1. u and v vectors in a vector space V.
H a subspace of V that contains u and v.
Why does H contain Span{u, v}?

Any subspace that contains u and v must also contain all scalar multiples of u and v and must also contain all sums of those scalar multiples. Thus, H must contain all linear combinations of u and v, or Span{u, v}.

3. $T: V \to W$ Prove range of T is subspace of W.

Def: Range of T has T(V) for some v in V.

i) 0 in V? (i) ✓

$T(0) = 0$ and 0 is an element of V so 0 should be an element of T(V).

ii) Let a and b be elements of T(V). There should be x and v such that $u = T(x)$ and $v = T(w)$. (ii) ✓

$u + v = T(x) + T(w) = T(x + w)$, which is an element of T(V)!

iii) is $c\vec{u}$ in W? (iii) ✓

assuming v is an element of T(V) and a is in R, $v = T(x)$ for some x that is an element of V.

$a \cdot v = a \cdot T(x) = T(ax) \to$ should be in W!

The range of T is a subspace of W as the zero vector is an element, $\vec{u} + \vec{v}$ is an element, and some constant times $\vec{v}$ is an element. Therefore, the range of T contains the zero vector, is closed under addition and scalar multiplication and thus T(V) is a subspace of W!

5. Set L.I. in P_3?

$1+2t^3,\ 2+t-3t^2,\ -t+2t^2-t^3$

$(1,0,0,2)$ $\quad(2,1,-3,0)\quad(0,-1,2,-1)$

$\begin{bmatrix}1\\0\\0\\2\end{bmatrix}\quad\begin{bmatrix}2\\1\\-3\\0\end{bmatrix}\quad\begin{bmatrix}0\\-1\\2\\-1\end{bmatrix}$

dim $P_3 = 4$

$$\begin{bmatrix}1&2&0\\0&1&-1\\0&-3&2\\2&0&-1\end{bmatrix}\xrightarrow{R_4-2\cdot R_1\to R_4}\begin{bmatrix}1&2&0\\0&1&-1\\0&-3&2\\0&-4&-1\end{bmatrix}$$

$R_3+3\cdot R_2\to R_3$

$$\begin{bmatrix}1&2&0\\0&1&-1\\0&0&1\\0&-4&-1\end{bmatrix}\xleftarrow{-1\cdot R_3}\begin{bmatrix}1&2&0\\0&1&-1\\0&0&-1\\0&-4&-1\end{bmatrix}$$

skipping some steps

dim $P_3 = 4$

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{bmatrix}$$

$\Rightarrow$ 3 pivot cols

dimen. of Col A is 3! $\Rightarrow$ vectors don't span P_3!

Free var $\Rightarrow$ nontriv sol $\Rightarrow$ not L.I.

Basis for $P_3 = \{1, t, t^2, t^3\}$

Col A 6.8

Cont'd 6a

$\boxed{\text{The polynomials are not linearly independent in } P_3 \text{ as the vectors are not linearly independent to begin with. A free variable implies a non-trivial solution in linear dependence.}}$

8. a) $\left\{ \begin{bmatrix} 3a-4p \\ 2p \\ a+1 \\ 2p+5a \end{bmatrix} : p, a \text{ [illegible]} \right\}$

$$\begin{matrix} 3a-4p+0 \\ 0a+2p+0 \\ 1a+0p+1 \\ 5a+2p+0 \end{matrix} \Rightarrow v_1 = \begin{bmatrix} 3 \\ 0 \\ 1 \\ 5 \end{bmatrix}, v_2 = \begin{bmatrix} -4 \\ 2 \\ 0 \\ 2 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$v_1 \neq 0$, v_2 is not a multiple of v_1 and v_3 is not a linear combination of v_1 and v_2 → linearly independent.

↳ Thus $\{v_1, v_2, v_3\}$ is a basis for the vector space!

$$\boxed{\left\{ \begin{bmatrix} 3 \\ 0 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}}$$

6. b) $\left\{ \begin{bmatrix} 2c-b \\ 3a-2b \\ 0 \\ a+4b+3c \end{bmatrix} a, b, c \text{ [illegible]} \right\}$

$2c-b=0$
$3a-2b=0$
$0=0$
$a+4b+3c=0$

$$\longrightarrow \begin{array}{cccc} a & b & c & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 0 & -1 & 2 & 0 \\ 3 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 4 & 3 & 0 \end{array}\right]$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 4 & 3 & 0 \\ 3 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \end{bmatrix} \xrightarrow{R_4 \leftrightarrow R_3} \begin{bmatrix} 1 & 4 & 3 & 0 \\ 3 & -2 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$-12 \quad -9$

$$\downarrow R_2 - 3 \cdot R_1 \to R_2$$

$$\begin{bmatrix} 1 & 4 & 3 & 0 \\ 0 & -14 & -9 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 4 & 3 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & -14 & -9 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\downarrow -1 \cdot R_2$$

$$\begin{bmatrix} 1 & 4 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & -14 & -9 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_3 + 14 \cdot R_2 \to R_3} \begin{bmatrix} 1 & 4 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & -37 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

[illegible]

~~[illegible]~~

$$\begin{bmatrix} 1 & 4 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & -37 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-\frac{1}{37} R_3} \begin{bmatrix} 1 & 4 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

; Sol keeps same steps

$H = \{(0,0,0)\}$

~~[illegible]~~

Only sol is trivial
only element of set is
zero vector!

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

No basis. Not a vector space as
the only solution is trivial and a
zero vector!

9. $A = \begin{bmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & -1 & 5 \\ 0 & -2 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

a) dim Col A?

Rank A = 2 ⇒ Rank A = dim Col A

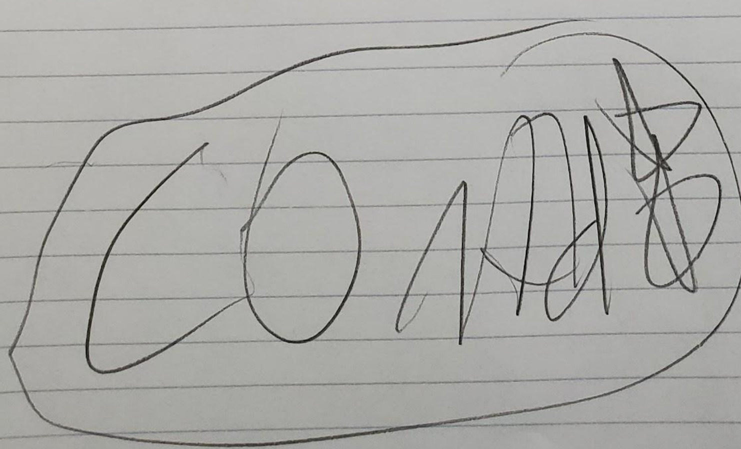
So, dim Col A = 2!

dim Nul A?

2 + dim Nul A = 4

So, dim Nul A = 2!

$\boxed{\text{dim Col } A = 2, \text{ dim Nul } A = 2}$



Cont'd #8

b) basis col A? → Look at pivots!

Basis col A: $\left\{ \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ -6 \end{bmatrix} \right\}$

basis Row A? → Non-zero rows of B!

Basis row A: $\{(1, 0, -1, 5), (0, -2, 5, -6)\}$

Basis Nul A? RREF!

$$\begin{bmatrix} 1 & 0 & -1 & 5 \\ 0 & -2 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_2} \begin{bmatrix} 1 & 0 & -1 & 5 \\ 0 & 1 & -2.5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 - 1x_3 + 5x_4 = 0$ → $x_1 = x_3 - 5x_4$

$1x_2 - 2.5x_3 + 3x_4 = 0$ → $x_2 = 2.5x_3 - 3x_4$

$0 = 0$

x_3 and x_4 are free!

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 - 5x_4 \\ 2.5x_3 - 3x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 2.5 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

Basis for Nul A: $\left\{ \begin{bmatrix} 1 \\ 2.5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

Cont'd 3*

[illegible]

$$\boxed{\begin{aligned} &\text{Basis for col } A: \left\{ \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ -6 \end{bmatrix} \right\} \\ &\text{Basis for row } A: \{(1, 0, -1, 5), (0, -2, 5, -6)\} \\ &\text{Basis for Nul } A: \left\{ \begin{bmatrix} 1 \\ 2.5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\} \end{aligned}}$$

10. If A is a 5×4 matrix, the largest possible dim. of Row A is 4 as in a $m \times n$ matrix, row vectors live in $\mathbb{R}^n$, so $\mathbb{R}^4$ in this case.

If A is a 4×5 matrix, the largest possible dim. of Row A is 5 as in a $m \times n$ matrix, row vectors live in $\mathbb{R}^n$, so $\mathbb{R}^5$ in this case.