Math 30, Tuesday 3/24/2020

noon class

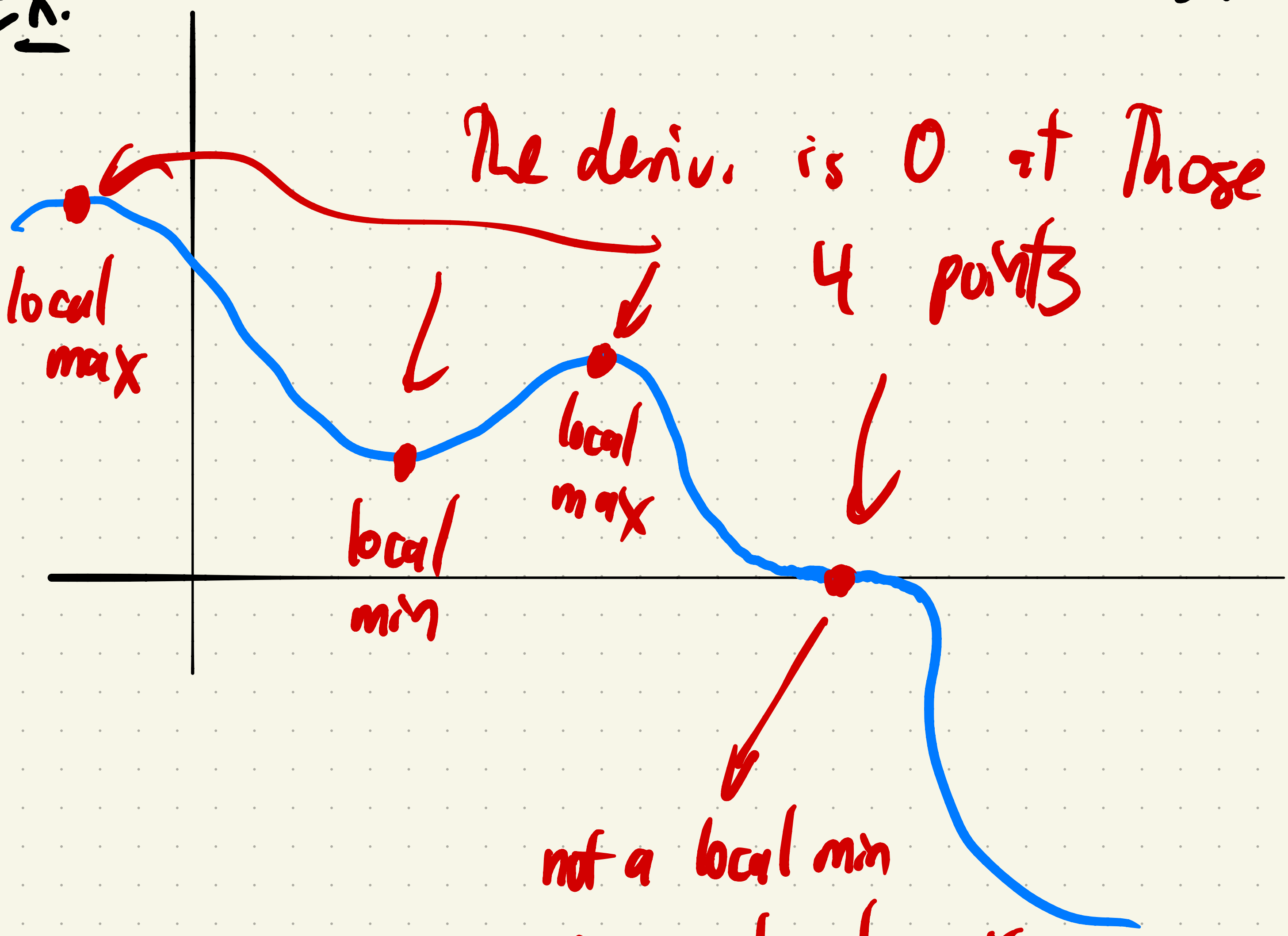
The Closed Interval Method

Last: Fermat's Theorem.

If you want local min & max, the candidates are the crit. pts

where $f'(x) = 0$ or doesn't exist.

Ex.



The deriv. is 0 at those 4 points

not a local min <u>or</u> a local max.

"a failed candidate"

Today: Global min & max

aka Absolute min & max

Now: look at critical points

and boundary points

includes boundary pts.

The Closed Interval Method.

To find the global max & min of f on $[a,b]$:

(1) Find the values of f at crit. pts

(2) " " boundary pts a & b

(3) compare & find biggest & smallest values of f.

Ex. Find global max & min. of

$$f(x) = x^4 + 8x^3 + 3$$

on $[-1,1]$.

1. Find values of f at crit. pts.

where $f' = 0$.

$$f'(x) = 4x^3 + 24x^2 = 4x^2(x+6)$$

crit. pts are $x=0$ ~~$x=-6$~~

$f(0) = 3$ value at crit. pt $x=0$

technically no because not in the interval $[-1,1]$.

Ex. Find global max & min. of
$$f(x) = x^4 + 8x^3 + 3$$
on $[-1,1]$.

2. Now find values of f at the boundary points $x=-1$ and $x=1$:

$$f(-1) = 1 - 8 + 3 = -4$$
$$f(1) = 1 + 8 + 3 = 12$$

3. Compare those values:

smallest $\leftarrow$ $f(-1) = -4$
$f(0) = 3$
biggest $\leftarrow$ $f(1) = 12$

These are the candidates for the global max & min.

global max: $f(1) = 12$, global min: $f(-1) = -4$.

I plotted using Wolfram Alpha:

MATH 30, 3/24-25/2020: THE CLOSED INTERVAL METHOD

Last time: Critical points and Fermat's theorem. If you want to find local max and min, the candidates are the critical points (you can ignore all the other points).

The Closed Interval Method. To find the *global* max and min values of f on $[a, b]$:

(1) Find the values of f at the critical points of f in (a, b).
(2) Find $f(a)$ and $f(b)$.
(3) Compare all these values, and take the biggest and smallest values of f.

Example. Find the global max and min of the function $f(x) = x^4 + 8x^3 + 3$ on the interval $[-1, 1]$.

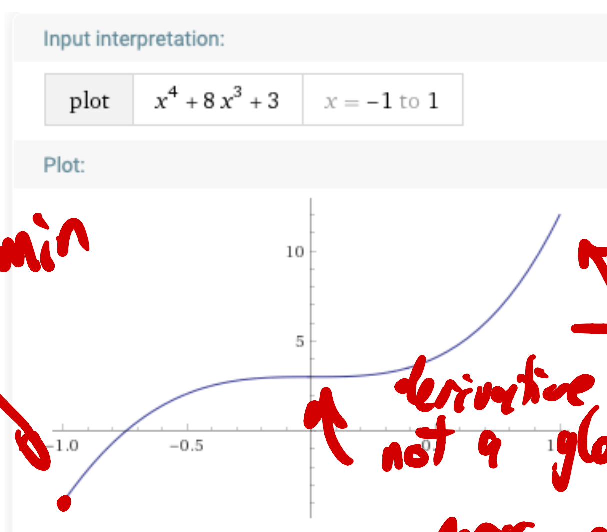
Solution. First, let's find the critical points: $f'(x) = 4x^3 + 24x^2 = 4x^2(x + 6)$. So the critical points are $x = 0$ and ... **not** $x = -6$ only because that point is not in the interval $[-1, 1]$.

Now compare the values of the function f at $x = -1, 0, 1$:

$$
\begin{aligned}
f(-1) &= -4 \\
f(0) &= 3 \\
f(1) &= 12.
\end{aligned}
$$

So we see that the global maximum is $f(1) = 12$, and the global minimum is $f(-1) = -4$.

For fun, let's look at the graph of the function on the given interval $[-1, 1]$:

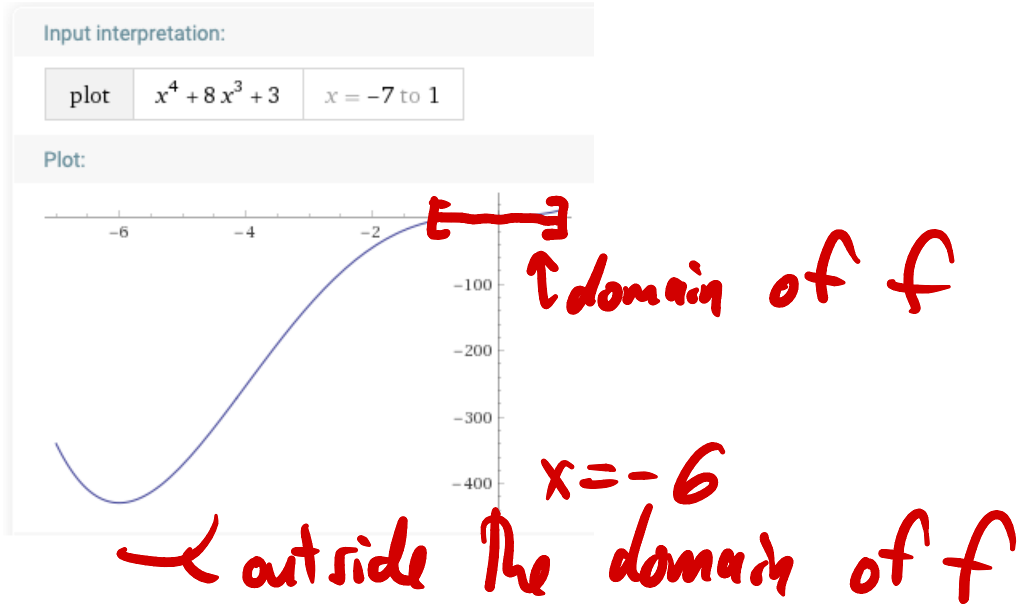


global min

global max

derivative is 0 at $x=0$ but not a global or local max or min.

Also for fun (even though I wasn't asking about this), let's look at the graph on the interval $[-7, 1]$:



Now it's your turn! Here are some "optimization and pessimization" problems: (You will probably need scratch paper...)

1. Find the point on the line $y = 2x - 3$ that is closest to the origin. Make a sketch.

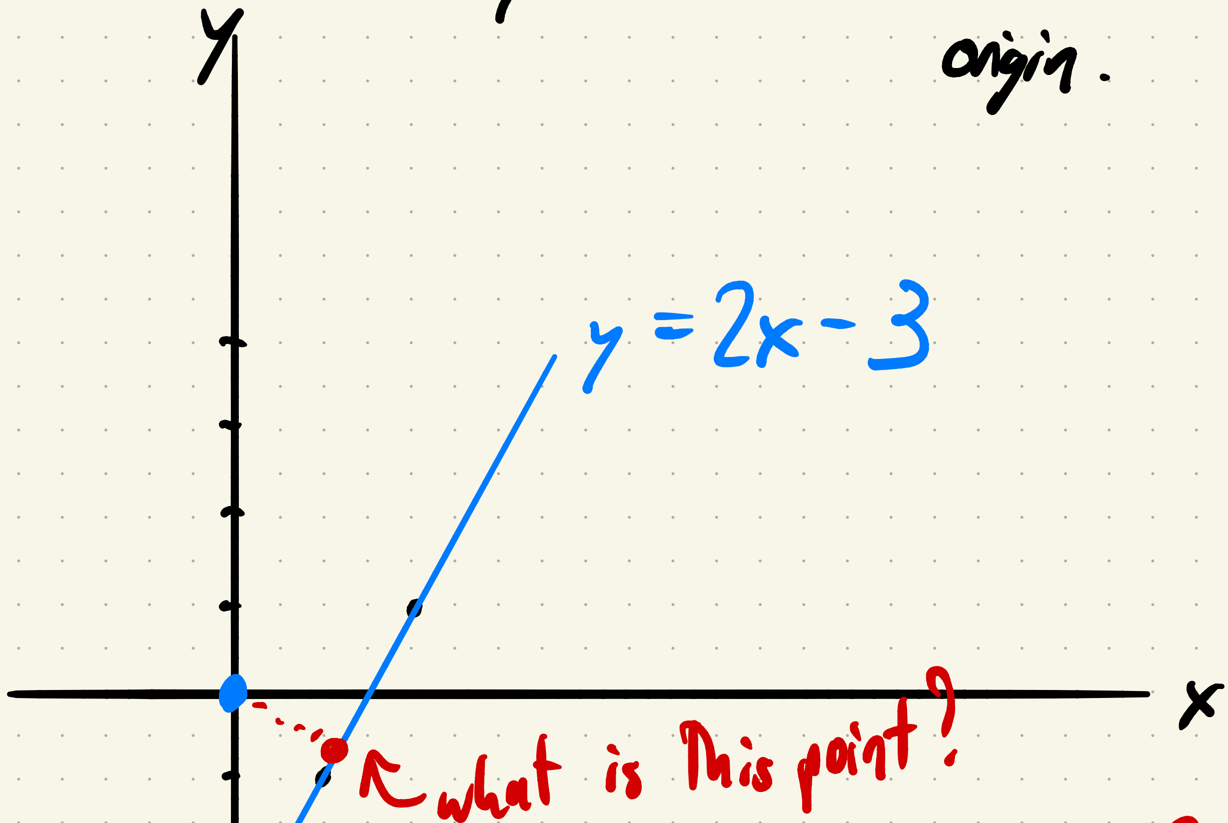
2. Prove that among all rectangles of a given perimeter (say they all have perimeter P), the square has the largest area.

3. Find the global max and min of the function $f(x) = x^5 + x + 1$ on the interval $[-1, 1]$.

4. Find the global max and min of the function $f(x) = x^3 - x^2 - 8x + 1$ on the interval $[-2, 2]$.

5. If 1200 cm^2 of cardboard is available to make a box with a square base and an open top, find the largest possible volume of the box.

1. Pt. on line $y = 2x - 3$ closest to the origin.



guess: $\left(\frac{5}{4}, \frac{-3}{4}\right)$?

dist. from (x,y) on line to $(0,0)$ is

$$(x,y) = (x, 2x-3)$$

$$d = \sqrt{x^2 + (2x-3)^2}$$ minimize this

Trick:

It's easier to minimize d^2:

$$f(x) = x^2 + (2x-3)^2$$

same x minimizes both!

$y = 2x - 3$

to minimize

(Zoom crashed while doing this...)

$f(x) = x^2 + (2x-3)^2$

find crit. pts: $f'(x) = 0$.

$f'(x) = 2x + 2(2x-3)\cdot 2$

chain rule!

$= 2x + 8x - 12$

$= 10x - 12$

This is zero when $10x = 12$

The only crit. pt. $\rightarrow$ $x = \frac{6}{5}$

So the y-value is

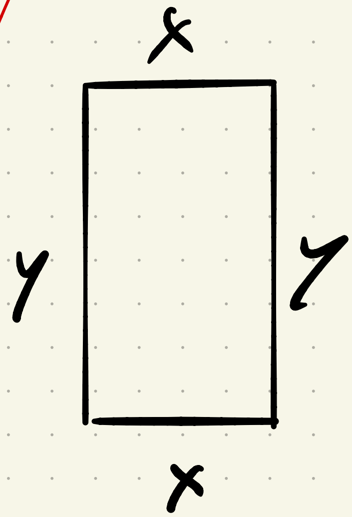
point on line closest to $(0,0)$

$y = 2\left(\frac{6}{5}\right) - 3 = \frac{12-15}{5}$

$(x,y) = \left(\frac{6}{5}, \frac{-3}{5}\right)$

$= \frac{-3}{5}$

2. Prove: among all rectangles w/ same perimeter P, the square has the largest area.



Perimeter is

$$P = 2x + 2y$$

Find when $A = xy$ is <u>biggest</u>.

But y is related to x by: $y = \frac{P-2x}{2}$

So

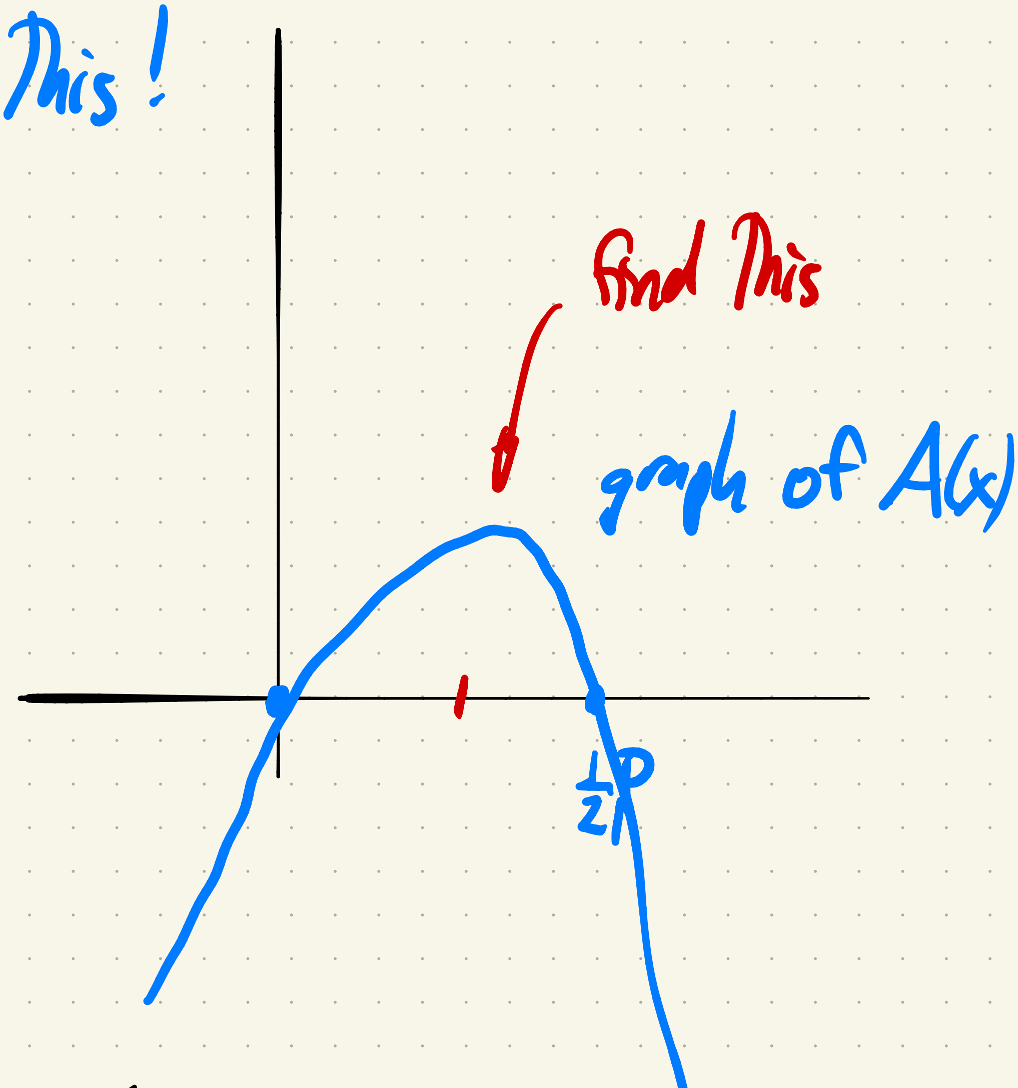
$$A(x) = x\left(\frac{P}{2} - x\right) = \frac{1}{2}Px - x^2$$

where is it biggest?

$$A(x) = x\left(\frac{P}{2} - x\right) = \frac{1}{2}Px - x^2$$

where is it biggest?

maximize this!



Find crit. pts: $A'(x) = \frac{1}{2}P - 2x$

This is zero when $x = \frac{1}{4}P$ square!

But remember $P = 2x + 2y = \frac{1}{2}P + 2y$

So also $y = \frac{1}{4}P$ So the max happens when $x = y$

4. Global max & min of
$$f(x) = x^3 - x^2 - 8x + 1$$
on $[-2, 2]$.

Closed Int. Method:

1. Look at values of f at crit. pts:
$$f'(x) = 3x^2 - 2x - 8$$
This is zero when (quad. formula):
$$x = \frac{2 \pm \sqrt{4 - 4(3)(-8)}}{6}$$
$\frac{32}{3}$
$$= \frac{2 \pm \sqrt{4 + 96}}{6}$$
$$= \frac{2 \pm 10}{6} = \frac{-8}{6} \text{ and } 2$$

max & min of

$f(x) = x^3 - x^2 - 8x + 1$

on $[-2, 2]$.

already on ddy

Crit. pts: $x = -\frac{4}{3}$ and $x = 2$.

also look at $x = -2$ (boundary pt.)

3. Compare

$f(-\frac{4}{3})$, $f(-2)$, $f(2)$

see which is biggest &

which is smallest.

The end.