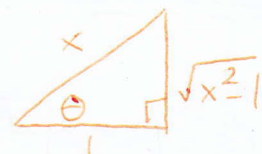
1 Trigonometric Substitutions

Evaluate each of the following integrals.

1. $\int \frac{\sqrt{x^2-1}}{x^4}dx$

Let $x = \sec\theta$

$dx = \sec\theta\tan\theta\,d\theta$



$$= \int \frac{\sqrt{\sec^2\theta-1}}{\sec^4\theta}\cdot\sec\theta\tan\theta\,d\theta = \int \frac{\sqrt{\tan^2\theta}}{\sec^3\theta}\cdot\tan\theta\,d\theta = \int \frac{\tan^2\theta}{\sec^3\theta}\,d\theta$$

$$= \int \frac{\sin^2\theta}{\cos^2\theta}\cdot\frac{\cos^3\theta}{1}\,d\theta = \int \sin^2\theta\cos\theta\,d\theta$$

Let $u = \sin\theta$

$du = \cos\theta\,d\theta$

$$= \int u^2\,du = \frac{1}{3}u^3 + C = \frac{1}{3}\sin^3\theta + C = \boxed{\frac{1}{3}\left(\frac{\sqrt{x^2-1}}{x}\right)^3 + C}$$

2. $\int_0^3 \frac{x}{\sqrt{36-x^2}}dx$

Let $x = 6\sin\theta$

$dx = 6\cos\theta\,d\theta$

$$= \int_0^{\pi/6} \frac{6\sin\theta}{\sqrt{36-36\sin^2\theta}}\cdot 6\cos\theta\,d\theta = \int_0^{\pi/6}\frac{36\sin\theta\cos\theta}{6\sqrt{1-\sin^2\theta}}\,d\theta = 6\int_0^{\pi/6}\sin\theta\,d\theta$$

$$= -6\cos\theta\Big|_0^{\pi/6} = -6\left(\frac{\sqrt{3}}{2}-1\right) = \boxed{6\left(1-\frac{\sqrt{3}}{2}\right) \approx 0.8038}$$

3. $\int_0^2 \frac{dt}{\sqrt{4+t^2}}dt$

Let $t = 2\tan\theta$

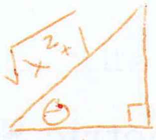
$dt = 2\sec^2\theta\,d\theta$

$$= \int_0^{\pi/4}\frac{2\sec^2\theta\,d\theta}{\sqrt{4+4\tan^2\theta}} = \int_0^{\pi/4}\frac{\sec^2\theta\,d\theta}{\sqrt{1+\tan^2\theta}} = \int_0^{\pi/4}\sec\theta\,d\theta$$

$$= \ln|\sec\theta+\tan\theta|\Big|_0^{\frac{\pi}{4}} = \boxed{\ln(\sqrt{2}+1)}$$

4. $\int \frac{x}{\sqrt{1+x^2}}dx$ Let $x = \tan\theta$

$dx = \sec^2\theta \, d\theta$

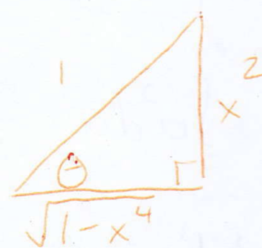


$$= \int \frac{\tan\theta \sec^2\theta \, d\theta}{\sqrt{1+\tan^2\theta}} = \int \frac{\tan\theta \sec^2\theta \, d\theta}{\sec\theta} = \int \sec\theta \tan\theta \, d\theta$$

$$= \sec\theta + C = \boxed{\sqrt{x^2+1} + C}$$

5. $\int x\sqrt{1-x^4}dx$ Let $x = \sqrt{\sin\theta} \rightarrow \sin\theta = x^2$

$dx = \frac{1}{2}(\sin\theta)^{-1/2}\cos\theta \, d\theta$



$$= \int \cancel{(\sin\theta)^{1/2}} \sqrt{1-\sin^2\theta}\left[\frac{1}{2}\cancel{(\sin\theta)^{-1/2}}\cos\theta\right]d\theta$$

$$= \frac{1}{2}\int \cos\theta\sqrt{1-\sin^2\theta}\, d\theta = \frac{1}{2}\int \cos^2\theta \, d\theta = \frac{1}{2}\int \left(\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= \frac{1}{4}\int (1+\cos 2\theta)\,d\theta = \frac{1}{4}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$

$$= \frac{1}{4}\left[\theta + \frac{1}{2}(2\sin\theta\cos\theta)\right] + C = \frac{1}{4}(\theta + \sin\theta\cos\theta) + C$$

$$= \boxed{\frac{1}{4}\left[\sin^{-1}(x^2) + x^2\sqrt{1-x^4}\right] + C}$$

2 Partial Fraction Decomposition

Evaluate each of the following integrals.

1. $\int \frac{3t-2}{t+1} dt$

$$\frac{3t-2}{t+1} = 3 - \frac{5}{t+1}$$

$$\int \frac{3t-2}{t+1} dt = \int \left(3 - \frac{5}{t+1}\right) dt = 3\int dt - 5\int \frac{dt}{t+1}$$

$$= \boxed{3t - 5\ln|t+1| + C}$$

2. $\int \frac{y}{(y+4)(2y-1)} dy$

$$\frac{y}{(y+4)(2y-1)} = \frac{A}{y+4} + \frac{B}{2y-1} \rightarrow \frac{y}{(y+4)(2y-1)} = \frac{4}{9(y+4)} + \frac{1}{9(2y-1)}$$

$$y = A(2y-1) + B(y+4)$$

y: $1 = 2A + B$ $\quad A = \frac{4}{9}$

1: $0 = -A + 4B$ $\quad B = \frac{1}{9}$

$$\int \frac{y}{(y+4)(2y-1)} dy = \int \left[\frac{4}{9(y+4)} + \frac{1}{9(2y-1)}\right] dy = \frac{4}{9}\int \frac{dy}{y+4} + \frac{1}{9}\int \frac{dy}{2y-1}$$

$$= \boxed{\frac{4}{9}\ln|y+4| + \frac{1}{18}\ln|2y-1| + C}$$

3. $\int_1^2 \frac{x^3+4x^2+x-1}{x^3+x^2}dx$

$$\frac{x^3+4x^2+x-1}{x^3+x^2} = 1 + \frac{3x^2+x-1}{x^3+x^2}$$

$$\frac{3x^2+x-1}{x^2(x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2} \rightarrow \frac{3x^2+x-1}{x^2(x+1)} = \frac{1}{x+1} + \frac{2x-1}{x^2}$$

$$3x^2+x-1 = Ax^2 + B(x^2+x) + C(x+1)$$

x^2: $3 = A+B$ $\quad A = 1$

x: $1 = B+C$ $\quad B = 2$

1: $-1 = C$ $\quad C = -1$

$$\int \frac{x^3+4x^2+x-1}{x^3+x^2}dx = \int\left(1+\frac{1}{x+1}+\frac{2x-1}{x^2}\right)dx = \int\left(1+\frac{1}{x+1}+\frac{2x}{x^2}-\frac{1}{x^2}\right)dx$$

$$= \boxed{x + \frac{1}{x} + \ln|x+1| + 2\ln|x| + C}$$

4. $\int \frac{z^2-z+6}{z^3+3z}dz$

$$\frac{z^2-z+6}{z(z^2+3)} = \frac{A}{z} + \frac{Bx+C}{z^2+3} \rightarrow \frac{z^2-z+6}{z(z^2+3)} = \frac{2}{z} - \frac{z+1}{z^2+3}$$

$$z^2-z+6 = A(z^2+3) + Bz^2 + Cz$$

z^2: $1 = A+B$ $\quad A = 2$

z: $-1 = C$ $\quad B = -1$

1: $6 = 3A$ $\quad C = -1$

$$\int \frac{z^2-z+6}{z^3+3z}dz = \int\left(\frac{2}{z} - \frac{z+1}{z^2+3}\right)dz = 2\int\frac{dz}{z} - \int\frac{z}{z^2+3}dz - \int\frac{dz}{z^2+3}$$

$$= \boxed{2\ln|z| - \frac{1}{2}\ln|z^2+3| - \frac{\sqrt{3}}{3}\tan^{-1}\left(\frac{\sqrt{3}x}{3}\right) + C}$$