1 Limits of Functions

1. The displacement (in centimeters) of a particle moving back and forth along a straight line is given by the equation of motion $s = 2\sin\pi t + 3\cos\pi t$, where t is measured in seconds.

(a) Find the average velocity during each time period:

i. $[1, 2]$ ii. $[1, 1.1]$

iii. $[1, 1.01]$ iv. $[1, 1.0001]$

(b) Estimate the instantaneous velocity of the particle when $t = 1$.

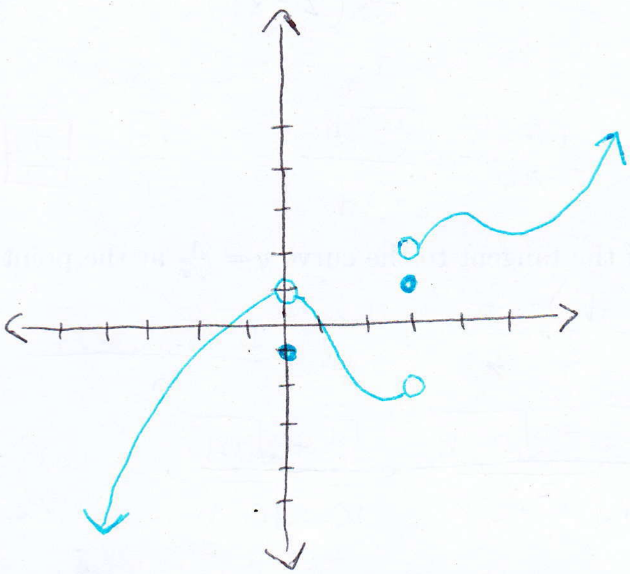
(a) $\frac{s(2)-s(1)}{2-1} = \frac{(2\sin 2\pi + 3\cos 2\pi)-(2\sin\pi + 3\cos\pi)}{1} = \boxed{6}$

$\frac{s(1.0001)-s(1)}{1.0001-1} = \frac{(2\sin 1.0001\pi + 3\cos 1.0001\pi)-(2\sin\pi + 3\cos\pi)}{0.0001} = \boxed{-6.28}$

(b) $s'(1) = -2\pi$

2. Sketch the graph of an example of a function f that satisfies all of the given conditions.

$$\lim_{x\to 0} f(x) = 1,\ \lim_{x\to 3^-} f(x) = -2,\ \lim_{x\to 3^+} f(x) = 2,\ f(0) = -1,\ f(3) = 1$$



3. Evaluate the limit, if it exists.

$$\lim_{t\to 0} \frac{\sqrt{1+t}-\sqrt{1-t}}{t}$$

$$\frac{\sqrt{1+t}-\sqrt{1-t}}{t} = \frac{(\sqrt{1+t}-\sqrt{1-t})(\sqrt{1+t}+\sqrt{1-t})}{t(\sqrt{1+t}+\sqrt{1-t})} = \frac{(1+t)-(1-t)}{t(\sqrt{1+t}+\sqrt{1-t})} = \frac{2}{\sqrt{1+t}+\sqrt{1-t}}$$

$$\lim_{t\to 0} \frac{\sqrt{1+t}-\sqrt{1-t}}{t} = \lim_{t\to 0} \frac{2}{\sqrt{1+t}+\sqrt{1-t}} = \frac{2}{\sqrt{1+0}+\sqrt{1-0}} = \boxed{1}$$

4. Use the Squeeze Theorem to show that $\lim_{x\to 0} \sqrt{x^3+x^2}\sin\frac{\pi}{x} = 0$.

$$-1 \le \sin\theta \le 1$$

$$-\sqrt{x^3+x^2} \le \sqrt{x^3+x^2}\sin\theta \le \sqrt{x^3+x^2}$$

$$\lim_{x\to 0} -\sqrt{x^3+x^2} \le \lim_{x\to 0} \sqrt{x^3+x^2}\sin\theta \le \lim_{x\to 0} \sqrt{x^3+x^2}$$

$$\boxed{0 \le \lim_{x\to 0} \sqrt{x^3+x^2}\sin\frac{\pi}{x} \le 0}$$

5. Find the limit, if it exists.

$$\lim_{x\to -\infty} \frac{\sqrt{1+4x^6}}{2-x^3}$$

$$\lim_{x\to -\infty} \frac{\sqrt{1+4x^6}}{2-x^3} = \lim_{x\to -\infty} \frac{\frac{1}{x^3}\sqrt{1+4x^6}}{\frac{1}{x^3}(2-x^3)} = \lim_{x\to -\infty} \frac{\sqrt{\frac{1}{x^6}(1+4x^6)}}{\frac{1}{x^3}(2-x^3)} = \lim_{x\to -\infty} \frac{\sqrt{\frac{1}{x^6}+4}}{\frac{2}{x^3}-1}$$

As $x \to \pm\infty$, $\frac{1}{x^6} \to 0$ and $\frac{2}{x^3} \to 0$, so

$$\lim_{x\to -\infty} \frac{\sqrt{\frac{1}{x^6}+4}}{\frac{2}{x^3}-1} = \lim_{x\to \infty} -\frac{\sqrt{\frac{1}{x^6}+4}}{\frac{2}{x^3}-1} = -\frac{\sqrt{4}}{-1} = \boxed{2}$$

6. Find the slope of the tangent to the curve $y = \frac{1}{\sqrt{x}}$ at the point $x = a$.

$$m = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{\frac{1}{\sqrt{a+h}} - \frac{1}{\sqrt{a}}}{h} = \lim_{h\to 0} \frac{\frac{\sqrt{a}-\sqrt{a+h}}{\sqrt{a^2+ah}}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{a}-\sqrt{a+h}}{h\sqrt{a^2+ah}} \cdot \frac{\sqrt{a}+\sqrt{a+h}}{\sqrt{a}+\sqrt{a+h}} = \lim_{h\to 0} \frac{-h}{h\sqrt{a^2+ah}(\sqrt{a}+\sqrt{a+h})}$$

$$= \lim_{h\to 0} \frac{-1}{\sqrt{a^2+ah}(\sqrt{a}+\sqrt{a+h})} = \frac{-1}{\sqrt{a^2}(\sqrt{a}+\sqrt{a})} = \boxed{-\frac{1}{a\sqrt{2a}}}$$

2 Derivatives of Functions

1. Differentiate the function given.

$$f(v) = \frac{\sqrt[3]{v} - 2ve^v}{v}$$

$f(v) = \frac{v^{1/3} - 2ve^v}{v} = v^{-2/3} - 2e^v$

$\boxed{f'(v) = -\frac{2}{3}v^{-5/3} - 2e^v}$

2. Find the equation of a normal line ot the parabola $y = x^2 - 1$ at the point $(-1, 0)$.

$m = y'(-1) = -2$

$m_\perp = \frac{1}{2}$

$(y - 0) = \frac{1}{2}(x+1)$

$\boxed{y = \frac{1}{2}x + \frac{1}{2}}$

3. Prove that $\frac{d}{dx}(\cot x) = -\csc^2 x$.

$\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) = \frac{\sin x \cdot \frac{d}{dx}\cos x - \cos x \cdot \frac{d}{dx}\sin x}{\sin^2 x}$

$= \frac{\sin x(-\sin x) - \cos x(\cos x)}{\sin^2 x}$

$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$

$= \frac{-1}{\sin^2 x} = \boxed{-\csc^2 x}$

4. An elastic band is hung on a hook and a mass is hung on the lower end of the band. When the mass is pulled downward, and then released, it vibrates vertically. The equation of motion is $s = 2\cos t + 3\sin t$ where $t \geq 0$, s is given in centimeters, and t in seconds.
(a) Find the velocity and acceleration at time t.
(b) When does the mass pass through the equilibrium point for the first time?
(c) How far from its equilibrium position does the mass travel?
(d) What is the maximum speed?

(a) $v(t) = -2\sin t + 3\cos t$
$a(t) = -2\cos t - 3\sin t$

(b) $s(t) = 0 \to 2\cos t + 3\sin t = 0 \to 2\cos t = -3\sin t$
$\boxed{t \approx 2.55 \text{ seconds}}$

(c) $v(t) = 0 \to -2\sin t + 3\cos t = 0 \to 3\cos t = 2\sin t$
$t \approx 0.98 \to s(0.98) \approx \boxed{3.61 \text{ centimeters}}$

(d) $a(t) = 0 \to -2\cos t - 3\sin t = 0 \to -2\cos t = 3\sin t$
$t \approx 2.55 \to v(2.55) \approx \boxed{3.61 \text{ cm/s}}$

5. Find the first and second derivative of $y = \sqrt{1 - \sec t}$.

$y = \sqrt{1-\sec t} = (1-\sec t)^{1/2}$

$$\boxed{y' = \frac{1}{2}(1-\sec t)^{-1/2}(-\sec t \tan t)}$$

$$\boxed{y'' = -\frac{\sec t}{2\sqrt{1-\sec t}}\left(\tan^2 t + \sec^2 t + \frac{\tan t}{2(1-\sec t)}\right)}$$

6. Find the first and second derivative of $\sin y + \cos x = 1$.

$\cos y \cdot y' - \sin x = 0 \to \boxed{y' = \frac{\sin x}{\cos y}}$

$$y'' = \frac{\cos x \cdot \cos y + \sin y \cdot y' \cdot \sin x}{\cos^2 y} \to \boxed{y'' = \frac{\cos x \cos^2 y + \sin y \sin^2 x}{\cos^3 y}}$$