MATH 30: LECTURE 27 SOLUTIONS

Say that g is the inverse function of f. By definition, this means $g(f(x)) = x$ for **every** x in the domain of f. This means the two sides are equal *as functions*, so they have the same derivatives. The Chain Rule then says:

$$g'(f(x))f'(x) = 1.$$

I recommend *not* memorizing the formula. Just use the Chain Rule from scratch every time!

1. Consider the function $g(y) = 3y + 2$. Call the inverse function $f(x)$, so that $g(f(x)) = x$ for all x. Find the value of $f'(4)$.

One way ("explicit differentiation"): Write $x = g(y)$ and $y = f(x)$. Then rewrite $x = 3y + 2$ to get $y = \frac{1}{3}x - \frac{2}{3}$. Now explicitly differentiate to get $f'(x) = \frac{dy}{dx} = \frac{1}{3}$. (The derivative is constant, so plugging in $x = 4$ makes no difference.)

Another way ("implicit differentiation"): Since they are inverses, $g(f(x)) = x$ for all x. Differentiate both sides to get

$$g'(f(x))f'(x) = 1 \qquad \text{for all } x.$$

But $g'(y) = 3$ so if you plug in $y = f(x)$ you still get 3 (it's a constant function). So we get $f'(x) = \frac{1}{3}$ as before.

2. Consider the function $g(y) = y^3 + 2y + 3$, Call the inverse function $f(x)$, so that $g(f(x)) = x$ for all x. What is $f(0)$? Now find the value of $f'(0)$.

The *explicit formula* for the inverse function f is complicated. It's much easier to use implicit differentiation. Since they are inverses, $g(f(x)) = x$ for all x. To find $f(0)$, keep in mind that it must satisfy $g(f(0)) = 0$. Clearly $g(-1) = 0$.

Since $g'(y) = 3y^2 + 2 > 0$ for all y, the function g is increasing, so it can only have $g(y) = 0$ at a single point, which again must be $y = -1$. This shows that $f(0) = -1$.

To find $f'(0)$, again differentiate both sides of the fundamental equation to get

$$g'(f(x))f'(x) = 1 \qquad \text{for all } x.$$

Now plug in $x = 0$ to get:

$$g'(-1)f'(0) = 1.$$

But here $g'(y) = 3y^2 + 2$, so $g'(-1) = 5$. So $f'(0) = \frac{1}{5}$.

3. Consider the function $g(y) = 7^y$ ("7 to the y"), whose inverse function is called $f(x) = \log_7 x$. Thus we have $g(f(x)) = x$ for all $x > 0$. What is $f(7)$? Now find the value of $f'(7)$.
   [In case you forgot the derivative of g, just remember that $7^y = (e^{\ln 7})^y = e^{y \ln 7}$.]

Similar to the previous problem, $f(7)$ must satisfy $g(f(7)) = 7$. The exponential function $g(y) = 7^y$ is increasing, so $y = 1$ is the only solution of $g(y) = 7$. This shows that $f(7) = 1$.

By the Chain Rule, $g'(y) = (\ln 7) \cdot 7^y$, and as usual $g'(f(x))f'(x) = 1$ for all x. Plug in $x = 7$ to get $g'(1)f'(7) = 1$. That is, $f'(7) = \frac{1}{7 \ln 7}$.