Math 30, Wednesday, May 6, 2020

Substitution, cont'd.

Q: F. T. of Calc. Part II:

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

That is, if you want to find $\int_a^b g(x)\,dx$,

it would be great to know an antiderivative

f of g: $\quad f'(x) = g(x)$.

Then $\int_a^b g(x)\,dx = f(b) - f(a)$

and done! ☺

IF you <u>can</u> do an integral by hand,

This is probably how you will do it...

Calc. II is all about This...

Substitution for Definite Integrals:

x starts at a and ends at b

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

"substitute": write the integral in terms of u (no x's!)

Let $\boxed{u = g(x)}$

Then $\frac{du}{dx} = g'(x)$

rewrite as:

$du = g'(x)\,dx$

I say to myself:

"when $x = a$, $u = g(a)$"

"when $x = b$, $u = g(b)$"

need to change the limits of integration!

Simple example to illustrate my point.

Ex. $\displaystyle\int_0^2 2x\,dx = ?$

For fun and practice, do a substitution:

relationship between x and u:

$\boxed{u = 2x}$

$\dfrac{du}{dx} = 2$

$du = 2dx$

lower limit of integration:

"when $x=0$, $u=0$"

upper limit of integration:

"when $x=2$, $u=4$"

So

$$\int_0^2 2x\,dx = \int_0^4 u \cdot \frac{1}{2}\,du$$

$$= \frac{1}{2}\int_0^4 u\,du$$

$dx = \frac{1}{2}du$

Summary:

$$\int_0^2 2x\,dx = \frac{1}{2}\int_0^4 u\,du$$

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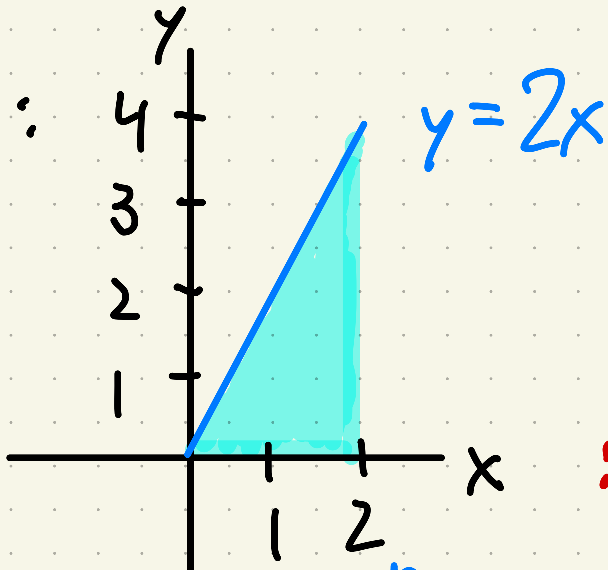
a picture helps explain:

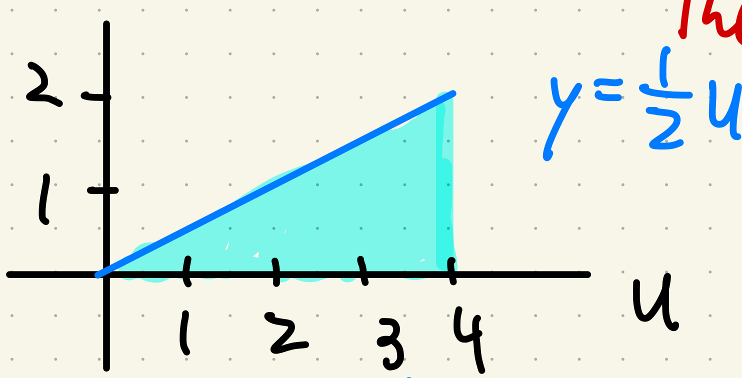
left side:



integral is this area

you see: they are the same areas only if I change the limits of integration!

right side:



integral is this area

Another example (more typical book problem):

$$\int_{\pi/4}^{\pi/2} \cot x \, dx = ?$$

kind of an art form...

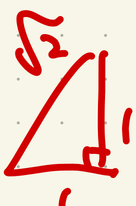
not obvious how to approach it...

rewrite it:

$$\int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin x} dx = \int_{\pi/4}^{\pi/2} \frac{1}{\sin x} \cos x \, dx$$

$$= \int_{1/\sqrt{2}}^{1} \frac{1}{u} du$$

Try $\boxed{u = \sin x}$



$du = \cos x \, dx$

"when $x = \frac{\pi}{4}$, $u = \frac{1}{\sqrt{2}}$"

"when $x = \frac{\pi}{2}$, $u = 1$"

an easier integral!

Summary:

$$\int_{\pi/4}^{\pi/2} \cot x \, dx = \int_{1/\sqrt{2}}^{1} \frac{1}{u} \, du \left(= \int_a^b f'(u)\, du \right.$$

$$= f(b) - f(a)$$

F.T. of Calc Part II.

$$= \Big[\ln|u|\Big]_{1/\sqrt{2}}^{1}$$

$$\ln\left(\frac{1}{\sqrt{2}}\right) = \ln\left(2^{-1/2}\right)$$

$$= \ln 1 - \ln\left(\frac{1}{\sqrt{2}}\right)$$

using properties of the logarithm...

$$= 0 + \ln\sqrt{2}$$

$$\boxed{= \frac{1}{2}\ln 2}$$

$$\ln(a^b) = b\ln a$$

The Fundamental Theorem of Calc., Part I:

"differentiating an integral":

$$\frac{d}{dx}\int_a^x g(t)\,dt = g(x)$$

The Fundamental Theorem of Calc., Part II:

"integrating a derivative":

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

Quiz 10: $$\frac{d}{dx}\int_0^x \sin(2t)\,dt = ?$$

Do it two ways

Let's redo the previous example without using substitution

remember, substitution is "undoing the Chain Rule"

Ex. $\displaystyle\int_{\pi/4}^{\pi/2} \cot x \, dx = \int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin x} dx$

Chain Rule (check!)

$\displaystyle = \int_{\pi/4}^{\pi/2} \frac{d}{dx} \ln|\sin x| \, dx$

F.T. of Calc. Part II

same as before! ☺

$\displaystyle = \ln\left|\sin\left(\frac{\pi}{2}\right)\right| - \ln\left|\sin\left(\frac{\pi}{4}\right)\right|$

$$\frac{d}{dx}\ln|\sin x| = ?$$

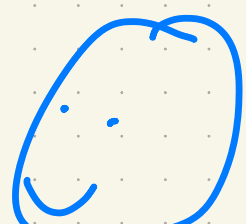
"outside function" is $f(u) = \ln|u|$

"inside function" is $g(x) = \sin x$

$$\ln|\sin x| = f(g(x))$$

$$\frac{d}{dx}\ln|\sin x| \overset{\text{chain Rule}}{=} f'(g(x))\cdot g'(x)$$

$$= \frac{1}{\sin x}\cos x$$



$$= \cot x$$

so $\int \cot x\,dx$ is the integral of a derivative! Good!

Quiz today: Due by 11:59pm.

tomorrow & Friday:

I'll answer Questions in class.

If it was a normal class,

I'd pass out the worksheet

Then walk around