2. Let $H$ and $K$ be subspaces of a vector space $V$. The **intersection** of $H$ and $K$, denoted by $H \cap K$, is the set of $\mathbf{v}$ in $V$ such that $\mathbf{v}$ belongs to both $H$ and $K$. Show that $H \cap K$ is a subspace of $V$.

2. i) $0$ in $H$? ✓

ii) for $\vec{u}, \vec{v}$ in $H$, $\vec{u}+\vec{v}$ in $H$? ✓

iii) $c\vec{u}$ in $H$? ✓

i) $0 \in H$ and $0 \in K$, so $0 \in H \cap K$!

ii) ~~let~~ $u \in H \cap K$ and $v \in H \cap K$

we ~~get that~~ $u, v \in H$ → so, $u+v \in H$ if $H$ is a subspace!

we ~~get that~~ $u, v \in K$ → so, $u+v \in K$ if $H$ is a subspace!

Thus, $u+v \in H \cap K$!

iii) let $u \in H \cap K$, let $c$ be a scalar

$u \in H$, and so $cu \in H$ since $H$ is a subspace

$u \in K$, and so $cu \in K$ since $K$ is a subspace

Thus,

$cu \in H \cap K$ ~~for~~ any scalar $c$, and $H \cap K$ is a subspace of $V$!

*My work above is messy, but I was able to prove that H∩K is a subspace of V! We know that H and K both have the zero vector of V if they are subspaces of V. As such, the zero vector would be in H∩K. Next, we had to show that u + v is in H∩K. We know that u and v are in H, so u + v would be in H since H is a subspace. We also know that u and v are in K, so u + v would be in K since K is a subspace. This gives us that u + v is in H∩K! Last, we had to show that for any scalar c and for u in H∩K, that cu is in H∩K. Since we know that u is in H, then cu is in H since H is a subspace. Since we know that u is in K, then cu is in K since H is a subspace. Thus, we get that cu is in H∩K, and we get that H∩K is a subspace of V! I don't know how I didn't get this before. I have done problems like this on the HW and quiz, so I should have been able to do this on the exam, but it's too late now.

4. Define $T : \mathbb{P}_2 \to \mathbb{R}^2$ by $T(\mathbf{p}) = \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix}$.

For example if $\mathbf{p}(t) = 3 + 4t + 5t^2$ then $T(\mathbf{p}) = \begin{bmatrix} 3 \\ 12 \end{bmatrix}$.

(a) Show that $T$ is a linear transformation.

4. $T: \mathbb{P}_2 \to \mathbb{R}^2$ by $T(p) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix}$

if $p(t) = 3 + 4t + 5t^2$, then $T(p) = \begin{bmatrix} 3 \\ 12 \end{bmatrix}$

a) i) Let $u$ and $v$ be arbitrary polynomials in $\mathbb{P}_2$!

$$T(u+v) = \begin{bmatrix} (u+v)(0) \\ (u+v)(1) \end{bmatrix} = \begin{bmatrix} u(0) + v(0) \\ u(1) + v(1) \end{bmatrix}$$

$$= \begin{bmatrix} u(0) \\ u(1) \end{bmatrix} + \begin{bmatrix} v(0) \\ v(1) \end{bmatrix} = T(u) + T(v)$$

ii) Let $c$ be any scalar

$$T(cp) = \begin{bmatrix} (cp)(0) \\ (cp)(1) \end{bmatrix} = c\begin{bmatrix} p(0) \\ p(1) \end{bmatrix} = cT(p)$$

$\boxed{\text{Thus, } T \text{ is a linear transformation!}}$

(b) Find a polynomial in the kernel of $T$.

b) kernel of T is set of all u in V such that $T(u)=0$.

$\hookrightarrow T(u) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

So, $p(0)=0$ and $p(1)=0$.

$\boxed{\text{The polynomial } p(t)=t(t-1) \text{ spans kernel of } T!}$

(c) What is the range of $T$?

c) For any vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ in $R^2$

the polynomial $p(t)=x_1+(x_2-x_1)t$ has $p(0)=x_1$ and $p(1)=x_2$

$\boxed{\text{So, we have that the range of } T \text{ is all of } R^2!}$

6. Let $H$ be a subspace of $M_{2\times 2}$ whose vectors are of the form $\begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$. Then, $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \right\}$ is a basis for $H$.

Find the coordinate vector of $\mathbf{v} = \begin{bmatrix} 7 & -1 \\ 0 & 0 \end{bmatrix}$ according to the basis, $\mathcal{B}$.

6. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $B = \left\{ \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \right\}$

$v = \begin{bmatrix} 7 & -1 \\ 0 & 0 \end{bmatrix}$

$c_1 \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 7 & -1 \\ 0 & 0 \end{bmatrix}$

~~Treat~~ As $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$?

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 7 \\ 0 & 1 & -1 & -1 \\ -1 & -1 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 7 \\ 0 & 1 & -1 & -1 \\ 0 & -1 & 3 & 7 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 7 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right] \longleftarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 7 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 2 & 6 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 7 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$\boxed{\text{So } [x]_B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}$

$\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} + \begin{bmatrix} 6 & -3 \\ 3 & 0 \end{bmatrix}$

$\downarrow$

$\begin{bmatrix} 7 & -1 \\ 0 & 0 \end{bmatrix}$ it works?

7. Let $H$ be the subspace of $M_{2\times 2}$ in Question 6 with the added restriction that $a + 2b + 3c = 0$. Find a basis for $H$, and state the dimension of $H$.

7. $\begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$ $B = \left\{ \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \right\}$

And $a+2b+3c=0$ Find basis for H
& state dim of H

$a=1$
$b=2$ $\quad H = \left\{ \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} : a+2b+3c=0 \right\}$
$c=3$

$a+2b+3c=0$

$\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} + 2\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + 3\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$ $\quad 6 \quad -6 \quad 3 \quad -3$

$\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \quad \begin{bmatrix} 6 & -3 \\ 3 & 0 \end{bmatrix}$
$v_1 \quad v_2 \quad v_3$

all linearly independent is neither H is combination of the 2!

$$\boxed{\text{Basis for } H: \left\{ \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ -2 & 2 \end{bmatrix}, \begin{bmatrix} 6 & -3 \\ 3 & 0 \end{bmatrix} \right\} \quad \text{Dimension of } H: 3}$$

8. Let $H$ be the set of all vectors of the forms given below and either find a basis for the vector space, or give an example to show that it is not a vector space.

(a) $\left\{ \begin{bmatrix} 3q-4p \\ 2p \\ q+1 \\ 2p+5q \end{bmatrix} : p, q \text{ are real} \right\}$

8\. a) $\left\{ \begin{bmatrix} 3q-4p \\ 2p \\ q+1 \\ 2p+5q \end{bmatrix} ; p, q \text{ are real} \right\}$

$\cancel{\begin{matrix} -4p+3q \\ 2p+0q \\ 0p+1q \\ 2p+5q \end{matrix}}$

Suppose $\begin{bmatrix} 3q-4p \\ 2p \\ q+1 \\ 2p+5q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$p = 0$

$q = -1$

$3(-1) - 4(0) = 0?$ X

$2(0) = 0$

$-1 + 1 = 0$

$2(0) + 5(-1) = 0?$ X

Not a vector space as 3q-4p should be 0 but based on what we got the p and q does are contradictory results

*Not a vector space as 3q - 4p should be 0, but based on what we got for p and q, there are contradictory results.

(b) $\left\{ \begin{bmatrix} 2c-b \\ 3a-2b \\ 0 \\ a+4b+3c \end{bmatrix} \ a, b, c \text{ are real} \right\}$

b) $\left\{ \begin{bmatrix} 2c-b \\ 3a-2b \\ 0 \\ a+4b+3c \end{bmatrix} \right.$ a, b, c are real $\}$

$2c - b = 0$
$3a - 2b = 0$
$0 = 0$
$a + 4b + 3c = 0$

$$\rightarrow \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \left[\begin{array}{ccc|c} 0 & -1 & 2 & 0 \\ 3 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 4 & 3 & 0 \end{array}\right] \end{array}$$

Cont'd 2★

Cont'd 2b

$$\left[\begin{array}{ccc|c} 0 & -1 & 2 & 0 \\ 3 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 4 & 3 & 0 \end{array}\right] \longrightarrow \begin{bmatrix} 1 & 4 & 3 & 0 \\ 3 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix} 1 & 4 & 3 & 0 \\ 3 & -2 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 - 3 \cdot R_1 \to R_2} \begin{bmatrix} 1 & 4 & 3 & 0 \\ 0 & -14 & -9 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 4 & 3 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & -14 & -9 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-1 \cdot R_2} \begin{bmatrix} 1 & 4 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & -14 & -9 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_3 + 14R_2 \to R_3} \begin{bmatrix} 1 & 4 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & -37 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-\frac{1}{37} \cdot R_3} \begin{bmatrix} 1 & 4 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{\text{skipped some steps}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Only sol is trivial

Not a vector space as the only solution is trivial and a zero vector!

*Not a vector space as the only solution is trivial and a zero vector!