1. Determine if the system has a nontrivial solution. Use as few operations as possible. (Use row notation to show your steps)

$$x_1 - 2x_2 + 3x_3 = 0$$
$$-2x_1 - 7x_2 + x_3 = 0$$
$$2x_1 - 4x_2 + 9x_3 = 0$$

Week 2 HW

1. $x_1 - 2x_2 + 3x_3 = 0$
$-2x_1 - 7x_2 + x_3 = 0$
$2x_1 - 4x_2 + 9x_3 = 0$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ -2 & -7 & 1 & 0 \\ 2 & -4 & 9 & 0 \end{array}\right] \leftarrow \text{get to echelon form!}$$

$$\downarrow R_2 + 2R_1 \to R_1$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & -11 & 7 & 0 \\ 2 & -4 & 9 & 0 \end{array}\right]$$

$$\xleftarrow{R_3 - 2\cdot R_1 \to R_3}$$

$$\begin{array}{cccc} & +4 & -6 & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & -11 & 7 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right]$$

→ Since there are no free variables, the system has no nontrivial solution

$\boxed{\text{No nontrivial solution!}}$

2. Following the example done in class, write the solution set of the given homogeneous system in parametric vector form.

$$2x_1 + 2x_2 + 4x_3 = 0$$
$$-4x_1 - 4x_2 - 8x_3 = 0$$
$$-3x_2 - 3x_3 = 0$$

2. $2x_1 + 2x_2 + 4x_3 = 0$
$-4x_1 - 4x_2 - 8x_3 = 0$
$-3x_2 - 3x_3 = 0$

$$\begin{bmatrix} 2 & 2 & 4 & 0 \\ -4 & -4 & -8 & 0 \\ 0 & -3 & -3 & 0 \end{bmatrix} \xrightarrow{R_2 + 2R_1 \to R_2} \begin{bmatrix} 2 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & -3 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 2 & 2 & 4 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-\frac{1}{3}R_2} \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 - R_2 \to R_1} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

x_3 is free

has nontrivial solution!

cont'd

cont'd

$x_1 + x_3 = 0$
$x_2 + x_3 = 0$
$0 = 0$

$\Rightarrow$ $x_1 = -x_3$
$x_2 = -x_3$
x_3 is free

$$\boxed{x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}}$$

3. Describe all solutions of Ax=0 in parametric vector form.

$$A = \begin{bmatrix} 1 & -3 & -8 & 5 \\ 0 & 1 & 2 & -4 \end{bmatrix}$$

3. $A = \begin{bmatrix} 1 & -3 & -8 & 5 \\ 0 & 1 & 2 & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -3 & -8 & 5 & 0 \\ 0 & 1 & 2 & -4 & 0 \end{bmatrix}$

$\downarrow R_1 + 3R_2 \to R_1$

$\begin{bmatrix} 1 & 0 & -2 & -7 & 0 \\ 0 & 1 & 2 & -4 & 0 \end{bmatrix}$

$x_1 - 2x_3 - 7x_4 = 0$

$x_2 + 2x_3 - 4x_4 = 0$

Basic variables: x_1 and x_2 Free variables: x_3 and x_4

$x_1 = 2x_3 + 7x_4$

$x_2 = -2x_3 + 4x_4$

$$\boxed{x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_3 + 7x_4 \\ -2x_3 + 4x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_3 \\ -2x_3 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 7x_4 \\ 4x_4 \\ 0 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 7 \\ 4 \\ 0 \\ 1 \end{bmatrix}}$$

4. Suppose the solution set of a certain system of linear equations can be described as x1 = 5 + 4x3, x2 = −2 − 7x3, with x3 free. Use vectors to describe this set as a line in R 3 . [Hint: Examine the notes on Part 6, Solutions of Non-homogeneous Systems.]

4. $x_1 = 5 + 4x_3$ Sol set of system of linear eq

$x_2 = -2 - 7x_3$ Use vectors to describe set as a line in R^3.

x_3 is free.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 + 4x_3 \\ -2 - 7x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix} + \begin{bmatrix} 4x_3 \\ -7x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ -7 \\ 1 \end{bmatrix} = p + x_3 v$$

The solution set is the line through $\begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix}$ in the direction of $\begin{bmatrix} 4 \\ -7 \\ 1 \end{bmatrix}$

5. Following the example done in class, write the solution set of the following system in parametric vector form. Give a geometric description of the solution set when compared to the solution in Exercise 2.

$$\begin{aligned} 2x_1 + 2x_2 + 4x_3 &= 8 \\ -4x_1 - 4x_2 - 8x_3 &= -16 \\ -3x_2 - 3x_3 &= 12 \end{aligned}$$

5. $$\begin{aligned} 2x_1 + 2x_2 + 4x_3 &= 8 \\ -4x_1 - 4x_2 - 8x_3 &= -16 \\ -3x_2 - 3x_3 &= 12 \end{aligned}$$

lets find general solution using row reduction RREF!

$$\begin{bmatrix} 2 & 2 & 4 & 8 \\ -4 & -4 & -8 & -16 \\ 0 & -3 & -3 & 12 \end{bmatrix} \xrightarrow{R_2 + 2\cdot R_1 \to R_2} \begin{bmatrix} 2 & 2 & 4 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & -3 & 12 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 2 & 2 & 4 & 8 \\ 0 & -3 & -3 & 12 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\frac{1}{2}\cdot R_1} \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -3 & 12 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{3}\cdot R_2} \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 - R_2 \to R_1} \begin{bmatrix} 1 & 0 & 1 & 8 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

cont'd

Cont'd

$$\begin{bmatrix} 1 & 0 & 1 & 8 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{matrix} x_1 + x_3 = 8 \\ x_2 + x_3 = -4 \\ 0 = 0 \end{matrix}$$

$$x_1 = 8 - x_3$$
$$x_2 = -4 - x_3$$
$$x_3 \text{ free}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

This is basically what we got in ~~exercise~~ 2 about translated by a solution $V_p = (8, -4, 0)$

6. Determine if the vectors are linearly independent. Justify your answer

$$\begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix}, \begin{bmatrix} 9 \\ 4 \\ -8 \end{bmatrix}$$

6. $\begin{bmatrix}5\\0\\0\end{bmatrix}, \begin{bmatrix}7\\2\\-6\end{bmatrix}, \begin{bmatrix}9\\4\\-8\end{bmatrix} \rightarrow \begin{bmatrix}5 & 7 & 9 & 0\\0 & 2 & 4 & 0\\0 & -6 & -8 & 0\end{bmatrix}$ REF!

$R_3 + 3 \cdot R_2 \rightarrow R_3$

$\begin{bmatrix}5 & 7 & 9 & 0\\0 & 2 & 4 & 0\\0 & 0 & 4 & 0\end{bmatrix}$

No free variables

So, only the trivial sol.

Vectors are linearly independent! There are no free variables which means that the homogeneous system only has a trivial solution. As such, it is linearly independent!

7. Determine if the columns of the matrix form a linearly independent set.

$$\begin{bmatrix}1 & 4 & -3 & 0\\-2 & -7 & 5 & 1\\-4 & -5 & 7 & 5\end{bmatrix}$$

7. $\begin{bmatrix} 1 & 4 & -3 & 0 \\ -2 & -7 & 5 & 1 \\ -4 & -5 & 7 & 5 \end{bmatrix}$ ← There are 3 rows, so there is a max of 3 pivot positions. There are 4 variables, but 1 must be free in this case.

$$\begin{bmatrix} 1 & 4 & -3 & 0 & 0 \\ -2 & -7 & 5 & 1 & 0 \\ -4 & -5 & 7 & 5 & 0 \end{bmatrix} \xrightarrow{R_2+2\cdot R_1 \to R_2} \begin{bmatrix} 1 & 4 & -3 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ -4 & -5 & 7 & 5 & 0 \end{bmatrix}$$

$$\downarrow R_3+4\cdot R_1 \to R_3$$

$$\begin{bmatrix} 1 & 4 & -3 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 6 & -6 & 0 \end{bmatrix} \xleftarrow{R_3-11\cdot R_2 \to R_3} \begin{bmatrix} 1 & 4 & -3 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 11 & -5 & 5 & 0 \end{bmatrix}$$

Free variable from 4th row, so it has a nontrivial solution!

So, the columns are linearly dependent or not linearly independent!

8. Find the value(s) of h for which the vectors are linearly dependent.

$$\begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ 7 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ h \end{bmatrix}$$

8. $\begin{bmatrix}2\\-2\\4\end{bmatrix}, \begin{bmatrix}4\\-6\\7\end{bmatrix}, \begin{bmatrix}-2\\2\\h\end{bmatrix}$ (v_1, v_2, v_3) $A=[v_1\ v_2\ v_3]$

$x=x$
$b=0$ $Ax=b$

$$\begin{bmatrix}2&4&-2\\-2&-6&2\\4&7&h\end{bmatrix}x=b \rightarrow \left[\begin{array}{ccc|c}2&4&-2&0\\-2&-6&2&0\\4&7&h&0\end{array}\right]$$

$$\xrightarrow{\frac{1}{2}\cdot R_1} \begin{bmatrix}1&2&-1&0\\-2&-6&2&0\\4&7&h&0\end{bmatrix} \xrightarrow{R_2+2\cdot R_1\rightarrow R_2} \begin{bmatrix}1&2&-1&0\\0&-2&0&0\\4&7&h&0\end{bmatrix}$$

$$\xrightarrow{-\frac{1}{2}\cdot R_2} \begin{bmatrix}1&2&-1&0\\0&1&0&0\\4&7&h&0\end{bmatrix} \xrightarrow{R_3-4\cdot R_1\rightarrow R_3} \begin{bmatrix}1&2&-1&0\\0&1&0&0\\0&-1&h+4&0\end{bmatrix}$$

$$\xrightarrow{R_3+R_2\rightarrow R_3} \begin{bmatrix}1&2&-1&0\\0&1&0&0\\0&0&h+4&0\end{bmatrix}$$

$\boxed{\text{Vectors are linearly dependent when } h=-4!}$

trivial sol when $h\neq -4$

9. Give an example showing that the following statement is false:
If v1, v2, v3 are in R3 and v3 is not a linear combination of v1 and v2, then {v1, v2, v3} is linearly independent.

False. This is like saying if {v1,v2,v3} is not linearly independent, then v3 is a linear combination of v1,v2. An indexed set of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. In other words, just because v3 is not a linear combination of v1 and v2, it does not mean that the other vectors won't be a linear combination of the other vectors.

For example, take v1 and v2 to all be multiples of one vector, but v3 would not be a multiple of that vector.

$$V_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, V_2 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, V_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

*Page that I used as reference below:

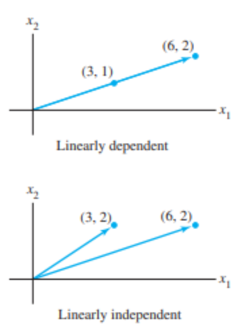

FIGURE 1

The arguments in Example 3 show that you can always decide *by inspection* when a set of two vectors is linearly dependent. Row operations are unnecessary. Simply check whether at least one of the vectors is a scalar times the other. (The test applies only to sets of *two* vectors.)

A set of two vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent if at least one of the vectors is a multiple of the other. The set is linearly independent if and only if neither of the vectors is a multiple of the other.

In geometric terms, two vectors are linearly dependent if and only if they lie on the same line through the origin. Figure 1 shows the vectors from Example 3.

Sets of Two or More Vectors

The proof of the next theorem is similar to the solution of Example 3. Details are given at the end of this section.

THEOREM 7 **Characterization of Linearly Dependent Sets**

An indexed set $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. In fact, if S is linearly dependent and $\mathbf{v}_1 \neq \mathbf{0}$, then some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v}_1, \dots, \mathbf{v}_{j-1}$.

Warning: Theorem 7 does *not* say that *every* vector in a linearly dependent set is a linear combination of the preceding vectors. A vector in a linearly dependent set may fail to be a linear combination of the other vectors. See Practice Problem 3.

EXAMPLE 4 Let $\mathbf{u} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}$. Describe the set spanned by $\mathbf{u}$ and $\mathbf{v}$, and explain why a vector $\mathbf{w}$ is in Span $\{\mathbf{u}, \mathbf{v}\}$ if and only if $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent.

SOLUTION The vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly independent because neither vector is a multiple of the other, and so they span a plane in $\mathbb{R}^3$. (See Section 1.3.) In fact,

10. For the following statements determine whether they are true or false. If false, then explain why it is false.
(a) A homogeneous system of equations can be inconsistent.

False. Since the zero vector is always a solution, a homogeneous system of equations can never be inconsistent. A system of linear equations is said to be homogeneous if it can be written in the form Ax = 0, where A is an mxn matrix and 0 is the zero vector in Rm. Such a system Ax

= 0 always hasat least one solution, namely, x = 0 (the zero vector in Rn). To be consistent means that it will have one or infinitely many solutions, so it will always be consistent.

1.5 SOLUTION SETS OF LINEAR SYSTEMS

Solution sets of linear systems are important objects of study in linear algebra. They will appear later in several different contexts. This section uses vector notation to give explicit and geometric descriptions of such solution sets.

Homogeneous Linear Systems

A system of linear equations is said to be **homogeneous** if it can be written in the form $A\mathbf{x} = \mathbf{0}$, where A is an $m \times n$ matrix and $\mathbf{0}$ is the zero vector in $\mathbb{R}^m$. Such a system $A\mathbf{x} = \mathbf{0}$ *always* has at least one solution, namely, $\mathbf{x} = \mathbf{0}$ (the zero vector in $\mathbb{R}^n$). This zero solution is usually called the **trivial solution**. For a given equation $A\mathbf{x} = \mathbf{0}$, the important question is whether there exists a **nontrivial solution**, that is, a nonzero vector $\mathbf{x}$ that satisfies $A\mathbf{x} = \mathbf{0}$. The Existence and Uniqueness Theorem in Section 1.2 (Theorem 2) leads immediately to the following fact.

> The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the equation has at least one free variable.

(b) If x is a nontrivial solution of Ax = 0, then every entry in x is nonzero.

False. A nontrivial solution of Ax = 0 is any nonzero x that satisfies the equation.

1.5 SOLUTION SETS OF LINEAR SYSTEMS

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Homogeneous Linear Systems

A system of linear equations is said to be **homogeneous** if it can be written in the form $A\mathbf{x} = \mathbf{0}$, where A is an $m \times n$ matrix and $\mathbf{0}$ is the zero vector in $\mathbb{R}^m$. Such a system $A\mathbf{x} = \mathbf{0}$ *always* has at least one solution, namely, $\mathbf{x} = \mathbf{0}$ (the zero vector in $\mathbb{R}^n$). This zero solution is usually called the **trivial solution**. For a given equation $A\mathbf{x} = \mathbf{0}$, the important question is whether there exists a **nontrivial solution**, that is, a nonzero vector $\mathbf{x}$ that satisfies $A\mathbf{x} = \mathbf{0}$. The Existence and Uniqueness Theorem in Section 1.2 (Theorem 2) leads immediately to the following fact.

> The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the equation has at least one free variable.

(c) The columns of a matrix A are linearly independent if the equation Ax = 0 has the trivial solution.

False. For every matrix A, Ax = 0 has the trivial solution. The columns of A are independent only if the equation has no solution other than the trivial solution.

Linear Independence of Matrix Columns

Suppose that we begin with a matrix $A = [\mathbf{a}_1 \ \cdots \ \mathbf{a}_n]$ instead of a set of vectors. The matrix equation $A\mathbf{x} = \mathbf{0}$ can be written as

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{0}$$

Each linear dependence relation among the columns of A corresponds to a nontrivial solution of $A\mathbf{x} = \mathbf{0}$. Thus we have the following important fact.

The columns of a matrix A are linearly independent if and only if the equation $A\mathbf{x} = \mathbf{0}$ has *only* the trivial solution. (3)

(d) The columns of any 4 × 5 matrix are linearly dependent.

True. If there are more vectors than entries in each vector, the set is linearly dependent.

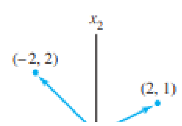
THEOREM 8

If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

$$\begin{array}{cc} & p \\ n & \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \end{array}$$

FIGURE 3
If $p > n$, the columns are linearly dependent.

PROOF Let $A = [\mathbf{v}_1 \ \cdots \ \mathbf{v}_p]$. Then A is $n \times p$, and the equation $A\mathbf{x} = \mathbf{0}$ corresponds to a system of n equations in p unknowns. If $p > n$, there are more variables than equations, so there must be a free variable. Hence $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution, and the columns of A are linearly dependent. See Fig. 3 for a matrix version of this theorem. ■



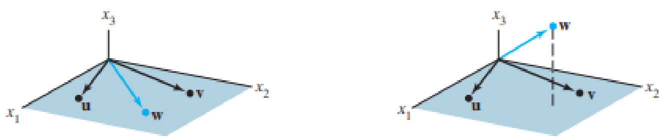
Warning: Theorem 8 says nothing about the case in which the number of vectors in the set does *not* exceed the number of entries in each vector.

(e) If u and v are linearly independent, and if w is in Span{u, v}, then {u, v, w} is linearly dependent.

True. See reference page below.

EXAMPLE 4 Let $\mathbf{u} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}$. Describe the set spanned by $\mathbf{u}$ and $\mathbf{v}$, and explain why a vector $\mathbf{w}$ is in Span $\{\mathbf{u}, \mathbf{v}\}$ if and only if $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent.

SOLUTION The vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly independent because neither vector is a multiple of the other, and so they span a plane in $\mathbb{R}^3$. (See Section 1.3.) In fact, Span $\{\mathbf{u}, \mathbf{v}\}$ is the x_1x_2-plane (with $x_3 = 0$). If $\mathbf{w}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$, then $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent, by Theorem 7. Conversely, suppose that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent. By Theorem 7, some vector in $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a linear combination of the preceding vectors (since $\mathbf{u} \neq \mathbf{0}$). That vector must be $\mathbf{w}$, since $\mathbf{v}$ is not a multiple of $\mathbf{u}$. So $\mathbf{w}$ is in Span $\{\mathbf{u}, \mathbf{v}\}$. See Fig. 2. ■



Linearly dependent, $\mathbf{w}$ in Span $\{\mathbf{u}, \mathbf{v}\}$

Linearly independent, $\mathbf{w}$ *not* in Span $\{\mathbf{u}, \mathbf{v}\}$

FIGURE 2 Linear dependence in $\mathbb{R}^3$.

Example 4 generalizes to any set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ in $\mathbb{R}^3$ with $\mathbf{u}$ and $\mathbf{v}$ linearly independent. The set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ will be linearly dependent if and only if $\mathbf{w}$ is in the plane spanned by $\mathbf{u}$ and $\mathbf{v}$.

(f) If three vectors in R3 lie in the same plane in R3, then they are linearly dependent.

True. If three vectors lie in the same plane, then only two are required such that their span gives the plane. So if there are 3 vectors in this plane given, then one must be a linear combination of the other two, making the set dependent. You could also use the reference material from part e to help explain this.

EXAMPLE 4 Let $\mathbf{u} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}$. Describe the set spanned by $\mathbf{u}$ and $\mathbf{v}$, and explain why a vector $\mathbf{w}$ is in Span $\{\mathbf{u}, \mathbf{v}\}$ if and only if $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent.

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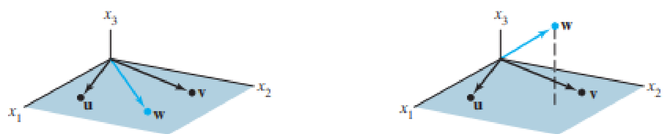

Linearly dependent, $\mathbf{w}$ in Span$\{\mathbf{u}, \mathbf{v}\}$

Linearly independent, $\mathbf{w}$ *not* in Span$\{\mathbf{u}, \mathbf{v}\}$

FIGURE 2 Linear dependence in $\mathbb{R}^3$.

Example 4 generalizes to any set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ in $\mathbb{R}^3$ with $\mathbf{u}$ and $\mathbf{v}$ linearly independent. The set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ will be linearly dependent if and only if $\mathbf{w}$ is in the plane spanned by $\mathbf{u}$ and $\mathbf{v}$.

(g) If v1, . . . , v4 are in R4 and v3 = 2v1 + v2, then {v1, v2, v3, v4} is linearly independent.

True. See theorem 7.

y independent

Sets of Two or More Vectors

The proof of the next theorem is similar to the solution of Example 3. Details are given at the end of this section.

THEOREM 7

Characterization of Linearly Dependent Sets

An indexed set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. In fact, if S is linearly dependent and $\mathbf{v}_1 \neq \mathbf{0}$, then some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.

Warning: Theorem 7 does *not* say that *every* vector in a linearly dependent set is a linear combination of the preceding vectors. A vector in a linearly dependent set may fail to be a linear combination of the other vectors. See Practice Problem 3.