P(a, n) = a \cdot P(a, n-1)$

I

is

$$P(a, n) = a^n$$

Base case

$$P(a, 0) = a^0 = 1$$

Inductive hypothesis

$$P(a, n) = a^n$$

Prove $P(a, n+1) = a^{n+1}$

we can express

$$P(a, n+1) = a \cdot P(a, n) \quad -(1)$$

by IH, $P(a, n) = a^n$, apply it above in (1)

$$P(a, n+1) = a \cdot a^n = a^{n+1}. \quad - \text{QED}$$

Prove $sum(n) = \frac{(n+1)n}{2}$ by recursion and Induction.

$$sum(n+1) = sum(n) + (n+1)$$

Base case: $sum(1) = sum(0) + 1 = 1$

Inductive step:

Hypothesis : $sum(n) = \frac{n \cdot (n+1)}{2}$

Prove: $sum(n+1) = \frac{(n+1)(n+2)}{2}$

By Recursion formula we have

$$sum(n+1) = sum(n) + (n+1)$$

$$= \frac{n \cdot (n+1)}{2} + (n+1)$$

$$= \frac{n \cdot (n+1)}{2} + \frac{2}{2} \cdot (n+1)$$

$$= \frac{n \cdot (n+1) + 2(n+1)}{2}$$

$$= \frac{(n+1)(n+2)}{2}$$ QED

Prove: The number of nodes in a perfect binary tree ~~of~~ of height h is $2^{h+1}-1$.

$$n(h) = 2^{h+1}-1$$

Base case

$$n(0) = 2^{0+1}-1 = 2^{1}-1 = 1$$

Inductive step: $n(h) = 2^{h+1}-1$

Prove $n(h+1) = 2^{(h+1)+1}-1 = 2^{h+2}-1$ — ①

we know by recursion

$$n(h+1) = 1 + n(h^{L}) + n(h^{R})$$

h^{L} = on the left

h^{R} = on the right

$$= 1 + \left(2^{h+1}-1\right) + \left(2^{h+1}-1\right)$$

$$= 1 + 2\left(2^{h+1}-1\right)$$

$$= 1 + 2 \cdot 2^{h+1} - 2$$

$$= 2^{h+2}-1$$

QED

Proof: TOH - consider N disks. Total moves needed is $2^N - 1$.

we have a recursion formula

$$M(n) = 2M(n-1) + 1$$

Base case: $M(1) = 1$ because $M(0) = 0$

Inductive step: we have

$$M(N) = 2^N - 1$$

Prove: $M(N+1) = 2^{N+1} - 1$

But
$$\begin{aligned} M(N+1) &= 2 \cdot M(N) + 1 \\ &= 2(2^N - 1) + 1 \\ &= 2^{N+1} - 2 + 1 \\ &= 2^{N+1} - 1 \end{aligned}$$

QED.

Prove : Sum of the first n cubes is

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + 3^3 + \cdots\cdots + n^3 = \left[\frac{(n+1)\,n}{2}\right]^2$$

by recussion and Induction

Base case $K = 1$ $\left[\frac{(K+1)K}{2}\right]^2 = \left[\frac{(1+1)1}{2}\right]^2 = 1^2 = 1$

Inductive step.

Assume it's true that for any K

$$1^3 + 2^3 + 3^3 + \cdots\cdots + K^3 = \left[\frac{(K+1)K}{2}\right]^2$$

Prove $1^3 + 2^3 + 3^3 + \cdots\cdots + K^3 + (K+1)^3$

$$\left[\frac{K^2(K+1)^2}{2^2}\right] + (K+1)^3 = \frac{(K+1)^2}{2^2}\left[K^2 + 2^2(K+1)\right]$$

$$= \frac{(K+1)^2}{2^2}\left[K^2 + 4K + 2\right]$$

$$= \frac{(K+1)^2}{2^2}(K+2)^2$$

$$= \left(\frac{(K+1)(K+2)}{2}\right)^2$$

— QED

Prove the sum of first n squares is $\frac{n(n+1)(2n+1)}{6}$ by Induction

ie $\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

ie $= 0^2 + 1^2 + 2^2 + \ldots\ldots + n^2$

Base case: $n = 0$

$$\sum_{i=0}^{0} i^2 = \frac{0(0+1)(2 \times 0+1)}{6} = 0$$

Induction step.

Assume the property holds for k.

$$\sum_{i=0}^{k} i^2 = \frac{k(k+1)(2k+1)}{6.}, \text{ ie}$$

Inductive hypothesis.

we need to prove this IH holds for $k+1$

$$\sum_{i=0}^{k+1} i^2 = \sum_{i=0}^{k} i^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= \frac{(k+1)}{6}\left[(k)(2k+1) + (k+1).6\right] = \frac{(k+1)}{6}\left[2k^2+7k+6\right]$$

$$= \frac{(k+1)}{6}\left[(n+2)(2n+3)\right] = \frac{(k+1)(h+2)(2(k+1)+1)}{6}$$

which is of the form $\frac{n.(n+1)(2n+1)}{6}$

QED