1. Define T(x) = Ax. Find a vector x whose image under T is b, and determine whether x is unique.

(a) $A = \begin{bmatrix} 1 & 0 & -3 \\ -3 & 1 & 6 \\ 2 & -2 & -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix}$

1.a) $A = \begin{bmatrix} 1 & 0 & -3 \\ -3 & 1 & 6 \\ 2 & -2 & -1 \end{bmatrix}$, $b = \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix}$

Define $T(x) = Ax$

$$T(x) = Ax = \begin{bmatrix} 1 & 0 & -3 \\ -3 & 1 & 6 \\ 2 & -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -3 & -2 \\ -3 & 1 & 6 & 3 \\ 2 & -2 & -1 & -1 \end{bmatrix} \xrightarrow{R_2 + 3\cdot R_1 \to R_2} \begin{bmatrix} 1 & 0 & -3 & -2 \\ 0 & 1 & -3 & -3 \\ 2 & -2 & -1 & -1 \end{bmatrix} \xrightarrow{R_3 - 2\cdot R_1 \to R_3} \begin{bmatrix} 1 & 0 & -3 & -2 \\ 0 & 1 & -3 & -3 \\ 0 & -2 & 5 & 3 \end{bmatrix}$$

$$\xrightarrow{R_3 + 2\cdot R_2 \to R_3} \begin{bmatrix} 1 & 0 & -3 & -2 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & -1 & -3 \end{bmatrix} \xrightarrow{-R_3} \begin{bmatrix} 1 & 0 & -3 & -2 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\xrightarrow{R_2 + 3\cdot R_3 \to R_2} \begin{bmatrix} 1 & 0 & -3 & -2 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow{R_1 + 3\cdot R_3 \to R_1} \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\boxed{x = \begin{bmatrix} 7 \\ 6 \\ 3 \end{bmatrix}, \text{ unique solution}}$$

(b) $A = \begin{bmatrix} 1 & -3 & 2 \\ 3 & -8 & 8 \\ 0 & 1 & 2 \\ 1 & 0 & 8 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 6 \\ 3 \\ 10 \end{bmatrix}$

1. b)

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 3 & -8 & 8 \\ 0 & 1 & 2 \\ 1 & 0 & 8 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 6 \\ 3 \\ 10 \end{bmatrix}$$

Define $T(x) = Ax$

$$T(x) = Ax = \begin{bmatrix} 1 & -3 & 2 \\ 3 & -8 & 8 \\ 0 & 1 & 2 \\ 1 & 0 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 3 \\ 10 \end{bmatrix}$$

$$\begin{array}{cccc} & 9 & 6 & 3 \end{array}$$

$$\begin{bmatrix} 1 & -3 & 2 & 1 \\ 3 & -8 & 8 & 6 \\ 0 & 1 & 2 & 3 \\ 1 & 0 & 8 & 10 \end{bmatrix} \xrightarrow{R_2 - 3 \cdot R_1 \to R_2} \begin{bmatrix} 1 & -3 & 2 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \\ 1 & 0 & 8 & 10 \end{bmatrix}$$

$$\xrightarrow{R_4 - R_1 \to R_4} \begin{bmatrix} 1 & -3 & 2 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 3 & 6 & 9 \end{bmatrix} \xrightarrow{R_3 - R_2 \to R_3} \begin{bmatrix} 1 & -3 & 2 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 6 & 9 \end{bmatrix}$$

$$\xrightarrow{R_4 - 3 \cdot R_2 \to R_4} \begin{bmatrix} 1 & -3 & 2 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 + 3 \cdot R_2 \to R_1} \begin{bmatrix} 1 & 0 & 8 & 10 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 + 8x_3 = 10$
$x_2 + 2x_3 = 3$

$x_1 = 10 - 8x_3$
$x_2 = 3 - 2x_3$
x_3 is free

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 - 8x_3 \\ 3 - 2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -8 \\ -2 \\ 1 \end{bmatrix}$$

[illegible]

Cont'd

General sol:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 - 8x_3 \\ 3 - 2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -8 \\ -2 \\ 1 \end{bmatrix}$$

One possible solution could be $x_3 = 0$ and $x = \begin{bmatrix} 10 \\ 3 \\ 0 \end{bmatrix}$

x is not unique!

2. Find all x in R^4 that are mapped into the zero vector by the transformation x → Ax.

$$A = \begin{bmatrix} 3 & 2 & 10 & -6 \\ 1 & 0 & 2 & -4 \\ 0 & 1 & 2 & 3 \\ 1 & 4 & 10 & 8 \end{bmatrix}$$

2. $A = \begin{bmatrix} 3 & 2 & 10 & -6 \\ 1 & 0 & 2 & -4 \\ 0 & 1 & 2 & 3 \\ 1 & 4 & 10 & 8 \end{bmatrix}$ Solve $Ax=0$

RREF

$$\begin{bmatrix} 3 & 2 & 10 & -6 & 0 \\ 1 & 0 & 2 & -4 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 1 & 4 & 10 & 8 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 0 & 2 & -4 & 0 \\ 3 & 2 & 10 & -6 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 1 & 4 & 10 & 8 & 0 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{3} \cdot R_2} \begin{bmatrix} 1 & 0 & 2 & -4 & 0 \\ 1 & 2/3 & 10/3 & -2 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 1 & 4 & 10 & 8 & 0 \end{bmatrix} \xrightarrow{R_1 - R_2 \to R_1} \begin{bmatrix} 0 & -2/3 & -4/3 & -2 & 0 \\ 1 & 2/3 & 10/3 & -2 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 1 & 4 & 10 & 8 & 0 \end{bmatrix}$$

$$\xrightarrow{R_4 - R_2 \to R_4} \begin{bmatrix} 0 & -2/3 & -4/3 & -2 & 0 \\ 1 & 2/3 & 10/3 & -2 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 10/3 & 20/3 & 10 & 0 \end{bmatrix} \xrightarrow{-\frac{3}{2} \cdot R_1} \begin{bmatrix} 0 & 1 & 2 & 3 & 0 \\ 1 & 2/3 & 10/3 & -2 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 10/3 & 20/3 & 10 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 2/3 & 10/3 & -2 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 10/3 & 20/3 & 10 & 0 \end{bmatrix}$$

$\frac{6}{3} - 2$

$$\xrightarrow{R_1 - \frac{2}{3} \cdot R_2 \to R_1} \begin{bmatrix} 1 & 0 & 2 & -4 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 10/3 & 20/3 & 10 & 0 \end{bmatrix}$$

$\frac{4}{3}$ $\quad \frac{10}{3} - \frac{4}{3} = \frac{6}{3}$

$$\longrightarrow \begin{bmatrix} 1 & 0 & 2 & -4 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 10/3 & 20/3 & 10 & 0 \end{bmatrix} \xrightarrow{R_3 - \frac{10}{3} \cdot R_2 \to R_3} \begin{bmatrix} 1 & 0 & 2 & -4 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

cont'd

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & -4 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{aligned} x_1 + 2x_3 - 4x_4 &= 0 \\ x_2 + 2x_3 + 3x_4 &= 0 \end{aligned}$$

$$\downarrow$$

$$x_1 = -2x_3 + 4x_4$$

$$x_2 = -2x_3 - 3x_4$$

x_3 is free

x_4 is free

$$\boxed{\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_3 + 4x_4 \\ -2x_3 - 3x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 4 \\ -3 \\ 0 \\ 1 \end{bmatrix}}$$

3. Let

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{y}_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \mathbf{y}_2 = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$$

and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps $\mathbf{e}_1$ into $\mathbf{y}_1$ and maps $\mathbf{e}_2$ into $\mathbf{y}_2$. Find the images of $\begin{bmatrix} 5 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

3. $e_1 \begin{bmatrix}1\\0\end{bmatrix}; e_2 = \begin{bmatrix}0\\1\end{bmatrix}, v_1 = \begin{bmatrix}2\\5\end{bmatrix}, v_2 = \begin{bmatrix}-1\\6\end{bmatrix}$

$x = \begin{bmatrix}5\\-3\end{bmatrix} = 5\begin{bmatrix}1\\0\end{bmatrix} - 3\begin{bmatrix}0\\1\end{bmatrix} = 5e_1 - 3e_2$

$T(x) = T(5e_1 - 3e_2) = 5T(e_1) - 3T(e_2) = 5v_1 - 3v_2$

$= 5\begin{bmatrix}2\\5\end{bmatrix} - 3\begin{bmatrix}-1\\6\end{bmatrix} = \begin{bmatrix}13\\7\end{bmatrix} \longrightarrow \boxed{\begin{bmatrix}13\\7\end{bmatrix}, \begin{bmatrix}2x_1 - x_2\\5x_1 + 6x_2\end{bmatrix}}$

For images of $\begin{bmatrix}x_1\\x_2\end{bmatrix} \rightarrow x = \begin{bmatrix}x_1\\x_2\end{bmatrix} = x_1\begin{bmatrix}1\\0\end{bmatrix} + x_2\begin{bmatrix}0\\1\end{bmatrix} = x_1e_1 + x_2e_2$

$T(x) = T(x_1e_1 + x_2e_2) = x_1T(e_1) + x_2T(e_2)$

$= x_1\begin{bmatrix}2\\5\end{bmatrix} + x_2\begin{bmatrix}-1\\6\end{bmatrix} = \begin{bmatrix}2x_1 - x_2\\5x_1 + 6x_2\end{bmatrix}$

4. Suppose vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ span $\mathbb{R}^n$ and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Suppose $T(\mathbf{v}_i) = \mathbf{0}$ for $i = 1, \ldots, p$. Show that T is the zero transformation. That is, show that if $\mathbf{x}$ is any vector in $\mathbb{R}^n$, then $T(\mathbf{x}) = \mathbf{0}$.

*Your explanation in class helped. I sure hope I got the right idea.

4. $V_1, V_2, \dots V_p$ span $\mathbb{R}^n$

$\{\vec{v}_1, \dots, \vec{v}_p\} = C$ $\quad T: \mathbb{R}^n \to \mathbb{R}^n$

Span$(C) = \mathbb{R}^m$

We have linear transformation going from R_n to R_n such that

$T(\vec{V}_1) = \vec{0}_n$

$\vec{w} \in R^n$

→ Because it spans all of R^n if we take any random vector then we can take the linear combination of $\{\vec{v}_1, \dots, \vec{v}_p\}$ and then it into any random vector

$\vec{w} = a_1 \vec{v}_1 + a_2 \vec{v}_2 + \dots a_p \vec{v}_p$

$T(\vec{w}) = T(a_1 \vec{v}_1 + a_2 \vec{v}_2 + \dots a_p \vec{v}_p)$

$T(\vec{w}) = T(a_1 \vec{v}_1) + T(a_2 \vec{v}_2) + \dots T(a_p \vec{v}_p)$

$T(\vec{w}) = a_1 T(\vec{v}_1) + a_2 T(\vec{v}_2) + \dots a_p T(\vec{v}_p)$

$T(\vec{w}) = a_1(0) + a_2(0) + \dots a_p(0)$

$T(\vec{w}) = 0 + 0 + \dots 0$

$T(\vec{w}) = 0$

So, for any $x \in R^n$, $\boxed{T(x) = 0}$

5. Compute $A - 5I_3$ and $(5I_3)A$, where $A = \begin{bmatrix} 5 & -1 & 3 \\ -4 & 3 & -6 \\ -3 & 1 & 2 \end{bmatrix}$.

5. Compute $A-5I_3$ and $(5I_3)A$ 2.1

$$A=\begin{bmatrix}5&-1&3\\-4&3&-6\\-3&1&2\end{bmatrix}$$

$$A-5I_3=\begin{bmatrix}5&-1&3\\-4&3&-6\\-3&1&2\end{bmatrix}-\begin{bmatrix}5&0&0\\0&5&0\\0&0&5\end{bmatrix}=\begin{bmatrix}0&-1&3\\-4&-2&-6\\-3&1&-3\end{bmatrix}$$

$$(5I_3)A=5(I_3A)\overset{=5A}{=}5\begin{bmatrix}5&-1&3\\-4&3&-6\\-3&1&2\end{bmatrix}=\begin{bmatrix}25&-5&15\\-20&15&-30\\-15&5&10\end{bmatrix}$$

$$\boxed{\begin{bmatrix}0&-1&3\\-4&-2&-6\\-3&1&-3\end{bmatrix},\ \begin{bmatrix}25&-5&15\\-20&15&-30\\-15&5&10\end{bmatrix}}$$

6. Suppose CA = In (the n × n identity matrix). Show that the equation Ax = 0 has only the trivial solution. Explain why A cannot have more columns than rows.

6. $CA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ row

Show $Ax=0$ has only a trivial sol.

If x satisfies $Ax=0$ then it could be said that $x=0$ as $CAx=C0=0$ and $Ix=0$. The equation $Ax=0$ can only has a trivial solution if $Ax=0$ has no free variables and this happens if every column of A is a pivot column. This would be the case if CA results in an identity matrix. Since each column is a pivot column, A would have to have at least as many rows as columns.

Col ≠ row

7. Let

$$A = \begin{bmatrix} 1 & 2 \\ 5 & 12 \end{bmatrix}, \mathbf{b}_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}, \mathbf{b}_3 = \begin{bmatrix} 2 \\ 6 \end{bmatrix}, \mathbf{b}_4 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

(a) Find A−1 and use it to solve Ax = b1, Ax = b2, Ax = b3, Ax = b4.

7. $A = \begin{bmatrix} 1 & 2 \\ 5 & 12 \end{bmatrix}$, $b_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, $b_2 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$, $b_3 = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$, $b_4 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

a) Find A^{-1} and use it to solve $Ax = b_1, \dots, Ax = b_4$.

$\det A = 1 \cdot 12 - 2 \cdot 5$

$$\begin{bmatrix} 1 & 2 \\ 5 & 12 \end{bmatrix}^{-1} = \frac{1}{1 \cdot 12 - 2 \cdot 5}\begin{bmatrix} 12 & -2 \\ -5 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 12 & -2 \\ -5 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -1 \\ -2.5 & .5 \end{bmatrix}$$

$$x = A^{-1}b_1 = \frac{1}{2}\begin{bmatrix} 12 & -2 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1(12) + 3(-2) \\ -1(-5) + 3(1) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -18 \\ 8 \end{bmatrix} = \begin{bmatrix} -9 \\ 4 \end{bmatrix}$$

$$x = A^{-1}b_2 = \frac{1}{2}\begin{bmatrix} 12 & -2 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -5 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1(12) - 5(-2) \\ 1(-5) - 5(1) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 22 \\ -10 \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \end{bmatrix}$$

$$x = A^{-1}b_3 = \frac{1}{2}\begin{bmatrix} 12 & -2 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 6 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2(12) + 6(-2) \\ 2(-5) + 6(1) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 12 \\ -4 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$$

$$x = A^{-1}b_4 = \frac{1}{2}\begin{bmatrix} 12 & -2 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 3(12) + 5(-2) \\ 3(-5) + 5(1) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 26 \\ -10 \end{bmatrix} = \begin{bmatrix} 13 \\ -5 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} -9 \\ 4 \end{bmatrix}, \begin{bmatrix} 11 \\ -5 \end{bmatrix}, \begin{bmatrix} 6 \\ -2 \end{bmatrix}, \begin{bmatrix} 13 \\ -5 \end{bmatrix}}$$

(b) The four equations in (a) can be solved by the same set of row operations, since the coefficient matrix is the same in each case. Solve the four equations in part (a) by row reducing the augmented matrix [A b1 b2 b3 b4]

b) $$[A\ b_1\ b_2\ b_3\ b_4] = \begin{bmatrix} 1 & 2 & -1 & 1 & 2 & 3 \\ 5 & 12 & 3 & -5 & 6 & 5 \end{bmatrix}$$

$$\xrightarrow{R_2 - 5 \cdot R_1 \to R_2} \begin{bmatrix} 1 & 2 & -1 & 1 & 2 & 3 \\ 0 & 2 & 8 & -10 & -4 & -10 \end{bmatrix} \xrightarrow{\frac{1}{2} \cdot R_2} \begin{bmatrix} 1 & 2 & -1 & 1 & 2 & 3 \\ 0 & 1 & 4 & -5 & -2 & -5 \end{bmatrix}$$

$$\xrightarrow{R_1 - 2R_2 \to R_1} \begin{bmatrix} 1 & 0 & -9 & 11 & 6 & 13 \\ 0 & 1 & 4 & -5 & -2 & -5 \end{bmatrix} \to \text{Gives us} \to \boxed{\begin{bmatrix} -9 \\ 4 \end{bmatrix}, \begin{bmatrix} 11 \\ -5 \end{bmatrix}, \begin{bmatrix} 6 \\ -2 \end{bmatrix}, \begin{bmatrix} 13 \\ -5 \end{bmatrix}}$$

8. Suppose P is invertible and A = P BP −1 . Solve for B in terms of A. Show your steps.

8. P is invertible, $A = PBP^{-1}$. Solve for B in terms of A

$$A = PBP^{-1} \rightarrow P^{-1}A = P^{-1}PBP^{-1} \rightarrow P^{-1}A = IBP^{-1}$$

$$P^{-1}A = BP^{-1} \rightarrow P^{-1}AP = BP^{-1}P \rightarrow P^{-1}AP = BI$$

$$\boxed{P^{-1}AP = B}$$

9. Use matrix algebra (use the matrix operations that we've learned) to show that if A is invertible and D satisfies AD = I, then D = A−1 . Show all the steps.

9. Show that A is invertible and D satisfies $AD = I$ then $D = A^{-1}$. Use matrix algebra!

$$AD = I \rightarrow A^{-1}AD = A^{-1}I \rightarrow ID = A^{-1}I$$

$$\boxed{D = A^{-1}}$$

10. Suppose AB = AC, where B and C are n × p matrices and A is invertible. Show that B = C. Show all your steps.

10. Suppose $AB = AC$, B and C are $n \times p$ matrices and A is invertible. Show that $B = C$

$$AB = AC \rightarrow A^{-1}AB = A^{-1}AC \rightarrow IB = A^{-1}AC$$

$$B = IC$$

$$\boxed{B = C}$$

11. If an n×n matrix A is invertible, then the columns of AT are linearly independent. Explain why.

THEOREM 8 **The Invertible Matrix Theorem**

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a. A is an invertible matrix.
b. A is row equivalent to the $n \times n$ identity matrix.
c. A has n pivot positions.
d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
e. The columns of A form a linearly independent set.
f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.
h. The columns of A span $\mathbb{R}^n$.
i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
j. There is an $n \times n$ matrix C such that $CA = I$.
k. There is an $n \times n$ matrix D such that $AD = I$.
l. A^T is an invertible matrix.

If A is an invertible nxn matrix, then it would follow that A^T is invertible by part l of the Invertible Matrix Theorem. According to part e, the columns of A form a linearly independent set. Also, according to the IMT, if a matrix is invertible its columns form a linearly independent set. Therefore, the columns of A^T are linearly independent.

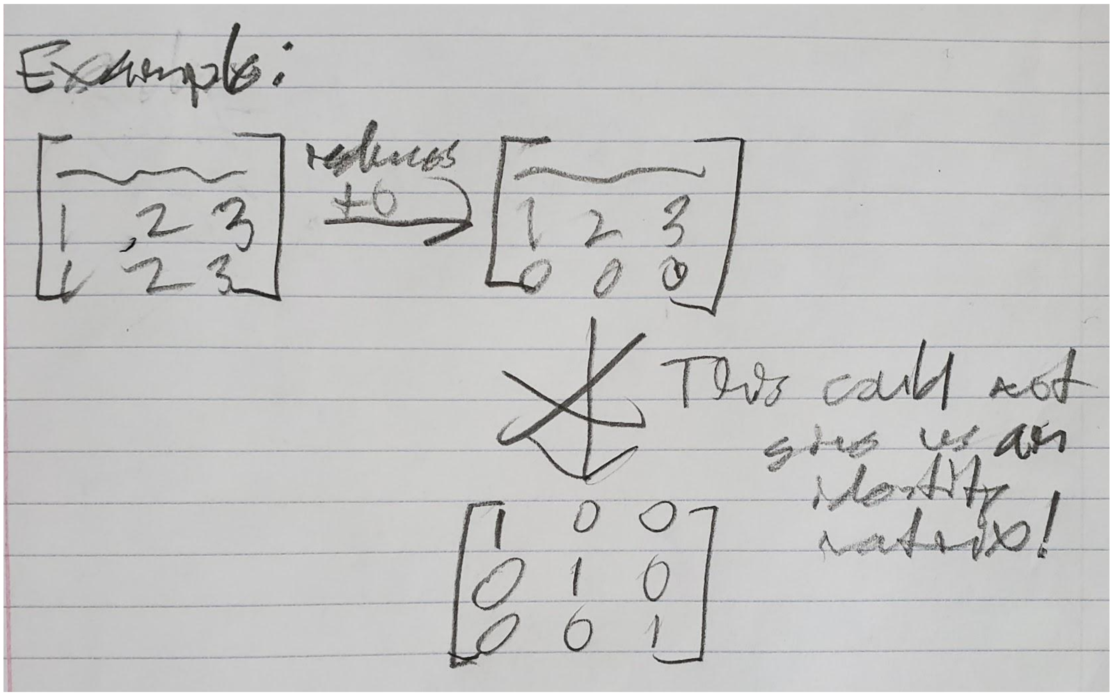
12. Can a square matrix with two identical rows be invertible? Why or why not? Do not just cite a theorem and move on, but you can build up to a theorem.

THEOREM 8 **The Invertible Matrix Theorem**

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a. A is an invertible matrix.
b. A is row equivalent to the $n \times n$ identity matrix.
c. A has n pivot positions.
d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
e. The columns of A form a linearly independent set.
f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.
h. The columns of A span $\mathbb{R}^n$.
i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
j. There is an $n \times n$ matrix C such that $CA = I$.
k. There is an $n \times n$ matrix D such that $AD = I$.
l. A^T is an invertible matrix.

If a matrix contains two identical rows, then it cannot be row reduced to an identity matrix because there would be a row of zeros. Such a matrix would not satisfy part b of the IMT, thus the matrix would not be invertible.



13. Let A and B be n × n matrices. Show that if AB is invertible, so is A. You cannot use Theorem 6(b), because you cannot assume that A and B are invertible. [Hint: There is a matrix W such that ABW = I, but why can we say that?]

THEOREM 8 **The Invertible Matrix Theorem**

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a. A is an invertible matrix.
b. A is row equivalent to the $n \times n$ identity matrix.
c. A has n pivot positions.
d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
e. The columns of A form a linearly independent set.
f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.
h. The columns of A span $\mathbb{R}^n$.
i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
j. There is an $n \times n$ matrix C such that $CA = I$.
k. There is an $n \times n$ matrix D such that $AD = I$.
l. A^T is an invertible matrix.

Assuming W is the inverse of AB, we get that ABW = I. We would also get that A(BW) = I. Part k of the IMT would be satisfied if we let D = BW. As such, A is invertible. The fact that A is square also helps.