1. Let W be the set of all vectors of the form $\begin{bmatrix} 2b+3c \\ -b \\ 2c \end{bmatrix}$, where b and c are arbitrary real numbers. Find vectors **u** and **v** such that W =Span{**u**,**v**}. Why does this show that W is a subspace of $\mathbb{R}^3$ (without doing any extra work)?

THEOREM 1 If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in a vector space V, then Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a subspace of V.

1. Let $W = \left\{ \begin{bmatrix} 2b+3c \\ -b \\ 2c \end{bmatrix} : b, c \in R \right\}$

This is a subspace of R^3, since

$$\begin{bmatrix} 2b+3c \\ -b \\ 2c \end{bmatrix} = \begin{bmatrix} 2b \\ -b \\ 0 \end{bmatrix} + \begin{bmatrix} 3c \\ 0 \\ 2c \end{bmatrix} = b\underbrace{\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}}_{u} + c\underbrace{\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}}_{v}$$

$u = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ $v = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$

$\boxed{W = \text{Span}\{u, v\}, \text{ where } u = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, v = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} \text{ by theorem 1 } W \text{ is a subspace of } R^3}$

2. Let W be the set of all vectors of the form $\begin{bmatrix} 2a-b \\ 3b-c \\ 3c-a \\ 3b \end{bmatrix}$, where a,b and c are arbitrary real numbers. Either find a set S of vectors that spans W or give an example to show that W is not a vector space.

2. $w = \begin{bmatrix} 2a-b \\ 3b-c \\ 3c-a \\ 3b \end{bmatrix} = \begin{bmatrix} 2a \\ 0 \\ -a \\ 0 \end{bmatrix} + \begin{bmatrix} -b \\ 3b \\ 0 \\ 3b \end{bmatrix} + \begin{bmatrix} 0 \\ -c \\ 3c \\ 0 \end{bmatrix}$

$= a\begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \end{bmatrix} + b\begin{bmatrix} -1 \\ 3 \\ 0 \\ 3 \end{bmatrix} + c\begin{bmatrix} 0 \\ -1 \\ 3 \\ 0 \end{bmatrix}$

$\boxed{S = \left\{ \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 3 \\ 0 \end{bmatrix} \right\}}$

3. For fixed positive integers m and n, the set $M_{m\times n}$ of all $m \times n$ matrices is a vector space, under the usual operations of addition of matrices and multiplication by real scalars.

Determine if the set H of all matrices of the form $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ is a subspace of $M_{2\times 2}$.

HW 6 → #3 $\quad M_{2\times 2}$

$V = V.S \quad H \leq V.$ $\quad \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = \vec{u}$

i) $\vec{0}$ in H $\longrightarrow \vec{u} + \vec{0} = \vec{u}$? ✓

ii) for $\vec{u}, \vec{v}$ in H, need $\vec{u} + \vec{v}$ in H

iii) c in R, $\vec{u}$ in H $\rangle$ $c\vec{u}$ in H. $\quad \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} + \begin{bmatrix} v_1 & x_2 \\ v_3 & x_4 \end{bmatrix}$

$$\begin{bmatrix} a+v_1 & b+x_2 \\ 0+v_3 & d+v_3 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$$

$a = b = d = 0$ $\quad \vec{u} = \begin{bmatrix} x_1 & x_2 \\ 0 & x_4 \end{bmatrix}$

i) $\vec{0}$ in H? ✓ $\quad \vec{u} + \vec{v} = \begin{bmatrix} \sim & \sim \\ 0 & \sim \end{bmatrix}$

$\vec{V} = \begin{bmatrix} v_1 & v_2 \\ 0 & v_4 \end{bmatrix}$

ii)? ✓

iii)? ✓

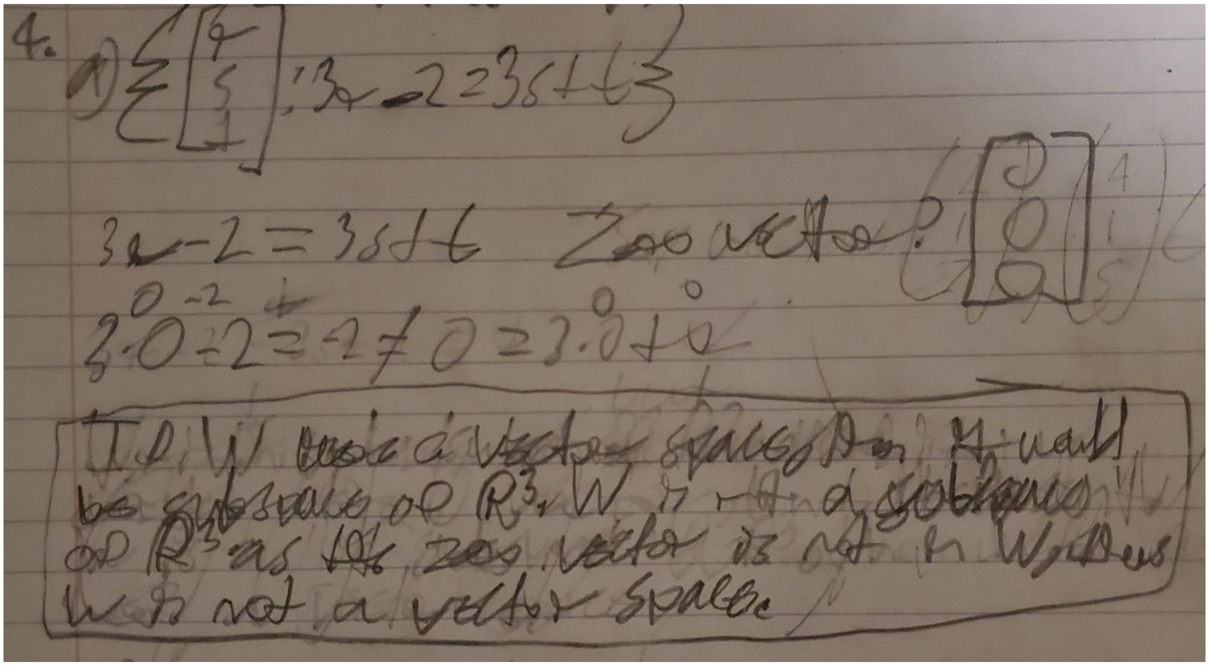
$$5\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \rightarrow \begin{bmatrix} 5\cdot a & 5\cdot b \\ 5\cdot 0 & 5\cdot d \end{bmatrix} \rightarrow \begin{bmatrix} 5\cdot a & 5\cdot b \\ 0 & 5\cdot d \end{bmatrix}$$

The set H of all matrices of the form $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ is a subspace as the zero matrix is in H, the sum of 2 upper triangular matrices is still upper triangular and the scalar multiple of an upper triangular matrix is still upper triangular.

*Your explanation in class really helped. My handwriting here is horrendous, but I basically said that it is because the zero matrix is in H, the sum of 2 upper triangular matrices is still upper triangular, and the scalar multiple of an upper triangular matrix would still be upper triangular.

4. Either use an appropriate theorem to show that the given set, W, is a vector space, or find a specific example to the contrary.

(a) $\left\{ \begin{bmatrix} r \\ s \\ t \end{bmatrix} : 3r - 2 = 3s + t \right\}$



4. a) $\left\{ \begin{bmatrix} r \\ s \\ t \end{bmatrix} : 3r - 2 = 3s + t \right\}$

$3r - 2 = 3s + t$ Zero vector? $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$3 \cdot 0 - 2 = -2 \neq 0 = 3 \cdot 0 + 0$

$\boxed{\text{If W were a vector space, then it would be subspace of } \mathbb{R}^3\text{. W is not a subspace of } \mathbb{R}^3 \text{ as the zero vector is not in W, thus W is not a vector space.}}$

DEFINITION

A **subspace** of a vector space V is a subset H of V that has three properties:

a. The zero vector of V is in H.[2]

b. H is closed under vector addition. That is, for each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u} + \mathbf{v}$ is in H.

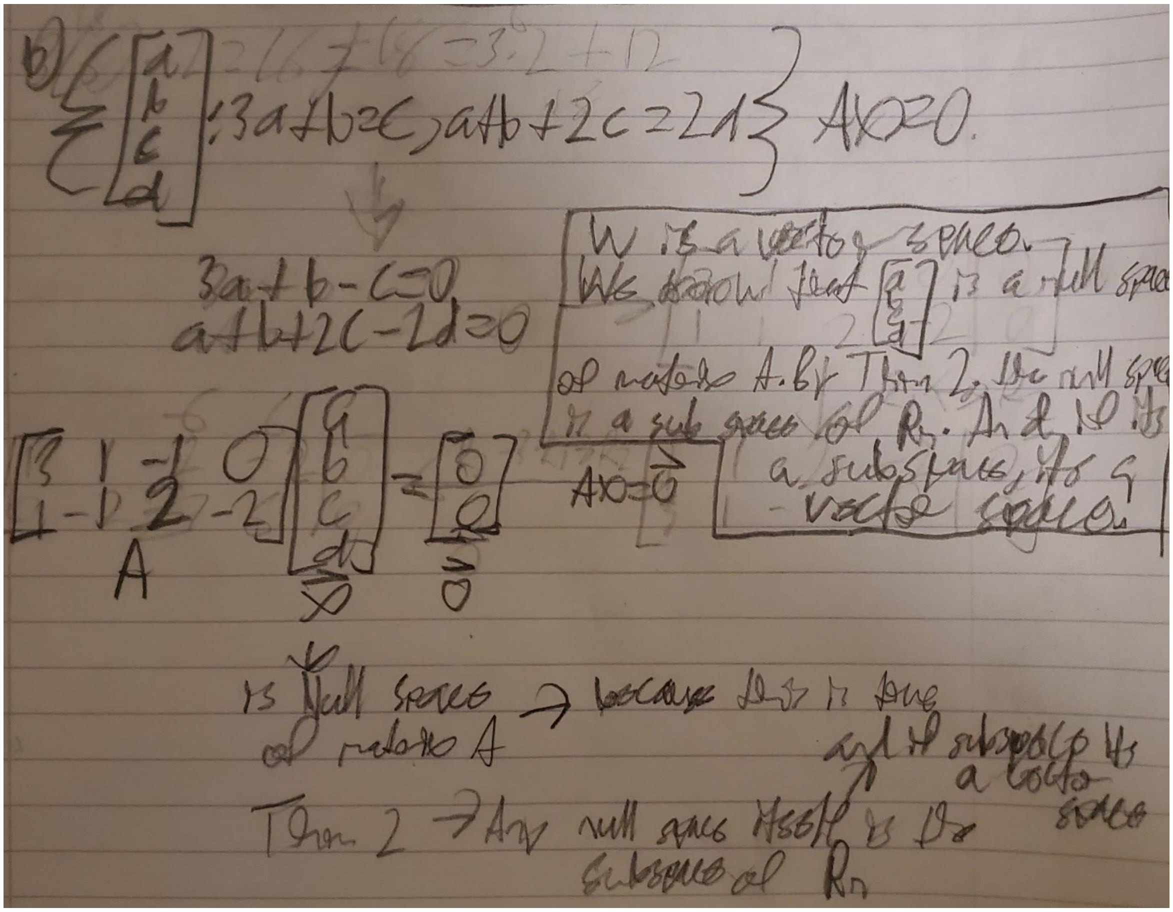
c. H is closed under multiplication by scalars. That is, for each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.

*My handwriting is bad, but here's what I wrote: If W were a vector space, then it would be a subspace of R3. W is not a subspace of R3 as the zero vector is not in W, thus W is not a vector space.

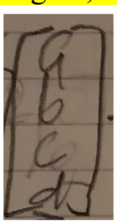
(b) $\left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : 3a + b = c, \;\; a + b + 2c = 2d \right\}$

THEOREM 2

The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions to a system $A\mathbf{x} = \mathbf{0}$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.



*Again, my handwriting is bad, but for the answer, I wrote: W is a vector space. We know that



is a null space of matrix A (*represented above). By theorem 2, the null space is a subspace of Rn. And, if it's a subspace, it's a vector space. Again, your explanation in class helped.

5. Find an explicit description of Nul A, by listing vectors that span the null space.

(a) $\begin{bmatrix} 1 & 2 & 4 & 0 \\ 0 & 1 & 3 & -2 \end{bmatrix}$

5.

a) $\begin{bmatrix} 1 & 2 & 4 & 0 \\ 0 & 1 & 3 & -2 \end{bmatrix}$ Find general sol for $Ax=0$

$$\begin{bmatrix} 1 & 2 & 4 & 0 & 0 \\ 0 & 1 & 3 & -2 & 0 \end{bmatrix} \xrightarrow{R_1 - 2\cdot R_2 \to R_1} \begin{bmatrix} 1 & 0 & -2 & 4 & 0 \\ 0 & 1 & 3 & -2 & 0 \end{bmatrix}$$

4

$1x_1 - 2x_3 + 4x_4 = 0$
$1x_2 + 3x_3 - 2x_4 = 0$

$x_1 = 2x_3 - 4x_4$
$x_2 = -3x_3 + 2x_4$
x_3 is free
x_4 is free

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_3 - 4x_4 \\ -3x_3 + 2x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

Spanning set for Nul A is:

$$\boxed{\left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}}$$

(b) $\begin{bmatrix} -1 & 3 & -4 & -3 & 1 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(Cont'd)

Find gen sol for $Ax=0$

(b) $\begin{bmatrix} -1 & 3 & -4 & -3 & 1 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 3 & -4 & -3 & 1 & 0 \\ 0 & 1 & -3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$\downarrow -R_1$

$\begin{bmatrix} 1 & 0 & -5 & 6 & -1 & 0 \\ 0 & 1 & -3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xleftarrow{R_1+3R_2} \begin{bmatrix} 1 & -3 & 4 & 3 & -1 & 0 \\ 0 & 1 & -3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$-9\ 3$

$1x_1 - 5x_3 + 6x_4 - 1x_5 = 0 \Rightarrow x_1 = 5x_3 - 6x_4 + 1x_5$

$1x_2 - 3x_3 + 1x_4 = 0 \qquad x_2 = 3x_3 - 1x_4$

x_3 is free

x_4 is free

x_5 is free

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5x_3 - 6x_4 + 1x_5 \\ 3x_3 - 1x_4 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3\begin{bmatrix} 5 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} -6 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Spanning set for Nul A is:

$$\boxed{\left\{ \begin{bmatrix} 5 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}}$$

6. Find A such that the given set is Col A: $\left\{ \begin{bmatrix} 2s+t \\ r-s+2t \\ 3r+s \\ 2r-s-t \end{bmatrix} \right\}$

6. Find A " ...

Col A: $\left\{ \begin{bmatrix} 2s+t \\ r-s+2t \\ 3r+s \\ 2r-s-t \end{bmatrix} \right\}$ Firstly write as set of linear combinations

$$r\begin{bmatrix} 0 \\ 1 \\ 3 \\ 2 \end{bmatrix} + s\begin{bmatrix} 2 \\ -1 \\ 1 \\ -1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 0 \\ 2 & -1 & -1 \end{bmatrix}\begin{bmatrix} r \\ s \\ t \end{bmatrix}$$

So the set is Col A where

$$\boxed{A = \begin{bmatrix} 0 & 2 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 0 \\ 2 & -1 & -1 \end{bmatrix}}$$

7. Given A and $\mathbf{w}$ below, determine if $\mathbf{w}$ is in Col A. Is $\mathbf{w}$ in Nul A?

$$A = \begin{bmatrix} 10 & -8 & -2 & -2 \\ 0 & 2 & 2 & -2 \\ 1 & -1 & 6 & 0 \\ 1 & 1 & 0 & -2 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 2 \end{bmatrix}$$

7. $A=\begin{bmatrix}10 & -8 & -2 & -2\\ 0 & 2 & 2 & -2\\ 1 & -1 & 6 & 0\\ 1 & 1 & 0 & -2\end{bmatrix}, W=\begin{bmatrix}2\\2\\0\\2\end{bmatrix}$

$$AW\begin{bmatrix}10 & -8 & -2 & -2\\ 0 & 2 & 2 & -2\\ 1 & -1 & 6 & 0\\ 1 & 1 & 0 & -2\end{bmatrix}\begin{bmatrix}2\\2\\0\\2\end{bmatrix} = \begin{bmatrix}10 & -8 & -2 & -2 & 2\\ 0 & 2 & 2 & -2 & 2\\ 1 & -1 & 6 & 0 & 0\\ 1 & 1 & 0 & -2 & 2\end{bmatrix}$$

$$\xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix}1 & -1 & 6 & 0 & 0\\ 0 & 2 & 2 & -2 & 2\\ 10 & -8 & -2 & -2 & 2\\ 1 & 1 & 0 & -2 & 2\end{bmatrix}$$

$$\xrightarrow{R_3 - 10\cdot R_1 \to R_3} \begin{bmatrix}1 & -1 & 6 & 0 & 0\\ 0 & 2 & 2 & -2 & 2\\ 0 & 2 & -62 & -2 & 2\\ 1 & 1 & 0 & -2 & 2\end{bmatrix}$$

$$\xrightarrow{R_4 - R_1 \to R_4} \begin{bmatrix}1 & -1 & 6 & 0 & 0\\ 0 & 2 & 2 & -2 & 2\\ 0 & 2 & -62 & -2 & 2\\ 0 & 2 & -6 & -2 & 2\end{bmatrix}$$

$$\xrightarrow{\frac{1}{2}R_2} \begin{bmatrix}1 & -1 & 6 & 0 & 0\\ 0 & 1 & 1 & -1 & 1\\ 0 & 2 & -62 & -2 & 2\\ 0 & 2 & -6 & -2 & 2\end{bmatrix}$$

$$\xrightarrow{R_1 + R_2 \to R_1} \begin{bmatrix}1 & 0 & 7 & -1 & 1\\ 0 & 1 & 1 & -1 & 1\\ 0 & 2 & -62 & -2 & 2\\ 0 & 2 & -6 & -2 & 2\end{bmatrix}$$

$$\xrightarrow{R_3 - 2\cdot R_2 \to R_3} \begin{bmatrix}1 & 0 & 7 & -1 & 1\\ 0 & 1 & 1 & -1 & 1\\ 0 & 0 & -64 & 0 & 0\\ 0 & 2 & -6 & -2 & 2\end{bmatrix}$$

HW #6

Cont'd*

$$\begin{bmatrix} 1 & 0 & 7 & -1 & 1 \\ 0 & 1 & 1 & -1 & 1 \\ 0 & 0 & -64 & 0 & 0 \\ 0 & 2 & -6 & -2 & 2 \end{bmatrix} \xrightarrow{R_4 - 2\cdot R_2 \to R_4} \begin{bmatrix} 1 & 0 & 7 & -1 & 1 \\ 0 & 1 & 1 & -1 & 1 \\ 0 & 0 & -64 & 0 & 0 \\ 0 & 0 & -8 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{64}\cdot R_3} \begin{bmatrix} 1 & 0 & 7 & -1 & 1 \\ 0 & 1 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -8 & 0 & 0 \end{bmatrix} \xrightarrow{R_4 + 8\cdot R_3 \to R_4} \begin{bmatrix} 1 & 0 & 7 & -1 & 1 \\ 0 & 1 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 - R_3 \to R_2} \begin{bmatrix} 1 & 0 & 7 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 - 7\cdot R_3 \to R_1} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \\ \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

There's a free variable, so it has infinitely many solutions and is consistent

w is in Col A

Cont'd

Cont'd

$$A\mathbf{w} = \begin{bmatrix} 10 & -8 & -2 & -2 \\ 0 & 2 & 2 & -2 \\ 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} ?$$

$$2\begin{bmatrix} 10 \\ 0 \\ 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} -8 \\ 2 \\ -1 \\ 1 \end{bmatrix} + 0\cancel{\begin{bmatrix} -2 \\ 2 \\ 0 \\ 0 \end{bmatrix}} + 2\begin{bmatrix} -2 \\ -2 \\ 0 \\ -2 \end{bmatrix}$$

$$\begin{array}{cccl} 20 & -16 & -4 & = 0 \\ 0 & 4 & -4 & = 0 \rightarrow \text{Yes!} \\ 2 & -2 & 0 & = 0 \\ 2 & 2 & -4 & = 0 \end{array}$$

So w is in Nul A.

$\boxed{\text{W is in both Nul A and Col A}}$

8. Determine whether the following sets are bases for $\mathbb{R}^3$. If the set is *not* a basis, determine which ones are linearly independent and which ones span $\mathbb{R}^3$. Justify your answers.

(a) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

DEFINITION

Let H be a subspace of a vector space V. An indexed set of vectors $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_p\}$ in V is a **basis** for H if

(i) $\mathcal{B}$ is a linearly independent set, and

(ii) the subspace spanned by $\mathcal{B}$ coincides with H; that is,

$$H = \text{Span}\,\{\mathbf{b}_1, \dots, \mathbf{b}_p\}$$

8. a) $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix} \rightarrow A = \begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}$

Looks like you can row reduce to get identity matrix

So it should be linearly independent!

$$\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix} \xrightarrow{R_1 - R_2 \to R_1} \begin{bmatrix}1&0&0\\0&1&1\\0&0&1\end{bmatrix} \xrightarrow{R_2 - R_3 \to R_2} \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$x_1 \; x_2 \; x_3$

linearly independent!

There are 3 pivot columns, so the vectors span $\mathbb{R}^3$. They are linearly independent too.

So, $\boxed{\text{The given set of vectors is a basis for } \mathbb{R}^3!}$

(b) $\begin{bmatrix}1\\0\\-3\end{bmatrix}, \begin{bmatrix}3\\1\\-4\end{bmatrix}, \begin{bmatrix}-2\\-1\\1\end{bmatrix}$

(b) $\begin{bmatrix}1\\0\\-3\end{bmatrix}, \begin{bmatrix}3\\1\\-4\end{bmatrix}, \begin{bmatrix}-2\\-1\\1\end{bmatrix} \longrightarrow \begin{bmatrix}1 & 3 & -2\\0 & 1 & -1\\-3 & -4 & 1\end{bmatrix}$

$\xrightarrow{R_3+3\cdot R_1 \to R_3} \begin{bmatrix}1 & 3 & -2\\0 & 1 & -1\\0 & 5 & -5\end{bmatrix}$

$\xrightarrow{R_3-5\cdot R_2 \to R_3} \begin{bmatrix}1 & 3 & -2\\0 & 1 & -1\\0 & 0 & 0\end{bmatrix}$

$\xrightarrow{R_1-3\cdot R_2 \to R_1} \begin{bmatrix}1 & 0 & 1\\0 & 1 & -1\\0 & 0 & 0\end{bmatrix}$ → There are only 2 pivot positions

↓

So the vectors do not span R_3

From LA

↓

So linearly dependent

So, $\boxed{\text{The given set of vectors do not form a basis for } R^3!}$

9. Find a basis for the space spanned by the given vectors $\mathbf{v}_1, ..., \mathbf{v}_5$.

$$\begin{bmatrix}1\\0\\-2\\3\end{bmatrix}, \begin{bmatrix}0\\1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\-2\\-8\\0\end{bmatrix}, \begin{bmatrix}2\\-1\\10\\3\end{bmatrix}, \begin{bmatrix}3\\-1\\-6\\9\end{bmatrix}$$

9. $\begin{bmatrix}1\\0\\-2\\3\end{bmatrix}, \begin{bmatrix}0\\1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\-2\\-8\\0\end{bmatrix}, \begin{bmatrix}2\\-1\\10\\3\end{bmatrix}, \begin{bmatrix}3\\-1\\-6\\9\end{bmatrix}$

$V_1 \quad V_2 \quad V_3 \quad V_4 \quad V_5$

Sort of like asking to find a basis for Col A where $A=[V_1 \ldots V_5]$

$$\begin{bmatrix}1&0&2&2&3\\0&1&-2&-1&-1\\-2&2&-8&10&-6\\3&3&0&3&9\end{bmatrix} \xrightarrow{R_3+2\cdot R_1 \to R_3} \begin{bmatrix}1&0&2&2&3\\0&1&-2&-1&-1\\0&2&-4&14&0\\3&3&0&3&9\end{bmatrix}$$

$$\xrightarrow{R_4-3\cdot R_1 \to R_4} \begin{bmatrix}1&0&2&2&3\\0&1&-2&-1&-1\\0&2&-4&14&0\\0&3&-6&-3&0\end{bmatrix} \xrightarrow{R_3-2\cdot R_2 \to R_3} \begin{bmatrix}1&0&2&2&3\\0&1&-2&-1&-1\\0&0&0&16&2\\0&3&-6&-3&0\end{bmatrix}$$

$$\xrightarrow{R_4-3\cdot R_2 \to R_4} \begin{bmatrix}1&0&2&2&3\\0&1&-2&-1&-1\\0&0&0&16&2\\0&0&0&0&3\end{bmatrix} \xrightarrow{\frac{1}{3}R_4} \begin{bmatrix}1&0&2&2&3\\0&1&-2&-1&-1\\0&0&0&16&2\\0&0&0&0&1\end{bmatrix}$$

$$\xrightarrow{R_3-2\cdot R_4 \to R_3} \begin{bmatrix}1&0&2&2&3\\0&1&-2&-1&-1\\0&0&0&16&0\\0&0&0&0&1\end{bmatrix} \xrightarrow{\frac{1}{16}R_3} \begin{bmatrix}1&0&2&2&3\\0&1&-2&-1&-1\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}$$

[illegible]

(cont'd)

$$\begin{bmatrix} 1 & 0 & 2 & 2 & 3 \\ 0 & 1 & -2 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 - R_3 \to R_2} \begin{bmatrix} 1 & 0 & 2 & 2 & 3 \\ 0 & 1 & -2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\downarrow R_1 - R_3 \to R_1$$

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xleftarrow{R_1 - R_4 \to R_1} \begin{bmatrix} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & -2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\downarrow R_2 - R_4 \to R_2$$

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Pivot columns at 1st, 2nd, 4th, 5th cols

So, a basis for the space spanned by the given vectors is

$$\boxed{\left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 10 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -6 \\ a \end{bmatrix} \right\} \text{ or } \{V_1, V_2, V_4, V_5\}}$$

10. Mark the following statements true or false. If false, give an explanation of why it is false.

(a) R 2 is a subspace of R 3.

EXAMPLE 8 The vector space $\mathbb{R}^2$ is *not* a subspace of $\mathbb{R}^3$ because $\mathbb{R}^2$ is not even a subset of $\mathbb{R}^3$. (The vectors in $\mathbb{R}^3$ all have three entries, whereas the vectors in $\mathbb{R}^2$ have only two.) The set

$$H = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} : s \text{ and } t \text{ are real} \right\}$$

is a subset of $\mathbb{R}^3$ that "looks" and "acts" like $\mathbb{R}^2$, although it is logically distinct from $\mathbb{R}^2$. See Fig. 7. Show that H is a subspace of $\mathbb{R}^3$.

SOLUTION The zero vector is in H, and H is closed under vector addition and scalar multiplication because these operations on vectors in H always produce vectors whose third entries are zero (and so belong to H). Thus H is a subspace of $\mathbb{R}^3$. ■

False. If you look at example 8 above, you can see that R 2 is not a subspace of R 3 because it isn't even a subset of R 3.

The issue could be that elements of R 2 look like:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

While elements of R 3 look like:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(b) The column space of A is the range of the mapping x → Ax.

Note that a typical vector in Col A can be written as $A\mathbf{x}$ for some $\mathbf{x}$ because the notation $A\mathbf{x}$ stands for a linear combination of the columns of A. That is,

$$\boxed{\text{Col } A = \{\mathbf{b} : \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \text{ in } \mathbb{R}^n\}}$$

The notation $A\mathbf{x}$ for vectors in Col A also shows that Col A is the *range* of the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$. We will return to this point of view at the end of the section.

True.

(c) The kernel of a linear transformation is a vector space.

DEFINITION

A **linear transformation** T from a vector space V into a vector space W is a rule that assigns to each vector $\mathbf{x}$ in V a unique vector $T(\mathbf{x})$ in W, such that

(i) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}$, $\mathbf{v}$ in V, and

(ii) $T(c\mathbf{u}) = cT(\mathbf{u})$ for all $\mathbf{u}$ in V and all scalars c.

The **kernel** (or **null space**) of such a T is the set of all $\mathbf{u}$ in V such that $T(\mathbf{u}) = \mathbf{0}$ (the zero vector in W). The **range** of T is the set of all vectors in W of the form $T(\mathbf{x})$ for some $\mathbf{x}$ in V. If T happens to arise as a matrix transformation—say, $T(\mathbf{x}) = A\mathbf{x}$ for some matrix A—then the kernel and the range of T are just the null space and the column space of A, as defined earlier.

It is not difficult to show that the kernel of T is a subspace of V. The proof is essentially the same as the one for Theorem 2. Also, the range of T is a subspace of W. See Fig. 2 and Exercise 30.

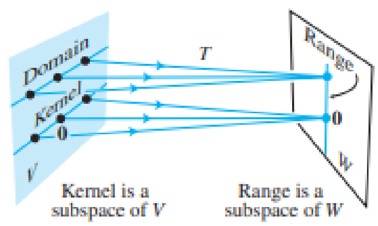


FIGURE 2 Subspaces associated with a linear transformation.

Subspaces

In many problems, a vector space consists of an appropriate subset of vectors from some larger vector space. In this case, only three of the ten vector space axioms need to be checked; the rest are automatically satisfied.

DEFINITION

A **subspace** of a vector space V is a subset H of V that has three properties:

a. The zero vector of V is in H.[2]

b. H is closed under vector addition. That is, for each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u}+\mathbf{v}$ is in H.

c. H is closed under multiplication by scalars. That is, for each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.

True.

(d) A single vector by itself is linearly dependent.

In this section we identify and study the subsets that span a vector space V or a subspace H as "efficiently" as possible. The key idea is that of linear independence, defined as in $\mathbb{R}^n$.

An indexed set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in V is said to be **linearly independent** if the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0} \tag{1}$$

has *only* the trivial solution, $c_1 = 0, \dots, c_p = 0$.[1]

The set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is said to be **linearly dependent** if (1) has a nontrivial solution, that is, if there are some weights, $c_1, \dots, c_p$, *not all zero*, such that (1) holds. In such a case, (1) is called a **linear dependence relation** among $\mathbf{v}_1, \dots, \mathbf{v}_p$.

Just as in $\mathbb{R}^n$, a set containing a single vector $\mathbf{v}$ is linearly independent if and only if $\mathbf{v} \neq \mathbf{0}$. Also, a set of two vectors is linearly dependent if and only if one of the vectors is a multiple of the other. And any set containing the zero vector is linearly dependent. The following theorem has the same proof as Theorem 7 in Section 1.7.

False. Looking at the explanation above, we know that it is linearly independent if and only if v≠0, but we don't know if that is the case or not, meaning it could be possible for it not to be linearly dependent.

(e) If H=Span{b1, ..., bp}, then {b1, ..., bp} is a basis for H.

DEFINITION

Let H be a subspace of a vector space V. An indexed set of vectors $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_p\}$ in V is a **basis** for H if

(i) $\mathcal{B}$ is a linearly independent set, and

(ii) the subspace spanned by $\mathcal{B}$ coincides with H; that is,

$$H = \text{Span}\{\mathbf{b}_1, \dots, \mathbf{b}_p\}$$

False. The set {b1, …, bp} would also have to be linearly independent according to the definition of a basis.

(f) The columns of an invertible n × n matrix form a basis for R n .

EXAMPLE 3 Let A be an invertible $n \times n$ matrix—say, $A = [\,\mathbf{a}_1 \;\; \cdots \;\; \mathbf{a}_n\,]$. Then the columns of A form a basis for $\mathbb{R}^n$ because they are linearly independent and they span $\mathbb{R}^n$, by the Invertible Matrix Theorem. ■

True.

(g) If f is a function in the vector space V of all real-valued functions on R and if f(t) = 0 for some t, then f is the zero vector in V .

EXAMPLE 5 Let V be the set of all real-valued functions defined on a set $\mathbb{D}$. (Typically, $\mathbb{D}$ is the set of real numbers or some interval on the real line.) Functions are added in the usual way: $\mathbf{f} + \mathbf{g}$ is the function whose value at t in the domain $\mathbb{D}$ is $\mathbf{f}(t) + \mathbf{g}(t)$. Likewise, for a scalar c and an $\mathbf{f}$ in V, the scalar multiple $c\mathbf{f}$ is the function whose value at t is $c\mathbf{f}(t)$. For instance, if $\mathbb{D} = \mathbb{R}$, $\mathbf{f}(t) = 1 + \sin 2t$, and $\mathbf{g}(t) = 2 + .5t$, then

$$(\mathbf{f} + \mathbf{g})(t) = 3 + \sin 2t + .5t \quad \text{and} \quad (2\mathbf{g})(t) = 4 + t$$

Two functions in V are equal if and only if their values are equal for every t in $\mathbb{D}$. Hence the zero vector in V is the function that is identically zero, $\mathbf{f}(t) = 0$ for all t, and the negative of $\mathbf{f}$ is $(-1)\mathbf{f}$. Axioms 1 and 6 are obviously true, and the other axioms follow from properties of the real numbers, so V is a vector space. ■

It is important to think of each function in the vector space V of Example 5 as a single object, as just one "point" or vector in the vector space. The sum of two vectors $\mathbf{f}$ and $\mathbf{g}$ (functions in V, or elements of *any* vector space) can be visualized as in Fig. 5, because this can help you carry over to a general vector space the geometric intuition you have developed while working with the vector space $\mathbb{R}^n$. See the *Study Guide* for help as you learn to adopt this more general point of view.

False. The zero vector in V is the function f whose values f(t) are zero for **all** t, not just some, but for all.

(h) If a finite set S of nonzero vectors spans a vector space V, then some subset of S is a basis for V .

THEOREM 5

The Spanning Set Theorem

Let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ be a set in V, and let $H = \text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$.

a. If one of the vectors in S—say, $\mathbf{v}_k$—is a linear combination of the remaining vectors in S, then the set formed from S by removing $\mathbf{v}_k$ still spans H.

b. If $H \neq \{\mathbf{0}\}$, some subset of S is a basis for H.

True. *You basically apply the theorem for V instead of H. Space V is nonzero because as was described, the spanning set uses nonzero vectors.