Math 30, Thursday April 9, 2020

L'Hôpital's Rule

L'Hôpital's Rule

French mathematician

also spelled "L'Hospital"

Another way to evaluate "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$" limits.

when you get this, not done — "do more work"!

Simple Ex. $\lim_{x \to 0} \frac{mx}{x}$ looks like "$\frac{0}{0}$"

$\parallel$ cancel x's:

$$\lim_{x \to 0} m = m$$

So "$\frac{0}{0}$" can be any number

doesn't make sense as an answer.

Ex. $\lim_{x \to 1} \frac{x^2-1}{x^2-x}$ looks like "$\frac{0}{0}$"

"I need to do more work"

can use algebra to factorize & cancel

$$= \lim_{x \to 1} \frac{(x+1)(x-1)}{x(x-1)} = \lim_{x \to 1} \frac{x+1}{x}$$

$$= \frac{2}{1} = \boxed{2}$$

New topic:

Ex. $\lim_{x \to 0} \frac{\tan x - x}{x^3}$ looks like $\frac{0}{0}$

but in this example "algebra + cancel" doesn't work.

Ex. $\lim_{x \to 0} \frac{\tan x - x}{x^3}$

can't factorize & cancel this time —

but we can use calculus to evaluate it.

In general, suppose we want to evaluate

$\lim_{x \to a} \frac{f(x)}{g(x)}$ where $f(a) = 0$

and $g(a) = 0$

again looks like $\frac{0}{0}$

"I need to do more work"

Problem. $\lim_{x \to a} \frac{f(x)}{g(x)} = ?$ where $f(a) = g(a) = 0$ and $g'(a) \neq 0$.

$"\frac{0}{0}"$ need to do more work.

Method:

$$= \lim_{x \to a} \frac{f(x) - f(a)}{g(x) - g(a)} \quad \left(\text{because } f(a) = g(a) = 0 \right)$$

$$= \lim_{x \to a} \frac{\left(\frac{f(x) - f(a)}{x - a} \right)}{\left(\frac{g(x) - g(a)}{x - a} \right)}$$

limit law

$$= \frac{f'(a)}{g'(a)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

This is L'Hôpital's Rule...

Summary: if $f(a) = g(a) = 0$ and $g'(a) \neq 0$

Then $\underbrace{\lim_{x \to a} \frac{f(x)}{g(x)}}_{\text{looks like } \text{"}\frac{0}{0}\text{"}} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

This is L'Hôpital's Rule.

Redo Example. $\lim_{x \to 1} \frac{x^2-1}{x^2-x} = ?$

Here $a=1$.

check conditions

$f(x) = x^2-1$ $\quad$ $g(x) = x^2 - x$

$f(1) = 0$ ✓ $\quad$ $g(1) = 0$ ✓

$g'(x) = 2x-1$

so $g'(1) = 1 \neq 0$. ✓

So I can use L'Hôpital's Rule

So L'Hôpital's Rule says

$$\lim_{x \to 1} \frac{x^2-1}{x^2-x} = \lim_{x \to 1} \frac{2x}{2x-1} = \frac{2}{1} = \boxed{2}$$

Same as before! ☺

Totally diff. way to do the problem.

Ex. $\lim_{x\to 1} \frac{\ln x}{\sin(\pi x)} = ?$

πx $\pi = 3.14...$

In this problem you can't "factorize & cancel."

You need to use l'Hôpital's Rule.

Check the conditions for l'Hôpital:

$f(x) = \ln x$ $\qquad$ $g(x) = \sin(\pi x)$

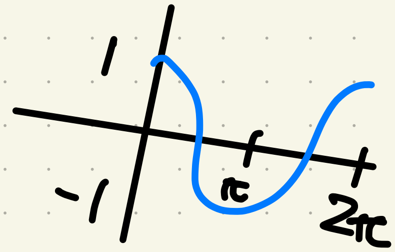
$f(1) = 0$ ✓ $\qquad$ $g(1) = 0$ ✓

$g'(x) = \pi\cos(\pi x)$ (Chain Rule)

Aside: $e^x = y \iff x = \ln y$ because they are inverses.

so $x = 0$ $\leftarrow$ $e^x = 1 \iff x = \ln 1$

Ex. $\lim\limits_{x\to 1} \dfrac{\ln x}{\sin(\pi x)} = ?$



Here $f(x) = \ln x \quad g(x) = \sin(\pi x)$

$f(1) = 0 \quad g(1) = 0$

chain Rule

$g'(x) = \pi\cos(\pi x)$

So $g'(1) = -\pi \neq 0$

So The conditions are satisfied —

can use L'Hôp.

So

$$\lim_{x\to 1} \frac{\ln x}{\sin(\pi x)} \overset{\text{L'Hôp}}{=} \lim_{x\to 1} \frac{\left(\frac{1}{x}\right)}{\pi\cos(\pi x)} = \frac{1}{-\pi}$$

$$= \boxed{-\frac{1}{\pi}}$$

Ex. $\lim_{x \to 0} \frac{1-\cos x}{x^2} = ?$

looks like "$\frac{0}{0}$" "I need to do more work"

can't "factorize & cancel" — need to use L'Hôpital.

Check the conditions:

$f(x) = 1 - \cos x$ $\quad g(x) = x^2$

$f(0) = 0$ ✓ $\quad g(0) = 0$ ✓

$g'(x) = 2x$

$g'(0) = 0$

uh oh – can't use L'Hôpital as it was written.

Ex. $\lim_{x\to 0} \frac{1-\cos x}{x^2} = ?$

But still: $f(x) = 1-\cos x$ $\quad g(x) = x^2$

$f'(x) = \sin x$ $\quad g'(x) = 2x$

can still do this

$$\ldots = \lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)}$$

$$= \lim_{x\to 0} \frac{\sin x}{2x} \quad \text{"}\frac{0}{0}\text{"}$$

need to do more work

$$= \frac{1}{2} \lim_{x\to 0} \frac{\sin x}{x} = \boxed{\frac{1}{2}}$$

Another way:

Ex. $\lim_{x \to 0} \frac{1-\cos x}{x^2} = \lim_{x \to 0} \frac{f(x)}{g(x)} \to$ "$\frac{0}{0}$"

$f(x) = 1-\cos x$

$g(x) = x^2$

$\overset{\text{L'Hop}}{=} \lim_{x \to 0} \frac{f'(x)}{g'(x)} \to$ still "$\frac{0}{0}$"

$f'(x) = \sin x$

$g'(x) = 2x$

can do L'Hop again!

$\overset{\text{L'Hop}}{=} \lim_{x \to 0} \frac{f''(x)}{g''(x)}$

$f''(x) = \cos x$

$g''(x) = 2$

$= \lim_{x \to 0} \frac{\cos x}{2} = \boxed{\frac{1}{2}}$

same!

Ex. $\lim_{x \to 0} \frac{\tan x - x}{x^3} = ?$ "$\frac{0}{0}$"

$f(x) = \tan x - x$

$g(x) = x^3$

$f'(x) = \sec^2 x - 1$

$g'(x) = 3x^2$

would get $\frac{0}{0}$ again...

So use L'Hôpital again...

$= \lim_{x \to 0} \frac{f'(x)}{g'(x)} = \lim_{x \to 0} \frac{f''(x)}{g''(x)}$

Let me show you a more clever way...

Ex. $\lim_{x\to 0} \frac{\tan x - x}{x^3}$ } looks like $\frac{0}{0}$

L'Hop

$$= \lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2}\left(\frac{\cos^2 x}{\cos^2 x}\right)$$

$$= \lim_{x\to 0} \frac{1-\cos^2 x}{3x^2\cos^2 x}$$

algebra

$$= \lim_{x\to 0} \left(\frac{1-\cos x}{x^2}\right)\left(\frac{1+\cos x}{3\cos^2 x}\right)$$

see prev. example

tricky: used a hard limit we already did

$$= \left(\frac{1}{2}\right)\left(\frac{2}{3}\right) = \boxed{\frac{1}{3}}$$

L'Hôp also works for limits where we get

$$\text{"}\frac{\infty}{\infty}\text{"}$$

"I need to do more work"

Still get

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)}$$

if $\lim_{x \to \infty} f(x) = \infty$

and $\lim_{x \to \infty} g(x) = \infty$

Ex. $\lim\limits_{x\to\infty} \dfrac{\ln x}{x^{1/3}} = ?$

$\ln x \to$ goes to ∞

$x^{1/3} \to$ goes to ∞

$$= \lim_{x\to\infty} \frac{\left(\frac{1}{x}\right)}{\frac{1}{3}x^{-2/3}}$$ Power Rule

$$= \lim_{x\to\infty} \frac{1}{x}\left(3x^{2/3}\right)$$

$$= \lim_{x\to\infty} \frac{3}{x^{1/3}} = 0$$

Questions?

Please typed notes for a

trick question —

where not allowed to use [illegible].