Math 30, Monday 3/23/2020

1pm class

Intro to Max/Min Problems

Past lectures: Related rates

impt. appl. of calc.

if time today: more examples

Today: another impt. application:

max/min problems

optimization problems

"best"

pessimization problems

"worst"

in business: maximize profit

in physics: minimize energy exerted

(lots of examples)

Definitions (see handout).

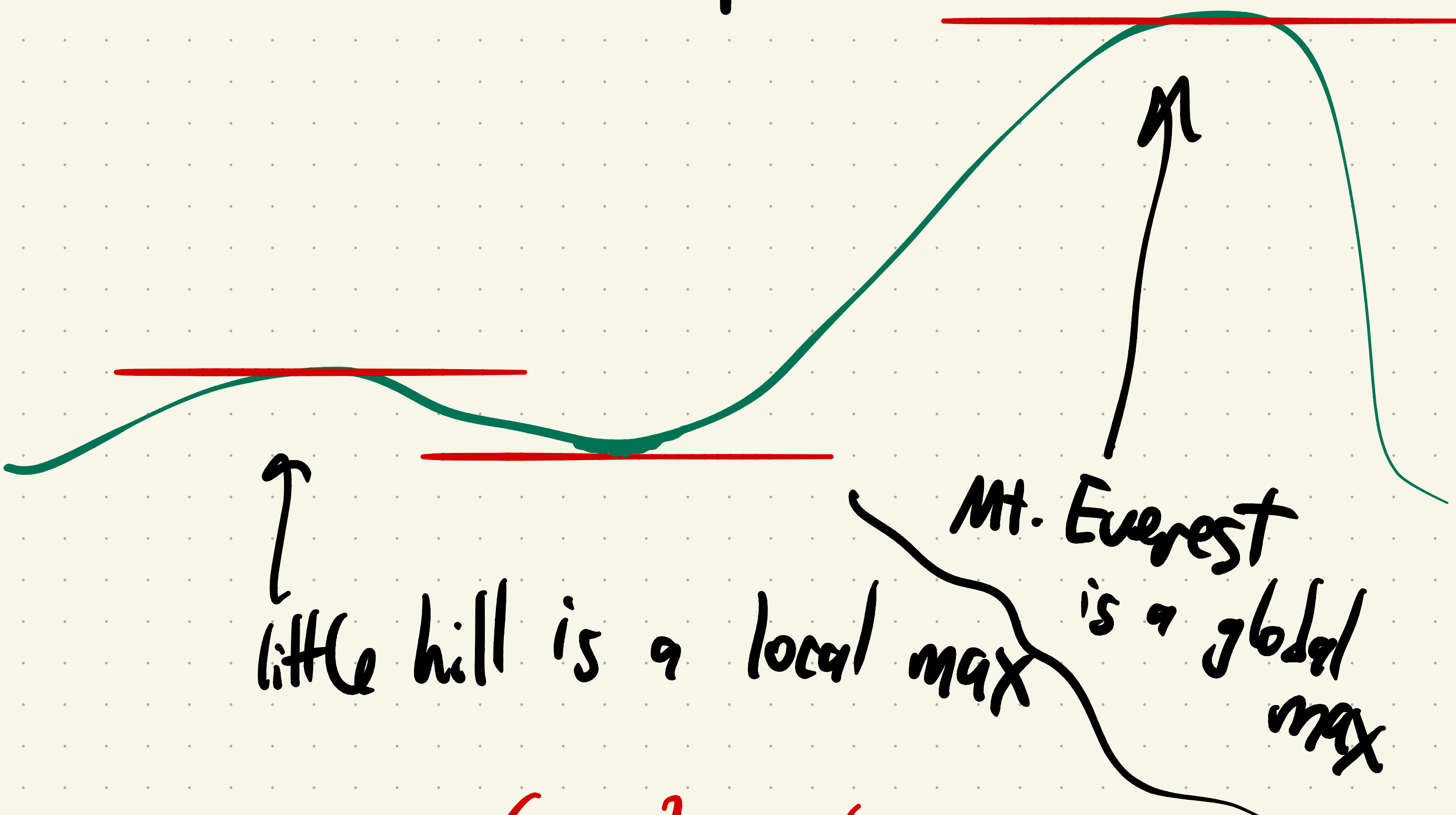
a function f has an absolute maximum at c (also call it a "global max") if

$$\underbrace{f(x)}_{\text{everything else is small}} \le \underbrace{f(c)}_{\text{biggest value}} \quad \text{for } \underline{\text{all}}\ x.$$

Def. Say f is defined on an interval $[a,b]$. We say f has a local max at c (if c is an interior point: $a<c<b$)

if $f(x) \le f(c)$ for all x near c.

Think of a landscape



Connection to calculus

Note: at local max & min

we have $f'(c) = 0$

MATH 30, 3/23/2020: INTRO TO MAXIMA AND MINIMA

Another Related Rates Review Problem (from Stewart's **Calculus**, #39): A plane flied horizontally at an altitude of 5 kilometers and passes directly over a tracking telescope on the ground. When the angle of elevation is $\pi/3$, this angle is decreasing at a rate of $\pi/6$ radians per minute. How fast is the plane traveling at that time?
Answer. $\frac{10\pi}{9}$ kilometers per minute.

Linear Approximation Review Problem. Use linear approximation to approximate the value of $\sqrt{3.9}$.
Answer. The approximate value is $79/40 = 1.975$. The true value is $1.97484\ldots$ Pretty good, right?

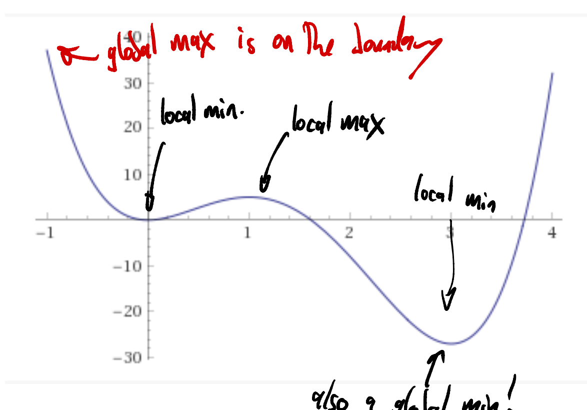
Because of coronavirus, we will skip the material on "differentials." It's the same as "linear approximation," but in different notation.

New topic: Maxima and Minima.

"Optimization Problems": What is the "best" way to do something? When is a quantity *maximized* or *minimized*? This shows up *a lot* in physics.

Definitions.
a function f has an **absolute maximum** (aka global maximum) at c if $f(x) \leq f(c)$ for all x.
a function f has an **absolute minimum** (aka global minimum) at c if $f(x) \geq f(c)$ for all x.
Definitions.
Let c be a point in the interior of the domain of f (not on the boundary of the domain).
a function f has an **local maximum** at c if $f(x) \leq f(c)$ for all x near c.
a function f has an **local minimum** at c if $f(x) \geq f(c)$ for all x near c.

Example. Consider $f(x) = 3x^4 - 16x^3 + 18x^2$ for $-1 \leq x \leq 4$.



There is a global maximum at $x = -1$, a local minimum at $x = 0$, a local maximum at $x = 1$, and a local *and* global minimum at $x = 3$. There is not a local max at $x = 4$ because it is a boundary point.

Definition. a **critical points** of f is a number c in the interior of the domain of f where

(1) $f'(c) = 0$ or
(2) $f'(c)$ does not exist.

In the above example, $x = -1$ and $x = 4$ are on the boundary of the domain, not in the interior.

"Fermat's Theorem." If f has a local max or local min at c, and if $f'(c)$ exists, then $f'(c) = 0$.

That is, if you want to find local max and min, the critical points are the possible candidates.

In the above example, $f(x) = 3x^4 - 16x^3 + 18x^2$ for $-1 \le x \le 4$, we have

$$f'(x) = 12x(x-1)(x-3),$$

so the critical points of f are $x = 0, 1, 3$. These are the only possible places where we can have local max or min. (Of course, we knew that, based on the picture.)

Here is a common min/max problem: what is the point on a line that is closest to a given point?

Example. Find the point on the line $y = 9 - 6x$ that is closest to the point $(-3, 1)$.

First of all, the distance from $(-3, 1)$ to a general point $(x, 9 - 6x)$ on the line is:

$$d(x) = \sqrt{(x+3)^2 + (8-6x)^2}.$$

We would like to find the value of x that *minimizes* this distance.

But this value of x is the *same* as the x that minimizes

$$f(x) = (x+3)^2 + (8-6x)^2,$$

a simpler function.

Using Fermat's theorem, we would like to find the critical points of f: that is, the points x where $f'(x) = 0$.

After some calculation, we find $x = \frac{45}{37}$, which corresponds to the point on the line $(\frac{45}{37}, \frac{63}{37})$.

Find the point on the line $y = 9 - 6x$ that is closest to the point $(-3, 1)$.

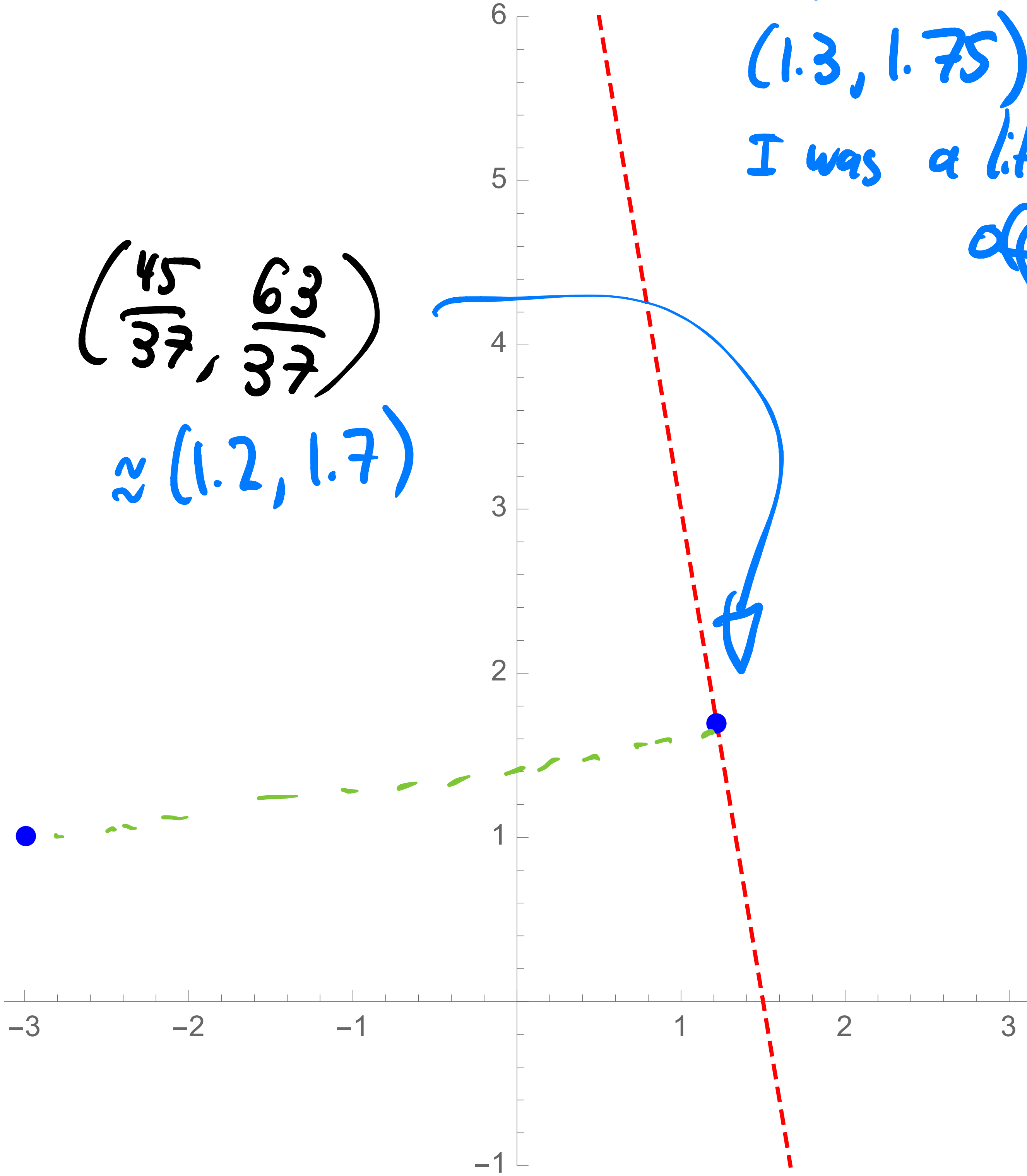
my guess was:

$(1.3, 1.75)$

I was a little off

$\left(\frac{45}{37}, \frac{63}{37}\right)$

$\approx (1.2, 1.7)$

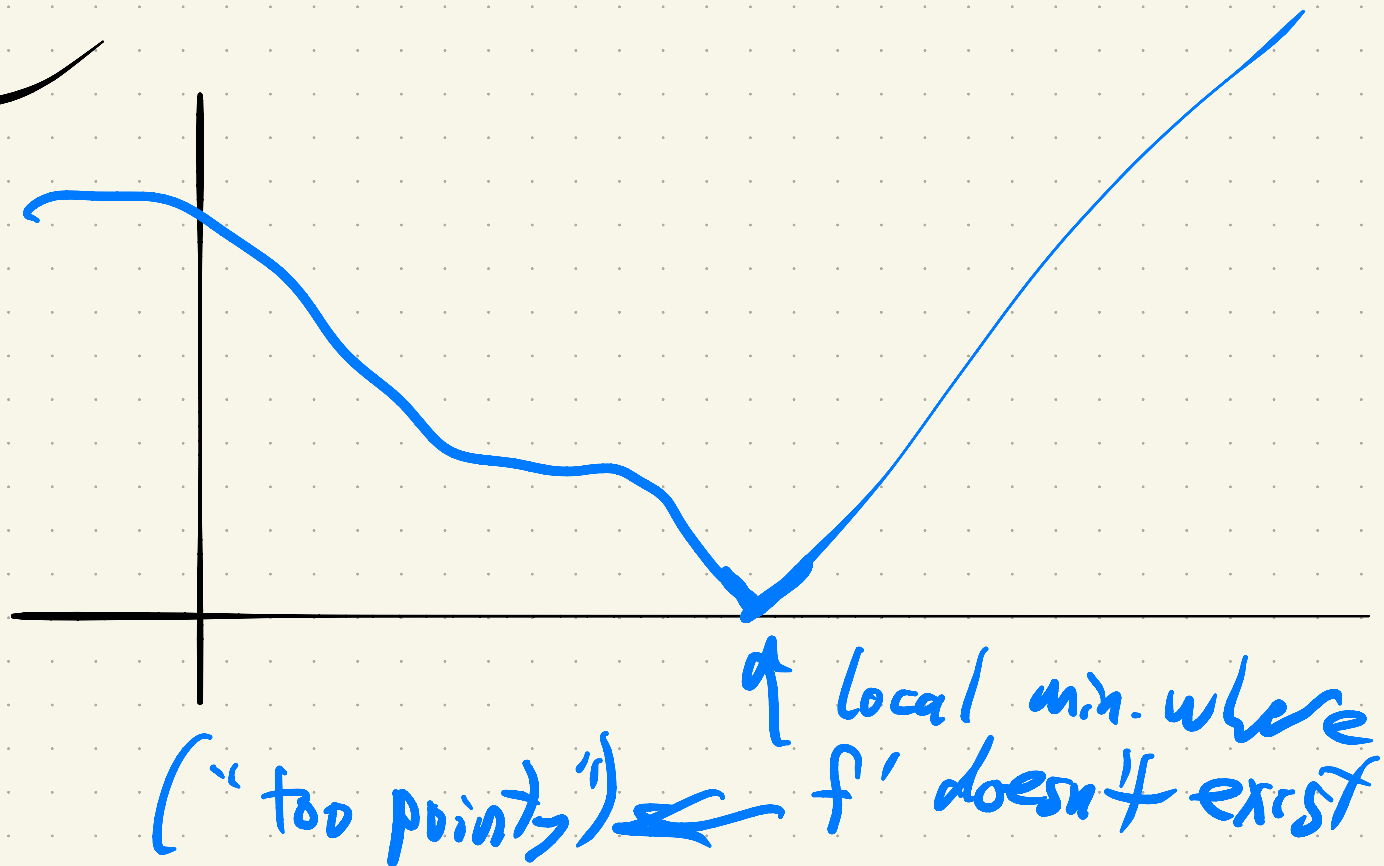


Note: pts where $f' = 0$ are the candidates for local max & min.

Def: a critical point of f is a point c where either:

(1) $f'(c) = 0$ or

(2) $f'(c)$ doesn't exist.



"Fermat's Thm": If f has a local max or min at c, and if $f'(c)$ exists, then $f'(c) = 0$.

This says: if you want local max/min, the candidates are the critical pts.

Prev. example. $f(x) = 3x^4 - 16x^3 + 18x^2$ find local max & min over $[-1, 4]$.

First find crit. pts: (power rule)

$$f'(x) = 12x^3 - 48x^2 + 36x$$

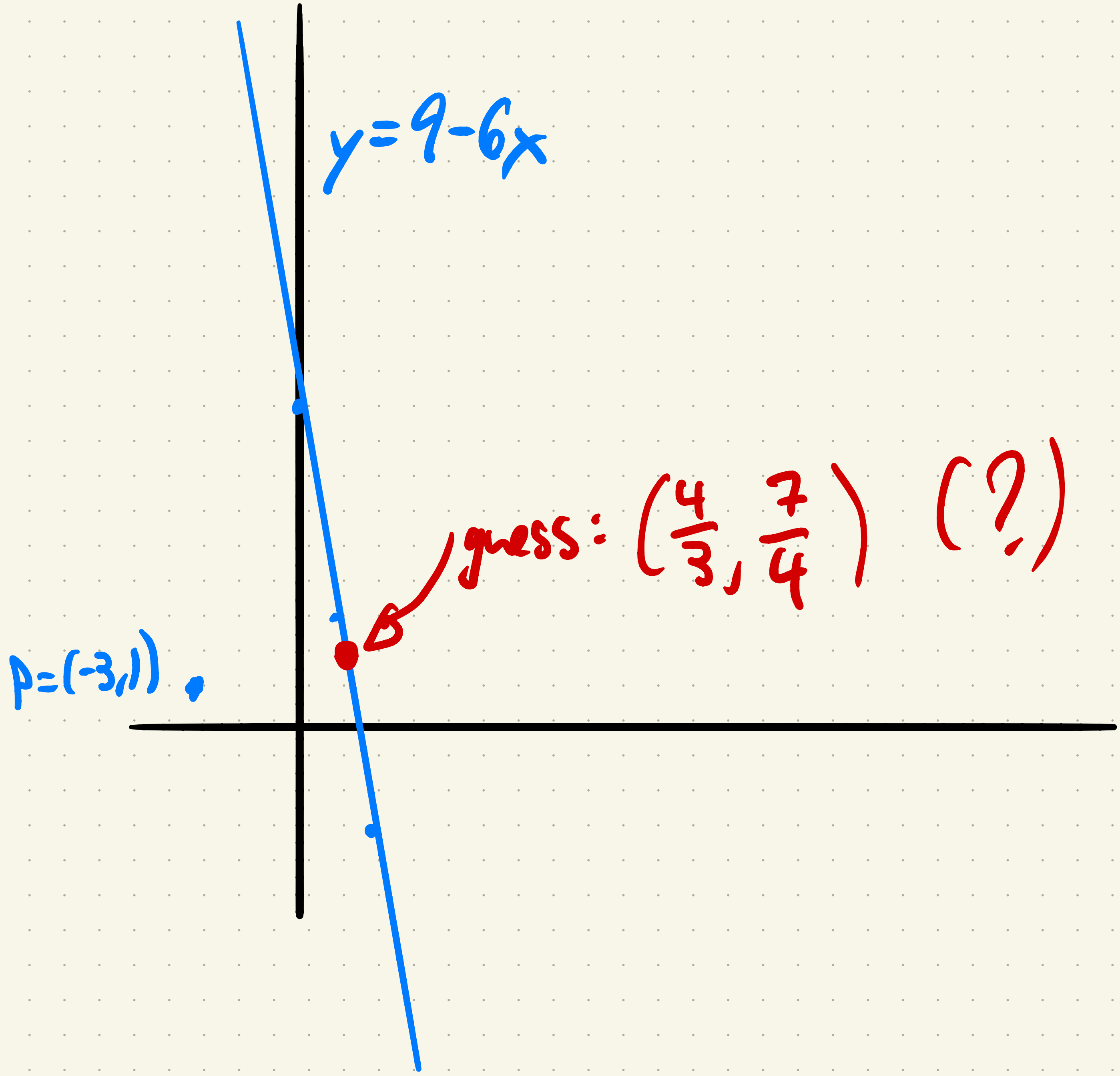
$$= 12x(x^2 - 4x + 3)$$

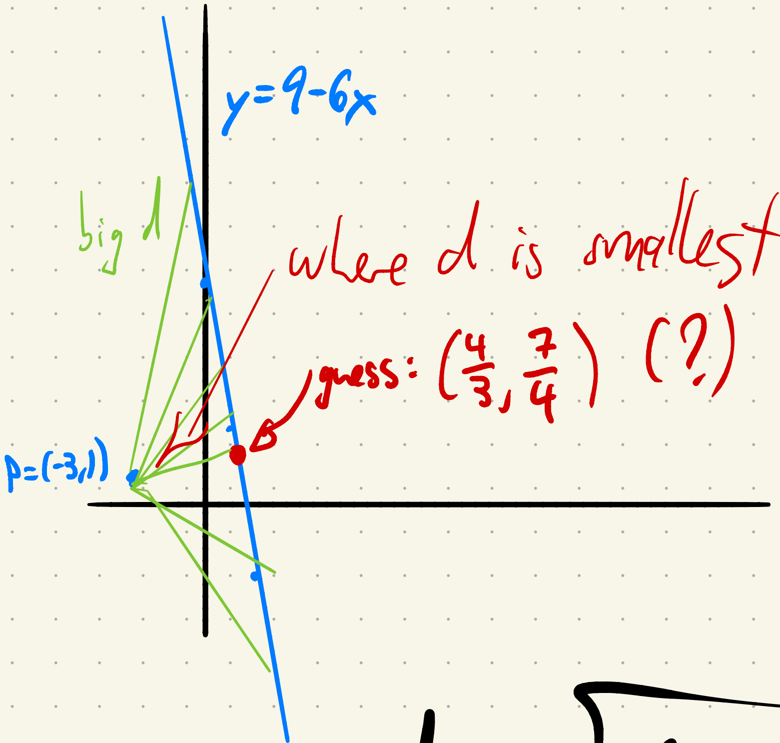
$$= 12x(x-1)(x-3)$$

crit. pts are $x = 0, 1, 3$

Compare with graph

Example. Find pt on the line $y = 9 - 6x$ closest to the pt. $(-3, 1)$.





For a point (x,y) on the line:

$(x, 9-6x)$

The dist. to $(-3,1)$ is:

$$d = \sqrt{(x+3)^2 + (9-6x-1)^2}$$

$$d = \sqrt{(x+3)^2 + (8-6x)^2}$$

Find x that makes this smallest.

Graph of d looks like:

(Dist. formula b/w (x_1, y_1) and (x_2, y_2)

is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$)

$$d = \sqrt{(x+3)^2 + (8-6x)^2}$$

Trick: instead of minimizing d,
it's easier to minimize d^2

$$f(x) = (x+3)^2 + (8-6x)^2$$

The x That minimizes d
is The same as The x minimizing f

minimize:

$$f(x) = (x+3)^2 + (8-6x)^2$$

<u>Step 1.</u> Find crit. pts.

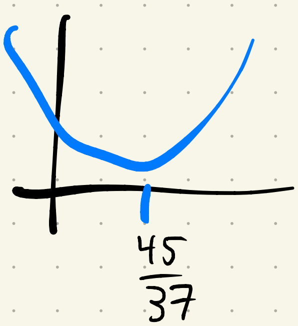
Chain Rule

$$f'(x) = 2(x+3) + 2(8-6x)\cdot(-6)$$

$$= 2x+6 - 12\cdot 2(4-3x)$$

$$= 2x+6 - 96 + 72x$$

(check my algebra)



$$= 74x - 90$$

only one critical pt:

$$x = \frac{90}{74} = \frac{45}{37}$$

So the pt. on the line happens when

$x = \frac{45}{37}$) The x value

$y = 9 - 6\left(\frac{45}{37}\right) = \frac{63}{37}$

Compare to graph

$(x,y) = \left(\frac{45}{37}, \frac{63}{37}\right) \rightarrow$ the point on the line $y = 9 - 6x$

see typed notes

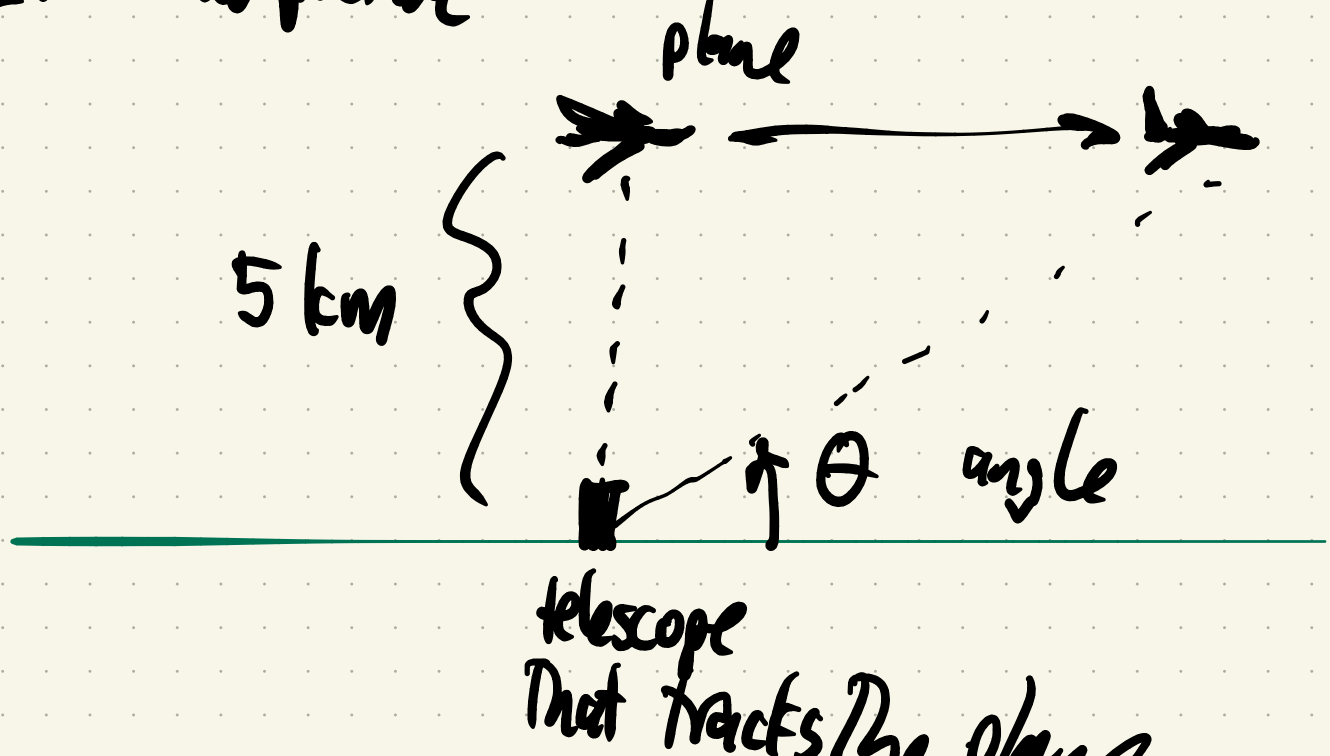
sol= of

$f'(x) = 0$

Impt. in applications:

Related rates on worksheet.

1. Read problem.
2. Draw picture

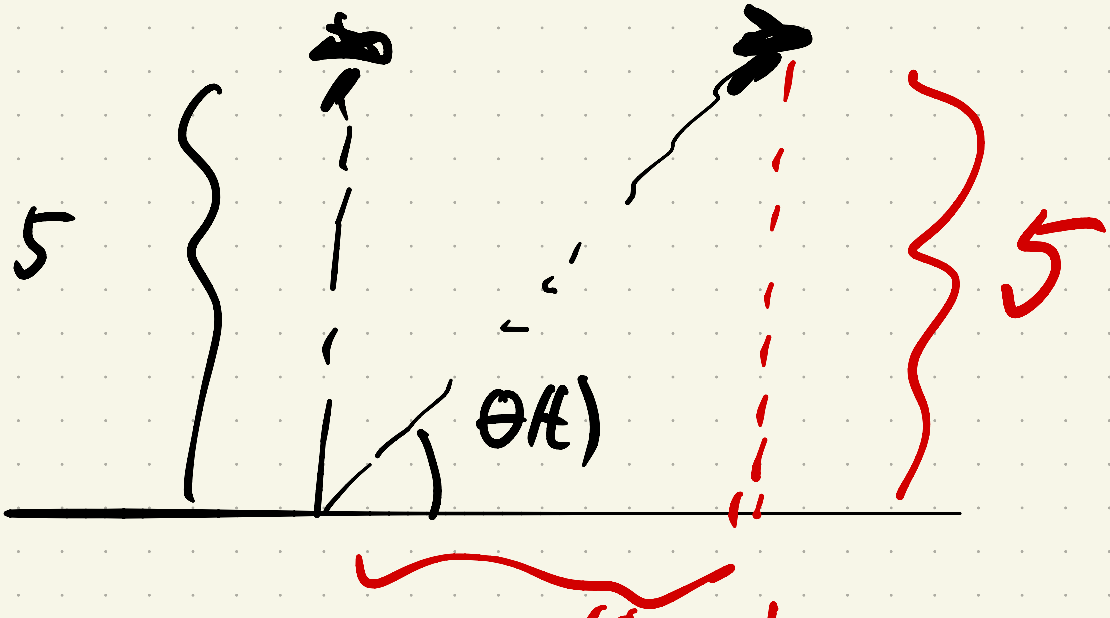


Q: at the time when $\theta = \frac{\pi}{3}$

what the speed of the plane?

We are given that at that time

$$\frac{d\theta}{dt} = \frac{-\pi}{6} \text{ rad/min}$$

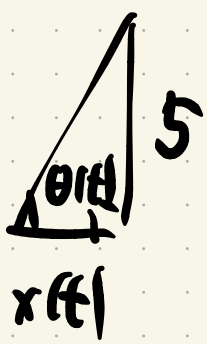


$x(t)$ horiz. pos. of plane.

Q: What is $\frac{dx}{dt}$?

speed of plane.

right triangle



relationship:

$$\boxed{\tan\theta(t) = \frac{5}{x(t)}}$$

key

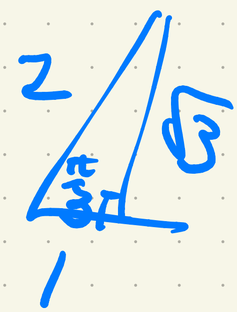
$$\boxed{\tan\theta(t) = \frac{5}{x(t)}}$$

$\theta(t)$ is a function of t

Now: Use Chain Rule:

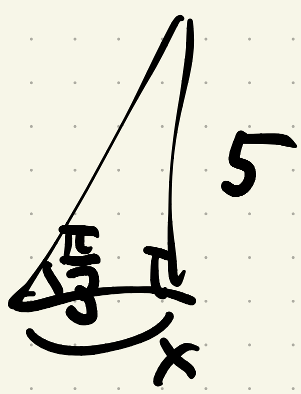
$$\sec^2(\theta(t)) \cdot \theta'(t) = \frac{-5}{x(t)^2} x'(t)$$

Now plug in $\theta(t) = \frac{\pi}{3}$



then $\frac{d\theta}{dt} = -\frac{\pi}{6}$

Also at that time,



$\tan\left(\frac{\pi}{3}\right) = \frac{5}{x}$

$\sqrt{3}$

at that time

$x = \frac{5}{\sqrt{3}}$ } plug that in

After plugging all that in, you get (check):

$$x'(t) = \frac{10\pi}{9} \frac{km}{min}.$$

The end!

Q's?

Quizzes/HW? TBD

Office Hours: TBD

email me Q's