1. (a)

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 5 \\ 2 & 1 & -3 & 13 \\ -1 & 1 & 0 & -8 \end{array}\right] \xrightarrow{R_2 - 2R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 5 \\ 0 & -3 & 3 & 3 \\ -1 & 1 & 0 & -8 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 5 \\ 0 & -3 & 3 & 3 \\ 0 & 3 & -3 & -3 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & -3 & 5 \\ 0 & -3 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 5 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 7 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 7 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 - x_3 = 7$
$x_2 - x_3 = -1$
$0 = 0$

$x_1 = 1 + x_3$
$x_2 = -1 + x_3$
x_3 is free

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$$

(b)

Cont'd *

Cont'd

(b)

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 2 & 1 & -3 & 0 \\ -1 & 1 & 0 & 0 \end{array}\right]$$

$\Rightarrow$ all the steps would be the same, but...

we get this!

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$1x_1 - 1x_3 = 0$
$1x_2 - 1x_3 = 0$
$0 = 0$

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$$

$x_1 = x_3$
$x_2 = x_3$
x_3 is free

3. i) 0 in H? ✓

ii) for $\vec{u}, \vec{v}$ in H, $\vec{u}+\vec{v}$ in H? ✓

iii) $c\vec{u}$ in H? ✓ $A = m \times n$ matrix

$Ax = 0$

i) $A \cdot 0 = 0$, and A can be any $m \times n$ matrix, so it would be implied that 0 is in X!

ii) Let $u \in A(x)$ and $v \in A(x)$

$A(u) = 0$ and $A(v) = 0$

So, $A(u+v) = 0$

iii) let c be in $\mathbb{R}$ and $u \in X$,

$c(u) \in \mathbb{R}^n$ and

$$A(cu) = c(A(u)) = c(0) = 0$$

Thus $cu \in V$ as well

The set of X's is a subspace of $\mathbb{R}^n$

The set of all X's is a subspace. We know this because the zero vector can be found in X as implied by AX = 0. For each u and v, we have that A(U) = 0 and A(V) =0, so A(U+V) = 0. For any scalar C and u in X, Cu would give us A(Cu) = C(A(u)) = C(0).

*My explanation might be weak, but keep in mind I was rushing. I kind of get the concept, but I need time to think.

4. $H = \left\{ \begin{bmatrix} -3b - a \\ 3a + 2c \\ a + 3b + 2c \\ 5b - 3c \end{bmatrix} : a, b, c \text{ are real} \right\}$

Subspace of $\mathbb{R}^4$?

Compose $\begin{bmatrix} -3b - a \\ 3a + 2c \\ a + 3b + 2c \\ 5b - 3c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ → If we set a, b, c to 0 all rows are 0 so it could be a subspace!

basis?

$$\begin{aligned} -1a - 3b + 0c &= 0 \\ 3a + 0b + 2c \\ 1a + 3b + 2c \\ 0a + 5b - 3c \end{aligned} \rightarrow v_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} -3 \\ 0 \\ 3 \\ 5 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 2 \\ 2 \\ -3 \end{bmatrix}$$

None appear to be a comb of other 2 so they are all LI!

Thus $\{v_1, v_2, v_3\}$ is a basis!

$$\boxed{\left\{ \begin{bmatrix} -1 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \\ -3 \end{bmatrix} \right\}}$$

5. $A = \begin{bmatrix} 3 & 5 & 5 & 2 & 0 \\ 1 & 0 & 2 & 2 & 1 \\ 1 & 1 & 1 & -2 & -2 \\ -2 & 0 & -4 & -2 & -2 \end{bmatrix}$

a)

$$\begin{bmatrix} 3 & 5 & 5 & 2 & 0 \\ 1 & 0 & 2 & 2 & 1 \\ 1 & 1 & 1 & -2 & -2 \\ -2 & 0 & -4 & -2 & -2 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 0 & 2 & 2 & 1 \\ 3 & 5 & 5 & 2 & 0 \\ 1 & 1 & 1 & -2 & -2 \\ -2 & 0 & -4 & -2 & -2 \end{bmatrix}$$

$$\xrightarrow{R_2 - 3R_1,\ R_3 - R_1,\ R_4 + 2R_1} \begin{bmatrix} 1 & 0 & 2 & 2 & 1 \\ 0 & 5 & -1 & -4 & -3 \\ 0 & 1 & -1 & -4 & -3 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & 2 & 1 \\ 0 & 1 & -1 & -4 & -3 \\ 0 & 5 & -1 & -4 & -3 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 2 & 2 & 1 \\ 0 & 1 & -1 & -4 & -3 \\ 0 & 0 & 4 & 16 & 12 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix}$$

4 pivot cols
↓
dim of Col A is 4
↓
Rank = 4

dim Nul A → 5 cols, but 4 pivots, so 1 has no pivot

dim Nul A = 1

$\boxed{\text{Rank } A = 4 \quad \dim \text{Nul } A = 1}$

b) Basis for Col A:
looking at echelon form, pivots in cols 1, 2, 3, 4, so vectors form basis

Basis → $\{v_1, v_2, v_3, v_4\}$

Row A:
Nonzero rows form basis!

Basis for row A: $\{(1, 0, 2, 2, 1), (0, 1, -1, -4, -2), (0, 0, 4, 16, 12), (0, 0, 0, 2, 0)\}$

Nul A: → Need RREF:

$$\begin{bmatrix} 1 & 0 & 2 & 2 & 1 \\ 0 & 1 & -1 & -4 & -2 \\ 0 & 0 & 4 & 16 & 12 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & 2 & 1 \\ 0 & 1 & -1 & -4 & -3 \\ 0 & 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -5 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \leftarrow \begin{bmatrix} 1 & 0 & 0 & -6 & -5 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$x_1 = 5$
$x_2 = 0$
$x_3 = 3$
$x_4 = 0$
~~x_5 free~~

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 3 \\ 0 \end{bmatrix}$$

Contd 8b

Cont'd 3b

Basis for col A: $\left\{ \begin{bmatrix} 3 \\ 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \\ 6 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ 1 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -2 \\ -2 \end{bmatrix} \right\}$

Basis for row A:

$\{(1, 0, 2, 3, 0), (0, 1, -1, 4, -3), (0, 0, 4, 16, 12)$
$, (0, 0, 0, 2, 0)\}$

Basis Nul A: $\begin{bmatrix} 5 \\ 0 \\ 3 \\ 2 \\ 10 \end{bmatrix}$

c) Nul A and Dros to Row A

6. $A = \begin{bmatrix} 3 & -4 & -1 \\ 0 & -1 & -1 \\ 0 & -4 & 2 \end{bmatrix}$ det $\begin{bmatrix} 3-\lambda & -4 & -1 \\ 0 & -1-\lambda & -1 \\ 0 & -4 & 2-\lambda \end{bmatrix}$

expansion across 1st col

$= 3-\lambda \cdot \det \begin{bmatrix} -1-\lambda & -1 \\ -4 & 2-\lambda \end{bmatrix}$

$(-1-\lambda)(2-\lambda)$ $\quad -2+\lambda-2\lambda+\lambda^2-(4)$

$\lambda^2-\lambda-6$ $\quad (\lambda+2)(\lambda-3)$

alg mul

Eigen: $-2, 3$

$\boxed{\text{A) multiplicity } m_1 = -2 \quad \text{mul } m_2 = 3}$

-2 $\begin{bmatrix} 3+2 & -4 & -1 \\ 0 & -1+2 & -1 \\ 0 & -4 & 2+2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\left[\begin{array}{ccc|c} 5 & -4 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -4 & 4 & 0 \end{array}\right]$ $\quad \left[\begin{array}{ccc|c} 5 & 0 & -5 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\left[\begin{array}{ccc|c} 5 & -4 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\quad \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

~~cont~~

$1x_1 - 1x_3 = 0$

$1x_2 - 1x_3 = 0$

x_3 free

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 3+3 & -4 & -1 \\ 0 & -1+3 & -1 \\ 0 & -4 & 2-3 \end{bmatrix} \quad \begin{bmatrix} 0 & -4 & -1 \\ 0 & -4 & -1 \\ 0 & -4 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -4 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$-4x_2 - 1x_3 = 0$

x_3 free

x_1 free

$\frac{-4x_2}{-4} = \frac{x_3}{-4}$

x_1 free

$x_2 = -\frac{1}{4}x_3$

x_3 free

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 1/-4 \\ 1 \end{bmatrix} + x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

b) Basis for -2: $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

Basis for 2: $\left\{ \begin{bmatrix} 0 \\ 1 \\ -4 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

2. a) $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ $\det(A - \lambda I) = \begin{bmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{bmatrix}$

$(1-\lambda)(3-\lambda) - 8$

$3 - \lambda - 3\lambda + \lambda^2 - 8$

de mult

$\lambda^2 - 4\lambda - 5$ $(\lambda + 1)(\lambda - 5)$

add mult

Eigenvals: $-1, 5$

basis?

$A + 1I = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$1x_1 + 2x_2 = 0$

x_2 free

$x_1 = -2x_2$ $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

x_2 free

$x_1 = x_2$

$-1x_1 = -1x_2$

$A - 5I = \begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \rightarrow -1x_1 + 1x_2 = 0$

x_2 free

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ P? → From basis!

$\begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}$ D? from eigenvals

$\begin{bmatrix} -1 & 0 \\ 0 & -5 \end{bmatrix}$

$\boxed{P = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}, D = \begin{bmatrix} -1 & 0 \\ 0 & -5 \end{bmatrix}}$

7 (b) $A = \begin{bmatrix} 3 & 1 & 0 \\ -1 & 5 & 0 \\ 0 & 0 & -4 \end{bmatrix}$ expand across ~~col 3~~

row 3]

$\det \begin{bmatrix} 3-\lambda & 1 & 0 \\ -1 & 5-\lambda & 0 \\ 0 & 0 & -4-\lambda \end{bmatrix}$

$\cancel{0 - 0} + (-4-\lambda) \cdot \det \begin{bmatrix} 3-\lambda & 1 \\ -1 & 5-\lambda \end{bmatrix}$

$(3-\lambda)(5-\lambda)$

$15 - 3\lambda - 5\lambda + \lambda^2 - (-1)$

$(\lambda^2 - 8\lambda + 16)(-4-\lambda)$

$-\lambda^3 + 8\lambda^2 - 16\lambda - 4\lambda^2 + 32\lambda - 64$

$-\lambda^3 + 4\lambda^2 + 16\lambda - 64$

8. $W = \text{Span}\{v_1, v_2\} \rightarrow$ orthon. basis:

$$v_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$$

orthog:

$$v_1 \cdot v_2 = \overset{2}{(1)(2)} + \overset{-8}{(-2)(4)} + \overset{6}{(2)(3)} = 0$$

$$\|v_1\| = \sqrt{1^2 + (-2)^2 + (2)^2}$$

$$= \sqrt{1 + 4 + 4}$$

$$\sqrt{9} = 3$$

$$\|v_2\| = \sqrt{2^2 + 4^2 + 3^2}$$

$$= \sqrt{4 + 16 + 9}$$

$$= \sqrt{29}$$

$$\boxed{\left\{ \frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \frac{1}{\sqrt{29}}\begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix} \right\}}$$

a.

$$v_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, v = \begin{bmatrix} 3 \\ 1 \\ 5 \\ 1 \end{bmatrix}$$

$$y \cdot v_1 = \underset{9}{3(3)} + \underset{1}{(1)(1)} + \underset{-5}{(5)(-1)} + \underset{1}{(1)(1)} = 6$$

$$v_1 \cdot v_1 = 9 + 1 + 1 + 1 = 12$$

$$v \cdot v_2 = \underset{3}{3(1)} + \underset{-1}{(1)(-1)} + \underset{5}{5(1)} + \underset{-1}{(1)(-1)} = 6$$

$$v_2 \cdot v_2 = 1 + 1 + 1 + 1 = 4$$

$$\hat{y} = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2$$

$$\frac{6}{12}\begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix} + \frac{6}{4}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix} + \frac{3}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 3/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{bmatrix} + \begin{bmatrix} 3/2 \\ -3/2 \\ 3/2 \\ -3/2 \end{bmatrix} = \begin{bmatrix} 6/2 \\ -2/2 \\ +2/2 \\ -2/2 \end{bmatrix} = \boxed{\begin{bmatrix} 3 \\ -1 \\ 1 \\ -1 \end{bmatrix}}$$

10

a) $A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 1 & 0 \\ 2 & 2 \end{bmatrix}$, $b = \begin{bmatrix} 0 \\ 4 \\ 6 \\ 2 \end{bmatrix}$

4×2 ~~$\times$~~ 4×1

no match

Cont ~~case~~

$A^T A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 1 & 0 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 10 & 3 \\ 3 & 6 \end{bmatrix}$

2×4 $\quad 4 \times 2$ $\quad 2 \times 2$

$a_{11} = (1 \cdot 1) + (2 \cdot 2) + (1 \cdot 1) + (2 \cdot 2) = 10$

$a_{12} = 1 + (-2) + 0 + 4 = 3$

$a_{21} = 1 + (-2) + 0 + 4 = 3$

$a_{22} = 1 + 1 + \cancel{0} + 4 = 6$

$A^T b = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 4 \\ 6 \\ 2 \end{bmatrix} = \begin{bmatrix} 18 \\ 0 \end{bmatrix}$

2×4 $\quad 4 \times 1$ $\quad 2 \times 1$

$a_{11} = 0 + 8 + 6 + 4 = 18$

$a_{21} = 0 - 4 + 0 + 4 = 0$

$\left[\begin{array}{cc|c} 10 & 3 & 18 \\ 3 & 6 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 0 & -4 & 18 \\ 3 & 6 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 3 & 6 & 0 \\ 0 & -9 & 18 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 6/3 & 0 \\ 0 & -1 & 2 \end{array}\right] \rightarrow \boxed{\begin{bmatrix} 0 \\ -2 \end{bmatrix}}$

(b) $A = \begin{bmatrix} 1 & 2 \\ 1 & 4 \\ 0 & -5 \\ -1 & 6 \end{bmatrix}, b = \begin{bmatrix} 3 \\ -1 \\ 2 \\ 0 \end{bmatrix}$

$A\,\cancel{x}\,2 \quad b = 4 \times 1$

no match

LS

$A^T A = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 2 & 4 & -5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 4 \\ 0 & -5 \\ -1 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 81 \end{bmatrix}$

$a_{11} = 1+1+0+1 = 3$

$a_{12} = 2+4+0-6 = 0$

$a_{21} = 2+4+0-6 = 0$

$a_{22} = 4+16+25+36 = 81$

$\left[\begin{array}{cc|c} 3 & 0 & 2 \\ 0 & 81 & -8 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 2/3 \\ 0 & 1 & -8/81 \end{array}\right] \rightarrow \boxed{\begin{bmatrix} 2/3 \\ -8/81 \end{bmatrix}}$

$A^T b = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 2 & 4 & -5 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -8 \end{bmatrix}$

$a_{11} = 3 - 1 + 0 + 0 = 2$

$a_{21} = 6 - 4 \overset{-14}{-10} + 0 = -8$