MATH 30, 4/6/2020: CONCAVITY/CONVEXITY

Last time, before Spring Break, we saw what the first derivative f' reveals about the shape of the curve $y = f(x)$. Now we will see what the *second* derivative f'' reveals.

Definition. If the graph of f lies *above* all its tangent lines on an interval, then it is called *concave up* (or "convex") on that interval. [Picture–it looks like a smile.]

Definition. If the graph of f lies *below* all its tangent lines on an interval, then it is called *concave down* on that interval. [Picture–it looks like a frown.]

Simple Example. $f(x) = x^2$ is concave up.
Note that $f'(x) = 2x$, so that f is decreasing when $x < 0$ and is increasing when $x > 0$.
Also note that f'' is the derivative of f', so it tells you where f' is increasing or decreasing. In this example, $f''(x) = 2$, so f' is *always increasing*. [Can you see this in the graph of f? The slope of the tangent line is always *increasing*.]

Simple Example. $f(x) = -x^2$. [Do it yourself.]

Theorem ("The Concavity Test").
(1) If $f''(x) > 0$ for all x in an interval I, then the graph is *concave up* on that interval.
(2) If $f''(x) < 0$... [You can probably guess...]

Proof. The precise mathematical proof uses the Mean Value Theorem. [Many things in calculus are proven using the Mean Value Theorem.] We will skip the details in the interest of time. The gist of it: Use the fact that $f''(x) > 0$ for all x to show that "the graph of f is above the tangent line."

If you ever forget "which is which" in the Concavity Test, just remember the two simple examples above: $f(x) = x^2$ looks like a smile, and $f(x) = -x^2$ looks like a frown.

The physics interpretation: Say that $x(t)$ represents position at time t. Then $x'(t)$ represents velocity and $x''(t)$ represents acceleration. Think about what "constant speed," "positive acceleration," and "negative acceleration" look like in the graph of $x(t)$. [Picture.]

You can also think in terms of Force=mass×acceleration. Then $F > 0$ means force is pushing *up* and $F < 0$ means force is pushing *down*.

Definition. An *inflection point* is a point where the graph changes concavity.

You actually hear this term in the news: for example, someone might say "we are at an inflection point in the war." It means that things might still be getting worse, but at a decreasing rate.

There is an inflection point wherever f'' changes sign. [Picture.]

Second derivatives are useful for max/min problems:

The Second Derivative Test.

(1) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$. [Picture.]
(2) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$. [Picture.]

Example. Find the local max and min of the function

$$f(x) = x^3 - 6x^2 + 10.$$

We first calculate $f'(x) = 3x^2 - 12x = 3x(x - 4)$, so the critical points are $x = 0$ and $x = 4$.

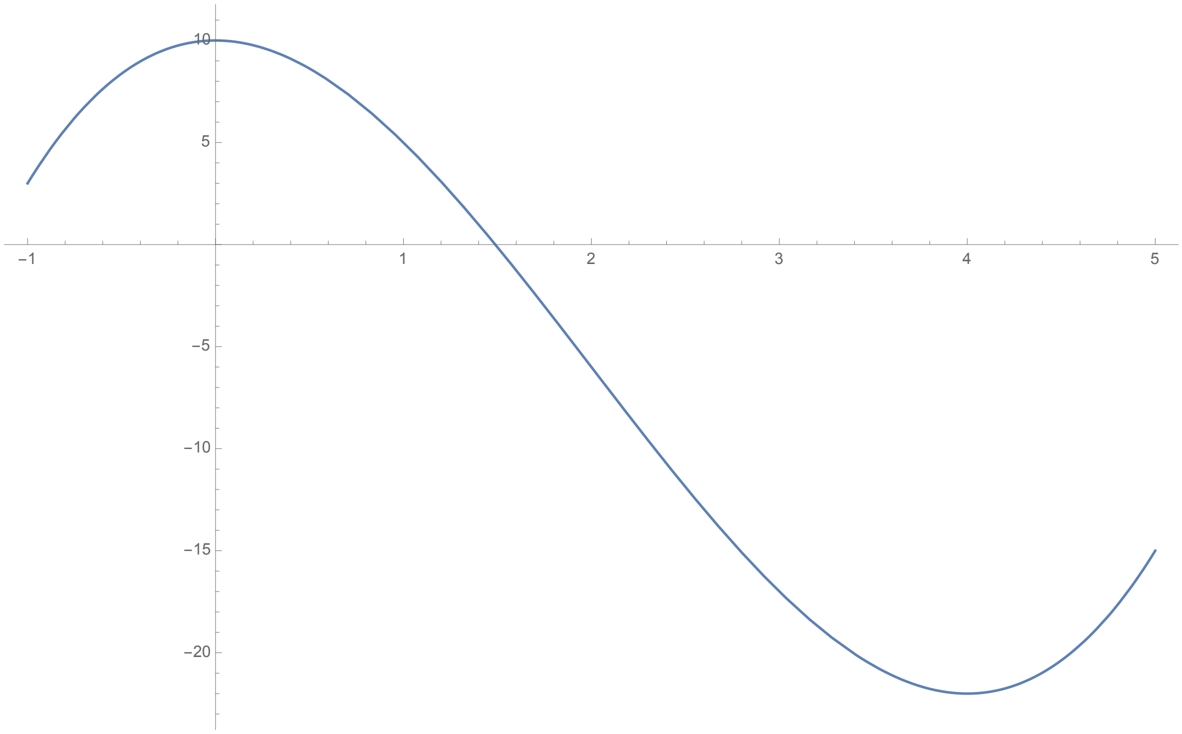
One way, using the Increasing/Decreasing Test: We note that $f'(x) > 0$ on $(-\infty, 0)$, $f'(x) < 0$ on $(0, 4)$, and $f'(x) > 0$ on $(4, \infty)$, so we know the function f is increasing, then decreasing, then increasing. It thus has a local max at $x = 0$ and a local min at $x = 4$.

Another way, using the Second Derivative Test: We have

$$f''(x) = 6x - 12.$$

Now check the concavity at the critical points: $f''(0) = -12 < 0$, which means that $x = 0$ is a local max, and $f''(4) = 12 > 0$, which means that $x = 4$ is a local min.

To help draw the picture, we note that the only inflection point is at $x = 2$: $f''(2) = 0$. You can see how the graph changes concavity there:



Remember, calculus helps us answer these questions *without* needing the picture!