HW 9

① P^{-1}: $\det(P) = (3-4) = -1$, $\text{adj}(P) = \begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix} \rightarrow P^{-1} = \frac{1}{-1}\begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$

$$A^4 = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 81 \end{bmatrix}\begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} -3 & 2 \\ 162 & -81 \end{bmatrix} = \begin{bmatrix} -3+324 & 2-162 \\ -6+486 & 4-243 \end{bmatrix} = \begin{bmatrix} 321 & -160 \\ 480 & -239 \end{bmatrix}$$

② a) $\begin{vmatrix} 1-\lambda & 3 \\ 4 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) - 12 = \lambda^2 - 3\lambda + 2 - 12 = \underline{\lambda^2 - 3\lambda - 10} = (\lambda-5)(\lambda+2) \rightarrow \lambda = -2, 5$

$D = \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix}$

✓ $\lambda = -2 \rightarrow \begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \rightarrow x_1 + x_2 = 0$, $\hookrightarrow x_1 = -x_2$ $\quad \vec{x} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} = x_2 \underline{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}$

✓ $\lambda = 5 \quad \begin{bmatrix} -4 & 3 \\ 4 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 4 & -3 \\ 0 & 0 \end{bmatrix} \rightarrow 4x_1 = 3x_2$, $x_1 = \frac{3}{4}x_2$ $\quad \vec{x} = \begin{bmatrix} 3/4\,x_2 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} 3/4 \\ 1 \end{bmatrix}$ $\hookrightarrow \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$\rightarrow P = \begin{bmatrix} -1 & 3 \\ 1 & 4 \end{bmatrix}$

P^{-1}: $\det P = -4 - 3 = -7$ $\quad \text{adj}(P) = \begin{bmatrix} 4 & -3 \\ -1 & -1 \end{bmatrix} \rightarrow P^{-1} = \begin{bmatrix} -4/7 & 3/7 \\ 1/7 & 1/7 \end{bmatrix} \rightarrow A = \begin{bmatrix} -1 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} -4/7 & 3/7 \\ 1/7 & 1/7 \end{bmatrix}$

b) $\lambda = 2 \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad x_1 = -x_2 - x_3$ $\quad \vec{x} = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\lambda = 5 \rightarrow \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \xrightarrow[2R_3 + R_1 \to R_1]{-R_3 + R_2 \to R_2} \begin{bmatrix} 0 & 3 & -3 \\ 0 & -3 & 3 \\ 1 & 1 & -2 \end{bmatrix} \xrightarrow[\frac{1}{3}R_1 \to R_1]{R_1 + R_2 \to R_2} \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 1 & 1 & -2 \end{bmatrix} \xrightarrow{-R_1 + R_3 \to R_3} \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}$

$x_2 - x_3 = 0 \rightarrow x_2 = x_3$

$x_1 - x_3 = 0 \rightarrow x_1 = x_3$

$\rightarrow \vec{x} = \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \quad P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

(3) A^{-1} is invertible since $A^{-1} \cdot A = I$. Since A is diagonalizable then $A = PDP^{-1}$ and since A is invertible then A^{-1} exists and,

$$A = PDP^{-1} \rightarrow A^{-1}A = A^{-1}(PDP^{-1}) \rightarrow I = A^{-1}PDP^{-1} \rightarrow I \cdot P = A^{-1}PD(P^{-1}P) \rightarrow P = A^{-1}PD$$

$$\rightarrow PD^{-1} = A^{-1}P(DD^{-1}) \rightarrow PD^{-1} = A^{-1}P \rightarrow PD^{-1}P^{-1} = A^{-1}(PP^{-1}) \rightarrow \underline{PD^{-1}P^{-1} = A^{-1}}$$

This shows that A^{-1} is digonalizable.

(4) $\vec{u} = \begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix} \rightarrow \|\vec{u}\| = \sqrt{(-6)^2 + (4)^2 + (-3)^2} = \sqrt{36 + 16 + 9} = \sqrt{61} \rightarrow \frac{1}{\|\vec{u}\|}\vec{u} = \underline{\begin{bmatrix} -6/\sqrt{61} \\ 4/\sqrt{61} \\ -3/\sqrt{61} \end{bmatrix}}$

(5) $\vec{u} - \vec{v} = \begin{bmatrix} 0-(-4) \\ -5-(-1) \\ 2-8 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ -6 \end{bmatrix} \rightarrow \|\vec{u} - \vec{v}\| = \sqrt{(4)^2 + (-4)^2 + (-6)^2} = \sqrt{(16) + (16) + 36} = \sqrt{68}$

(6) Since $\vec{y}$ is orthogonal to $\vec{u}$ and $\vec{v}$ they $\vec{y} \cdot \vec{u} = 0$ and $\vec{y} \cdot \vec{v} = 0$.

Thus $(\vec{u} + \vec{v}) \cdot \vec{y} = \vec{u} \cdot \vec{y} + \vec{v} \cdot \vec{y} = 0 + 0 = 0$, thus $\vec{y}$ is orthogonal to $\vec{u} + \vec{v}$.

(7) Suppose $\vec{y} \in W$, then $\vec{y} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots c_p\vec{v}_p$ for some scalars c_i $(i = 1, \ldots p)$.

Then $\vec{x} \cdot \vec{y} = \vec{x} \cdot (c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p) \underset{\text{dot product properties}}{=} c_1(\vec{x} \cdot \vec{v}_1) + c_2(\vec{x} \cdot \vec{v}_2) + \cdots + c_p(\vec{x} \cdot \vec{v}_p)$

Since $\vec{x} \cdot \vec{v}_i = 0$ for all $i = \{1, \ldots p\}$ $\leftarrow$ $=$ $c_1(0) + c_2(0) + \cdots + c_p(0) = 0$.

Therefore, $\vec{x} \cdot \vec{y} = 0$ and since $\vec{y}$ was arbitrary vector in W, then $\vec{x}$ is orthogonal to every vector in W.

8) a) $\vec{u}_1 \cdot \vec{u}_2 = (2\cdot 6) + (-3\cdot 4) = 12 - 12 = 0 \rightarrow$ orthogonal set

$\vec{x} = \frac{\vec{x}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\vec{u}_1 + \frac{\vec{x}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}\vec{u}_2 \rightarrow \vec{x}\cdot\vec{u}_1 = (18+21) = 39$, $\vec{u}_1\cdot\vec{u}_1 = (4+9) = 13$ | $\vec{x}\cdot\vec{u}_2 = [54 - 28] = 26$, $\vec{u}_2\cdot\vec{u}_2 = (36+16) = 52$

$\vec{x} = \frac{39}{13}\vec{u}_1 + \frac{26}{52}\vec{u}_2 = 3\vec{u}_1 + \frac{1}{2}\vec{u}_2$

b) $\vec{u}_1\cdot\vec{u}_2 = (6-6+0) = 0$, $\vec{u}_1\cdot\vec{u}_3 = (3-3+0) = 0$, $\vec{u}_2\cdot\vec{u}_3 = (2+2-4) = 0 \rightarrow$ orthogonal set

$\vec{x}\cdot\vec{u}_1 = (15+9+0) = 24$, $\vec{u}_1\cdot\vec{u}_1 = (9+9+0) = 18$ | $\vec{x}\cdot\vec{u}_2 = (10-6-1) = 3$, $\vec{u}_2\cdot\vec{u}_2 = (4+4+1) = 9$ | $\vec{x}\cdot\vec{u}_3 = (5-3+4) = 6$, $\vec{u}_3\cdot\vec{u}_3 = (1+1+16) = 18$

$\vec{x} = \frac{24}{18}\vec{u}_1 + \frac{3}{9}\vec{u}_2 + \frac{6}{18}\vec{u}_3 = \frac{4}{3}\vec{u}_1 + \frac{1}{3}\vec{u}_2 + \frac{1}{3}\vec{u}_3$

9) $\vec{y} = \begin{bmatrix}1\\7\end{bmatrix}$, $\vec{u} = \begin{bmatrix}-4\\2\end{bmatrix} \rightarrow \vec{y}\cdot\vec{u} = (-4+14) = 10$, $\vec{u}\cdot\vec{u} = (16+4) = 20 \rightarrow \text{proj}_L\vec{y} = \hat{y} = \frac{10}{20}\vec{u} = \frac{1}{2}\begin{bmatrix}-4\\2\end{bmatrix} = \boxed{\begin{bmatrix}-2\\1\end{bmatrix}}$

$L = \text{Span}\{\vec{u}\}$

10) $\vec{y} = \begin{bmatrix}2\\3\end{bmatrix}$, $\vec{u} = \begin{bmatrix}4\\-7\end{bmatrix} \rightarrow \vec{y}\cdot\vec{u} = (8-21) = -13$, $\vec{u}\cdot\vec{u} = (16+49) = 65 \rightarrow \text{proj}_L\vec{y} = \hat{y} = \frac{-13}{65}\vec{u} = -\frac{1}{5}\begin{bmatrix}4\\-7\end{bmatrix} = \begin{bmatrix}-4/5\\7/5\end{bmatrix}$

$L = \text{Span}\{\vec{u}\}$

$\vec{z} = \vec{y} - \hat{y} = \begin{bmatrix}2\\3\end{bmatrix} - \begin{bmatrix}-4/5\\7/5\end{bmatrix} = \begin{bmatrix}14/5\\8/5\end{bmatrix} \rightarrow \vec{y} = \hat{y} + z = \underbrace{\begin{bmatrix}-4/5\\7/5\end{bmatrix}}_{\alpha\vec{u}} + \underbrace{\begin{bmatrix}14/5\\8/5\end{bmatrix}}_{\perp\vec{u}}$

11) $\vec{y} = \begin{bmatrix}-3\\9\end{bmatrix}$, $\vec{u} = \begin{bmatrix}1\\2\end{bmatrix} \rightarrow \vec{y}\cdot\vec{u} = (-3+18) = 15$, $\vec{u}\cdot\vec{u} = (1+4) = 5 \rightarrow \hat{y} = \frac{15}{5}\vec{u} = 3\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}3\\6\end{bmatrix}$

$\vec{y} - \hat{y} = \begin{bmatrix}-3\\9\end{bmatrix} - \begin{bmatrix}3\\6\end{bmatrix} = \begin{bmatrix}-6\\3\end{bmatrix} \rightarrow \|\vec{y} - \hat{y}\| = \sqrt{(-6)^2 + (3)^2} = \sqrt{36+9} = \sqrt{45}$