1 Interal Test

1. Use the integral test to determine whether or not the series converges.

$u = 3x-1$
$du = 3du$

$$\sum_{n=1}^{\infty} \frac{1}{(3n-1)^4}, \quad f(x) = \frac{1}{(3x-1)^4}$$

$$\int_1^{\infty} f(x)dx = \int_1^{\infty} \frac{1}{(3x-1)^4}dx \quad \frac{1}{3}\int_2^{\infty} \frac{1}{u^4}du = \frac{1}{3}\cdot \lim_{t\to\infty}\int_2^t u^{-4}du$$

$$= \frac{1}{3}\cdot \lim_{t\to\infty}\left(-\frac{1}{3}u^{-3}\Big|_2^t\right) = -\frac{1}{9}\lim_{t\to\infty}\left(\frac{1}{t^3}-\frac{1}{8}\right) = \frac{1}{72}$$

By integral test, $\sum_{n=1}^{\infty}\frac{1}{(3n-1)^4}$ is convergent

2. Use the integral test to determine whether or not the series converges.

$u = \ln x$
$du = \frac{1}{x}dx$

$$\sum_{n=2}^{\infty} \frac{1}{n\ln n}$$

$$\int_2^{\infty} f(x)dx = \int_2^{\infty}\frac{1}{x\ln x}dx = \int_{\ln 2}^{\infty}\frac{1}{u}du = \int_{\ln 2}^{\infty}u^{-1}du = \lim_{t\to\infty}\int_{\ln 2}^{t}u^{-1}du$$

$$= \lim_{t\to\infty}\left(\ln u\Big|_{\ln 2}^{t}\right) = \lim_{t\to\infty}\left(\ln t - \ln(\ln 2)\right) \to \infty$$

By integral test, $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$ divergent

3. Use the integral test to determine whether or not the series converges.

Partial Fraction Decomp.

$$\sum_{n=1}^{\infty}\frac{1}{n^2+n^3}$$

$$\int_1^{\infty} f(x)dx = \int_1^{\infty}\frac{1}{x^2+x^3}dx = \int_1^{\infty}\left(-\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x+1}\right)dx$$

$$= \lim_{t\to\infty}\int_1^t\left(-\frac{1}{x}+x^{-2}+\frac{1}{x+1}\right)dx = \lim_{t\to\infty}\left(-\ln x\Big|_1^t\right)+\lim_{t\to\infty}\left(-\frac{1}{x}\Big|_1^t\right)+\lim_{t\to\infty}\left(\ln x+1\right.$$

$$= \lim_{t\to\infty}\left[\ln\left(\frac{x+1}{x}\right)-\frac{1}{x}\right]_1^t = \lim_{t\to\infty}\left[\left(\ln\left(\frac{t+1}{t}\right)-\frac{1}{t}\right)-(\ln 2-1)\right] = 1-\ln 2$$

By integral test, $\sum_{n=1}^{\infty}\frac{1}{n^2+n^3}$ is convergent

2 Comparison Test

1. Determine the convergence of the series given below. If it converges, find the limit.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{3n^4+1}}$$

$$\{a_n\} = \frac{1}{\sqrt[3]{3n^4+1}} < \frac{1}{\sqrt[3]{3n^4}} = \frac{1}{\sqrt[3]{3}\, n^{4/3}}$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{3n^4+1}} < \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{3}\, n^{4/3}} = \frac{1}{\sqrt[3]{3}} \sum_{n=1}^{\infty} \frac{1}{n^{4/3}} \text{ (converges by p-test)}$$

$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{3n^4+1}}$ is convergent by comparison test

2. Determine the convergence of the series given below. If it converges, find the limit.

$$\sum_{k=1}^{\infty} \frac{k \sin^2 k}{1+k^3}$$

$$\{b_n\} = \frac{k \sin^2 k}{1+k^3} \leq \frac{k}{1+k^3} < \frac{k}{k^3} = \frac{1}{k^2}$$

$$\sum_{k=1}^{\infty} \frac{k \sin^2 k}{1+k^3} < \sum_{k=1}^{\infty} \frac{1}{k^2} \text{ (converges by p-test)}$$

$\sum_{k=1}^{\infty} \frac{k \sin^2 k}{1+k^3}$ is convergent by comparison test

3. Determine the convergence of the series given below. If it converges, find the limit.

$$\sum_{k=1}^{\infty} \frac{\ln k}{k}$$

Notice for $k \geq 3$, $\ln k > 1$, so

$$\{c_k\} = \frac{\ln k}{k} > \frac{1}{k}, \quad k \geq 3$$

$$\sum_{k=3}^{\infty} \frac{\ln k}{k} > \sum_{k=1}^{\infty} \frac{1}{k} \text{ (diverges by p-test)}$$

$\sum_{k=1}^{\infty} \frac{\ln k}{k}$ is divergent by comparison test

3 Alternating Series Test

1. Determine whether or not the series shown below converges:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{n^3+4} = (-1)^{n+1} b_n$$

where $\{b_n\} = \frac{n^2}{n^3+4}$

1) $b_{n+1} \leq b_n$ for all n

2) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n^2}{n^3+4} = 0$

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{n^3+4}$ is convergent by alternating series test

2. Determine whether or not the series shown below converges:

$$\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{\pi}{n}\right)$$

Notice for $\{b_n\} = \{-1, 0, \frac{1}{2}, \frac{\sqrt{2}}{2}, \ldots\}$ is not decreasing so we cannot use alternating series test.

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \cos\left(\frac{\pi}{n}\right) \neq 0$, so $\lim_{n\to\infty} (-1)^n \cos\left(\frac{\pi}{n}\right) \neq 0$

$\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{\pi}{n}\right)$ is divergent

3. Determine the convergence of $-\frac{2}{5} + \frac{4}{6} - \frac{6}{7} + \frac{8}{8} - \frac{10}{9} + \cdots$

$\sum_{n=1}^{\infty} \frac{(-1)^n \cdot 2n}{n+4}$: Notice for $\{b_n\} = \frac{2n}{n+4}$, $\lim_{n\to\infty} b_n = 2$, so we cannot use alternating series test.

$\lim_{n\to\infty} \frac{(-1)^n \cdot 2n}{n+4}$ DNE

$\sum_{n=1}^{\infty} \frac{(-1)^n \cdot 2n}{n+4}$ is divergent

4. Determine the convergence of $\frac{2}{3} - \frac{2}{5} + \frac{2}{7} - \frac{2}{9} + \cdots$

$\sum_{n=1}^{\infty} \frac{2 \cdot (-1)^{n+1}}{2n+1}$ 1) $b_{n+1} \leq b_n$ for all n, 2) $\lim_{n\to\infty} \frac{2}{2n+1} = 0$

$\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{2n+1}$ is convergent by alternating series test