1.

$$f(x,y) = \begin{cases} \frac{1}{6}(x^3+3y^2) & 0<x<2,\ 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

X # of hours 1st branch
Y # of hours 2nd branch

$E(X) = \frac{7}{5},\ Var(X) = \frac{54}{225}$

a) find marginal density of Y

$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$

$= \int_0^2 \frac{1}{6}(x^3+3y^2)\,dx$

$\frac{1}{6}\int_0^2 (x^3+3y^2)\,dx \to \left(\frac{x^4}{4}+3y^2x\right)\Big|_0^2$

$\frac{2^4}{4} + 3y^2 2 \to \frac{16}{4}+6y^2 \to 4+6y^2$

$\boxed{f_Y(y) = 4+6y^2,\ 0<y<1}$

b) Find $Var(Y)$ $\quad \sigma_Y^2 = E(Y^2)-(E(Y))^2$

$E(Y) = \int_0^1 y(4+6y^2)\,dy \to \int (4y+6y^3)$

$\frac{7}{2} = 2+\frac{3}{2} \qquad \leftarrow \left(\frac{4y^2}{2}+\frac{6y^4}{4}\right)\Big|_0^1$

$E(Y^2) = \int_0^1 y^2(4+6y^2)\,dy \to \int 4y^2+6y^4 \to \frac{4y^3}{3}+\frac{6y^5}{5}\Big|_0^1$

$\frac{4}{3}+\frac{6}{5} = \frac{20}{15}+\frac{18}{15} = \frac{38}{15}$ Cont'd

Cont'd

$\sigma_Y^2 = \frac{38}{25} - \left(\frac{7}{2}\right)^2$

$\frac{38}{25} - \frac{49}{4} = \frac{152}{100} - \frac{1225}{100} = -\frac{1073}{100}$

$\boxed{\sigma_Y^2 = -\frac{1073}{100}}$

c) Are X and Y independent?

$\frac{1}{6}(x^3+3y^2) = (4+6y^2)(x^3+1)$

$f_X(x) = \int_0^1 \frac{1}{6}(x^3+3y^2)\,dy$

$\left(yx^3 + \frac{3y^3}{3}\right)\Big|_0^1 = \frac{1}{6}(x^3+1)$

No because the multiplication of the marginal density of Y and X is not equal to the probability density function of X and Y.

d) Find $\rho_{X,Y}$ $\quad \rho_{XY} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$

$\cancel{\int_0^1 \int_0^2 \frac{1}{6}(x^3+3y^2)\,dx\,dy}$

$\cancel{\frac{x^4}{4} + 3y^2x \Big|_0^2} = \int_0^1 (4+6y^2)\,dy$

$\cancel{4y + \frac{6y^3}{3}\Big|_0^1 = 4 + \frac{6}{3}} = 4+2 = 6 \; \frac{1}{6}$

Cont'd

cont'd

$E(Y) = \frac{7}{2}$ $E(X) = \frac{7}{5}$

$Var(X) =$

$Cov(X,Y) = E(X,Y) - E(X)\cdot E(Y)$

$$= \frac{94}{120} - \overbrace{\left(\frac{7}{5}\right)\left(\frac{7}{2}\right)}^{49/10} = \frac{94}{120} - \frac{98}{20} = -\frac{4}{20} = \boxed{-4.14}$$

$\int_0^1 \int_0^2 xy \frac{1}{6}(x^3 + 3xy^2)\, dx\, dy$

$\int \frac{1}{6}(x^4 y + 3x y^3) \Rightarrow \frac{x^5}{5} y + \frac{3x^2}{2} y^3 \Big|_0^2$

$\int_0^1 \left(\frac{32}{5} y + \frac{12}{2} y^3\right) dy$

$\frac{32}{5}\frac{y^2}{2} + 6\frac{y^4}{4}\Big|_0^1$

$\frac{32}{10} + \frac{6}{4} = \frac{64}{20} + \frac{30}{20} = \frac{94}{20} \cdot \frac{1}{6}$

$$\boxed{\frac{-4.14}{\left(\sqrt{\frac{59}{225}}\right)\left(\sqrt{\frac{1073}{100}}\right)}}$$

X and Y are not that correlated strongly as their abs value is close to 0, which might indicate they are weakly correlated.

e) Find $Var(X - 2Y)$

$= \sigma_x^2 + 2^2 \sigma_y^2 + 2(1)(2)\, Cov(X,Y)$

$= \frac{59}{225} + 4\left(\frac{1073}{100}\right) + 4\left(-\frac{1}{5}\right) = \boxed{42.38}$

2.

X \ Y	0	1	2	$P_X(x)$
0	0.25	0.10	0.20	0.55
1	0.10	0.25	0.10	0.45
$P_Y(y)$	0.35	0.35	0.3	1

a) Find marginal pmfs for X and Y

$$\boxed{\begin{array}{ll} P(X=0) = 0.55 & P(Y=0) = 0.35 \\ P(X=1) = 0.45 & P(Y=1) = 0.35 \\ & P(Y=2) = 0.3 \end{array}}$$

b) X and Y independent?

$P(X, Y) = P_X(X) P_Y(Y)$?

$0.55 * 0.35$

$P(X=0, Y=0) = P(X=0) \cdot P(Y=0)$?

$0.25 \neq 0.55 * 0.35$

No, they are not independent because probability of X and Y equaling 0 is not the same as the product of the probability of X being 0 and probability of Y being 0.

Cont'd

e) Find $Cov(X_0, Y)$ $Cov(X_0, Y) = \mu_{X_0Y} - \mu_{X_0}\mu_Y$

$\mu_{X_0Y} = (1)(1)(0.25) + (1)(2)(0.10)$
$= 0.45$

$\mu_{X_0} = \cancel{0(0.55)} + 1(0.45) = 0.45$

$\mu_Y = \cancel{0(0.35)} + 1(0.35) + 2(0.3) = 0.95$

$0.45 - (0.45)(0.95)$

$0.45 - 0.4275 = \boxed{0.0225}$

f) Find variance of $X_0 - Y$

$E(X_0)$ $\sigma_{X_0}^2 = E(X_0^2) - [E(X_0)]^2$

$E(X_0) = 0.45$

$\sigma_{X_0}^2 + \sigma_Y^2 - 2\,Cov(X_0, Y)$

$E(X_0^2) = 1^2(0.45) = 0.45$

$0.2475 + 0.6475 - 2(0.0225)$

$E(Y) = 0.95$

$= \boxed{0.85}$

$E(Y^2) = 1^2(0.35) + 2^2(0.3) = 2$
$0.35 + 4(0.3) = 1.55$

$\sigma_{X_0}^2 = 0.45 - (0.45)^2 = 0.2475$

$\sigma_Y^2 = 1.55 - (0.95)^2 = 0.6475$