Proofs (10 points)

1. Bubblesort is a popular, but inefficient, sorting algorithm. It works by repeatedly swapping adjacent elements that are out of order.

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BUBBLESORT(A)
1  for i = 1 to A.length − 1
2      for j = A.length downto i + 1
3          if A[j] < A[j − 1]
4              exchange A[j] with A[j − 1]
```

(a) [1 point] Let A' denote the output of BUBBLESORT(A). To prove that BUBBLESORT is correct, we need to prove that it terminates and that

$$A'[1] \leq A'[2] \leq \cdots \leq A'[n]$$

where $n = A.length$. In order to show that BUBBLESORT actually sorts, what else do we need to prove?

Solution: We need to show that the elements of A' form a permutation of the elements of A.

(b) [2 points] State precisely a loop invariant for the **for** loop in lines 2–4, and prove that this loop invariant holds. Your proof should use the structure of the loop invariant proof.

Solution:

Loop invariant: At the start of each iteration of the for loop of lines 2–4, $A[j] = \min\{A[k] : j \leq k \leq n\}$ and the subarray $A[j \dots n]$ is a permutation of the values that were in $A[j \dots n]$ at the time that the loop started.

Initialization: Initially, $j = n$, and the subaray $A[j \dots n]$ consists of single element $A[n]$. The loop invariant trivially holds.

Maintenance: Consider an iteration for a given value of j. By the loop invariant, $A[j]$ is the smallest value in $A[j \dots n]$. Lines $3-4$ exchange $A[j]$ and $A[j-1]$ if $A[j]$ is less than $A[j-1]$, and so $A[j-1]$ will be the smallest value in $A[j-1 \dots n]$ afterward. Since the only change to the subarray $A[j-1 \dots n]$ is this possible exchange, and the subarray $A[j \dots n]$ is a permutation of the values that were in $A[j \dots n]$ at the time that the loop started, we see that $A[j-1 \dots n]$ is a permutation of the values that were in $A[j-1 \dots n]$ at the time that the loop started. Decrementing j for the next iteration maintains the invariant.

Termination: The loop terminates when j reaches i. By the statement of the loop invariant, $A[i] = \min\{A[k] : i \leq k \leq n\}$ and $A[i \dots n]$ is a permutation of the values that were in $A[i \dots n]$ at the time that the loop started.

(c) [3 points] Using the termination condition of the loop invariant proved in part (b), state a loop invariant for the **for** loop in lines 1–4 that will allow you to prove inequality above. Your proof should use the structure of the loop invariant proof.

Solution:

Loop invariant: At the start of each iteration of the **for** loop of lines 1–4, the subarray $A[1 \ldots i-1]$ consists of the $i-1$ smallest values originally in $A[1 \ldots n]$, in sorted order, and $A[i \ldots n]$ consists of the $n-i+1$ remaining values originally in $A[1 \ldots n]$.

Initialization: Before the first iteration of the loop, $i = 1$. The subarray $A[1 \ldots i-1]$ is empty, and so the loop invariant vacuously holds.

Maintenance: Consider an iteration for a given value of i. By the loop invariant, $A[1 \ldots i-1]$ consists of the i smallest values in $A[1 \ldots n]$, in sorted order. Part (b) showed that after executing the for loop of lines 2–4, $A[i]$ is the smallest value in $A[i \ldots n]$, and so $A[1 \ldots i]$ is now the i smallest values originally in $A[1 \ldots n]$, in sorted order. Moreover, since the for loop of lines 2–4 permutes $A[i \ldots n]$, the subarray $A[i+1 \ldots n]$ consists of the $n-i$ remaining values originally in $A[1 \ldots n]$.

Termination: The **for** loop of lines 1–4 terminates when $i = n$, so that $i-1 = n-1$. By the statement of the loop invariant, $A[1 \ldots i-1]$ is the entire array $A[1 \ldots n]$, and it consists of the original array $A[1 \ldots n]$, in sorted order.

2. Prove by induction.

   (a) [2 points] For all $n \geq 1$, $\sum_{i=1}^{n}(2i-1) = n^2$.

Solution:

Basis: Suppose $n = 1$. Then $\sum_{i=1}^{n}(2i-1) = \sum_{i=1}^{1}(2i-1) = 2-1 = 1 = 1^2$.

Inductive step: Suppose the equality holds for $n = k$ (induction hypothesis: $\sum_{i=1}^{k}(2i-1) = k^2$). We show it holds for $n = k+1$.

Set $n = k+1$. Then

$$\begin{aligned} \sum_{i=1}^{k+1}(2i-1) &= \left[\sum_{i=1}^{k}(2i-1)\right] + [2(k+1)-1] \\ &= k^2 + [2(k+1)-1] && \text{based on induction hypothesis} \\ &= k^2 + 2k + 1 \\ &= (k+1)^2, \end{aligned}$$

which is what we wanted to prove.

   (b) [2 points] For all $n \geq 0$, $\sum_{i=0}^{n} x^i = \left(1 - x^{n+1}\right)/(1-x)$.

Solution:

Basis: Suppose $n = 0$. Then $\sum_{i=0}^{n} x^i = \sum_{i=0}^{0} x^i = 1 = (1-x)/(1-x) = \left(1 - x^{n+1}\right)/(1-x)$.

Inductive step: Suppose the equality holds for $n = k-1$ (induction hypothesis: $\sum_{i=0}^{k-1} x^i = \left(1 - x^k\right)/(1-x)$). We show it holds for $n = k$.

Set $n = k$. Then

$$\begin{aligned}
\sum_{i=0}^{k} x^i &= \left[\sum_{i=0}^{k-1} x^i\right] + x^k \\
&= \left(1 - x^k\right)/(1-x) + x^k && \text{based on induction hypothesis} \\
&= \left(1 - x^k + x^k(1-x)\right)/(1-x) \\
&= \left(1 - x^k + x^k - x^{k+1}\right)/(1-x) \\
&= \left(1 - x^{k+1}\right)/(1-x),
\end{aligned}$$

which is what we wanted to prove.