Math 30, Wednesday April 29, 2020

1pm class

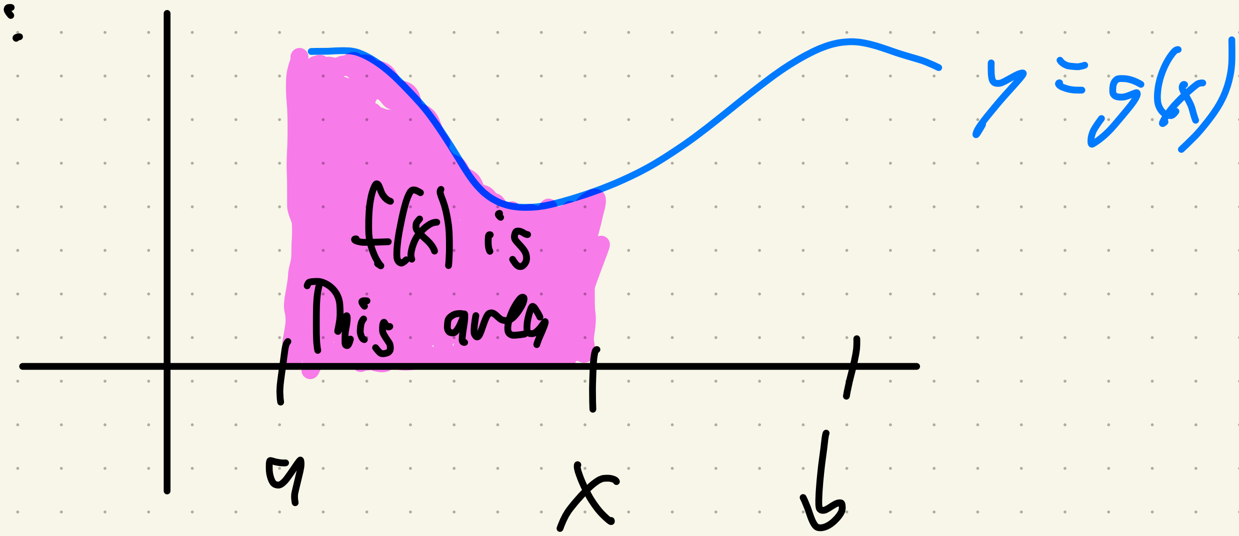
Integrals and Averages

Fund. Thm. of Calc. Part I:

Given a function $g(x)$, define a function

$$f(x) = \int_a^x g(t)\,dt$$

Picture:

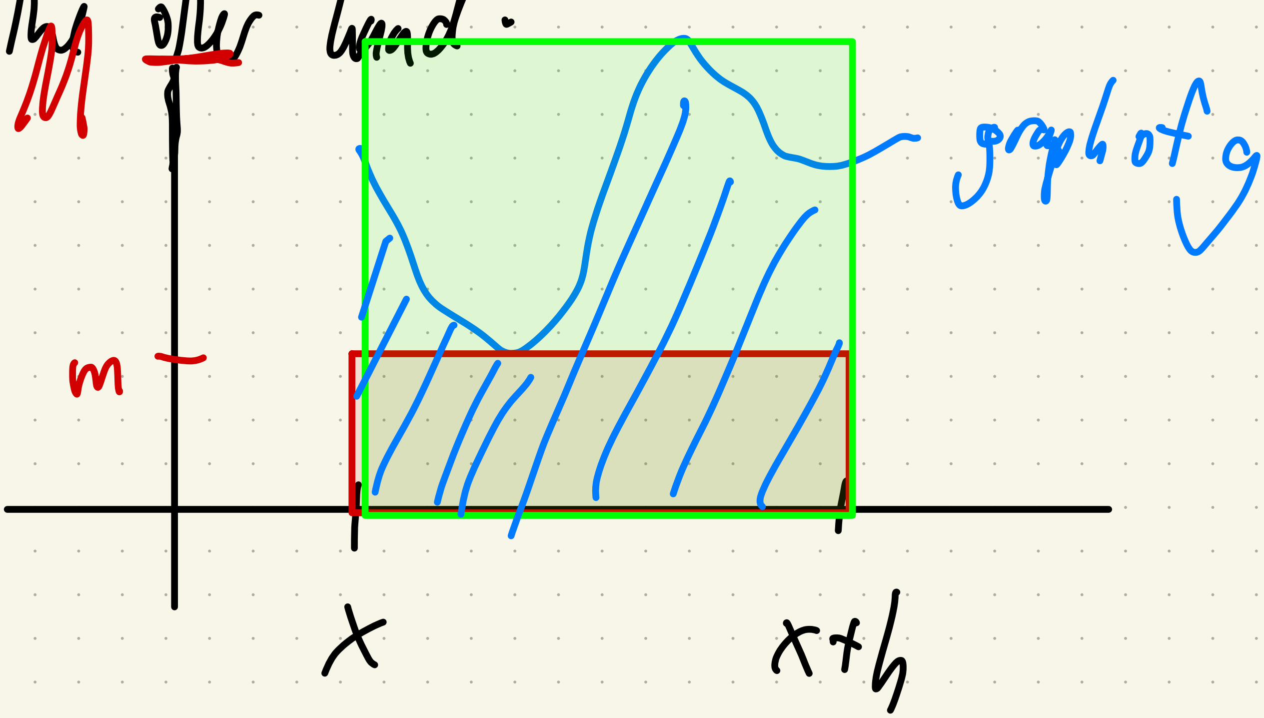


On the one hand,

last time

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} g(t)\,dt$$

On the other hand:



Let m be the min. of g on $[x, x+h]$
and M be the max. of g on $[x, x+h]$

Compare areas:

$$\underbrace{mh}_{\text{small red rectangle}} \leq \underbrace{\int_x^{x+h} g(t)\,dt}_{\text{area under blue curve}} \leq \underbrace{Mh}_{\text{large green rectangle}}$$

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divide by h:

$$m \le \frac{1}{h}\int_x^{x+h} g(t)\,dt \le M$$

let $h \to 0$: m and M both approach $g(x)$

So by Squeeze Theorem

$$\lim_{h\to 0} \frac{1}{h}\int_x^{x+h} g(t)\,dt = g(x).$$

Comparing the parts of the proof:

$$f'(x) = \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} g(t)\,dt$$

$$\parallel$$

$$g(x)$$

So:

$$f'(x) = g(x).$$

Summary: Given $g(x)$, the function

$$f(x) = \int_a^x g(t)\,dt \text{ satisfies } f'(x) = g(x).$$

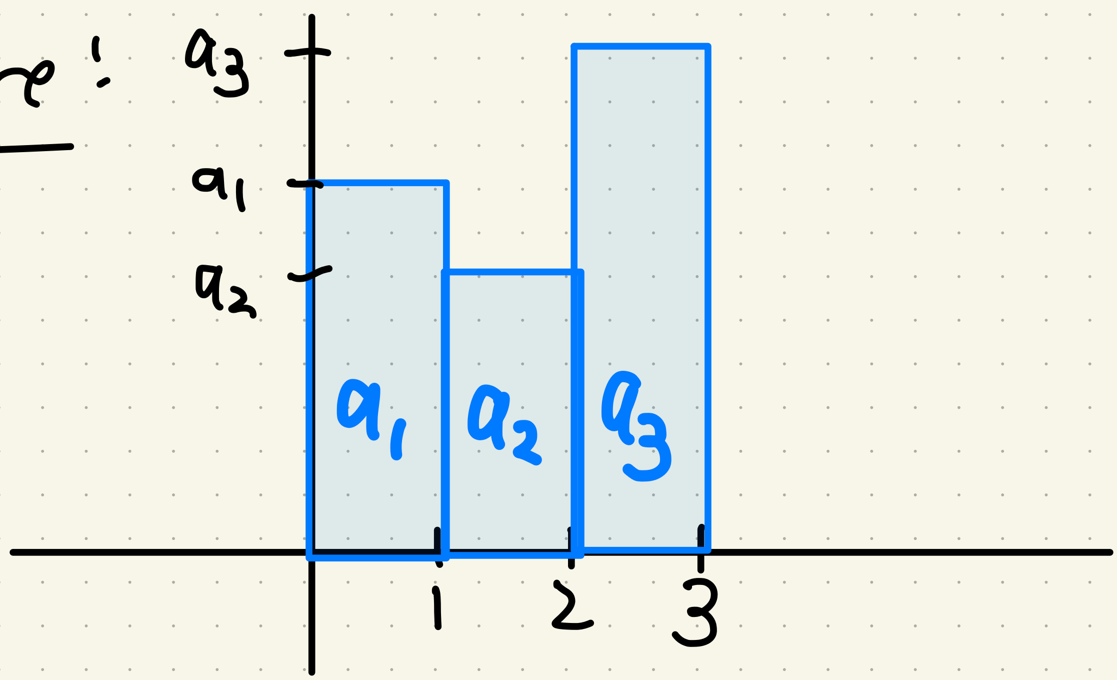
Another way to understand it:

in terms of <u>averages</u>.

<u>Review</u>: The average of three numbers a_1, a_2, and a_3 is:

$$\frac{a_1 + a_2 + a_3}{3}$$

<u>Picture</u>:



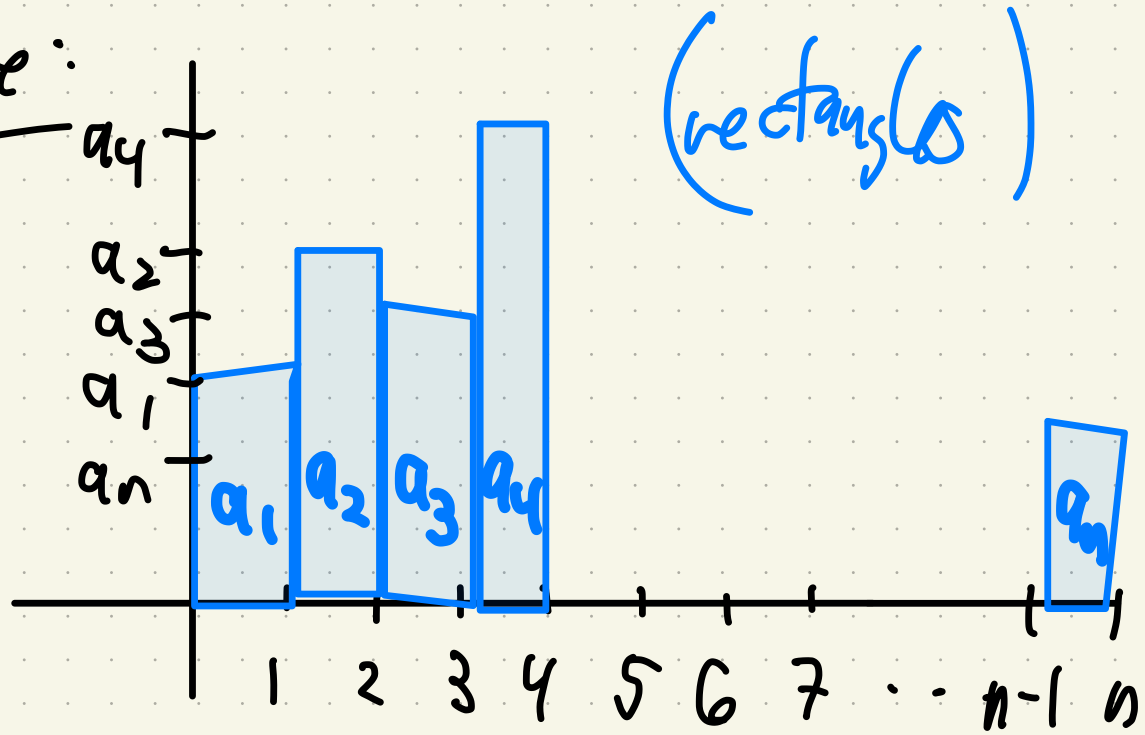
The average is the total area divided by the length of the interval.

in general, the average of n numbers

$a_1, a_2, a_3, \ldots, a_n$ is

$$\frac{a_1 + a_2 + a_3 + \cdots + a_n}{n}$$

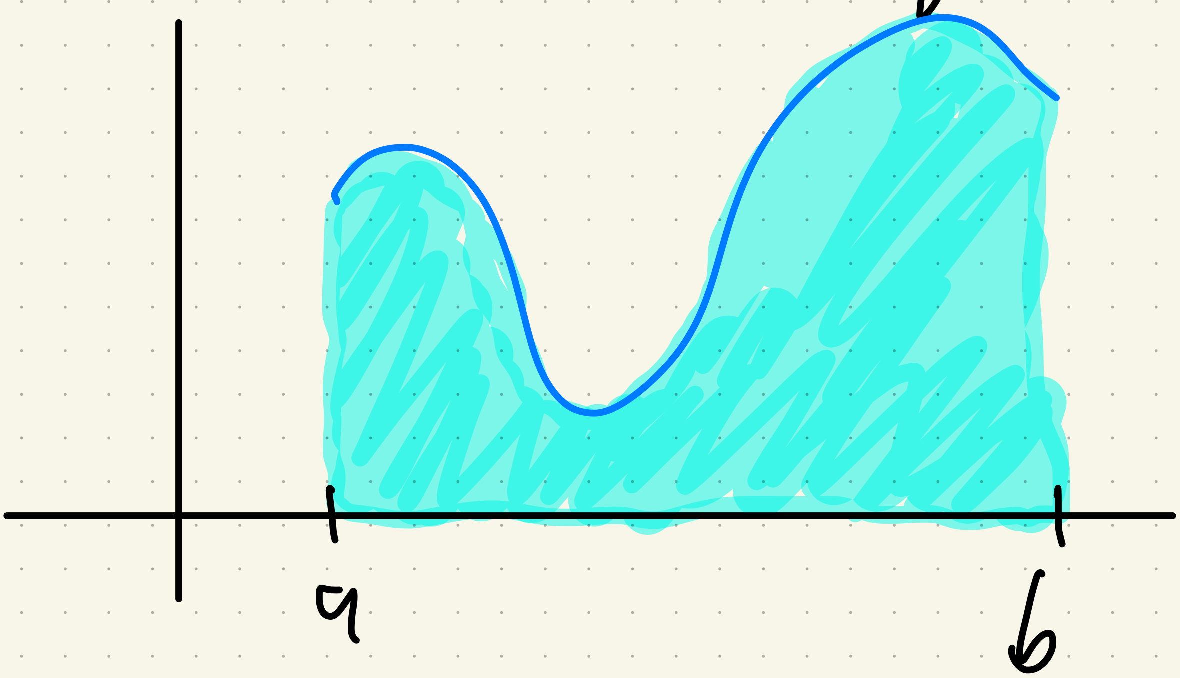
Picture:



The average is the total area divided by the length of the interval.

In more generality,

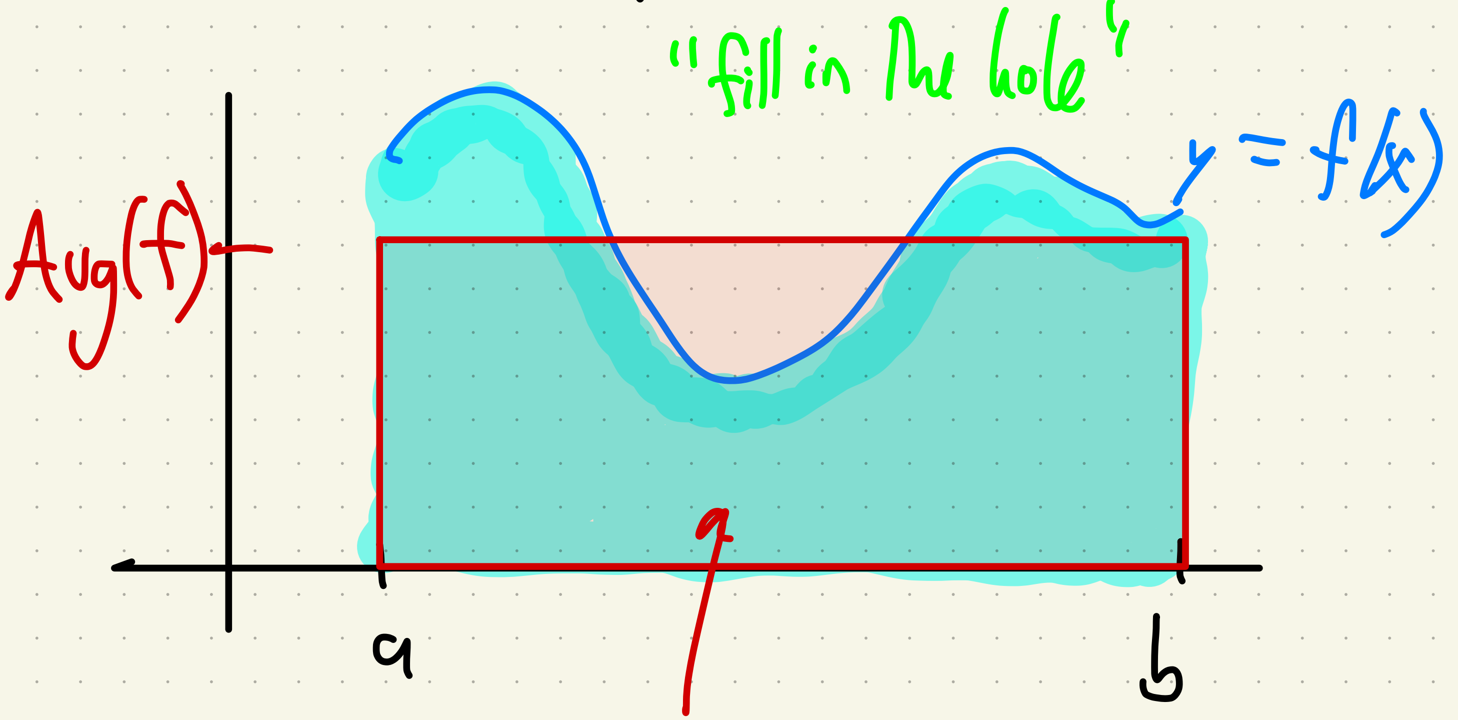
The average of a piecewise continuous function f over an interval $[a,b]$ is again the total area divided by length of interval:



$$\text{Avg}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

another way to see it:

The average is the height of the rectangle with the same area.



red rectangle has same area as area under curve

$$\int_a^b f(x)\,dx = \underbrace{(b-a)}_{\text{base}} \cdot \underbrace{\frac{1}{b-a}\int_a^b f(x)\,dx}_{\text{height}}$$

Let's revisit the proof of the F.T. of Calc.
Part I

in view of averages.

The F.T. of Calc. Part I.

If g is continuous on $[a,b]$,

then the function

$$f(x) \stackrel{\text{def}}{=} \int_a^x g(t)\,dt, \quad a \le x \le b$$

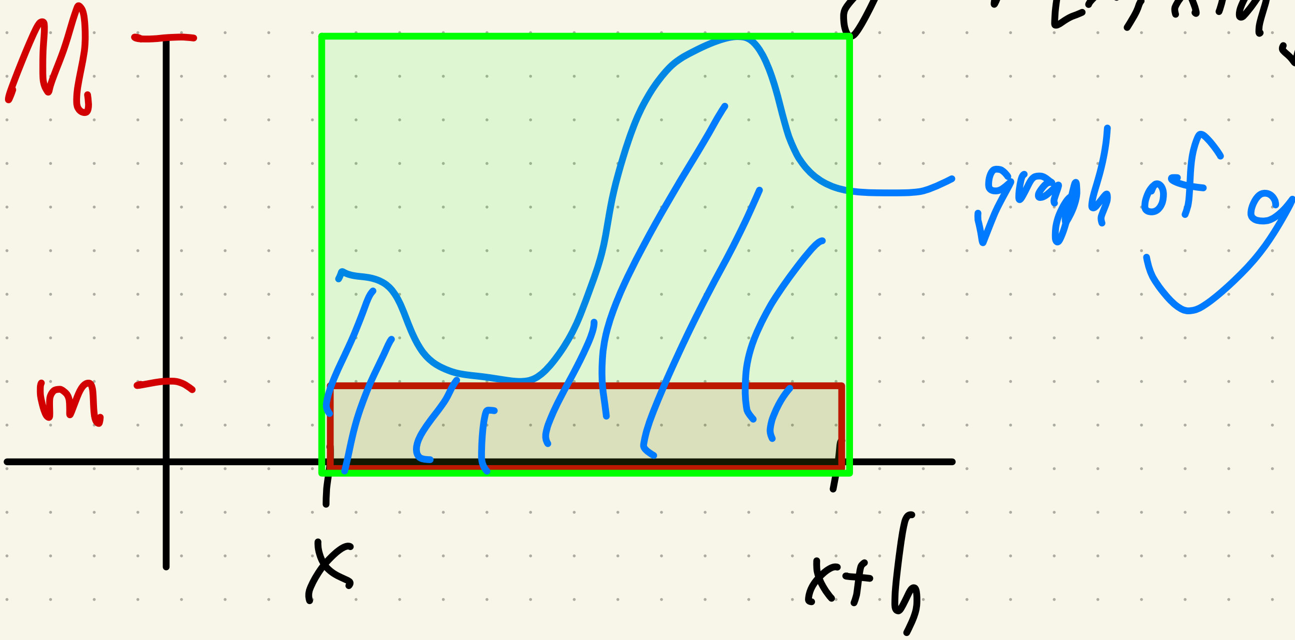
is an antiderivative of g:

$$f'(x) = g(x).$$

Explanation: let $h>0$:

let m be the minimum of g on $[x, x+h]$

let M be the maximum of g on $[x, x+h]$.



again compare areas:

$$mh \le \int_x^{x+h} g(t)\,dt \le Mh$$

Div. by h:

$$m \le \frac{1}{h}\int_x^{x+h} g(t)\,dt \le M$$

The avg. of g on $[x, x+h]$

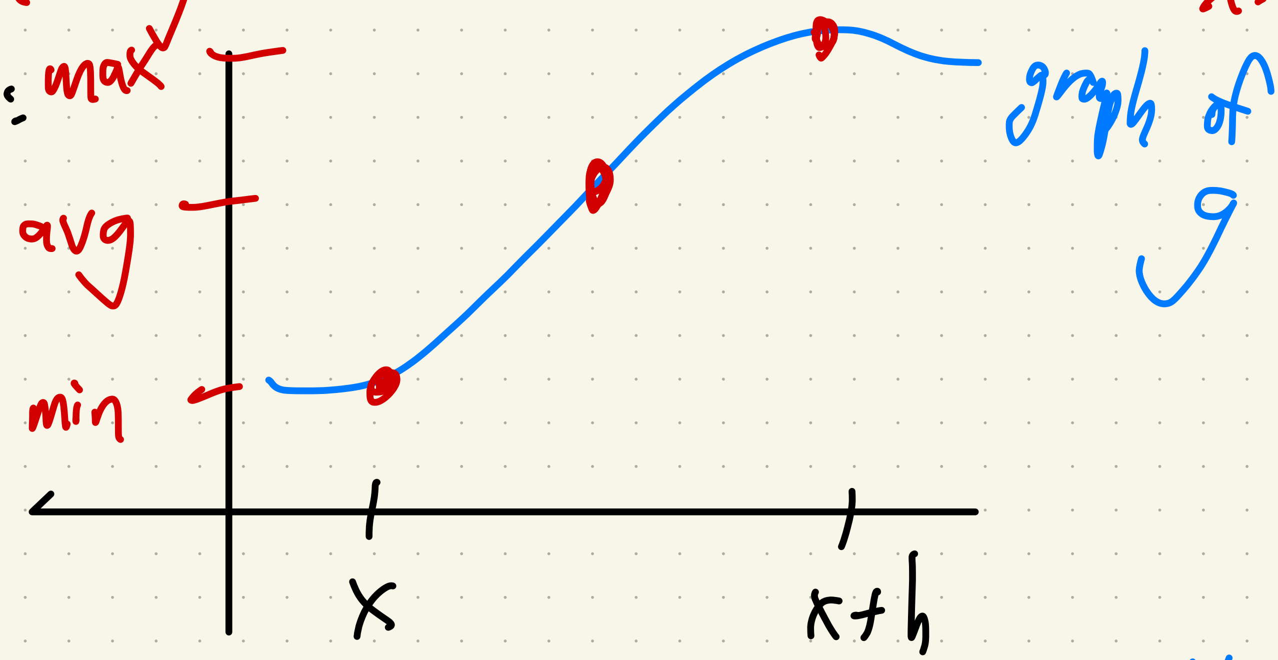
That is,

$$\underbrace{m}_{\text{The min. of } g \text{ on } [x,x+h]} \le \underbrace{\frac{1}{h}\int_x^{x+h} g(t)\,dt}_{\text{The avg. of } g \text{ on } [x,x+h]} \le \underbrace{M}_{\text{The max. of } g \text{ on } [x,x+h]}$$

makes sense!

The avg is between the min & max.

Picture:



Let $h \to 0$... Then m, M, and avg all approach $g(x)$

So $$\lim_{h\to 0} \frac{1}{h}\int_x^{x+h} g(t)\,dt = g(x)$$

Combine w/ fact that

$$f'(x) = \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} g(t)\,dt$$

(see beginning of notes)

So $$f'(x) = g(x)$$

Summary: all antiderivatives of $g(x)$ are of the form

$$f(x) = \int_a^x g(t)\,dt + C$$

"can find antiderivatives by integrating."

For that reason, we use the notation

$\int g(x)\,dx$ to denote the family of all antiderivatives of g.

an "indefinite integral"

Ex. $\int x^3\,dx$ represents the family of all antiderivatives of $g(x) = x^3$.

So

$$\int x^3\,dx = \frac{1}{4}x^4 + C$$

Quiz on Friday: $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{j=1}^{n} f(x_j^*)\Delta x$

Quiz 9 §4.10 – antiderivs

§5.1 – definite integrals & area

see Canvas

§5.2 – Riemann sums

(definition of integral

Antiderivatives: See April 20 notes
& onward

See you tomorrow!