1. Let W be the subspace spanned by the $\mathbf{u}$'s and write $\mathbf{y}$ as the sum of a vector in W and a vector orthogonal to W.

$$\mathbf{y} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}$$

1.

$$y = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}, \; u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \; u_2 = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}$$

The orthogonal projection of y onto W is

$$\hat{y} = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2$$

$$y \cdot u_1 = \underset{-1}{(-1)1} + \underset{4}{(4)1} + \underset{3}{(3)1} = 6$$

$$u_1 \cdot u_1 = (1)(1) + (1)(1) + (1)(1) = 3$$

$$y \cdot u_2 = \overset{1}{(-1)(-1)} + \overset{12}{(4)(3)} + \overset{-6}{(3)(-2)} = 7$$

$$u_2 \cdot u_2 = \overset{1}{(-1)(-1)} + \overset{9}{(3)(3)} + \overset{4}{(-2)(-2)} = 14$$

$$= \frac{6}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{7}{14}\begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix} = \frac{84}{42}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{21}{42}\begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 84/42 \\ 84/42 \\ 84/42 \end{bmatrix} + \begin{bmatrix} -21/42 \\ 63/42 \\ -42/42 \end{bmatrix} = \begin{bmatrix} 63/42 \\ 147/42 \\ 42/42 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 7/2 \\ 1 \end{bmatrix}$$

cont'd

(cont'd)

$$y-\hat{y} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} - \begin{bmatrix} 3/2 \\ 7/2 \\ 1 \end{bmatrix} = \begin{bmatrix} -5/2 \\ 1/2 \\ 2 \end{bmatrix}$$

$$\boxed{y = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 7/2 \\ 1 \end{bmatrix} + \begin{bmatrix} -5/2 \\ 1/2 \\ 2 \end{bmatrix}}$$

2. Find the closest point to $\mathbf{y}$ in the subspace W spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$.

$$\mathbf{y} = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 13 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix}$$

2. $v = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 13 \end{bmatrix}, u_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}, u_2 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix}$

$$\hat{v} = \frac{v \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{v \cdot u_2}{u_2 \cdot u_2} u_2$$

$$v \cdot u_1 = \overset{3}{(3)1} + \overset{2}{(-1)(-2)} + \overset{-1}{(1)(-1)} + \overset{26}{(13)2} = 30$$

$$u_1 \cdot u_1 = \overset{1}{(1)(1)} + \overset{4}{(-2)(-2)} + \overset{1}{(-1)(-1)} + \overset{4}{(2)(2)} = 10$$

$$v \cdot u_2 = \overset{-12}{(3)(-4)} + \overset{-1}{(-1)(1)} + \overset{0}{(1)(0)} + \overset{39}{(13)(3)} = 26$$

$$u_2 \cdot u_2 = \overset{16}{(-4)(-4)} + \overset{1}{(1)(1)} + \overset{0}{(0)0} + \overset{9}{(3)(3)} = 26$$

cont'd

Cont'd

$$\hat{y} = \frac{y\cdot u_1}{u_1\cdot u_1}u_1 + \frac{y\cdot u_2}{u_2\cdot u_2}u_2$$

$$= \frac{30}{10}\begin{bmatrix}1\\-2\\-1\\2\end{bmatrix} + \frac{26}{26}\begin{bmatrix}-4\\1\\0\\3\end{bmatrix} = 3\begin{bmatrix}1\\-2\\-1\\2\end{bmatrix} + 1\begin{bmatrix}-4\\1\\0\\3\end{bmatrix}$$

$$= \begin{bmatrix}3\\-6\\-3\\6\end{bmatrix} + \begin{bmatrix}-4\\1\\0\\3\end{bmatrix} = \boxed{\begin{bmatrix}-1\\-5\\-3\\9\end{bmatrix}}$$

3. Find the best approximation to $\mathbf{z}$ by vectors of the form $c_1\mathbf{v}_1 + c_2\mathbf{v}_2$.

$$\mathbf{z} = \begin{bmatrix}3\\-7\\2\\3\end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix}2\\-1\\-3\\1\end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix}1\\1\\0\\-1\end{bmatrix}$$

3. $z = \begin{bmatrix} 3 \\ -7 \\ 2 \\ 3 \end{bmatrix}, V_1 = \begin{bmatrix} 2 \\ -1 \\ -3 \\ 1 \end{bmatrix}, V_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$

Orthogonal? → vectors $u \cdot v = 0$

$V_1 \cdot V_2 = (2)1 + (-1)1 + (-3)0 + (1)-1 = 0$

So, Vectors V_1 and V_2 are orthogonal

Closest point in span$\{V_1, V_2\}$ to z is

$\hat{z} = \frac{z \cdot V_1}{V_1 \cdot V_1} V_1 + \frac{z \cdot V_2}{V_2 \cdot V_2} V_2$

cont'd

(cont'd)

$$z \cdot v_1 = \overset{6}{(3)2} + \overset{7}{(-7)(-1)} + \overset{-6}{(2)(-3)} + \overset{3}{(3)(1)} = 10$$

$$v_1 \cdot v_1 = \overset{4}{(2)(2)} + \overset{1}{(-1)(-1)} + \overset{9}{(-3)(-3)} + \overset{1}{(1)(1)} = 15$$

$$z \cdot v_2 = \overset{3}{(3)1} + \overset{-7}{(-7)(1)} + \overset{0}{(2)(0)} + \overset{-3}{(3)(-1)} = -7$$

$$v_2 \cdot v_2 = \overset{1}{1\cdot 1} + \overset{1}{1\cdot 1} + \overset{0}{0\cdot 0} + \overset{1}{(-1)(-1)} = 3$$

$$= \frac{10}{15}\begin{bmatrix}2\\-1\\-3\\1\end{bmatrix} + \frac{-7}{3}\begin{bmatrix}1\\1\\0\\-1\end{bmatrix} = \frac{2}{3}\begin{bmatrix}2\\-1\\-3\\1\end{bmatrix} + \frac{-7}{3}\begin{bmatrix}1\\1\\0\\-1\end{bmatrix}$$

$$= \begin{bmatrix}4/3\\-2/3\\-6/3\\2/3\end{bmatrix} + \begin{bmatrix}-7/3\\-7/3\\0\\7/3\end{bmatrix} = \begin{bmatrix}-3/3\\-9/3\\-6/3\\9/3\end{bmatrix} = \boxed{\begin{bmatrix}-1\\-3\\-2\\3\end{bmatrix}}$$

4. Let $\mathbf{y} = \begin{bmatrix}4\\8\\1\end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix}2/3\\1/3\\2/3\end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix}-2/3\\2/3\\1/3\end{bmatrix}$, and $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$.

(a) Let $U = [\mathbf{u}_1 \ \mathbf{u}_2]$. Compute U^TU and UU^T.

4. $y = \begin{bmatrix} 4 \\ 8 \\ 1 \end{bmatrix}, u_1 = \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix}, u_2 = \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$, and $W = \text{Span}\{u_1, u_2\}$

a) Let $U = [u_1 \; u_2]$. Compute U^TU and UU^T.

$$U = \begin{bmatrix} 2/3 & -2/3 \\ 1/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix} \quad U^T = \begin{bmatrix} 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \end{bmatrix}$$

$$U^TU = \begin{bmatrix} 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 2/3 & -2/3 \\ 1/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$2 \times 3 \quad 3 \times 2$

match

size of product!

$$a_{11} = \overset{4/9}{(2/3)(2/3)} + \overset{1/9}{(1/3)(1/3)} + \overset{4/9}{(2/3)(2/3)} = 9/9 = 1$$

$$a_{12} = \overset{-4/9}{(2/3)(-2/3)} + \overset{2/9}{(1/3)(2/3)} + \overset{2/9}{(2/3)(1/3)} = 0$$

$$a_{21} = \overset{-4/9}{(-2/3)(2/3)} + \overset{2/9}{(2/3)(1/3)} + \overset{2/9}{(1/3)(2/3)} = 0$$

$$a_{22} = \overset{4/9}{(-2/3)(-2/3)} + \overset{4/9}{(2/3)(2/3)} + \overset{1/9}{(1/3)(1/3)} = 9/9 = 1$$

Cont'd

(cont'd)

$$UU^T = \underset{3\times 2}{\begin{bmatrix} 2/3 & -2/3 \\ 1/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix}} \underset{2\times 3}{\begin{bmatrix} 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \end{bmatrix}} = \underset{3\times 3}{\begin{bmatrix} 8/9 & -2/9 & 2/9 \\ -2/9 & 5/9 & 4/9 \\ 2/9 & 4/9 & 8/9 \end{bmatrix}}$$

$$a_{11} = \overset{4/9}{(2/3 \times 2/3)} + \overset{4/9}{(-2/3 \times -2/3)} = 8/9$$

$$a_{12} = \overset{2/9}{(2/3 \times 1/3)} + \overset{-4/9}{(-2/3 \times 2/3)} = -2/9$$

$$a_{13} = \overset{4/9}{(2/3 \times 2/3)} + \overset{-2/9}{(-2/3 \times 1/3)} = 2/9$$

$$a_{21} = \overset{2/9}{(1/3 \times 2/3)} + \overset{-4/9}{(2/3 \times -2/3)} = -2/9$$

$$a_{22} = \overset{1/9}{(1/3 \times 1/3)} + \overset{4/9}{(2/3 \times 2/3)} = 5/9$$

$$a_{23} = \overset{2/9}{(1/3 \times 2/3)} + \overset{2/9}{(2/3 \times 1/3)} = 4/9$$

$$a_{31} = \overset{4/9}{(2/3 \times 2/3)} + \overset{-2/9}{(1/3 \times -2/3)} = 2/9$$

$$a_{32} = \overset{2/9}{(2/3 \times 1/3)} + \overset{2/9}{(1/3 \times 2/3)} = 4/9$$

$$a_{33} = \overset{4/9}{(2/3 \times 2/3)} + \overset{1/9}{(1/3 \times 1/3)} = 5/9$$

$$\boxed{U^T U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad UU^T = \begin{bmatrix} 8/9 & -2/9 & 2/9 \\ -2/9 & 5/9 & 4/9 \\ 2/9 & 4/9 & 8/9 \end{bmatrix}}$$

(b) Compute $\text{proj}_W \mathbf{y}$ and $(UU^T)\mathbf{y}$.

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contd 2A

b) Compute $\text{proj}_W v$ and $(UU^T)v$.

$$\hat{y} = \frac{v\cdot u_1}{u_1\cdot u_1}u_1 + \frac{v\cdot u_2}{u_2\cdot u_2}u_2$$

$$v\cdot u_1 = (\overset{8/3}{4\times 2/3}) + (\overset{8/3}{8\times 1/3}) + (\overset{2/3}{1\times 2/3}) = 18/3$$

$$u_1\cdot u_1 = (\overset{4/9}{2/3\times 2/3}) + (\overset{1/9}{1/3\times 1/3}) + (\overset{4/9}{2/3\times 2/3}) = 9/9 = 1$$

$$v\cdot u_2 = (\overset{-8/3}{4\times -2/3}) + (\overset{16/3}{8\times 2/3}) + (\overset{1/3}{1\times 1/3}) = 9/3 = 3$$

$$u_2\cdot u_2 = (\overset{4/9}{-2/3\times -2/3}) + (\overset{4/9}{2/3\times 2/3}) + (\overset{1/9}{1/3\times 1/3}) = 9/9 = 1$$

$$= \underset{6}{\frac{18/3}{1}} \cdot \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix} + \frac{3}{1}\begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 36/9 \\ 18/9 \\ 36/9 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 36/9 \\ 18/9 \\ 36/9 \end{bmatrix} + \begin{bmatrix} -18/9 \\ 18/9 \\ 9/9 \end{bmatrix} = \begin{bmatrix} 18/9 \\ 36/9 \\ 45/9 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$$

contd

Cont'd

$$(UU^T)v = \begin{bmatrix} 8/9 & -2/9 & 2/9 \\ -2/9 & 5/9 & 4/9 \\ 2/9 & 4/9 & 5/9 \end{bmatrix} \begin{bmatrix} 4 \\ 8 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$$

$3\times 3 \quad 3\times 1 \quad 3\times 1$

$$a_{11} = \overset{32/9}{(8/9 \times 4)} + \overset{-16/9}{(-2/9 \times 8)} + \overset{2/9}{(2/9 \times 1)} = 18/9 = 2$$

$$a_{21} = \overset{-8/9}{(-2/9 \times 4)} + \overset{40/9}{(5/9 \times 8)} + \overset{4/9}{(4/9 \times 1)} = 36/9 = 4$$

$$a_{31} = \overset{8/9}{(2/9 \times 4)} + \overset{32/9}{(4/9 \times 8)} + \overset{5/9}{(5/9 \times 1)} = 45/9 = 5$$

$$\boxed{\text{Proj}_W v = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}, \ (UU^T)v = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}}$$

5. Find a least-squares solution of $A\mathbf{x} = \mathbf{b}$ by (a) constructing the normal equations for $\hat{\mathbf{x}}$ and (b) solving for $\hat{\mathbf{x}}$. And (c) compute the least squares error associated with the least-squares solution.

   (1)

$$A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$$

1) $A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$

a) $A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix}$

$2 \times 3 \quad 3 \times 2 \quad 2 \times 2$

match → we can multiply!

size of product!

$a_{11} = (1 \times 1) + (1 \times 1) + (1 \times 1) = 3$

$a_{12} = (1 \times 3) + (1 \times -1) + (1 \times 1) = 3$

$a_{21} = (3 \times 1) + (-1 \times 1) + (1 \times 1) = 3$

$a_{22} = (3 \times 3) + (-1 \times -1) + (1 \times 1) = 11$

$A^T b = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$

2×1

$2 \times 3 \quad 3 \times 1$

match!

size of product!

$a_{11} = (1 \times 5) + (1 \times 1) + (1 \times 0) = 6$

$a_{21} = (3 \times 5) + (-1 \times 1) + (1 \times 0) = 14$

Cont'd

cont'd

$A^TAx = A^Tb$: normal equations!

$$\begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$$

b) solve for x:

A^TA invertible?

↳ nxn matrix so we ask is it row equivalent to I_n

$$\begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix} \xrightarrow{\frac{1}{3}R_1} \begin{bmatrix} 1 & 1 \\ 3 & 11 \end{bmatrix} \xrightarrow{R_2 - 3\cdot R_1 \to R_2} \begin{bmatrix} 1 & 1 \\ 0 & 8 \end{bmatrix}$$

$$\downarrow \tfrac{1}{8}R_2$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xleftarrow{R_1 - R_2 \to R_1} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

so it is row equivalent to I_n!

Since A^TA is invertible and 2x2, it is faster to do this below!

$$(A^TA)^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \frac{1}{3(11) - (-3)(-3)}\begin{bmatrix} 11 & -3 \\ -3 & 3 \end{bmatrix} = \frac{1}{33-9}\begin{bmatrix} 11 & -3 \\ -3 & 3 \end{bmatrix} = \frac{1}{24}\begin{bmatrix} 11 & -3 \\ -3 & 3 \end{bmatrix}$$

cont'd 3*

HW #10

cont'd 3b

So we do this:

$$\hat{x} = (A^T A)^{-1} A^T b$$

$$= \frac{1}{24}\begin{bmatrix} 11 & -3 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} 6 \\ 14 \end{bmatrix} = \frac{1}{24}\begin{bmatrix} 24 \\ 24 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \leftarrow \hat{x}$$

2x2 2x1 — match — size of product; 2x1

$$a_{11} = (\overset{66}{11 \times 6}) + (\overset{-42}{-3 \times 14}) = 24$$

$$a_{21} = (\overset{-18}{-3 \times 6}) + (\overset{42}{3 \times 14}) = 24$$

c) Find least squares error associated with the least-squares solution

$$A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}, \quad \hat{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$b - A\hat{x} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

cont'd

cont'd

$$\begin{bmatrix}1&3\\1&-1\\1&1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}4\\0\\2\end{bmatrix}$$

$3\times2 \quad 2\times1 \quad 3\times1$

matches

size of product!

$a_{11} = (1)(1) + (3)(1) = 4$

$a_{21} = (1)(1) + (-1)(1) = 0$

$a_{31} = (1)(1) + (1)(1) = 2$

$$\begin{bmatrix}5\\1\\0\end{bmatrix} - \begin{bmatrix}4\\0\\2\end{bmatrix} = \begin{bmatrix}1\\1\\-2\end{bmatrix}$$

$\mathbf{b} - A\hat{x}$

$\|\mathbf{b} - A\hat{x}\|$

$= \sqrt{(1)^2 + (1)^2 + (-2)^2}$

$= \sqrt{1+1+4}$

$= \sqrt{6}$

$$\boxed{\begin{array}{l} \text{a) } \begin{bmatrix}3&3\\3&11\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}6\\14\end{bmatrix} \\ \text{b) } \hat{x} = \begin{bmatrix}1\\1\end{bmatrix} \\ \text{c) } \sqrt{6} \end{array}}$$

(2)

$$A = \begin{bmatrix}1&-2\\-1&2\\0&3\\2&5\end{bmatrix}, \mathbf{b} = \begin{bmatrix}3\\1\\-4\\2\end{bmatrix}$$

2) $A = \begin{bmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{bmatrix}, b = \begin{bmatrix} 3 \\ 1 \\ -4 \\ 2 \end{bmatrix}$

a) $A^T A = \begin{bmatrix} 1 & -1 & 0 & 2 \\ -2 & 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 6 \\ 6 & 42 \end{bmatrix}$

$2 \times 4 \quad 4 \times 2 \quad 2 \times 2$

match

size of product!

$a_{11} = \overset{1}{(1 \times 1)} + \overset{1}{(-1 \times -1)} + \overset{0}{(0 \times 0)} + \overset{4}{(2 \times 2)} = 6$

$a_{12} = \overset{-2}{(1 \times -2)} + \overset{-2}{(-1 \times 2)} + \overset{0}{(0 \times 3)} + \overset{10}{(2 \times 5)} = 6$

$a_{21} = \overset{-2}{(-2 \times 1)} + \overset{-2}{(2 \times -1)} + \overset{0}{(3 \times 0)} + \overset{10}{(5 \times 2)} = 6$

$a_{22} = \overset{4}{(-2 \times -2)} + \overset{4}{(2 \times 2)} + \overset{9}{(3 \times 3)} + \overset{25}{(5 \times 5)} = 42$

cont'd

Cont'd

$$A^T b = \begin{bmatrix} 1 & -1 & 0 & 2 \\ -2 & 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \end{bmatrix}$$

$2 \times 4 \quad 4 \times 1 \quad 2 \times 1$

match

size of product!

$$a_{11} = \overset{3}{(1 \times 3)} + \overset{-1}{(-1 \times 1)} + \overset{0}{(0 \times -4)} + \overset{4}{(2 \times 2)} = 6$$

$$a_{21} = \overset{-6}{(-2 \times 3)} + \overset{2}{(2 \times 1)} + \overset{-12}{(3 \times -4)} + \overset{10}{(5 \times 2)} = -6$$

$A^T A x = A^T b$:

normal equations!

$$\begin{bmatrix} 6 & 6 \\ 6 & 42 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \end{bmatrix}$$

b) Solve for $\hat{x}$:

$A^T A$ invertible?

$$\begin{bmatrix} 6 & 6 \\ 6 & 42 \end{bmatrix} \xrightarrow{\frac{1}{6} R_1} \begin{bmatrix} 1 & 1 \\ 6 & 42 \end{bmatrix} \xrightarrow{R_2 - 6R_1 \to R_2} \begin{bmatrix} 1 & 1 \\ 0 & 36 \end{bmatrix} \xrightarrow{\frac{1}{36} R_2} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 - R_2 \to R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Yes, $A^T A$ is invertible!

Cont'd 5b

cont'd 5b

Since A^TA is invertible and 2x2, it is faster to do this below!

$$(A^TA)^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} 42 & -6 \\ -6 & 6 \end{bmatrix} \qquad \frac{1}{6\cdot 42 - 6\cdot 6}\begin{bmatrix} 42 & -6 \\ -6 & 6 \end{bmatrix} = \frac{1}{252-36}\begin{bmatrix} 42 & -6 \\ -6 & 6 \end{bmatrix}$$

$$\frac{1}{216}\begin{bmatrix} 42 & -6 \\ -6 & 6 \end{bmatrix}$$

So, we do this:

$$x = (A^T A)^{-1} A^T b$$

$$= \frac{1}{216}\begin{bmatrix} 42 & -6 \\ -6 & 6 \end{bmatrix}\begin{bmatrix} 6 \\ -6 \end{bmatrix} = \frac{1}{216}\begin{bmatrix} 288 \\ -72 \end{bmatrix} = \begin{bmatrix} 288/216 \\ -72/216 \end{bmatrix}$$

2x2 2x1

match

size of solution

$$a_{11} = (\overset{252}{42 \times 6}) + (\overset{36}{-6 \times -6}) = 288$$

$$a_{21} = (\overset{-36}{-6 \times 6}) + (\overset{-36}{6 \times -6}) = -72$$

cont'd

cont'd

Q Find least squares error associated with the least-squares solution

$$A=\begin{bmatrix}1&-2\\-1&2\\0&3\\2&5\end{bmatrix},\ b=\begin{bmatrix}3\\1\\-4\\2\end{bmatrix},\ \hat{x}=\begin{bmatrix}288/216\\-72/216\end{bmatrix}$$

$$b-A\hat{x}=\begin{bmatrix}3\\1\\-4\\2\end{bmatrix}-\begin{bmatrix}1&-2\\-1&2\\0&3\\2&5\end{bmatrix}\begin{bmatrix}288/216\\-72/216\end{bmatrix}$$

$$\underset{4\times2}{\begin{bmatrix}1&-2\\-1&2\\0&3\\2&5\end{bmatrix}}\underset{2\times1}{\begin{bmatrix}288/216\\-72/216\end{bmatrix}}=\underset{4\times1}{\begin{bmatrix}2/3\\-2/3\\1\\13/3\end{bmatrix}}$$

4/3, −1/3

matches

size of product!

$$a_{11}=(1\times4/3)+(-2\times-1/3)=6/3$$

4/3, +2/3

$$a_{21}=(-1\times4/3)+(2\times-1/3)=-6/3$$

−4/3, −2/3

$$a_{31}=(0\times4/3)+(3\times-1/3)=-1$$

0, −1

$$a_{41}=(2\times4/3)+(5\times-1/3)=3/3$$

8/3, −3/3

$$\begin{bmatrix}3\\1\\-4\\2\end{bmatrix}-\begin{bmatrix}2\\-2\\-1\\1\end{bmatrix}=\begin{bmatrix}1\\3\\-3\\1\end{bmatrix}$$

6/3

$b-A\hat{x}$

$\|b-A\hat{x}\|$

$=\sqrt{(1)^2+(3)^2+(-3)^2+(1)^2}$

$=\sqrt{1+9+9+1}$

$=\sqrt{20}$

$\hookrightarrow 2\sqrt{5}$

cont'd

Cont'd 5

a) $\begin{bmatrix} 6 & 6 \\ 6 & 42 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \end{bmatrix}$

b) $\hat{x} = \begin{bmatrix} 4/3 \\ -1/3 \end{bmatrix}$

c) $2\sqrt{5}$

6. Find (a) the orthogonal projection of **b** onto Col A and (b) a least-squares solution of $A\mathbf{x} = \mathbf{b}$. [*Hint:* Don't work too hard here. Investigate the columns of A.]

(1)

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \\ 1 & 2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$$

6. 1) $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \\ 1 & 2 \end{bmatrix}, b = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$

a) Columns of A orthogonal?

$a_1 \cdot a_2 = 0?$

$a_1 \cdot a_2 = \underbrace{(1)(2)}_{2} + \underbrace{(-1)(4)}_{-4} + \underbrace{(1)(2)}_{2} = 0$

Yes, columns of A are orthogonal!

So, the orthogonal projection of b onto Col A is given by:

$\hat{b} = \frac{b \cdot a_1}{a_1 \cdot a_1} a_1 + \frac{b \cdot a_2}{a_2 \cdot a_2} a_2$

Cont'd

Cont'd

$$b \cdot a_1 = (3)(1) + (-1)(-1) + (5)(1) = 9$$

$$a_1 \cdot a_1 = (1)(1) + (-1)(-1) + (1)(1) = 3$$

$$b \cdot a_2 = (3)(2) + (-1)(4) + (5)(2) = 12$$

$$a_2 \cdot a_2 = (2)(2) + (4)(4) + (2)(2) = 24$$

$$\hat{b} = \frac{9}{3}\begin{bmatrix}1\\-1\\1\end{bmatrix} + \frac{12}{24}\begin{bmatrix}2\\4\\2\end{bmatrix} = 3\begin{bmatrix}1\\-1\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}2\\4\\2\end{bmatrix}$$

$$= \begin{bmatrix}3\\-3\\3\end{bmatrix} + \begin{bmatrix}1\\2\\1\end{bmatrix} = \begin{bmatrix}4\\-1\\4\end{bmatrix}$$

b) Least-squares sol of $Ax = b$

We can solve $A\hat{x} = \hat{b}$ now that we know $\hat{b}$ but this is trivial as we know what weights to place on the cols of A to get $\hat{b}$ from the work above!

$$3\begin{bmatrix}1\\-1\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}2\\4\\2\end{bmatrix} \qquad \hat{x} = \begin{bmatrix}3\\1/2\end{bmatrix}$$

Use these!

$$\boxed{\text{a) } \hat{b} = \begin{bmatrix}4\\-1\\4\end{bmatrix} \quad \text{b) } \hat{x} = \begin{bmatrix}3\\1/2\end{bmatrix}}$$

Cont'd →

(2)

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ 5 \\ 6 \\ 6 \end{bmatrix}$$

2) $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix}, b = \begin{bmatrix} 2 \\ 5 \\ 6 \\ 6 \end{bmatrix}$

a) $a_1 \cdot a_2 = (1)(1) + (1)(0) + (0)(1) + (-1)(1) = 0$

$a_1 \cdot a_3 = (1)(0) + (1)(-1) + (0)(1) + (-1)(-1) = 0$

$a_2 \cdot a_3 = (1)(0) + (0)(-1) + (1)(1) + (1)(-1) = 0$

So, columns of A are orthogonal!

$b \cdot a_1 = (2)(1) + (5)(1) + (6)(0) + (6)(-1) = 1$

$a_1 \cdot a_1 = (1)(1) + (1)(1) + (0)(0) + (-1)(-1) = 3$

$b \cdot a_2 = (2)(1) + (5)(0) + (6)(1) + (6)(1) = 14$

$a_2 \cdot a_2 = (1)(1) + (0)(0) + (1)(1) + (1)(1) = 3$

$b \cdot a_3 = (2)(0) + (5)(-1) + (6)(1) + (6)(-1) = -5$

$a_3 \cdot a_3 = (0)(0) + (-1)(-1) + (1)(1) + (-1)(-1) = 3$

$b = \frac{b \cdot a_1}{a_1 \cdot a_1} a_1 + \frac{b \cdot a_2}{a_2 \cdot a_2} a_2 + \frac{b \cdot a_3}{a_3 \cdot a_3} a_3$

Cont'd

Cont'd

$$\hat{b} = \frac{1}{3}\begin{bmatrix}1\\1\\0\\-1\end{bmatrix} + \frac{14}{3}\begin{bmatrix}1\\0\\1\\1\end{bmatrix} + \frac{-5}{3}\begin{bmatrix}0\\-1\\1\\-1\end{bmatrix}$$

$$= \begin{bmatrix}1/3\\1/3\\0\\-1/3\end{bmatrix} + \begin{bmatrix}14/3\\0\\14/3\\14/3\end{bmatrix} + \begin{bmatrix}0\\5/3\\-5/3\\5/3\end{bmatrix} = \begin{bmatrix}15/3\\6/3\\9/3\\18/3\end{bmatrix} = \begin{bmatrix}5\\2\\3\\6\end{bmatrix}$$

b)

$$\frac{1}{3}\begin{bmatrix}\cdot\\\cdot\\\cdot\end{bmatrix} + \frac{14}{3}\begin{bmatrix}\cdot\\\cdot\\\cdot\end{bmatrix} + \frac{-5}{3}\begin{bmatrix}\cdot\\\cdot\\\cdot\end{bmatrix} \qquad \hat{x} = \begin{bmatrix}1/3\\14/3\\-5/3\end{bmatrix}$$

$$\boxed{\text{a) } \hat{b} = \begin{bmatrix}5\\2\\3\\6\end{bmatrix} \qquad \text{b) } \hat{x} = \begin{bmatrix}1/3\\14/3\\-5/3\end{bmatrix}}$$

7. Let A be an $m \times n$ matrix such that $A^T A$ is invertible. Show that the columns of A are linearly independent. [*Careful:* You may NOT assume that A is invertible; it may not even be square.]

Linear Independence of Matrix Columns

Suppose that we begin with a matrix $A = [\mathbf{a}_1 \quad \cdots \quad \mathbf{a}_n]$ instead of a set of vectors. The matrix equation $A\mathbf{x} = \mathbf{0}$ can be written as

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{0}$$

Each linear dependence relation among the columns of A corresponds to a nontrivial solution of $A\mathbf{x} = \mathbf{0}$. Thus we have the following important fact.

> The columns of a matrix A are linearly independent if and only if the equation $A\mathbf{x} = \mathbf{0}$ has *only* the trivial solution. (3)

THEOREM 8

The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a. A is an invertible matrix.
b. A is row equivalent to the $n \times n$ identity matrix.
c. A has n pivot positions.
d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
e. The columns of A form a linearly independent set.
f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.
h. The columns of A span $\mathbb{R}^n$.
i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
j. There is an $n \times n$ matrix C such that $CA = I$.
k. There is an $n \times n$ matrix D such that $AD = I$.
l. A^T is an invertible matrix.

THEOREM

The Invertible Matrix Theorem (continued)

Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

m. The columns of A form a basis of $\mathbb{R}^n$.

n. $\operatorname{Col} A = \mathbb{R}^n$

o. $\dim \operatorname{Col} A = n$

p. $\operatorname{rank} A = n$

q. $\operatorname{Nul} A = \{\mathbf{0}\}$

r. $\dim \operatorname{Nul} A = 0$

PROOF Statement (m) is logically equivalent to statements (e) and (h) regarding linear independence and spanning. The other five statements are linked to the earlier ones of the theorem by the following chain of almost trivial implications:

$$(g) \Rightarrow (n) \Rightarrow (o) \Rightarrow (p) \Rightarrow (r) \Rightarrow (q) \Rightarrow (d)$$

Statement (g), which says that the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$, implies statement (n), because $\operatorname{Col} A$ is precisely the set of all $\mathbf{b}$ such that the equation $A\mathbf{x} = \mathbf{b}$ is consistent. The implications (n) $\Rightarrow$ (o) $\Rightarrow$ (p) follow from the definitions of *dimension* and *rank*. If the rank of A is n, the number of columns of A, then $\dim \operatorname{Nul} A = 0$, by the Rank Theorem, and so $\operatorname{Nul} A = \{\mathbf{0}\}$. Thus (p) $\Rightarrow$ (r) $\Rightarrow$ (q).

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Also, statement (q) implies that the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, which is statement (d). Since statements (d) and (g) are already known to be equivalent to the statement that A is invertible, the proof is complete. ■

7. Let A be an $m \times n$ matrix such that A^TA is invertible. Show that the columns of A are linearly independent. [Careful: You may not assume that A is invertible; it may not even be square.]

We know that the columns of matrix A are linearly independent if and only if the equation $Ax=0$ has only the trivial solution. Assuming that $Ax=0$, we would need to prove $x=0$,
$A^TAx = A^T0 = 0$
Given A^TA is invertible, by the IMT, $Ax=0$ would only have the trivial solution, this would imply that x is 0 and it would follow that the columns of A are linearly independent.

*Using the excerpts above, I answered the following:

We know that the columns of matrix A are linearly independent if only if the equation Ax = 0 has only the trivial solution. Assuming that Ax = 0, we would need to prove x = 0.

$A^TAx = A^T0 = 0$

Given A^TA is invertible, by the IMT, Ax = 0 would only have the trivial solution. This would imply that x is 0 and it would follow that the columns of A are linearly independent.