Problems:Exponential distribution

1. The time to repair a machine is exponentially distributed random variable with mean 2.
   (a) What is the probability the repair takes more than $2h$.
   (b) What is the probability that the repair takes more than $5h$ given that it takes more than $3h$.

2. The lifetime of a radio is exponentially distributed with mean 5 years. If Ted buys a 7 year-old radio, what is the probability it will be working 3 years later?

3. A doctor has appointments at 9 and $9:30$. The amount of time each appointment lasts is exponential with mean 30 min. What is the expected amount of time after $9:30$ until the second patient has completed his appointment?

4. Copy machine 1 is in use now. Machine 2 will be turned on at time t. Suppose that the machines fail at rate i. What is the probability that machine 2 is the first to fail?

Solutions

1. Let T be the time of completion of repair. we are given $P(T \leq t) = 1 - e^{-t/2}$. So $P(T \geq t) = e^{-t/2}$. (Here t is taken to be in hrs.) So
   (a)probability repair takes more than $2hrs$ equals $e^{-2/2} = e^{-1}$,
   (b)probability repair takes more than $5hrs$ given that it takes more than $3hrs$, is $P(T > 5, T > 3)/P(T > 3) = e^{-5/2}/e^{-3/2} = e^{-1}$.

2. $P[T > 10 \mid T > 7] = P[T > 3] = e^{-3/5}$.

3. Let T_2 denote the sum of the two exponential distributions τ_1, τ_2, with mean $1/\lambda$. T_2 has the probability distribution which has the density function $f_2(t) = \lambda^2 te^{-\lambda t}$. By integrating we get,
$$P(T_2 \leq t) = 1 - e^{-\lambda t} - \lambda te^{-\lambda t}.$$
   So
$$P(T_2 > t) = e^{-\lambda t} + \lambda te^{-\lambda t}.$$
   The expected value of T_2 is obtained by integrating the expression for $P(T_2 > t)$. This yields $2/\lambda = 60$. This expected value $= E[T_2 \mid T_2 < 30] \times P(T_2 < 30) + E[T_2 \mid T_2 > 30] \times P(T_2 > 30)$. However if $E[T_2 \mid T_2 > 30]$ is asked for, then the answer is $\int_{30}^{\infty} P(T_2 > t)dt/P(T_2 > 30)$.

4. Probability that copy machine 2 fails first
$$= Pr\{machine\ 1\ does\ not\ fail\ before\ t,\ machine\ 2\ is\ the\ first\ to\ fail\ after\ t\}$$
$$= P(2\ fails\ first\ after\ t \mid 1\ does\ not\ fail\ before\ T) \times P(1\ does\ not\ fail\ before\ t).$$
   After time t, the machines follow the distribution $P(T_i \leq t + \tau) = 1 - e^{-\lambda_i \tau}, i = 1, 2$.
   The probability that, after time t, $T_2 < T_1$ is $\int_0^{\infty} f_{T_2}(\tau)P(T_1 > \tau)d\tau = \lambda_2/(\lambda_1 + \lambda_2)$.
   The probability that 1 does not fail before t is $e^{-\lambda_1 t}$.
   So Probability that copy machine 2 fails first $= \lambda_2/(\lambda_1 + \lambda_2) \times e^{-\lambda_1 t}$.