Math 30, Friday, 3/20/2020

1pm class

Questions? Comments?

For now:

HW is "honesty policy"

In future: submit on Canvas?

For now: have fun working on problems!

Now: review notes -

#5 from worksheet

MATH 30, 3/20/2020: LINEAR APPROXIMATION

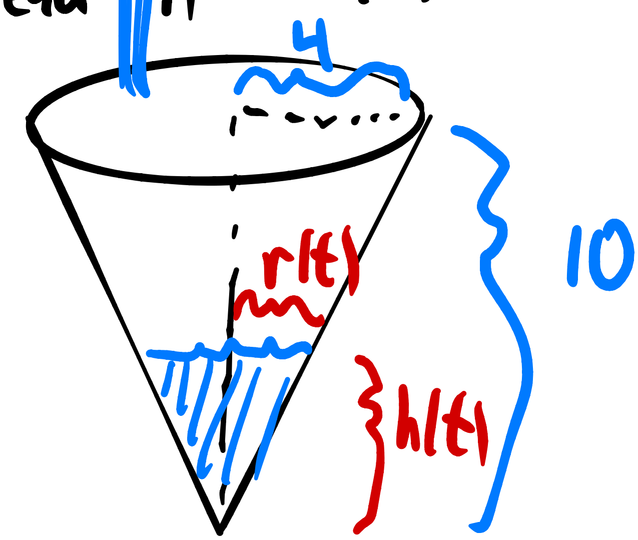
We will start with a **related rates** example, for review:

1. Read the problem carefully.
2. Draw a picture if possible.
3. Introduce notation.
4. Express the given information mathematically.
5. Write an equation that relates the various quantities.
6. Use the Chain Rule.
7. Substitute into the resulting equation and solve for the related rate.

Example. Water is being poured into an inverted cone (vertex down) of radius 4 inches and height 10 inches at a rate of 3 cubic inches per second. Find the rate at which the water level is rising when the depth of the water over the vertex is 6 inches.

1. read it - ok!

2.



4. Given rate of change of water vol:

$$\frac{dV}{dt} = 3\,\frac{in^3}{sec}.$$

3. $V(t)$ be the vol. at time t

$r(t)$ " radius "

$h(t)$ " height "

Next topic: Another cool, smart math shortcut.

Main idea: If a function f is differentiable at $x = a$, then the tangent line is the *best linear approximation* near that point. [Sketch.] Zoom in! The curve looks flat to a bug, just like how the Earth looks flat to us on the surface. The tangent line really is a good linear approximation (the best!).

This will look really cool on my tablet!

Remember: the tangent line through $(a, f(a))$ has slope $f'(a)$, so its equation is

$$y = f(a) + f'(a)(x - a).$$

(It's a function of x!)

Check: It has the right slope ✓ It goes through $(a, f(a))$ ✓

The tangent line gives a good approximation, so

$$f(x) \approx f(a) + f'(a)(x - a)$$

for x close to a.

Example. What is the decimal expansion for the side length of a square of area 125? [Sketch.]

Solution. Here the relevant function is $f(x) = \sqrt{x}$. We know $\sqrt{121} = 11$, and 121 is pretty close to 125, so use $a = 121$ as a "base point."

Using the above formula, we have

$$f(125) \approx f(121) + f'(121)(125 - 121).$$

That is,

$$\sqrt{125} \approx 11 + \frac{4}{2 \cdot 11} = 11 + \frac{2}{11}.$$

Using long division, we can find $\frac{2}{11} = 0.1818\ldots$, so

$$\sqrt{125} \approx 11.18.$$

Now compare this with your calculator:

$$\sqrt{125} = 11.18033\ldots$$

We were pretty close!

For review, see the "pop quiz" on differentiation of inverse functions and derivatives of exponentials and logarithms.

Given $\frac{dV}{dt} = 3 \frac{in^3}{sec}$

5. Eq = relating V, r, h:

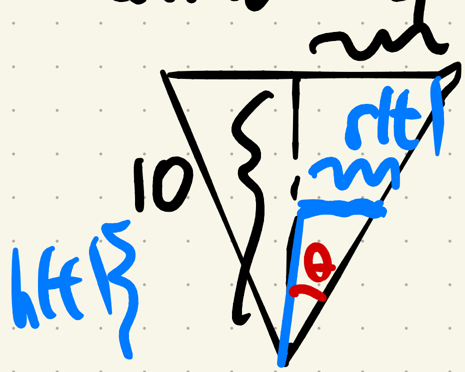
vol. of cone (look it up)

$$V = \frac{1}{3}\pi r^2 h$$

$$V(t) = \frac{1}{3}\pi r(t)^2 h(t)$$ as time changes.

Also, we have a relationship b/w r & h:

Sideview of cone:



similar triangles:

$$\tan\theta = \frac{4}{10} = \frac{r(t)}{h(t)}$$

$$\tan\theta = \frac{4}{10} = \frac{r(t)}{h(t)}$$ So $r(t) = \frac{2}{5}h(t)$

at all times.

So insert into

$$V(t) = \frac{1}{3}\pi r(t)^2 h(t):$$

$$V(t) = \frac{1}{3}\pi\left(\frac{2}{5}h(t)\right)^2 h(t)$$

$$= \frac{1}{3}\cdot\frac{4}{25}\pi\, h(t)^3$$

That is,

$$V(t) = \frac{4}{75}\pi\, h(t)^3$$

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Step 6. Use Chain Rule:

differentiate:

Chain Rule!

$$\frac{dV}{dt} = \frac{4\pi}{75}\left(3h(t)^2 h'(t)\right)$$

$$= \frac{12\pi}{75} h(t)^2 h'(t)$$

$$V'(t) = \frac{4\pi}{25} h(t)^2 h'(t)$$

$$V'(t) = \frac{4\pi}{25} h(t)^2 h'(t)$$

rate of change of vol.

rate of change of height of water.

Recall the Q:

What is $\frac{dh}{dt}$ at the moment when $h = 6$?

So plug in $h(t) = 6$

remember: we are given that

$$\frac{dV}{dt} = 3 \, \frac{\text{in}^3}{\text{sec}}$$

$$V'(t) = \frac{4\pi}{25} h(t)^2 h'(t)$$

Plug in $\frac{dV}{dt} = 3$, $h = 6$:

square it

$$3 = \frac{4\pi}{25} \cdot 36 \cdot h'(t)$$

simplify:

$$1 = \frac{4\pi}{25} \cdot 12 \cdot h'(t)$$

Final answer

so $\boxed{h'(t) = \frac{25}{48\pi} \ \frac{\text{inches}}{\text{sec}}}$

key: $V = \frac{1}{3}\pi r^2 h$ relates the quantities

Then diff. using Chain Rule.

Extra: Using geometry we got

$$V(t) = \underbrace{\frac{4}{75}\pi}_{\text{is a constant.}} h(t)^3$$

diff. both sides

Remember:

$$\frac{d}{dt} h(t)^3 = \ldots$$

The inside function: $h(t)$

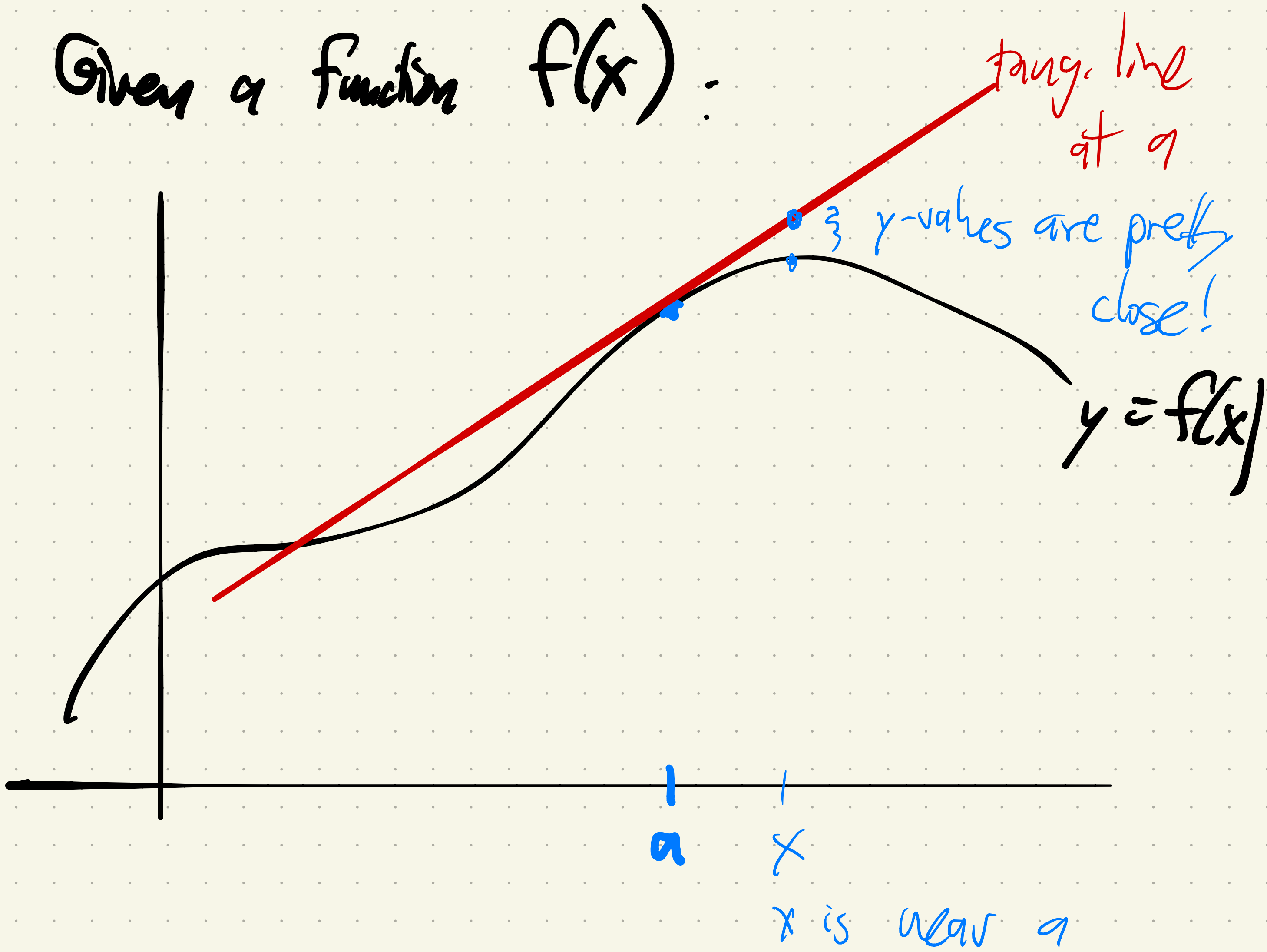
The outside fn: x^3

plug in

$$\frac{d}{dt} h(t)^3 = \underbrace{3(h(t))^2 \cdot h'(t)}_{\text{Chain Rule.}}$$ to get $h(t)^3$

Next: Linear approximation.

Given a function $f(x)$:



Can use the tang. line at a to get a good approx. near a.

Eq^{n} of tang. line:

$$y = f'(a)\cdot(x-a) + f(a)$$

Check: correct slope $f'(a)$ ✓

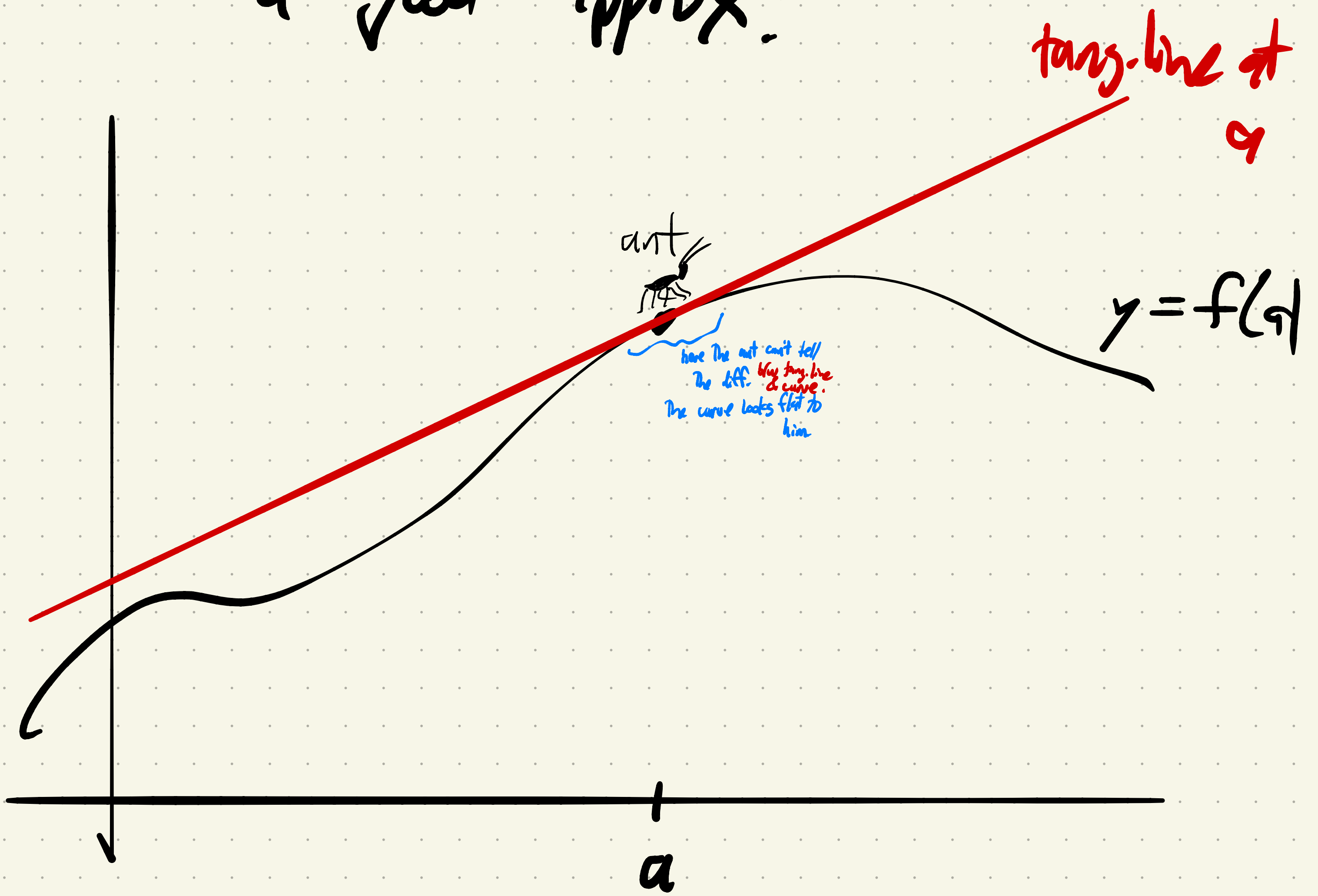
goes through $(a, f(a))$ ✓

So for x near a:

$$f(x) \approx \underbrace{f(a) + f'(a)\cdot(x-a)}_{\text{y-value of tangent line.}}$$

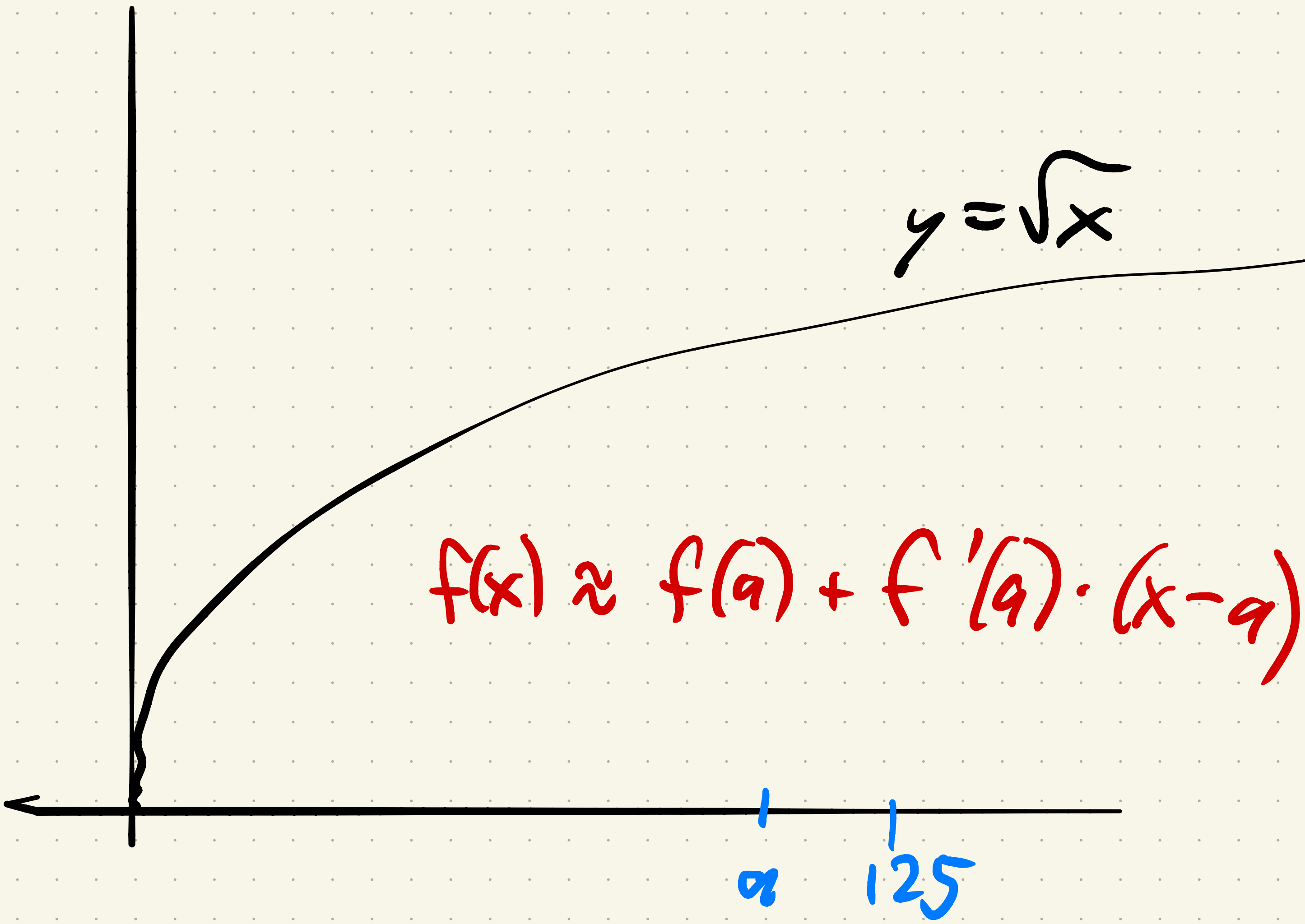
$f(x)$: y-value of f.

a good approx!



"Party Trick": Approx. $\sqrt{125}$.

Picture: The relevant f is
$$f(x) = \sqrt{x}$$



Pick "a" so that $f(a)$ is easy to find and a is near $x=125$.

Good choice: $a=121$ because it's near 125 and $\sqrt{121}=11$.

$$f(x) \approx f(a) + f'(a) \cdot (x-a)$$

use $x = 125$ and $a = 121$.

$$f(x) = \sqrt{x} = x^{1/2}$$

$$f'(x) = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}}.$$

$$\sqrt{125} \approx \sqrt{121} + \frac{1}{2\sqrt{121}}(125-121)$$

Simplify:

$$\sqrt{125} \approx 11 + \frac{1}{22}(4) = 11 + \frac{2}{11}$$

lin. approx.

$$\begin{array}{r} 0.181818\ldots \\ 11\overline{)\,2.0000} \\ \underline{11} \\ 88 \end{array}$$

(check)

So $\sqrt{125} \approx 11.1818\ldots$

That is, the linear approx gives:

$$\sqrt{125} \approx 11.1818\ldots$$

Compare to calculator:

$$\sqrt{125} = 11.1803\ldots$$

Very close!

Good approximation

Tang. line helps approximation.

I'll Think about exams
and HW/quizzes.

For now, have fun learning!!

last Q's?

email me.