1. Check whether the given eigenvalue or eigenvector works for the given matrix.

(a) Is $\lambda = -3$ an eigenvalue of $A = \begin{bmatrix} -1 & 4 \\ 6 & 9 \end{bmatrix}$? Why or why not?

1. a)

$\lambda = -3$ an eigen value of $A = \begin{bmatrix} -1 & 4 \\ 6 & 9 \end{bmatrix}$?

-3 is eigenvalue iff $Ax = -3x$ has non trivial solution!

$Ax = -3x$ is equivalent to $Ax + 3x = 0$

or

$(A+3I)x = 0$

$A + 3I = \begin{bmatrix} -1 & 4 \\ 6 & 9 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 12 \end{bmatrix}$

$2 \cdot 3 \quad 4 \cdot 3$

$6 \quad 12$

So $(A+3I)x = 0$ has nontrivial sol!

Row 2 is Row 1 multiplied by 3, so there is an obvious linear dependence

$\boxed{-3 \text{ is an eigenvalue of } A!}$

(b) Is $\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$ an eigenvector of $A = \begin{bmatrix} 3 & 6 & 7 \\ 3 & 2 & 7 \\ 5 & 6 & 4 \end{bmatrix}$? If so, find the eigenvalue.

b) Is $\underset{u}{\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}}$ an eigenvector of $A = \begin{bmatrix} 3 & 6 & 7 \\ 3 & 2 & 7 \\ 5 & 6 & 4 \end{bmatrix}$?

$$Au = \begin{bmatrix} 3 & 6 & 7 \\ 3 & 2 & 7 \\ 5 & 6 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = 1\begin{bmatrix} 3 \\ 3 \\ 5 \end{bmatrix} - 2\begin{bmatrix} 6 \\ 2 \\ 6 \end{bmatrix} + 2\begin{bmatrix} 7 \\ 7 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \\ 1 \end{bmatrix}$$

$3 - 12 + 14 = 5$

$3 - 4 + 14 = 13$

$5 - 12 + 8 = 1$

$$\begin{bmatrix} 5 \\ 13 \\ 1 \end{bmatrix} \neq \lambda \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

$\boxed{\text{No, } \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \text{ is not an eigenvector of } A!}$

(c) Is $\lambda = 1$ an eigenvalue of $A = \begin{bmatrix} 4 & -2 & 3 \\ 0 & -1 & 3 \\ -1 & 2 & -2 \end{bmatrix}$? If so, find one corresponding eigenvector.

c) $\lambda = 1$ eigenvalue of $A = \begin{bmatrix} 4 & -2 & 3 \\ 0 & -1 & 3 \\ -1 & 2 & -2 \end{bmatrix}$?

$Ax = 1x$

$Ax - 1x = 0$

$(A - 1I)x = 0$

Nontriv sol? Invertible?

$$A - 1I = \begin{bmatrix} 4 & -2 & 3 \\ 0 & -1 & 3 \\ -3 & 2 & -2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 3 \\ 0 & -2 & 3 \\ -3 & 2 & -3 \end{bmatrix}$$

Row reduce to $[(A - 1I)\ 0]$

$$\begin{bmatrix} 3 & -2 & 3 & 0 \\ 0 & -2 & 3 & 0 \\ -3 & 2 & -3 & 0 \end{bmatrix} \xrightarrow{R_3 + R_1 \to R_3} \begin{bmatrix} 3 & -2 & 3 & 0 \\ 0 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 - R_2 \to R_1} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\frac{1}{3}R_1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{2}R_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -3/2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Free var, so it has a non triv solution!

YAY

cont'd

Because it has a nontrivial sol., 1 is an eigenvalue!

Any nonzero sol of $(A-I)x=0$ is an eigenvector

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -3/2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{array}{l} x_1 = 0 \\ x_2 - \frac{3}{2}x_3 = 0 \quad x_2 = \frac{3}{2}x_3 \\ x_3 \text{ is free} \end{array}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{3}{2}x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 3/2 \\ 1 \end{bmatrix}$$

If $x_3 = 2$, then we get $\begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$

$\boxed{\text{1 is an eigenvalue of A! One corresponding eigenvector could be: } \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}}$

2. For the following, find a basis for the eigenspace corresponding to each listed eigenvalue.

(a) $A = \begin{bmatrix} 1 & -3 \\ -4 & 5 \end{bmatrix}$, and $\lambda = -1, 7$.

2. a) $A = \begin{bmatrix} 1 & -3 \\ -4 & 5 \end{bmatrix}$ and $\lambda = -1, 7$.

(-1) $A - (-1)I \Rightarrow A + 1I$

row reduce for $(A + 1I)x = 0$!

$$A + 1I = \begin{bmatrix} 1 & -3 \\ -4 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -3 & 0 \\ -4 & 6 & 0 \end{bmatrix} \xrightarrow{R_2 + 2R_1 \to R_2} \begin{bmatrix} 2 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & -3/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 - 3/2\, x_2 = 0$

x is free

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3/2\, x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$$

$x_1 = 3/2\, x_2$

x_2 is free

(7) $A - 7I$

$$A - 7I = \begin{bmatrix} 1 & -3 \\ -4 & 5 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} -6 & -3 \\ -4 & -2 \end{bmatrix}$$

$$\begin{bmatrix} -6 & -3 & 0 \\ -4 & -2 & 0 \end{bmatrix} \xrightarrow{-\frac{1}{6}R_1} \begin{bmatrix} 1 & 1/2 & 0 \\ -4 & -2 & 0 \end{bmatrix} \xrightarrow{R_2 + 4R_1 \to R_2} \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Cont'd

Cont'd

$\begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow x_1 + \frac{1}{2}x_2 = 0 \Rightarrow x_1 = -\frac{1}{2}x_2$

x_2 is free

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$

$\lambda = -1$: Assuming $x_2 = 2$, a basis could be $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$\lambda = 7$: Assuming $x_2 = 2$, a basis could be $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

(b) $A = \begin{bmatrix} 4 & 0 & -1 \\ 3 & 0 & 3 \\ 2 & -2 & 5 \end{bmatrix}$, and $\lambda = 3$.

(b) $A = \begin{bmatrix} 4 & 0 & -1 \\ 3 & 0 & 3 \\ 2 & -2 & 5 \end{bmatrix}$, and $\lambda = 3$

$$A - 3I = \begin{bmatrix} 4 & 0 & -1 \\ 3 & 0 & 3 \\ 2 & -2 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 3 & -3 & 3 \\ 2 & -2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 3 & -3 & 3 & 0 \\ 2 & -2 & 2 & 0 \end{bmatrix} \xrightarrow{\frac{1}{3} R_2} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 1 & -1 & 1 & 0 \\ 2 & -2 & 2 & 0 \end{bmatrix}$$

$$\xrightarrow{R_3 - 2 \cdot R_2 \to R_3} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 - R_1 \to R_2} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{-1 \cdot R_2} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{matrix} 1x_1 - 1x_3 = 0 \\ 1x_2 - 2x_3 = 0 \\ x_3 \text{ is free} \end{matrix} \to \begin{matrix} x_1 = x_3 \\ x_2 = 2x_3 \\ x_3 \text{ is free} \end{matrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ 2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\boxed{\text{Assuming } x_3 = 1, \text{ a basis could be } \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}}$$

3. Construct an example of a 2×2 matrix with only one distinct eigenvalue.

An **echelon matrix** (respectively, **reduced echelon matrix**) is one that is in echelon form (respectively, reduced echelon form). Property 2 says that the leading entries form an *echelon* ("steplike") pattern that moves down and to the right through the matrix. Property 3 is a simple consequence of property 2, but we include it for emphasis.

The "triangular" matrices of Section 1.1, such as

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

are in echelon form. In fact, the second matrix is in reduced echelon form. Here are additional examples.

THEOREM 1 The eigenvalues of a triangular matrix are the entries on its main diagonal.

3. Construct an example of a 2x2 matrix with only one distinct eigenvalue.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

From my understanding, triangular matrices result when all of the entries above or below the main diagonal are zero,

This matrix is triangular and so the eigenvalues are the entries on the main diagonal. The diagonal only has values of 1, so only 1 would be the eigenvalues of the matrix shown!

*For the explanation on the side, I just wrote that from my understanding, triangular matrices result when all of the entries above or below the main diagonal are zero. For the answer, I wrote that this matrix is triangular and so the eigenvalues are the entries on its main diagonal. The diagonal only has values of 1, so only 1 would be the eigenvalue of the matrix shown! I got this from the excerpts I posted above and your explanation in class helped a bit!

4. Let λ be an eigenvalue of an invertible matrix A. Show that λ^{-1} is an eigenvalue of A^{-1}. [*Hint:* Suppose a nonzero $\mathbf{x}$ satisfies $A\mathbf{x} = \lambda\mathbf{x}$.]

DEFINITION An **eigenvector** of an $n \times n$ matrix A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$; such an $\mathbf{x}$ is called an *eigenvector corresponding to λ*.[1]

Recall that the multiplicative inverse of a number such as 5 is 1/5 or 5^{-1}. This inverse satisfies the equations

$$5^{-1} \cdot 5 = 1 \quad \text{and} \quad 5 \cdot 5^{-1} = 1$$

The matrix generalization requires *both* equations and avoids the slanted-line notation (for division) because matrix multiplication is not commutative. Furthermore, a full generalization is possible only if the matrices involved are square.[1]

An $n \times n$ matrix A is said to be **invertible** if there is an $n \times n$ matrix C such that

$$CA = I \quad \text{and} \quad AC = I$$

where $I = I_n$, the $n \times n$ identity matrix. In this case, C is an **inverse** of A. In fact, C is uniquely determined by A, because if B were another inverse of A, then $B = BI = B(AC) = (BA)C = IC = C$. This unique inverse is denoted by A^{-1}, so that

$$A^{-1}A = I \quad \text{and} \quad AA^{-1} = I$$

nontrivial solution: A nonzero solution of a homogeneous equation or system of homogeneous equations.

A system of linear equations is said to be **homogeneous** if it can be written in the form $A\mathbf{x} = \mathbf{0}$, where A is an $m \times n$ matrix and $\mathbf{0}$ is the zero vector in $\mathbb{R}^m$. Such a system $A\mathbf{x} = \mathbf{0}$ *always* has at least one solution, namely, $\mathbf{x} = \mathbf{0}$ (the zero vector in $\mathbb{R}^n$). This zero solution is usually called the **trivial solution**. For a given equation $A\mathbf{x} = \mathbf{0}$, the important question is whether there exists a **nontrivial solution**, that is, a nonzero vector $\mathbf{x}$ that satisfies $A\mathbf{x} = \mathbf{0}$. The Existence and Uniqueness Theorem in Section 1.2 (Theorem 2) leads immediately to the following fact.

4. Let λ be an eigenvalue of an invertible matrix A. Show that λ^{-1} is an eigenvalue of A^{-1}. [Hint: Suppose a nonzero x satisfies $Ax = \lambda x$]

$$Ax = \lambda x$$

λ is a number or scalar and thus we can take its multiplicative inverse!

$$A^{-1}Ax = A^{-1}\lambda x$$

$$Ix = A^{-1}\lambda x$$

$$x = A^{-1}\lambda x$$

$$x = \lambda A^{-1}x$$

← Commutative property possible since λ is a scalar!

$$\cancel{\frac{\lambda x}{\lambda} = \frac{\lambda A^{-1}x}{\lambda}}$$

$$\frac{1}{\lambda}x = A^{-1}x$$

$$\lambda^{-1}x = A^{-1}x$$

λ^{-1} is an eigenvalue of A^{-1}! We know this as a scalar λ is called an eigenvalue of A if there is a nontrivial (*or in this case nonzero) solution x such that $Ax = \lambda x$.

*For the answer, I basically wrote that λ^{-1} is an eigenvalue of A^{-1}. We know this as a scalar λ is called an eigenvalue of A if there is a nontrivial (*or in this case nonzero) solution x such that $Ax = \lambda x$. Again, I got some help from the book. As you can see, I posted the excerpts above.

5. Find the characteristic polynomial and the real eigenvalues of the matrices below.

(a) $\begin{bmatrix} -4 & -1 \\ 6 & 1 \end{bmatrix}$

5.

a) $\begin{bmatrix} -4 & -1 \\ 6 & 1 \end{bmatrix}$ $A - \lambda I$?

$\det(A - \lambda I) = \det \begin{bmatrix} -4-\lambda & -1 \\ 6 & 1-\lambda \end{bmatrix}$

$= (-4-\lambda)(1-\lambda) - 6(-1) = \lambda^2 + 3\lambda + 2$

$-4 + 4\lambda - \lambda + \lambda^2 + 6$

$+2 + 3\lambda + \lambda^2$

$\lambda^2 + 3\lambda + 2 \rightarrow (\lambda + 2)(\lambda + 1)$

values that mult to form?
values that add to form?

$\lambda + 2 = 0 \qquad \lambda + 1 = 0$

$\lambda = -2 \qquad \lambda = -1$

Characteristic polynomial: $\lambda^2 + 3\lambda + 2$

Eigenvalues: -2, -1 -1

(b) $\begin{bmatrix} 5 & 3 \\ -4 & 4 \end{bmatrix}$

(b) $\begin{bmatrix} 5 & 3 \\ -4 & 4 \end{bmatrix}$ $\quad \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$

$\det(A - \lambda I) = \det \begin{bmatrix} 5-\lambda & 3 \\ -4 & 4-\lambda \end{bmatrix}$ $\quad x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$-(-12)$ add 3 mult?

$= (5-\lambda)(4-\lambda) - (3)(-4) = \lambda^2 - 9\lambda + 32$

$a \quad b \quad c$

$20 - 5\lambda - 4\lambda + \lambda^2$

use quadratic formula!

$\lambda^2 - 9\lambda + 32$

$20 - 9\lambda + \lambda^2 + 12$

$\dfrac{-(-9) \pm \sqrt{(-9)^2 - 4(1)(32)}}{2(1)}$

$\dfrac{9 \pm \sqrt{81 - 4(32)}}{2} \rightarrow \dfrac{9 \pm \sqrt{81 - 128}}{2}$

$\dfrac{9 \pm \sqrt{(-47)}}{2} \rightarrow$ not a real number!

So A has no real eigenvalues

Characteristic polynomial: $\lambda^2 - 9\lambda + 32$
No real eigenvalues!

6. Find the characteristic polynomial of the following matrices.[*Note:* Finding the characteristic polynomial of a 3×3 matrix is not easy to do with just row operations because the variable λ is involved.]

(a) $\begin{bmatrix} 4 & 0 & -1 \\ 0 & 4 & -1 \\ 1 & 0 & 2 \end{bmatrix}$

6. a) $\begin{bmatrix} 4 & 0 & -1 \\ 0 & 4 & -1 \\ 1 & 0 & 2 \end{bmatrix}$ $\quad 8 - 4\lambda - 2\lambda + \lambda^2$

-6λ

$$\det(A-\lambda I) = \det\begin{bmatrix} 4-\lambda & 0 & -1 \\ 0 & 4-\lambda & -1 \\ 1 & 0 & 2-\lambda \end{bmatrix}$$

Calculate expansion along 1st row

$$= 4-\lambda \cdot \det\begin{bmatrix} 4-\lambda & -1 \\ 0 & 2-\lambda \end{bmatrix} - \cancel{0 \cdot \det\begin{bmatrix} 0 & -1 \\ 1 & 2-\lambda \end{bmatrix}} + (-1)\det\begin{bmatrix} 0 & 4-\lambda \\ 1 & 0 \end{bmatrix}$$

$$= 4-\lambda\left((4-\lambda)(2-\lambda) - (-1)\cdot(0)\right) + (-1)\left((0)(0) - (4-\lambda)(1)\right)$$

$$= 4-\lambda\left(\lambda^2 - 6\lambda + 8\right) + (4-\lambda)$$

$$= 4\lambda^2 - 24\lambda + 32 - \lambda^3 + 6\lambda^2 - 8\lambda + (4-\lambda)$$

$$= -\lambda^3 + 10\lambda^2 - 33\lambda + 36$$

$$\boxed{\text{Characteristic polynomial: } -\lambda^3 + 10\lambda^2 - 33\lambda + 36}$$

(b) $\begin{bmatrix} -1 & 0 & 2 \\ 3 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} -1 & 0 & 2 \\ 3 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$ $-1+\cancel{\lambda\lambda}+\lambda^2$

5.2 #12

$\det(A-\lambda I) = \det\begin{bmatrix} -1-\lambda & 0 & 2 \\ 3 & 1-\lambda & 0 \\ 0 & 1 & 2-\lambda \end{bmatrix}$

Cofactor expansion along 3rd row

$= \cancel{0\cdot\det\begin{bmatrix} 0 & 2 \\ 1-\lambda & 0 \end{bmatrix}} - 1\cdot\det\begin{bmatrix} (-1-\lambda) & 2 \\ 3 & 0 \end{bmatrix} + 2-\lambda\cdot\det\begin{bmatrix} (-1-\lambda) & 0 \\ 3 & (1-\lambda) \end{bmatrix}$

$= -1(\cancel{(-1-\lambda)(0)} - 2(3)) + 2-\lambda((-1-\lambda)(1-\lambda) - \cancel{0(3)})$

$= -1(-6) + 2-\lambda(-1+\lambda^2)$ $\quad -2+2\lambda^2+\lambda-\lambda^3$

$= 6 + 2-\lambda(-1+\lambda^2)$

$= 6 - 2 + 2\lambda^2 + \lambda - \lambda^3$

$= -\lambda^3 + 2\lambda^2 + \lambda + 4$

$\boxed{\text{Characteristic Polynomial: } -\lambda^3+2\lambda^2+\lambda+4}$

7. Show that if A and B are similar, then $\det A = \det B$.

Similarity

The next theorem illustrates one use of the characteristic polynomial, and it provides the foundation for several iterative methods that *approximate* eigenvalues. If A and B are $n \times n$ matrices, then A **is similar to** B if there is an invertible matrix P such that $P^{-1}AP = B$, or, equivalently, $A = PBP^{-1}$. Writing Q for P^{-1}, we have $Q^{-1}BQ = A$. So B is also similar to A, and we say simply that A and B **are similar**. Changing A into $P^{-1}AP$ is called a **similarity transformation**.

THEOREM 4

If $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicities).

PROOF If $B = P^{-1}AP$, then

$$B - \lambda I = P^{-1}AP - \lambda P^{-1}P = P^{-1}(AP - \lambda P) = P^{-1}(A - \lambda I)P$$

Using the multiplicative property (b) in Theorem 3, we compute

$$\begin{aligned} \det(B - \lambda I) &= \det[P^{-1}(A - \lambda I)P] \\ &= \det(P^{-1}) \cdot \det(A - \lambda I) \cdot \det(P) \end{aligned} \tag{2}$$

Since $\det(P^{-1}) \cdot \det(P) = \det(P^{-1}P) = \det I = 1$, we see from equation (2) that $\det(B - \lambda I) = \det(A - \lambda I)$. ■

WARNINGS:

1. The matrices
$$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$
are not similar even though they have the same eigenvalues.
2. Similarity is not the same as row equivalence. (If A is row equivalent to B, then $B = EA$ for some invertible matrix E.) Row operations on a matrix usually change its eigenvalues.

THEOREM 3

Properties of Determinants

Let A and B be $n \times n$ matrices.

a. A is invertible if and only if $\det A \neq 0$.

b. $\det AB = (\det A)(\det B)$.

c. $\det A^T = \det A$.

d. If A is triangular, then $\det A$ is the product of the entries on the main diagonal of A.

e. A row replacement operation on A does not change the determinant. A row interchange changes the sign of the determinant. A row scaling also scales the determinant by the same scalar factor.

7. Show that if A and B are similar then $\det A = \det B$.

If A and B are similar then there exists a matrix such that

$$A = P^{-1}BP$$

From thm. 3, we get that

$$\det A = \det(P^{-1}BP)$$

$$\det A = (\det P^{-1})(\det B)(\det P)$$

$$\det A = (\det B)(\det P^{-1})(\det P)$$

$$\det A = (\det B)(\det P^{-1}P)$$

$$\det A = (\det B)(\det I) \qquad (\det I) = 1$$

$$\det A = (\det B)\cdot 1$$

We have that $\det A = \det B$ from thm. 3(b) and from the proof shown for thm. 4.

*We have that det A = det B from theorem 3(b) and from the proof shown for theorem 4.