Math 30, Monday April 6, 2020

1pm class

Concavity / Convexity

Exam 3: Friday April 17 (next week).

lecture schedule on public webpage
is not valid (can't seem to change it
from home)

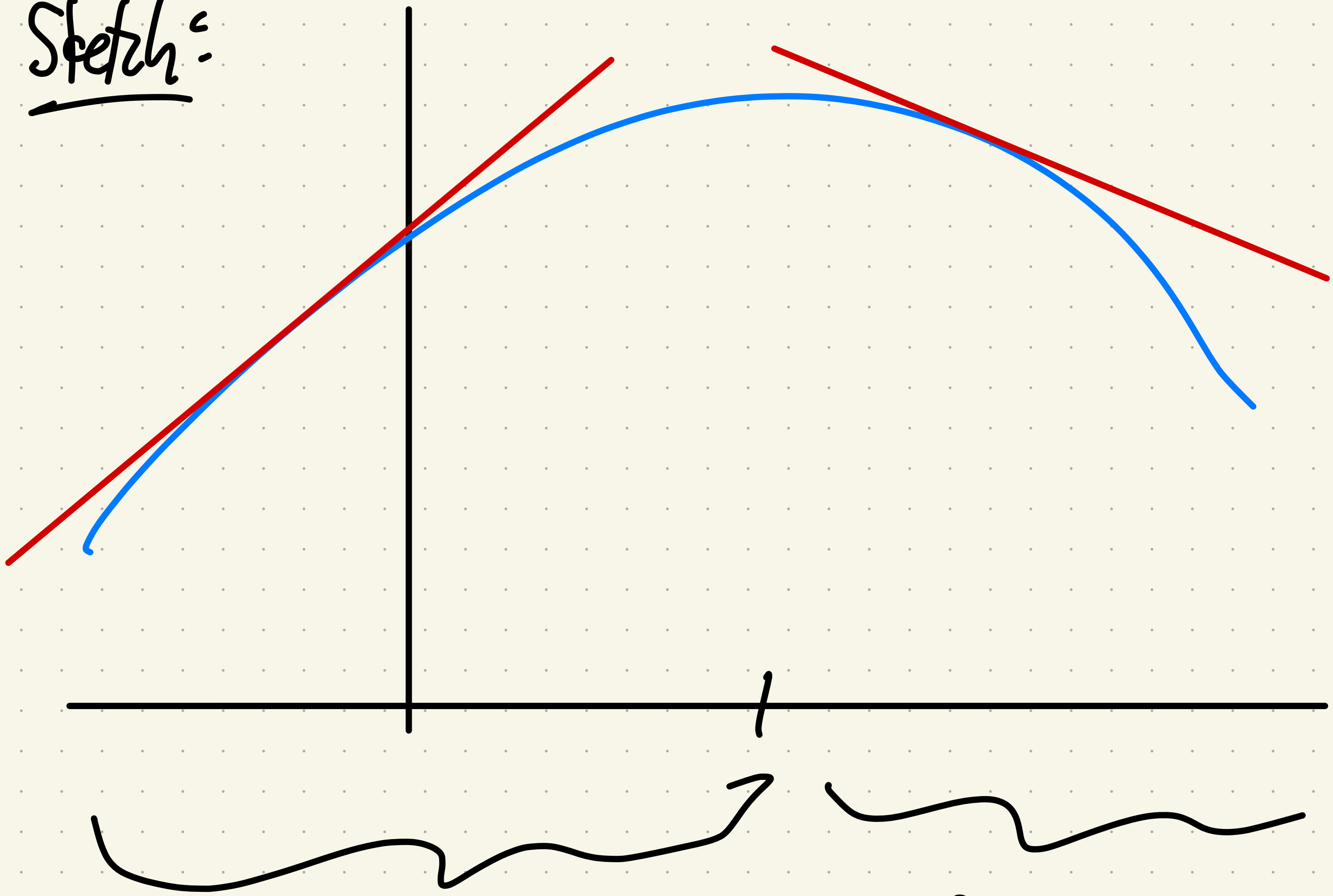
Before Spring Break:

What $f'(x)$ tells us about the graph of f:

If $f'(x) > 0$ on an interval (a,b)
Then f is increasing on (a,b)

If $f'(x) < 0$ on (a,b)
Then f is decreasing on (a,b).

Sketch:



$f' > 0 \iff f$ is increasing

$f' < 0 \iff f$ is decreasing

Now: What does the second deriv. f'' tell us about the graph of f.

Recall: f'' is the deriv. of f'

so $f''>0$ means f' is increasing

f' goes from very neg. to slightly neg.

to zero to slightly pos.

to very pos.

slope of graph of f

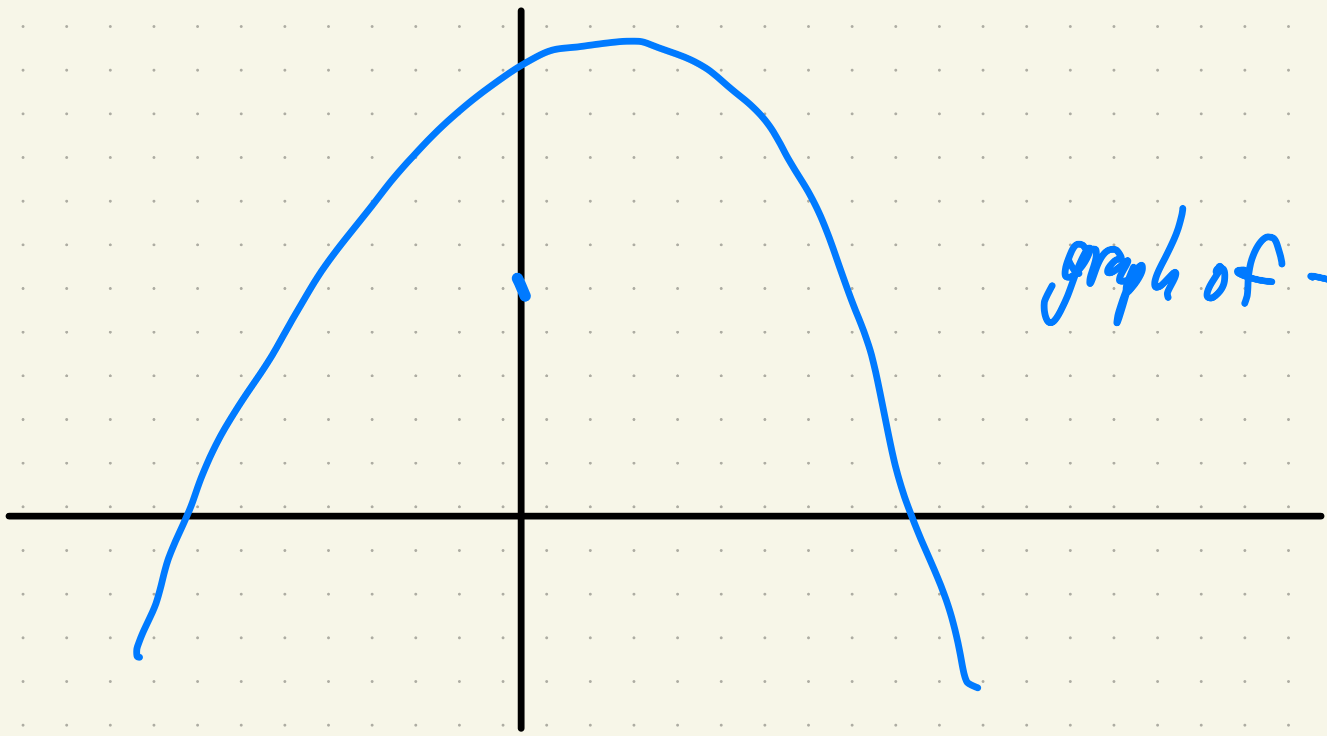
f' is increasing

$\Updownarrow$

$f''>0$

Recall: f'' is the deriv. of f'

so $f'' < 0$ means f' is decreasing



f' goes from very positive to slightly pos.

to zero to slightly neg.

to very negative

slope of graph of f

f' is decreasing

$\Updownarrow$

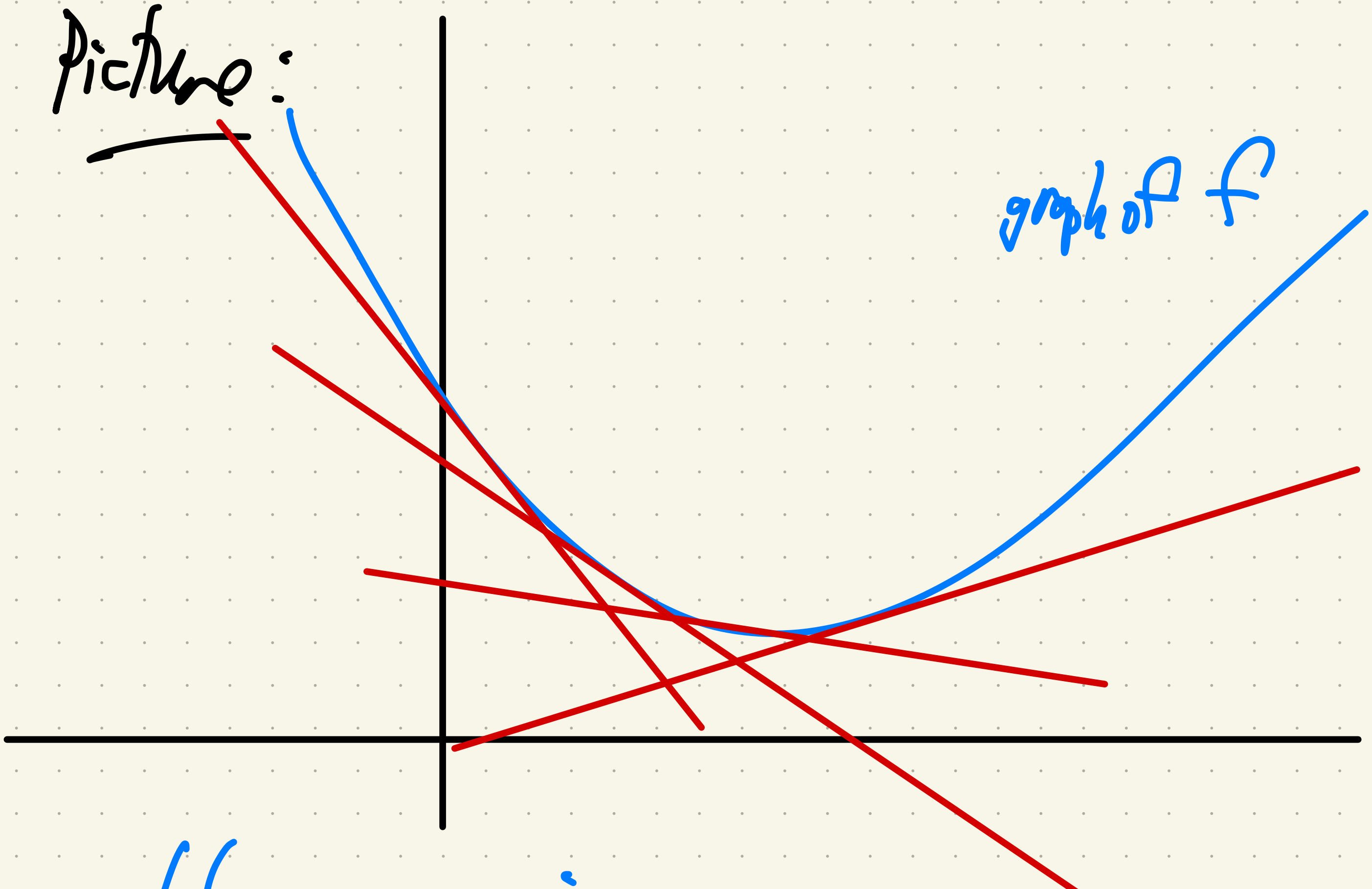
$f'' < 0$

That was background for intuition.

Start over following typed notes:

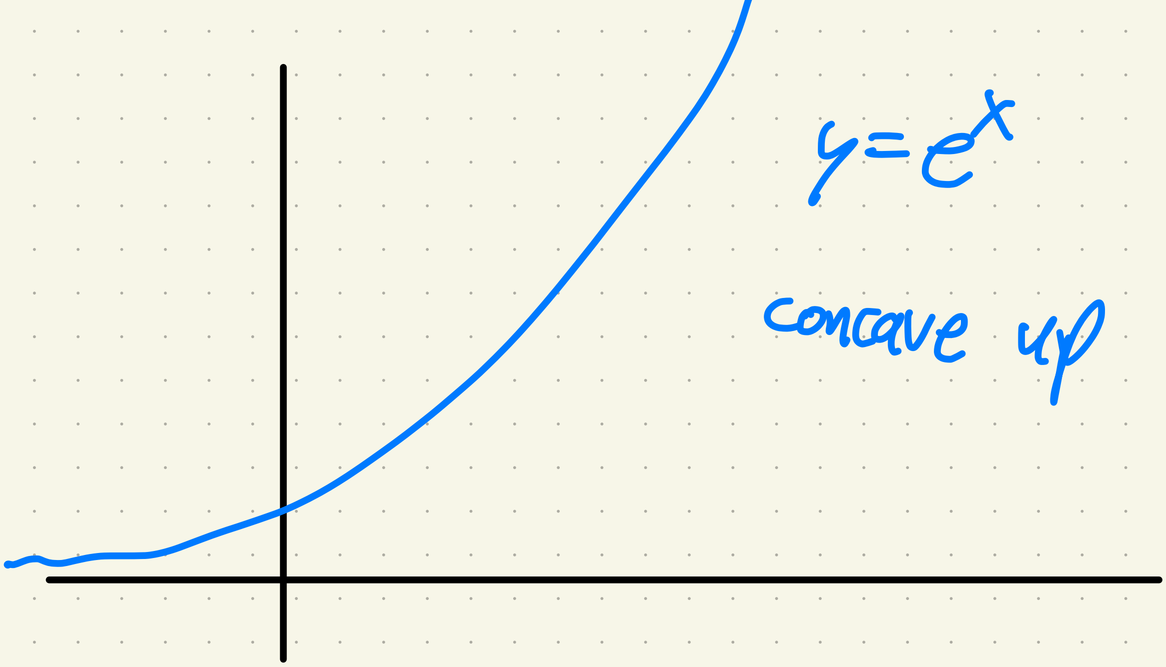
Def: If the graph of f lies above all of its tangent lines, we say it is concave up.

Picture:

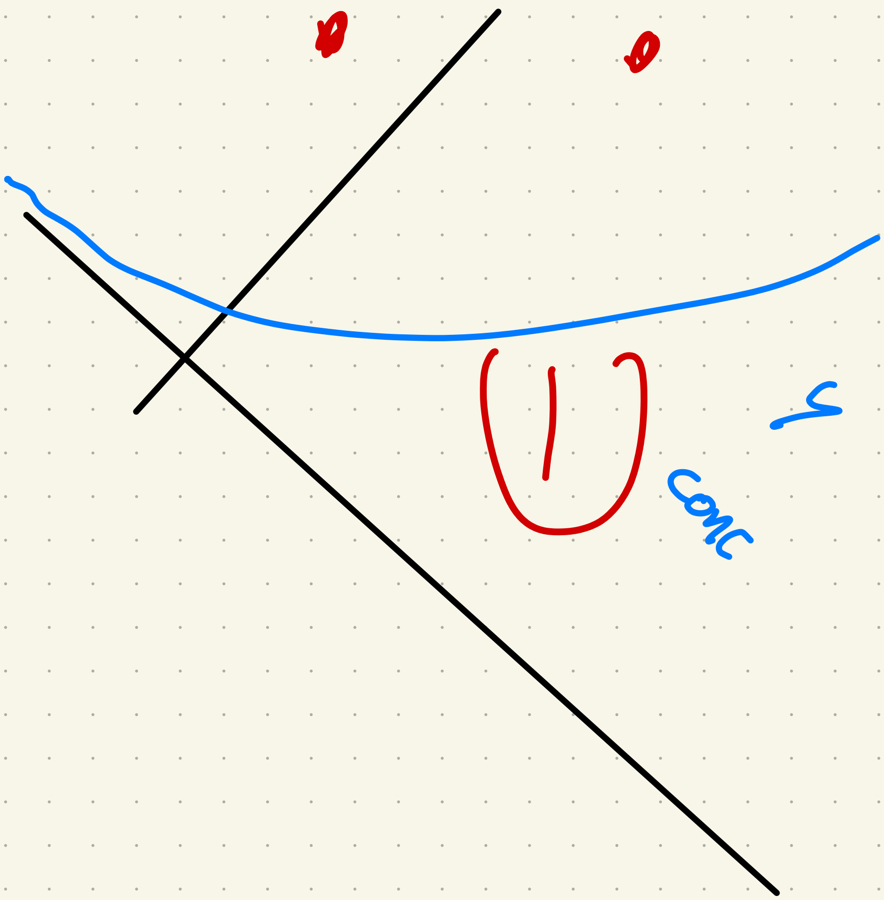


blue curve is concave up.

Looks like a smile. ◡



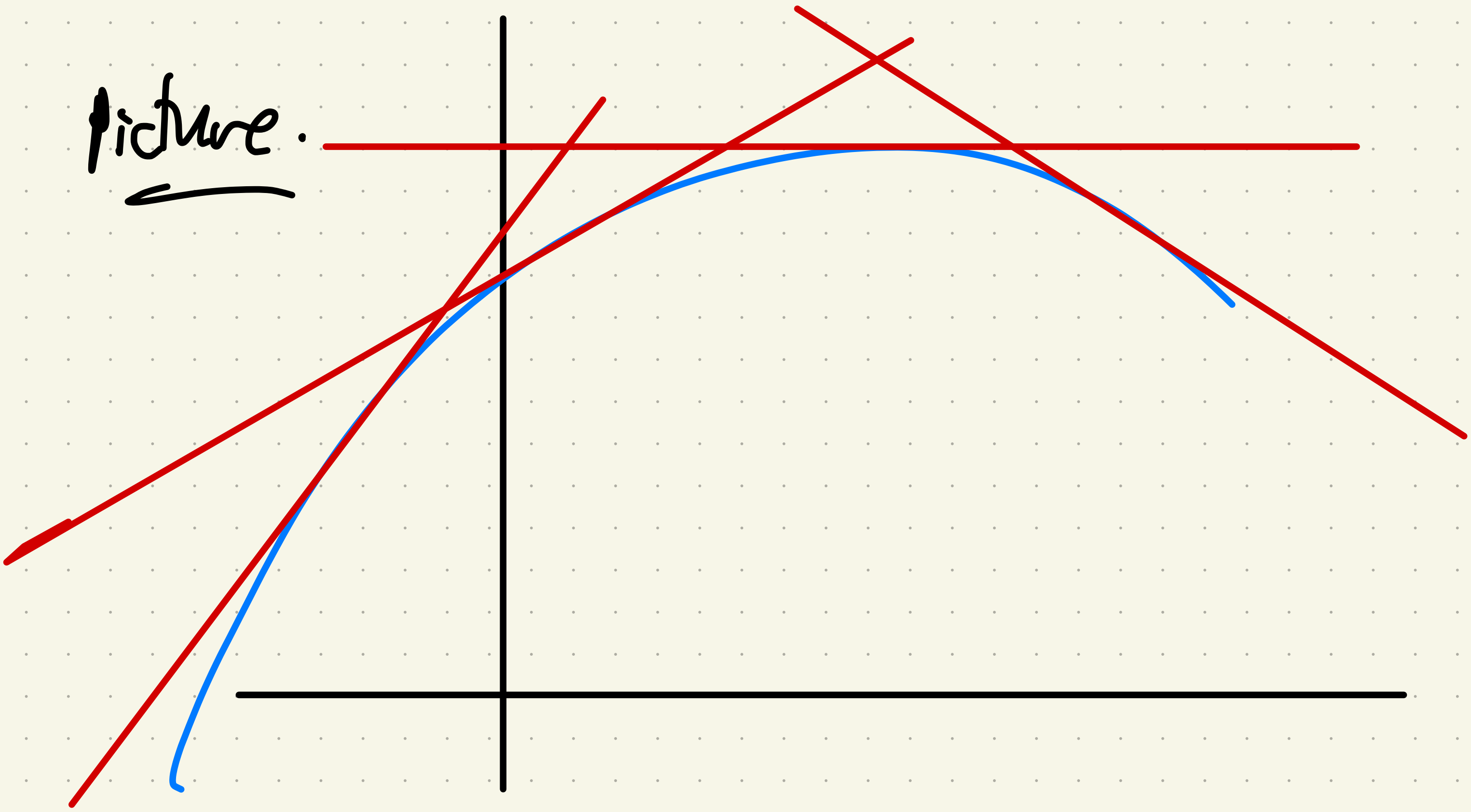
$y = e^x$
concave up



y
conc

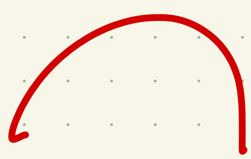
Def: If the graph of f lies below all of its tangent lines, we say it is concave down.

Picture.

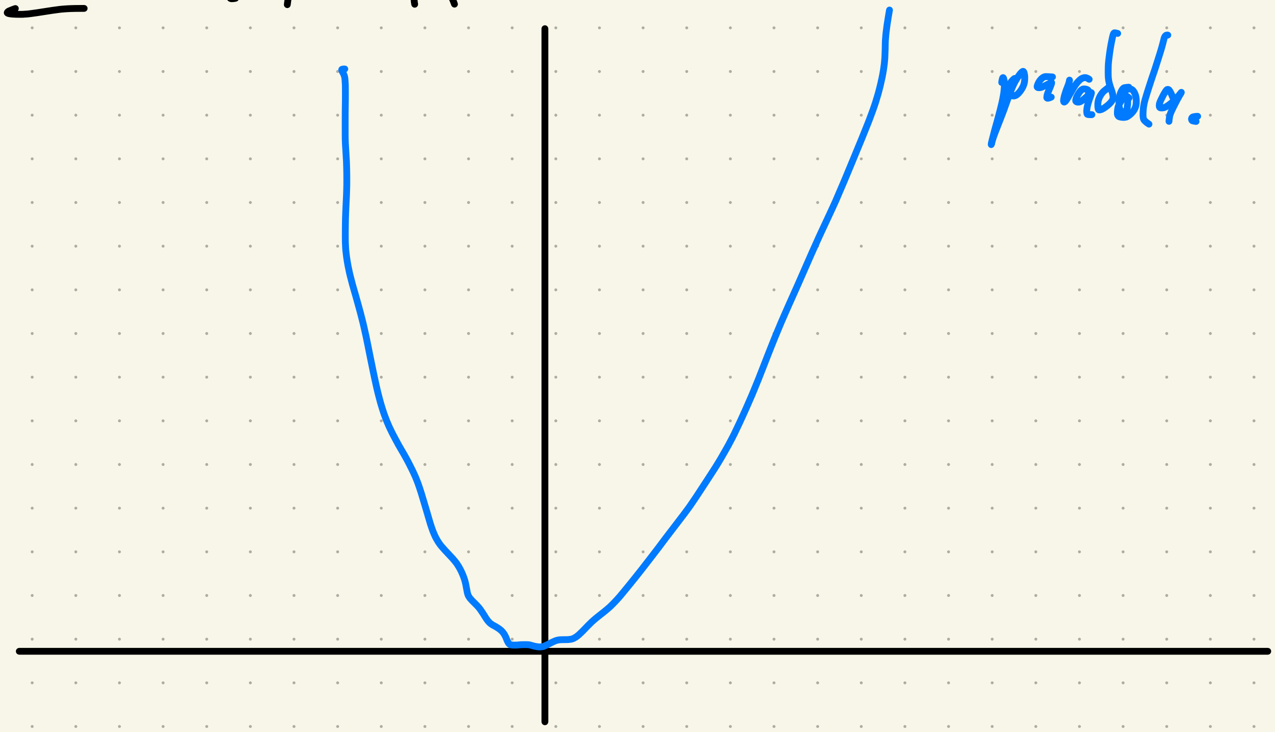


blue curve is concave down.

looks like a frown



Ex. $f(x) = x^2$



slope is increasing from neg. to positive.

$f'(x) = 2x$ is the slope at x

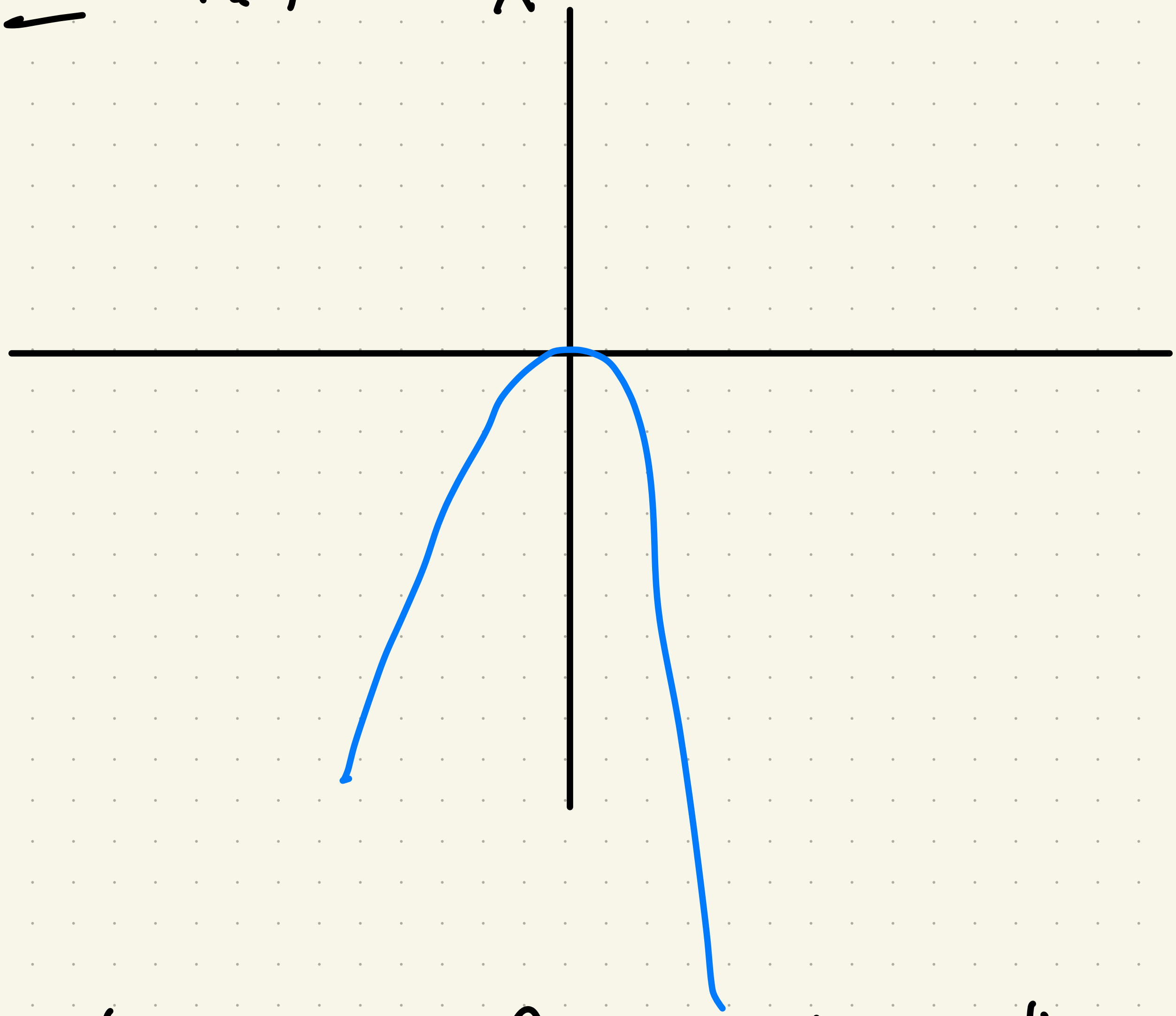
$f''(x) = 2 > 0$ is the rate of increase of the slope

$f'' > 0$

$\Updownarrow$

concave up.

Ex. $f(x) = -x^2$



slope is going from pos. to negative

$f'(x) = -2x$ is the slope

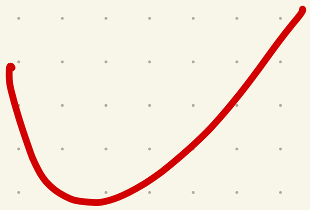
$f''(x) = -2 < 0$ is the rate of decrease of the slope

$f'' < 0 \Leftrightarrow$ concave down.

Theorem. ("The Concavity Test").

(1) if $f''(x) > 0$ on an interval (a,b)

Then the graph of f is

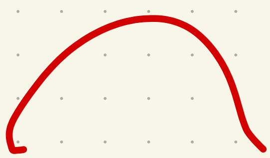


concave <u>up</u> on (a,b).

(like $f(x) = x^2$)

(2) if $f''(x) < 0$ on (a,b)

Then the graph of f is



concave <u>down</u> on (a,b).

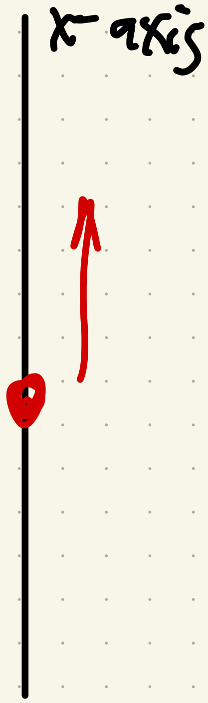
(like $f(x) = -x^2$)

Proof. Precise proof would use the MVT, but we'll skip it (see the book if you want).

Physics interpretation.

Throw a ball straight up.

Denote position at time t by $x(t)$



Then $x'(t)$ is velocity at time t

and $x''(t)$ is acceleration at time t.

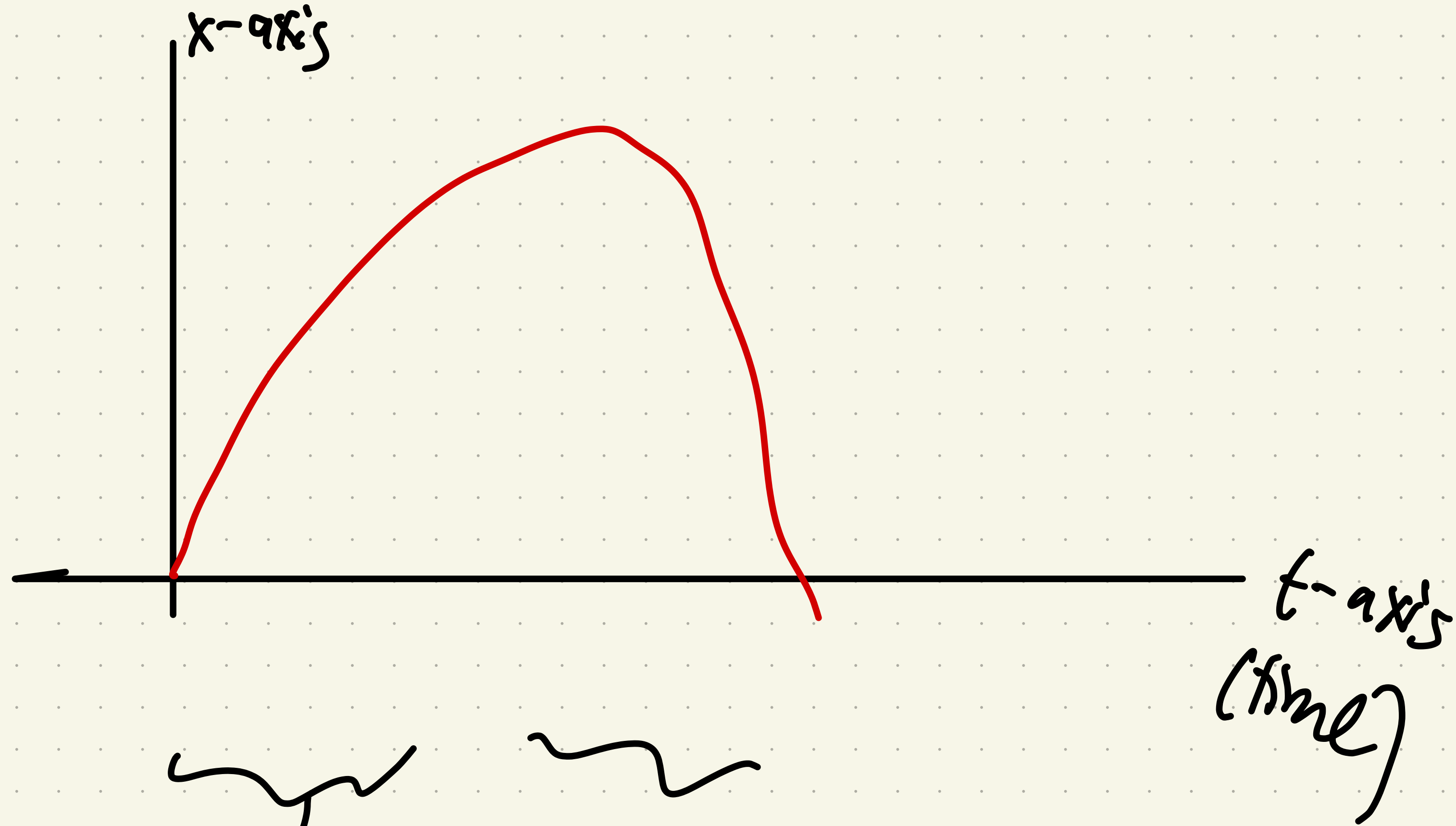
Also: $F = ma$ (force)

mass   accel.

So $x''(t) > 0$ means force is pushing up

$x''(t) < 0$ means force is pushing down

In this case the force is gravity pushing down so $x''(t) < 0$



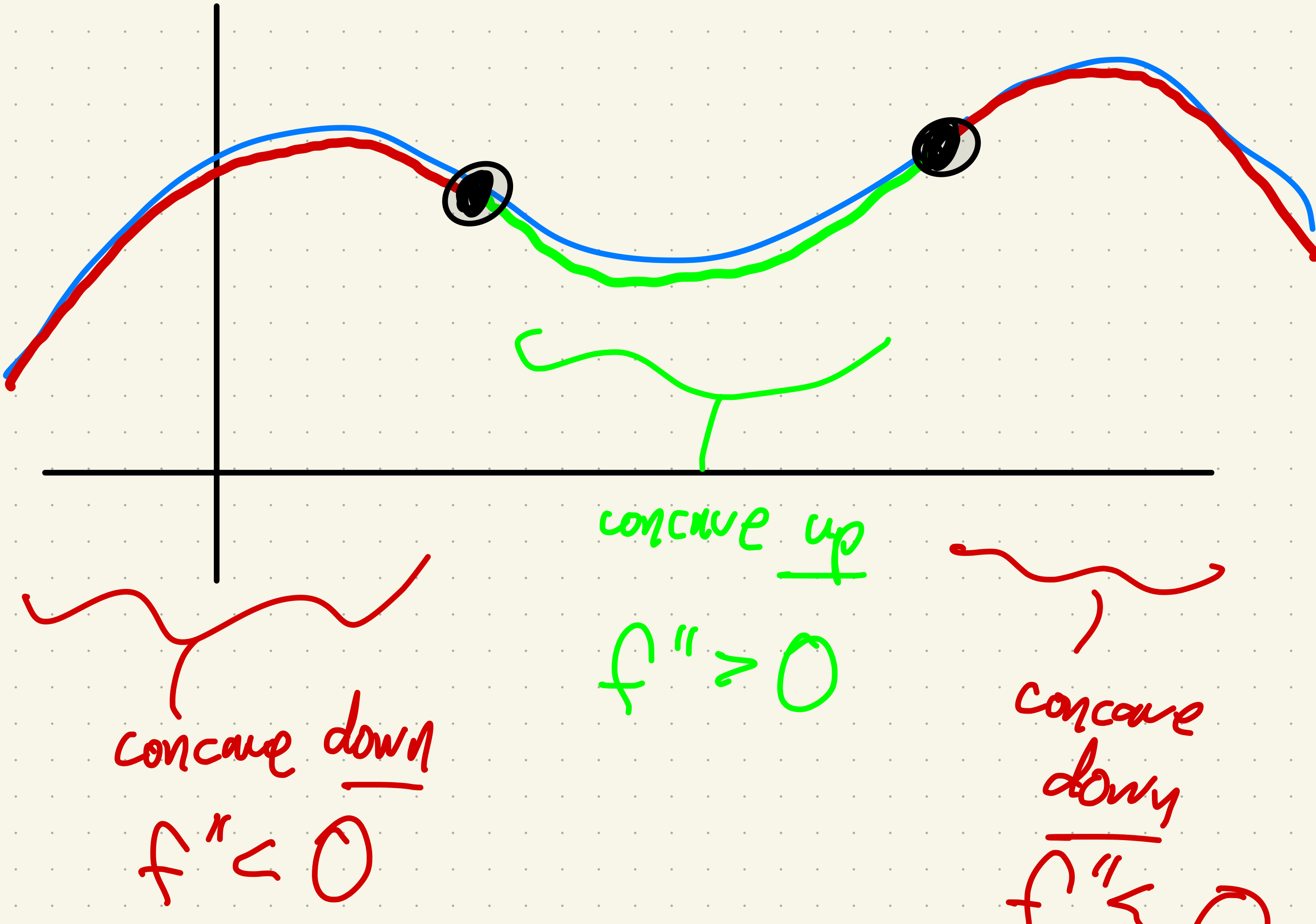
velocity is > 0

velocity is < 0

but accel is always negative

and graph is concave down.

Def: an <u>inflection point</u> is a point where the graph changes concavity.



<u>Theorem</u> ("Second deriv. test")

(1) if $f'(c) = 0$ and $f''(c) > 0$

then f has a local min. at $x = c$.

(2) if $f'(c)$ and $f''(c) < 0$

Then f has a local max at $x = c$.

Example. Find the local max & min of $f(x) = x^3 - 6x^2 + 10$.

One way: Use the Increasing/Decreasing Test (see notes)

typed

Another way: Use the Second Deriv. Test:

$$f'(x) = 3x^2 - 12x = 3x(x-4)$$

crit. pts are $x = 0, x = 4$

$$f''(x) = 6x - 12$$

$f(x) = x^3 - 6x^2 + 10$

$f'(x) = 3x^2 - 12x = 3x(x-4)$

crit. pts are $x=0$, $x=4$

$f''(x) = 6x - 12$

$f''(0) = -12 < 0$

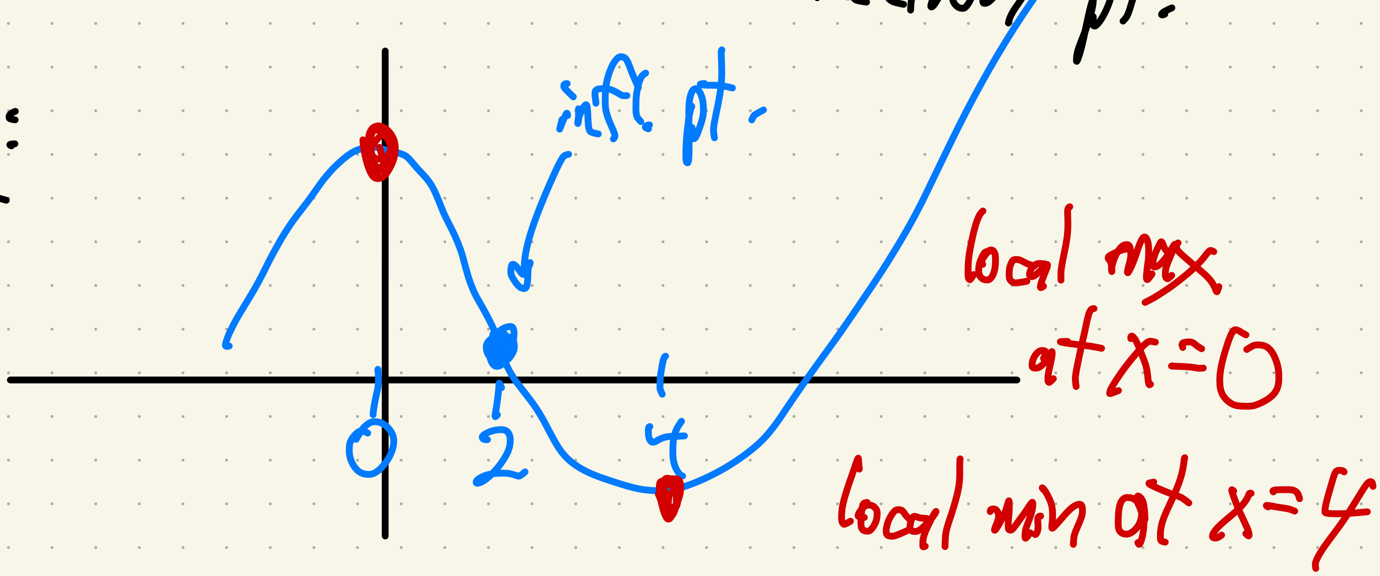
concave down

$f''(4) = 12 > 0$

concave up.

$f'' = 0$ at $x = 2 \Rightarrow$ inflection pt.

Graph:



Compare to Computer Graph:

MATH 30, 4/6/2020: CONCAVITY/CONVEXITY

Last time, before Spring Break, we saw what the first derivative f' reveals about the shape of the curve $y = f(x)$. Now we will see what the *second* derivative f'' reveals.

Definition. If the graph of f lies *above* all its tangent lines on an interval, then it is called *concave up* (or "convex") on that interval. [Picture–it looks like a smile.]

Definition. If the graph of f lies *below* all its tangent lines on an interval, then it is called *concave down* on that interval. [Picture–it looks like a frown.]

Simple Example. $f(x) = x^2$ is concave up.
Note that $f'(x) = 2x$, so that f is decreasing when $x < 0$ and is increasing when $x > 0$.
Also note that f'' is the derivative of f', so it tells you where f' is increasing or decreasing. In this example, $f''(x) = 2$, so f' is *always increasing*. [Can you see this in the graph of f? The slope of the tangent line is always *increasing*.]

Simple Example. $f(x) = -x^2$. [Do it yourself.]

Theorem ("The Concavity Test").
(1) If $f''(x) > 0$ for all x in an interval I, then the graph is *concave up* on that interval.
(2) If $f''(x) < 0$... [You can probably guess...]

Proof. The precise mathematical proof uses the Mean Value Theorem. [Many things in calculus are proven using the Mean Value Theorem.] We will skip the details in the interest of time. The gist of it: Use the fact that $f''(x) > 0$ for all x to show that "the graph of f is above the tangent line."

If you ever forget "which is which" in the Concavity Test, just remember the two simple examples above: $f(x) = x^2$ looks like a smile, and $f(x) = -x^2$ looks like a frown.

The physics interpretation: Say that $x(t)$ represents position at time t. Then $x'(t)$ represents velocity and $x''(t)$ represents acceleration. Think about what "constant speed," "positive acceleration," and "negative acceleration" look like in the graph of $x(t)$. [Picture.]

You can also think in terms of Force=mass$\times$acceleration. Then $F > 0$ means force is pushing *up* and $F < 0$ means force is pushing *down*.

Definition. An *inflection point* is a point where the graph changes concavity.

You actually hear this term in the news: for example, someone might say "we are at an inflection point in the war." It means that things might still be getting worse, but at a decreasing rate.

There is an inflection point wherever f'' changes sign. [Picture.]

Second derivatives are useful for max/min problems:

The Second Derivative Test.

(1) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$. [Picture.]
(2) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$. [Picture.]

Example. Find the local max and min of the function

$$f(x) = x^3 - 6x^2 + 10.$$

We first calculate $f'(x) = 3x^2 - 12x = 3x(x-4)$, so the critical points are $x = 0$ and $x = 4$.

One way, using the Increasing/Decreasing Test: We note that $f'(x) > 0$ on $(-\infty, 0)$, $f'(x) < 0$ on $(0, 4)$, and $f'(x) > 0$ on $(4, \infty)$, so we know the function f is increasing, then decreasing, then increasing. It thus has a local max at $x = 0$ and a local min at $x = 4$.

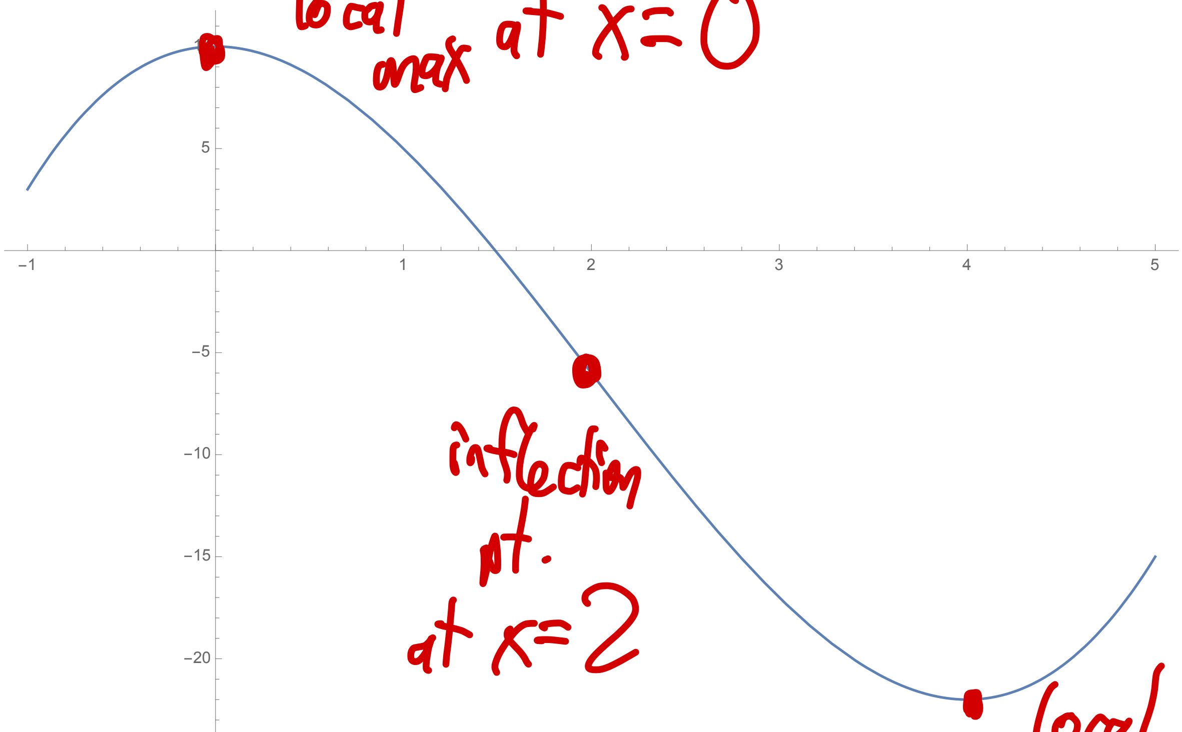
Another way, using the Second Derivative Test: We have

$$f''(x) = 6x - 12.$$

Now check the concavity at the critical points: $f''(0) = -12 < 0$, which means that $x = 0$ is a local max, and $f''(4) = 12 > 0$, which means that $x = 4$ is a local min.

To help draw the picture, we note that the only inflection point is at $x = 2$: $f''(2) = 0$. You can see how the graph changes concavity there:

local max at $x=0$



inflection pt. at $x=2$

local min

Remember, calculus helps us answer these questions *without* needing the picture!

at $x=4$

Turn in Quiz before 11:59pm tonight ✓

upload to Canvas.

on linear approximation.