Math 30, Friday April 10, 2020

L'Hôpital, cont'd

1pm Class

Questions?

Today's Quiz: Get info from

f' } incr/decr

and f'' } concavity

Exam 3 review problems are posted

and sol= to Quiz 7

✓

my final answer:

$$h'(t) = \frac{-125}{8\pi} \text{ cm/sec.}$$

Last time: if you see "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$"

use l'Hôpital's Rule.

Also use it if you see "$\underbrace{0 \cdot \infty}$"

Then you can rewrite as $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and use l'Hôp as usual.

Ex. $\lim\limits_{x \to \infty} \underbrace{\frac{1}{x} \cdot mx}$ where $m > 0$ is a positive constant

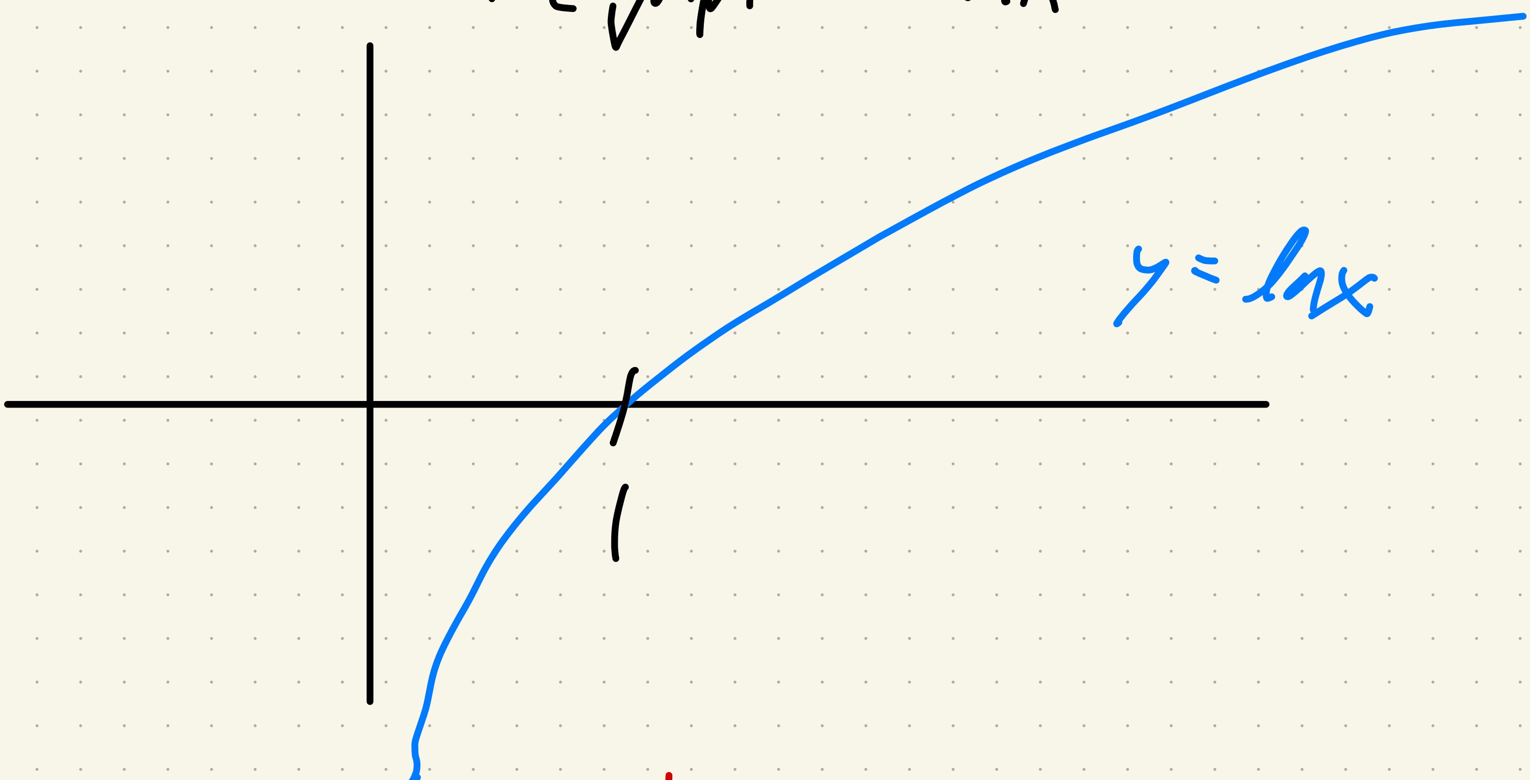
looks like $0 \cdot \infty$

rewrite as $\lim\limits_{x \to \infty} \frac{mx}{x} = m$

"$0 \cdot \infty$" can be any number — not an accepted answer.

Ex. $\lim_{x \to 0^+} x \ln x = ?$

Remember the graph of $\ln x$:



$$y = \ln x \overset{\text{same thing}}{\Longleftrightarrow} e^{y} = x$$

b/c ln and exp are inverses

as $x \to 0^+$

$x \to 0^+$

That means $y \to -\infty$

Ex (cont'd). $\lim_{x \to 0^+} \underbrace{x}_{\text{goes to } 0} \underbrace{\ln x}_{\text{goes to } -\infty}$

so it looks like "$0 \cdot (-\infty)$"

can rewrite as:

$$= \lim_{x \to 0^+} \frac{\ln x}{\left(\frac{1}{x}\right)}$$

now looks like "$\frac{-\infty}{\infty}$" so can use L'Hôpital as usual

$$\left(\lim_{x \to 0^+} \frac{f(x)}{g(x)} = \lim_{x \to 0^+} \frac{f'(x)}{g'(x)} \right)$$

$$= \lim_{x \to 0} \frac{\left(\frac{1}{x}\right)}{\left(\frac{-1}{x^2}\right)} \cdots$$

Again:

$$\lim_{x\to 0^+} x\ln x \overset{\text{rewriting}}{=} \lim_{x\to 0^+} \frac{\ln x}{\left(\frac{1}{x}\right)} \quad \text{“}\frac{-\infty}{\infty}\text{”}$$

$$\overset{\text{L'Hôp}}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{\left(-\frac{1}{x^2}\right)}$$

$$\overset{\text{simplify}}{=} \lim_{x\to 0^+} (-x) = \boxed{0}$$

"$0\cdot\infty$" can be rewritten as "$\frac{\infty}{\infty}$"

Then use L'Hôpital.

Q: $\frac{d}{dx} \frac{1}{x} = ?$

Fastest way: rewrite as, then use Power Rule

$$\frac{d}{dx} x^{-1} = (-1) x^{-2}$$

$$= \frac{-1}{x^2}$$

Ex. $\lim_{x \to \infty} \underbrace{x^3 e^{-x^2}}_{} = ?$

looks like "$\infty \cdot 0$"

rewrite as:

$$= \lim_{x \to \infty} \frac{x^3}{e^{x^2}}$$

now it looks like "$\frac{\infty}{\infty}$"

so use l'Hôpital as usual.

$$= \lim_{x \to \infty} \frac{3x^2}{2xe^{x^2}} \quad \text{(Chain Rule)}$$

simplify

$$= \lim_{x \to \infty} \frac{3x}{2e^{x^2}}$$

again looks like "$\frac{\infty}{\infty}$"

so use l'Hôpital again

Start over, faster:

$$\lim_{x\to\infty} \frac{x^3}{e^{x^2}} \overset{\text{L'Hop}}{=} \lim_{x\to\infty} \frac{3x^2}{2xe^{x^2}}$$

$$\overset{\text{simplif}}{=} \lim_{x\to\infty} \frac{3x}{2e^{x^2}}$$

$$\overset{\text{L'Hop}}{=} \lim_{x\to\infty} \frac{3}{4xe^{x^2}} \quad \text{(Chain Rule)}$$

$$= \boxed{0}$$

Q: why $2xe^{x^2}$ in the denom?

$$\lim_{x\to\infty} \frac{x^3}{e^{x^2}} = ?$$

$f(x) = x^3$ $g(x) = e^{x^2}$

$f'(x) = 3x^2$ $g'(x) = e^{x^2}(2x)$

by Chain Rule

x^2 is the "inside function"

and e^y is the "outside function"

e^{x^2}

$$\frac{d}{dx} e^{h(x)} = ?$$

$h(x)$ is the "inside function"

e^y is the "outside function"

Chain Rule:

$$= e^{h(x)} \cdot h'(x)$$

In our example, $h(x) = x^2$

Worksheet:

Can use L'Hôpital's Rule to sketch functions

MATH 30, 4/10/2020: L'HÔPITAL, CONT'D.

Last time: When you get "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$" in a limit, it means "I need to do more work." At the beginning of the semester, we factorized and canceled. Now we have a new technique: L'Hôpital's Rule.

We can also use L'Hôpital's Rule for limits of *products* where we get the indeterminate form "$0 \cdot \infty$," which is also meaningless.

Example. $\lim_{x \to 0^+} x \ln x =$? It looks like "$0 \cdot \infty$," which really means "I need to do more work."

We rewrite it as:

$$\begin{aligned}
\lim_{x \to 0^+} x \ln x &= \lim_{x \to 0^+} \frac{\ln x}{\left(\frac{1}{x}\right)} && \text{but now it looks like "}\tfrac{-\infty}{\infty}\text{,"} \\
&= \lim_{x \to 0^+} \frac{(1/x)}{(-1/x^2)} && \text{which means we can use L'Hôpital's Rule as before} \\
&= \lim_{x \to 0^+} (-x) = 0.
\end{aligned}$$

Example. $\lim_{x \to \infty} x^3 e^{-x^2} =$? Again it looks like "$\infty \cdot 0$," which means "I need to do more work." Rewrite it as:

$$\begin{aligned}
\lim_{x \to \infty} x^3 e^{-x^2} &= \lim_{x \to \infty} \frac{x^3}{e^{x^2}} && \text{but now it looks like "}\frac{\infty}{\infty}\text{,"} \\
&= \lim_{x \to \infty} \frac{3x^2}{2xe^{x^2}} && \text{which means we can use L'Hôpital's Rule} \\
&= \lim_{x \to \infty} \frac{3x}{2e^{x^2}} \\
&= \lim_{x \to \infty} \frac{3}{4xe^{x^2}} = 0 && \text{and use it again....}
\end{aligned}$$

Application: We can use L'Hôpital's Rule to help with curve sketching.

On the next page I have a worksheet for you to try.

Please try these on your own...

(1) Sketch the graph of the function $f(x) = (x^2+x+1)e^{-x}$. Start by finding critical points, where it is increasing & decreasing, where it has local max & min, and what it does as $x \to \pm\infty$.

(2) Sketch the graph of the function $f(x) = \sqrt{x}\,\ln x$ over $[0, \infty)$. Start by finding critical points, where it is increasing & decreasing, where it has local max & min, and what it does as $x \to 0$ and $x \to \infty$.

[You will need another piece of paper. ☺]

#1. Sketch $f(x) = (x^2+x+1)e^{-x}$

rewrite as:

$$f(x) = \frac{x^2+x+1}{e^x}$$

To see where incr/decr., find $f'(x)$:

Quotient Rule:

$$f'(x) \overset{\text{Q.R.}}{=} \frac{(2x+1)e^x - e^x(x^2+x+1)}{e^{2x}}$$

$$= \frac{e^x(2x+1-x^2-x-1)}{e^{2x}}$$

$$= \frac{e^x(-x^2+x)}{e^{2x}}$$

$$\frac{e^x x(-x+1)}{e^x e^x}$$

$$= \frac{x(1-x)}{e^x}$$

where pos.?

where neg.?

$$f(x) = \frac{x^2 + x + 1}{e^x}$$

$$f'(x) = \frac{x(1-x)}{e^x}$$

denom. is always positive

when $x<0$: numerator is neg × pos. = neg.

so $f'<0$, so f is decreasing.

when $0<x<1$: numerator is pos × pos.

so $f'>0$ so f is increasing.

when $x>1$: numerator is pos × neg = neg.

so $f'<0$ so f is decreasing.

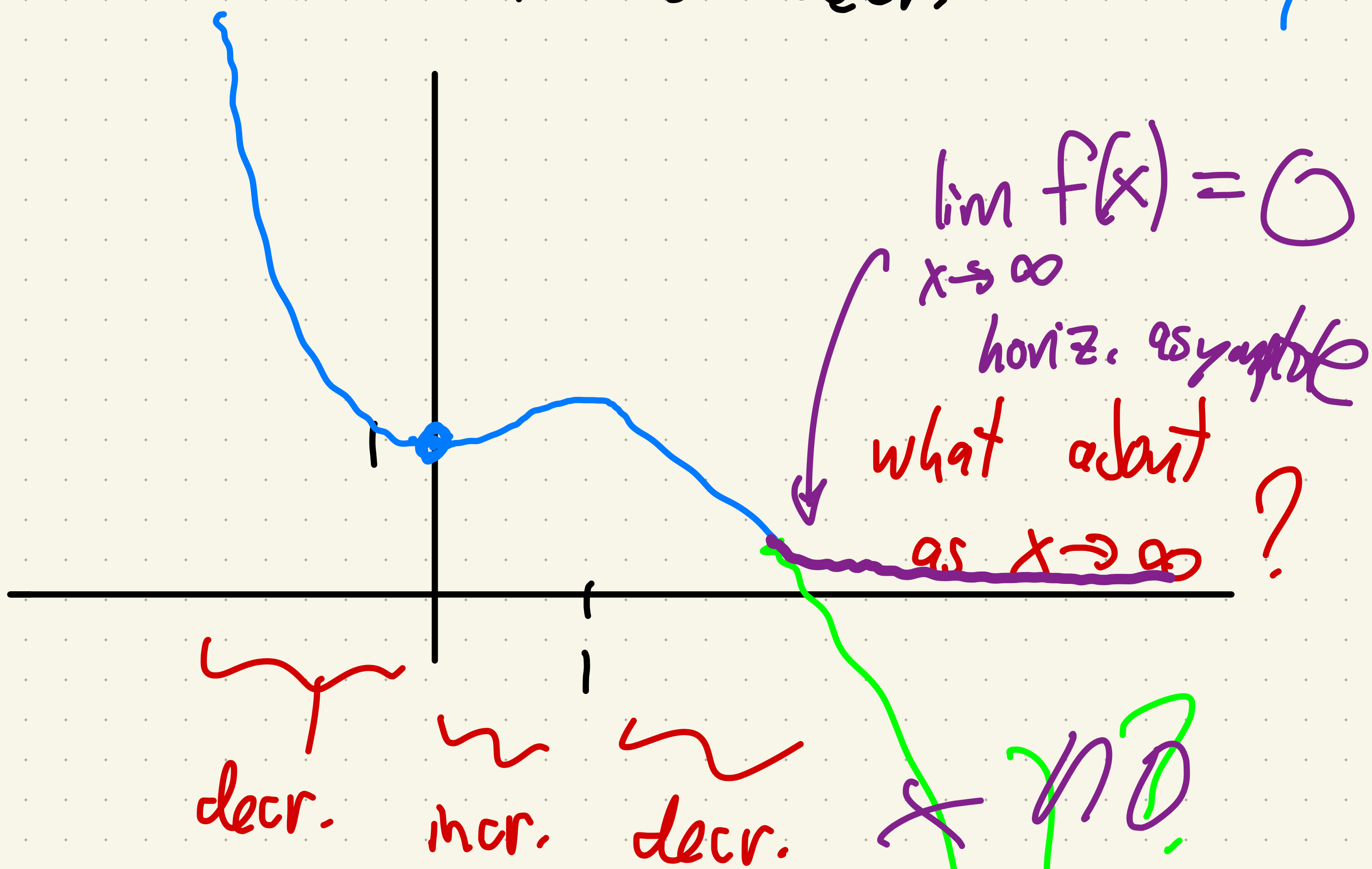
Summary: $f(x) = \dfrac{x^2+x+1}{e^x}$

if $x<0$: f is decr.

if $0<x<1$: f is incr.

if $x>1$: f is decr.

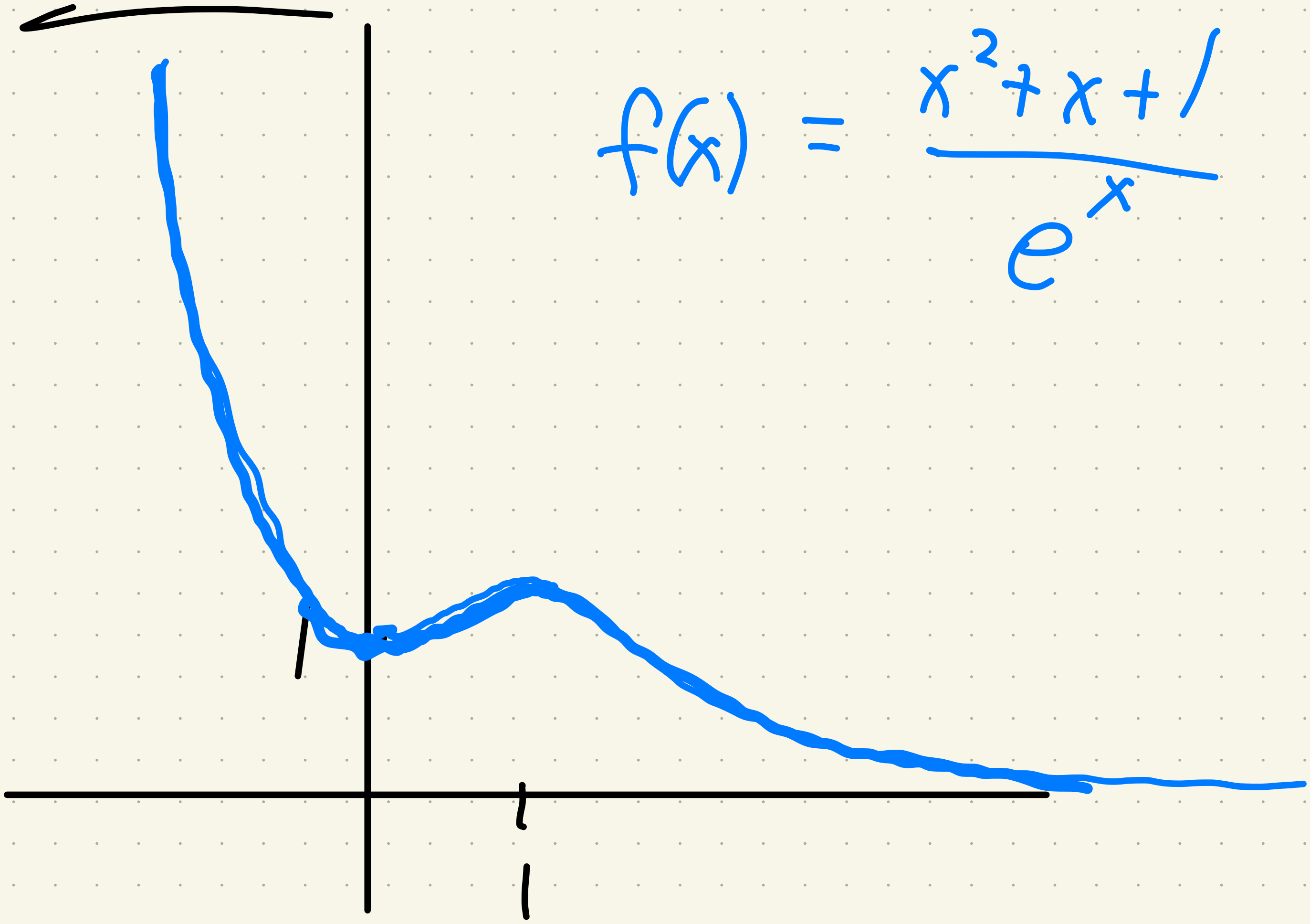
$f(0) = \dfrac{1}{1} = 1$



$$\lim_{x\to\infty} \frac{x^2+x+1}{e^x} \overset{\text{L'Hop}}{=} \lim_{x\to\infty} \frac{2x+1}{e^x} \overset{\text{L'Hop}}{=} \lim_{x\to\infty} \frac{2}{e^x} = 0$$

"$\frac{\infty}{\infty}$" "$\frac{\infty}{\infty}$"

Final sketch:



$$f(x) = \frac{x^2 + x + 1}{e^x}$$

Please try them on your own...

Then see my sol$^{\underline{n}}$s.

MATH 30, 4/10/2020: MORE CURVE SKETCHING: SOLUTIONS

(1) Sketch the graph of the function $f(x) = (x^2+x+1)e^{-x}$. Start by finding critical points, where it is increasing & decreasing, where it has local max & min, and what it does as $x \to \pm\infty$.

By the Product Rule we have $f'(x) = x(1-x)e^{-x}$, so f has critical points at $x = 0$ and $x = 1$. We see that f is decreasing for $x < 0$, increasing for $0 < x < 1$, and decreasing again for $x > 1$. So we see that f has a local minimum at $x = 0$ and a local maximum at $x = 1$.

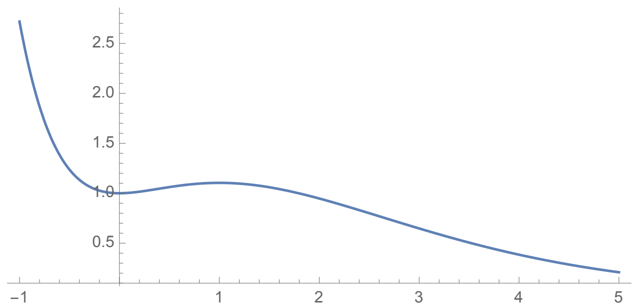
Alternatively, by the Product Rule we can calculate $f''(x) = (x^2 - 3x + 1)e^{-x}$, which shows $f''(0) > 0$ (so that f is concave up at $x = 0$) and $f''(1) < 0$ (so that f is concave down at $x = 1$). [You can check that f has inflection points at $x = \frac{3\pm\sqrt{5}}{2}$.]

Using L'Hôpital's Rule two times we see that

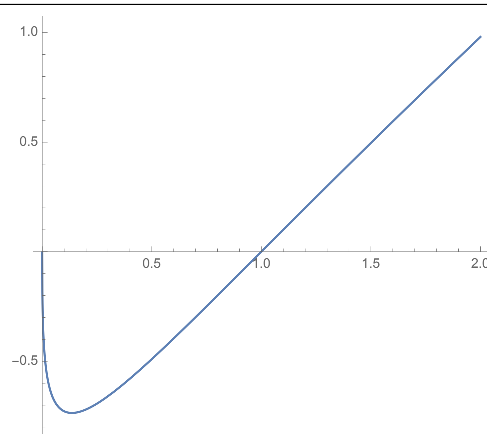
$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2+x+1}{e^x} = \lim_{x\to\infty} \frac{2x+1}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0,$$

which shows that f has a horizontal asymptote as $x \to \infty$.
Also, as $x \to -\infty$ the function goes to $+\infty$.

To illustrate all of that work, here is the graph:



(2) Sketch the graph of the function $f(x) = \sqrt{x}\,\ln x$ over $[0, \infty)$. Start by finding critical points, where it is increasing & decreasing, where it has local max & min, and what it does as $x \to 0$ and $x \to \infty$.



As $x \to \infty$, the function goes to ∞, but more and more slowly ($f'(x) \to 0$ as $x \to \infty$).

Last Q's?

Quiz #8 due tonight by 11:59 pm.

inflection pts: where concavity changes.

For ex., $f''>0$ means concave up

$f''<0$ means concave down

So when it changes from $f''>0$ to $f''<0$ you have an infl. pt.

typed notes & Zoom stuff

is on Canvas under

"Files"