Math 30, Wednesday April 8, 2020

1pm class

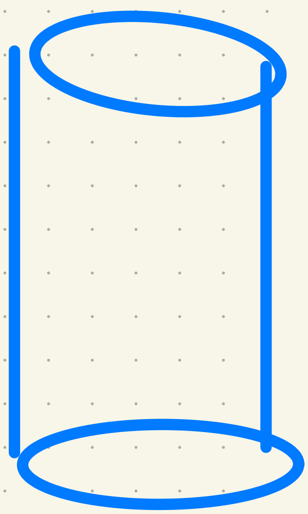
Worksheet on Concavity, etc.

See today's Quiz on Related Rates —

due by 11:59 pm $\underbrace{}_{\text{at night}}$

Questions?

circular cylinder



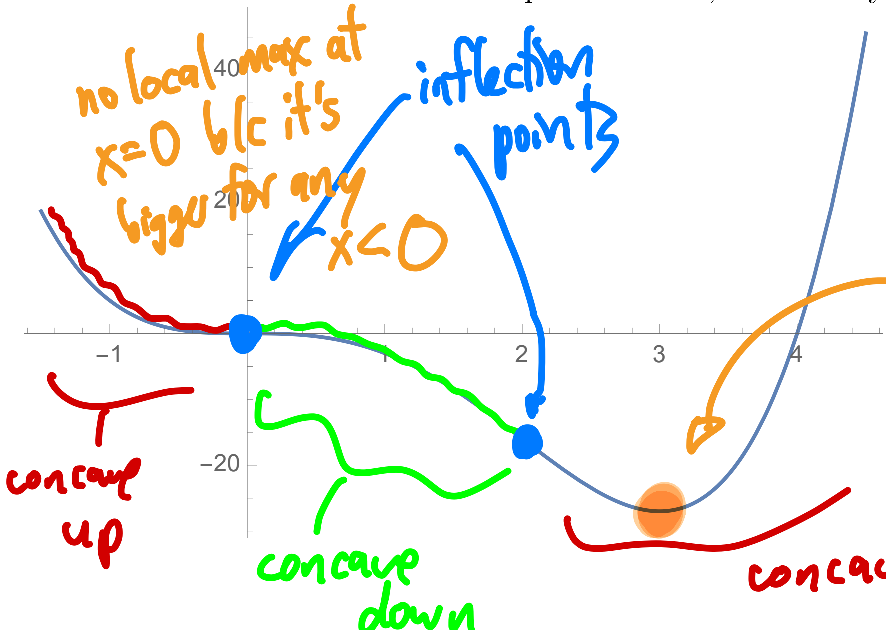
I'll grade quizzes today & tomorrow

#1. First, using the graph:

inflection points: where the graph changes concavity

MATH 30, 4/7-8/2020: WORKSHEET ON FIRST AND SECOND DERIV.S

(1) Discuss the graph of $f(x) = x^4 - 4x^3$ with respect to concavity, points of inflection, and local maxima and minima. You can use the picture below, but check your answers using calculus.



local min, no local max.

(2) Sketch the graph of a function satisfying the following conditions:
$f'(0) = f'(2) = f'(4) = 0$,
$f'(x) > 0$ if $x < 0$ or $2 < x < 4$,
$f'(x) < 0$ if $0 < x < 2$ or $x > 4$,
$f''(x) > 0$ if $1 < x < 3$,
$f''(x) < 0$ if $x < 1$ or $x > 3$.

Now check using calculus:

(3) For the following functions, find the intervals on which f is increasing or decreasing. Find the local maxima and minima of f. Find the intervals of concavity and the inflection points. Then make a rough sketch of the graph.
(a) $f(x) = 4x^3 + 3x^2 - 6x + 1$

(b) $g(x) = \sin x + \cos x$ for $0 \leq x \leq 2\pi$.

(c) $h(x) = x^4 e^{-x}$.

(d) $p(x) = 2 + 2x^2 - x^4$.

(4) Find the local maxima and minima of $f(x) = \dfrac{x^2}{x-1}$ using both the First Derivative Test ("Increasing/Decreasing") and the Second Derivative Test ("Concavity"). Which do you prefer?

#1. $f(x) = x^4 - 4x^3$.

Try doing without the graph.

To see where incr./decr: use $f'(x)$

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$$

critical pts: $x=0$ and $x=3$

where $f'=0$

when $x<3$: $f'(x) = \underbrace{4x^2}_{\text{positive}}\underbrace{(x-3)}_{\text{negative}} \leq 0$

decreasing on $(-\infty, 0)$

and on $(0,3)$.

where $f'<0$

so f is decreasing.

when $x>3$:

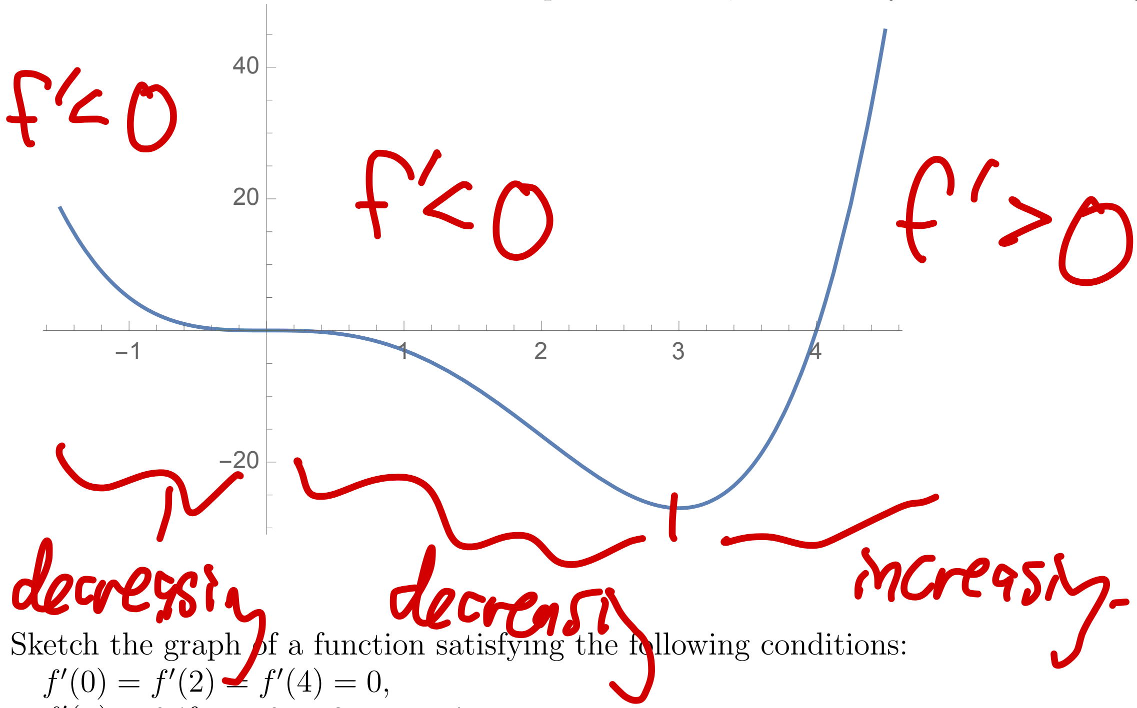
$f'(x) = \underbrace{4x^2}_{\text{pos.}}\underbrace{(x-3)}_{\text{pos.}} > 0$

so f is increasing on $(3, \infty)$

This agrees w/ the graph

MATH 30, 4/7-8/2020: WORKSHEET ON FIRST AND SECOND DERIV.S

(1) Discuss the graph of $f(x) = x^4 - 4x^3$ with respect to concavity, points of inflection, and local maxima and minima. You can use the picture below, but check your answers using calculus.



(2) Sketch the graph of a function satisfying the following conditions:
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$f'(x) > 0$ if $x < 0$ or $2 < x < 4$,
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(3) For the following functions, find the intervals on which f is increasing or decreasing. Find the local maxima and minima of f. Find the intervals of concavity and the inflection points. Then make a rough sketch of the graph.
(a) $f(x) = 4x^3 + 3x^2 - 6x + 1$

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(d) $p(x) = 2 + 2x^2 - x^4$.

(4) Find the local maxima and minima of $f(x) = \dfrac{x^2}{x-1}$ using both the First Derivative Test ("Increasing/Decreasing") and the Second Derivative Test ("Concavity"). Which do you prefer?

#1 cont'd. $f(x) = x^4 - 4x^3$

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$$

for concavity:

$$f''(x) = 12x^2 - 24x = 12x(x-2)$$

for $x<0$: $f''(x) = \underbrace{12x}_{\text{neg}}\underbrace{(x-2)}_{\text{neg}} > 0$

so: f is concave up on $(-\infty, 0)$

for $0<x<2$: $f''(x) = \underbrace{12x}_{\text{pos}}\underbrace{(x-2)}_{\text{neg}} < 0$

so: f is concave down on $(0,2)$

$x>2$: $f''>0$ so f is concave up on $(2,\infty)$

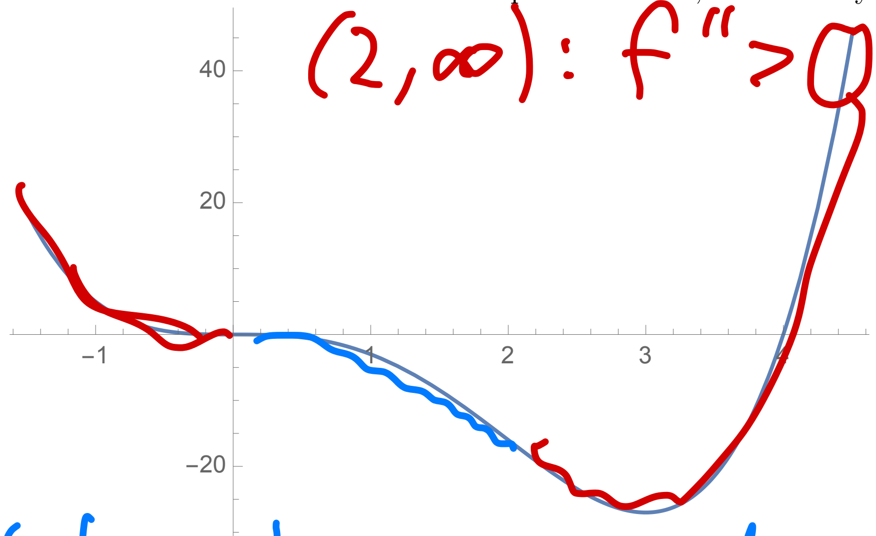
Summary: $(-\infty, 0)$: $f'' > 0$ concave up

MATH 30, 4/7-8/2020: WORKSHEET ON FIRST AND SECOND DERIV.S

$(0,2)$: $f'' < 0$ concave down

(1) Discuss the graph of $f(x) = x^4 - 4x^3$ with respect to concavity, points of inflection, and local maxima and minima. You can use the picture below, but check your answers using calculus.

$(2, \infty)$: $f'' > 0$ concave up



inflection pts: $x=0$ and $x=2$

(2) Sketch the graph of a function satisfying the following conditions:

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local max & min:

(3) For the following functions, find the intervals on which f is increasing or decreasing. Find the local maxima and minima of f. Find the intervals of concavity and the inflection points. Then make a rough sketch of the graph.

(a) $f(x) = 4x^3 + 3x^2 - 6x + 1$.

One way: use incr./decr.

(b) $g(x) = \sin x + \cos x$ for $0 \le x \le 2\pi$.

(c) $h(x) = x^4 e^{-x}$.

f is decr. from $(-\infty, 3)$

(d) $p(x) = 2 + 2x^2 - x^4$.

Then incr. from $(3, \infty)$

(4) Find the local maxima and minima of $f(x) = \dfrac{x^2}{x-1}$ using both the First Derivative Test ("Increasing/Decreasing") and the Second Derivative Test ("Concavity"). Which do you prefer?

So it has a local min at $x=3$

Another way to find local max/min:

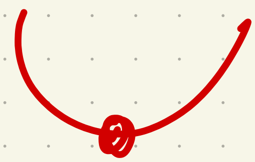
Find crit pts. Here: $\underbrace{x = 0, 3}_{\text{candidates for local max/min.}}$

$$f''(x) = 12x^2 - 24x = 12x(x-2)$$

plug crit pts into f'':

$f''(0) = 0$ no information

$\underbrace{f''(3) = 36 > 0}_{\text{so } f \text{ is concave up there}}$



so f has a local min at $x = 3$

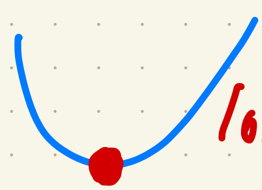
& $f'(3) = 0$
and $f''(3) > 0$

Method for finding local max/min using concavity:

First find crit. pts (where $f' = 0$
&
tang. line is horizontal)

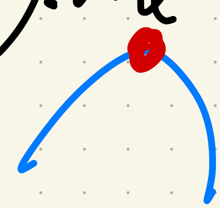
Then plug crit. pts into f'':

if $f''(c) > 0$: concave up
and tang. line is horiz:
so it looks like



local min

if $f''(c) < 0$: concave down
and tang. line is horiz
so it looks like:



local max.

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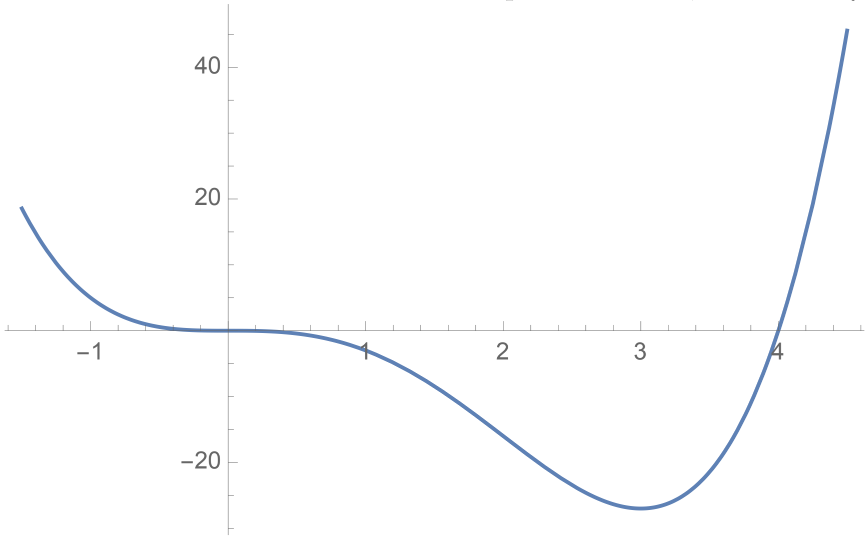

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#4, (d) $p(x) = 2 + 2x^2 - x^4$.

(4) Find the local maxima and minima of $f(x) = \dfrac{x^2}{x-1}$ using both the First Derivative Test ("Increasing/Decreasing") and the Second Derivative Test ("Concavity"). Which do you prefer?

#4. $f(x) = \dfrac{x^2}{x-1}$

try sketching the graph using incr/decr. and concavity.

f' tells us where f is incr. or decr:

$$f'(x) = \frac{2x(x-1) - x^2}{(x-1)^2} \quad \text{(Quotient Rule)}$$

$$= \frac{x^2 - 2x}{(x-1)^2} = \frac{x(x-2)}{(x-1)^2}$$

note: $\lim\limits_{x \to 1^-} f(x) = \dfrac{1}{0^-} = -\infty$

$\lim\limits_{x \to 1^+} f(x) = \dfrac{1}{0^+} = \infty$

vertical asymptote at $x = 1$.

$$f'(x) = \frac{x(x-2)}{(x-1)^2} \left(= \frac{x^2-2x}{(x-1)^2} \right)$$

for $x<0$: $f'>0$ $\rightarrow$ f is incr. $\left(\frac{\text{neg} \times \text{neg}}{\text{pos}} \right)$

for $0<x<1$: $f'<0$ $\rightarrow$ f is decr. $\left(\frac{\text{pos} \times \text{neg}}{\text{pos}} \right)$

for $1<x<2$: $f'<0$ $\rightarrow$ f is decr. $\left(\frac{\text{pos} \times \text{neg}}{\text{pos}} \right)$

for $x>2$: $f'>0$ $\rightarrow$ f is incr. $\left(\frac{\text{pos} \times \text{pos}}{\text{pos}} \right)$

$$f(x) = \frac{x^2}{x-1}$$

we found: $\lim_{x\to 1^-} f(x) = -\infty$, $\lim_{x \to 1^+} f(x) = +\infty$

f is incr. on $(-\infty, 0)$, decr. on $(0,1)$ and $(1,2)$, incr. on $(2,\infty)$

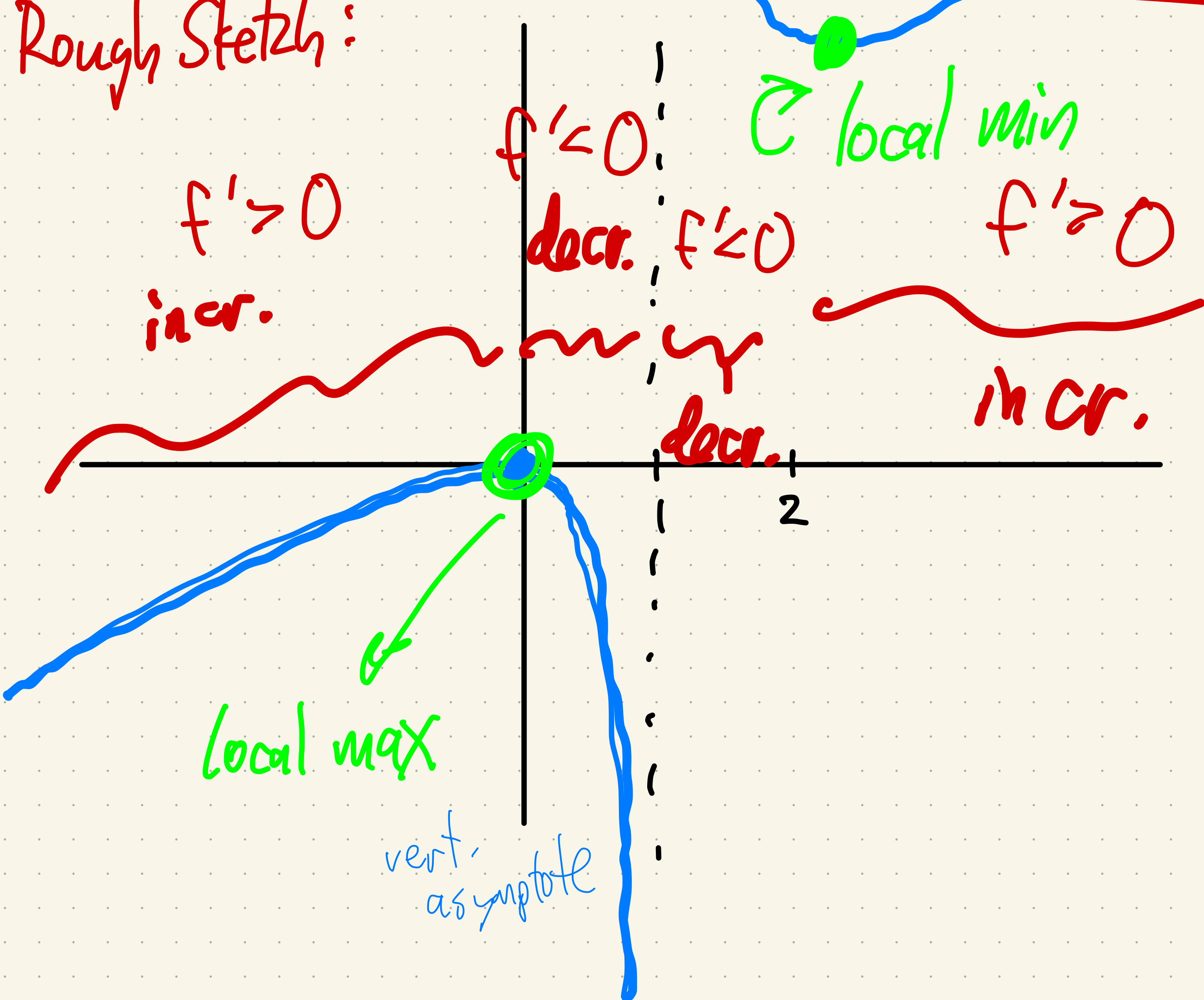
Summary.

$$f(x) = \frac{x^2}{x-1}$$

we found: $\lim_{x \to 1^-} f(x) = -\infty$, $\lim_{x \to 1^+} f(x) = +\infty$

f is incr. on $(-\infty, 0]$, decr. on $(0, 1)$ and $(1, 2]$ incr. on $(2, \infty)$

Rough Sketch:



I did all that with calculus —

no computer!

For fun: compare w/ computer graph:

WolframAlpha

plot x^2/(x-1) from -2 to 3

Input interpretation

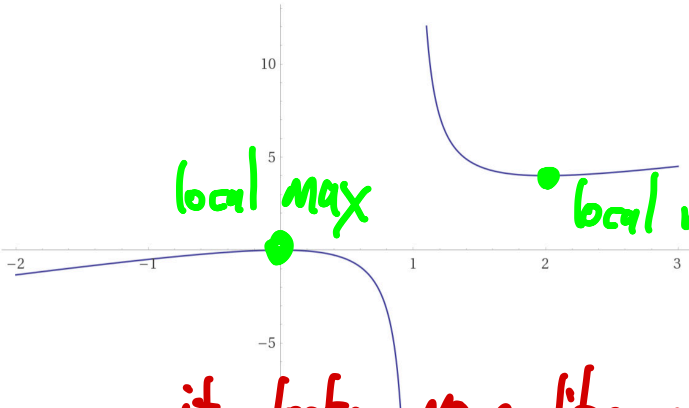
plot	$\frac{x^2}{x-1}$	$x = -2$ to 3

Result

(arc length is infinite)

Plot

looks like my graph, right?



it looks more like mine if done to scale

Check concavity:

$$f(x) = \frac{x^2}{x-1}$$

$$f'(x) \overset{\text{Q.R.}}{=} \frac{2x(x-1) - x^2}{(x-1)^2} = \frac{x^2 - 2x}{(x-1)^2}$$

$$f''(x) \overset{\text{Q.R.}}{=} \frac{(2x-2)(x-1)^2 - 2(x-1)x(x-2)}{(x-1)^4}$$

added later

$$= \frac{2(x-1)^3 - 2x(x-1)(x-2)}{(x-1)^4}$$

ran out of time

where $f'' > 0$: concave up

where $f'' < 0$: concave down.

Added later:

$$f''(x) = \frac{2(x-1)^3 - 2x(x-1)(x-2)}{(x-1)^4}$$

$$= \frac{2(x-1)\left[(x-1)^2 - x(x-2)\right]}{(x-1)^4}$$

$$= \frac{2(x-1)\left[x^2-2x+1-x^2+2x\right]}{(x-1)^4}$$

$$= \frac{2(1)}{(x-1)^3} = \frac{2}{(x-1)^3}$$

for $x<1$: $f''<0$ so concave down

for $x>1$: $f''>0$ so concave up.

This agrees with our graph.

That's all for today!

Please turn in quiz

by 11:59pm.

Goodbye!