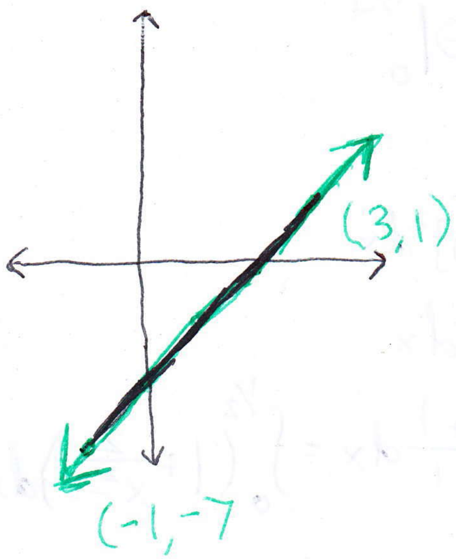
1 Arc Length

For numbers 1-4 of the problems listed below, sketch the graph of the curve on the interval being described. Then find the length of the curve on that given interval using the arc length formula.

1. $y = 2x - 5$ for the interval $-1 \le x \le 3$. (Check your answer by noting the curve is a straight line, and applying the distance formula)



Using distance formula:

$$L = \sqrt{(-1-3)^2 + (-7-1)^2}$$

$$L = \sqrt{16+64}$$

$$\boxed{L = 4\sqrt{5}}$$

Using arc length formula:

$$L = \int_{-1}^{3} \sqrt{1 + (2)^2}\, dx$$

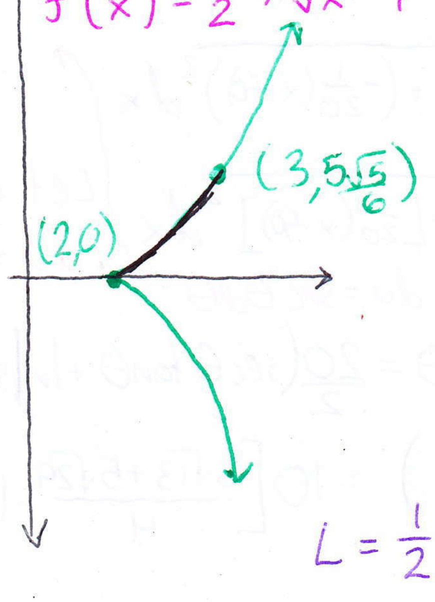
$$L = \int_{-1}^{3} \sqrt{5}\, dx$$

$$L = \sqrt{5}\, x \Big|_{-1}^{3} = 4\sqrt{5}$$

2. $36y^2 = (x^2 - 4)^3$ with $2 \le x \le 3$ and $y \ge 0$.

$$f(x) = \frac{1}{6}(x^2-4)^{3/2}$$

$$f'(x) = \frac{1}{2} x \sqrt{x^2-4}$$



$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

$$L = \int_2^3 \sqrt{1 + \left(\frac{1}{2} x \sqrt{x^2-4}\right)^2}\, dx$$

$$L = \int_2^3 \sqrt{1 + \frac{1}{4} x^2 (x^2-4)}\, dx$$

$$L = \int_2^3 \sqrt{1 + \frac{1}{4} x^4 - x^2}\, dx$$

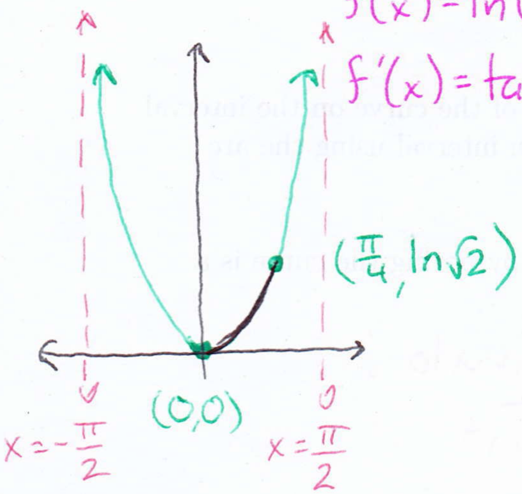
$$L = \int_2^3 \frac{1}{2} \sqrt{(x^2-2)^2}\, dx$$

$$L = \frac{1}{2} \int_2^3 (x^2-2)\, dx = \frac{1}{2}\left(\frac{1}{3} x^3 \Big|_2^3 - 2x \Big|_2^3\right) = \boxed{\frac{19}{6} - 1}$$

3. $y = \ln(\sec\theta)$ on the interval $[0, \pi/4]$.

$f(x) = \ln(\sec\theta)$

$f'(x) = \tan\theta$



$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$$

$$L = \int_0^{\pi/4} \sqrt{1 + \tan^2\theta}\,d\theta$$

$$L = \int_0^{\pi/4} \sec\theta\,d\theta$$

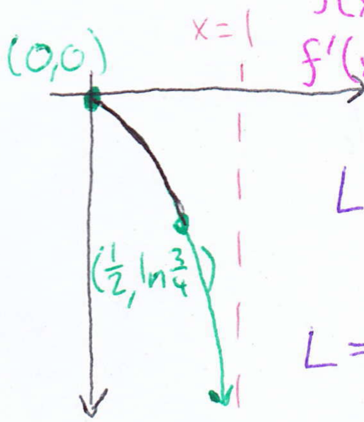
$$L = \ln|\sec\theta + \tan\theta|\Big|_0^{\pi/4}$$

$$L = \boxed{\ln(\sqrt{2} + 1)}$$

4. $f(x) = \ln(1 - x^2)$ for $0 \le x \le \frac{1}{2}$.

$f(x) = \ln(1 - x^2)$

$f'(x) = \frac{2x}{x^2 - 1}$



$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$$

$$L = \int_0^{1/2} \sqrt{1 + \left(\frac{2x}{x^2 - 1}\right)^2}\,dx$$

$$L = \int_0^{1/2} \sqrt{\frac{x^4 + 2x^2 + 1}{x^4 - 2x^2 + 1}}\,dx = \int_0^{1/2} \frac{x^2 + 1}{x^2 - 1}\,dx = \int_0^{1/2}\left(1 + \frac{2}{x^2 - 1}\right)dx$$

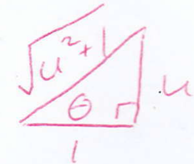
$$L = \int_0^{1/2}\left(1 + \frac{1}{x-1} - \frac{1}{x+1}\right)dx$$

$$L = x\Big|_0^{1/2} + \ln(x-1)\Big|_0^{1/2} - \ln(x+1)\Big|_0^{1/2} = \boxed{\frac{1}{2} - \ln(1.5)}$$ $\times$

5. A steady wind blows a kite due west. The kite's height above ground from horizontal position $x = 0$ to $x = 80$ feet is given by $y = 150 - \frac{1}{40}(x - 50)^2$. Find the total distance traveled by the kite.

$f(x) = 150 - \frac{1}{40}(x - 50)^2$

$f'(x) = -\frac{1}{20}(x - 50)$



$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$$

$$L = \int_0^{80} \sqrt{1 + \left(-\frac{1}{20}(x - 50)\right)^2}\,dx$$

$$L = \int_0^{80} \sqrt{1 + \frac{1}{400}(x - 50)^2}\,dx = \int_0^{80} \sqrt{1 + \left[\frac{1}{20}(x - 50)\right]^2}\,dx$$

Let $u = \frac{1}{20}(x - 50)$

$du = \frac{1}{20}dx$

Let $u = \tan\theta$, $du = \sec^2\theta\,d\theta$

$$L = 20\int_{-5/2}^{3/2} \sqrt{1 + u^2}\,du = 20\int \sec^3\theta\,d\theta = \frac{20}{2}(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta$$

$$L = 10\left(u\sqrt{u^2+1}\Big|_{-5/2}^{3/2} + \ln(\sqrt{u^2+1} + u)\Big|_{-5/2}^{3/2}\right) = 10\left[\frac{3\sqrt{13} + 5\sqrt{29}}{4} + \ln\left(\frac{3+\sqrt{13}}{2}\right) - \ln\left(\frac{\sqrt{29}}{2}\right.\right.$$

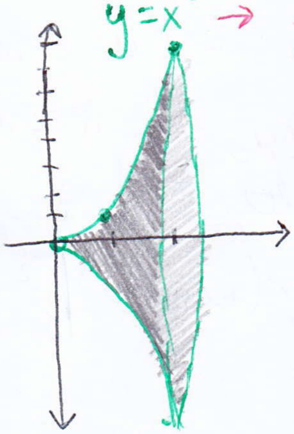
$$\boxed{L \approx 122.776\text{ft}}$$

2 Areas of a Surface of Revolution

For numbers 1-4 of the problems listed below, sketch a diagram of the solid figure being described. Then find the surface area of the solid generated.

1. The curve $y = x^3$ on the interval $0 \le x \le 2$ rotated about x-axis.

$y = x^3 \rightarrow \frac{dy}{dx} = 3x^2$



$$S = \int_0^2 2\pi x^3 \sqrt{1 + (3x^2)^2}\, dx$$

$$S = \int_0^2 2\pi x^3 \sqrt{1 + 9x^4}\, dx$$

Let $u = 1 + 9x^4$

$du = 36x^3 dx$

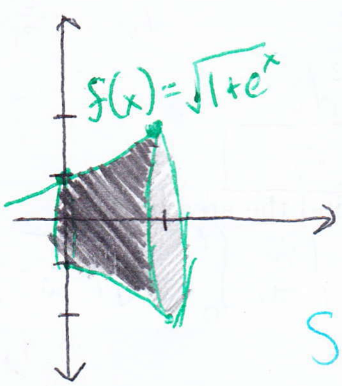
$$S = \frac{2\pi}{36} \int_0^2 36x^3 \sqrt{1 + 9x^4}\, dx = \frac{\pi}{18} \int_1^{145} u^{1/2}\, du$$

$$S = \frac{\pi}{18}\left(\frac{2}{3} u^{3/2} \Big|_1^{145}\right) = \frac{\pi}{27}\left(145\sqrt{145} - 1\right)$$

$$\boxed{S \approx 203.04}$$

2. The curve $f(x) = \sqrt{1 + e^x}, 0 \le x \le 1$ rotated about the x-axis.

$$f'(x) = \frac{e^x}{2\sqrt{1 + e^x}}$$



$$S = \int_0^1 2\pi \sqrt{1 + e^x} \cdot \sqrt{1 + \left(\frac{e^x}{2\sqrt{1 + e^x}}\right)^2}\, dx$$

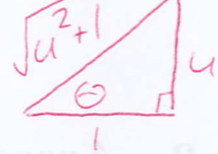
$$S = 2\pi \int_0^1 \sqrt{(1 + e^x)\left(1 + \frac{e^{2x}}{4(1 + e^x)}\right)}\, dx$$

$$S = 2\pi \int_0^1 \sqrt{1 + e^x + \frac{1}{4} e^{2x}}\, dx = 2\pi \int_0^1 \frac{1}{2} \sqrt{e^{2x} + 4e^x + 4}\, dx$$

$$S = \pi \int_0^1 \sqrt{(e^x + 2)^2}\, dx = \pi \int_0^1 (e^x + 2)\, dx$$

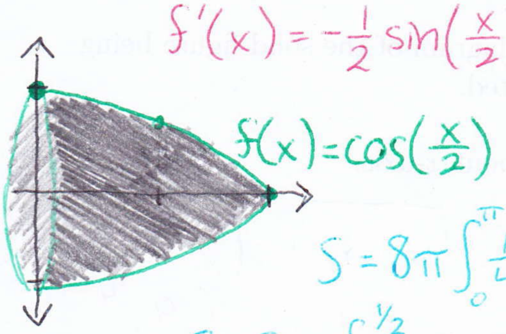
$$S = \pi(e^x + 2x)\Big|_0^1 = \pi(e - 1 + 2) = \pi(e + 1)$$

$$\boxed{S \approx 11.68}$$



3. The curve $y = \cos(\frac{x}{2})$ on the interval $[0, \pi]$ rotated about the x-axis.

$f'(x) = -\frac{1}{2}\sin\left(\frac{x}{2}\right)$



$$S = \int_0^{\pi} 2\pi \cos\left(\frac{x}{2}\right)\sqrt{1+\left(-\frac{\sin(\frac{x}{2})}{2}\right)^2}\,dx$$

$$S = 2\pi \int_0^{\pi} \cos\left(\frac{x}{2}\right)\sqrt{1+\frac{1}{4}\sin^2\left(\frac{x}{2}\right)}\,dx$$

$$S = 8\pi \int_0^{\pi} \frac{1}{4}\cos\left(\frac{x}{2}\right)\sqrt{1+\frac{1}{4}\sin^2\left(\frac{x}{2}\right)}\,dx$$

Let $u = \frac{1}{2}\sin\left(\frac{x}{2}\right)$

$du = \frac{1}{4}\cos\left(\frac{x}{2}\right)dx$

$$S = 8\pi \int_0^{1/2} \sqrt{1+u^2}\,du$$

Let $u = \tan\theta$, $du = \sec^2\theta\,d\theta$

$$= 8\pi \int \sqrt{1+\tan^2\theta}\cdot\sec^2\theta\,d\theta = 8\pi\int \sec^3\theta\,d\theta$$

$$S = 8\pi\left[\frac{1}{2}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right)\right] = 4\pi\left(u\sqrt{u^2+1}\,\Big|_0^{1/2} + \ln\left(\sqrt{u^2+1}+u\right)\Big|_0^{1/2}\right)$$

$$S = 4\pi\left[\frac{1}{2}\cdot\frac{\sqrt{5}}{2} + \ln\left(\frac{\sqrt{5}}{4}+\frac{1}{2}\right)\right] \approx \boxed{7.745}$$

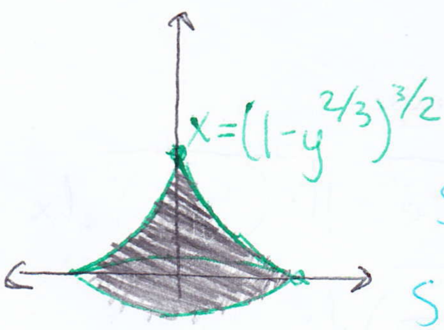
4. $x^{2/3} + y^{2/3} = 1$ rotated about the y-axis on the interval $0 \le y \le 1$.

$x = (1-y^{2/3})^{3/2}$

$\frac{dx}{dy} = -\frac{(1-y^{2/3})^{1/2}}{y^{1/3}}$

$$S = \int_0^1 2\pi(1-y^{2/3})^{3/2}\left[1+\left(-\frac{(1-y^{2/3})^{1/2}}{y^{1/3}}\right)^2\right]^{1/2}dy$$

$$S = 2\pi\int_0^1 (1-y^{2/3})^{3/2}\left(1+\frac{1-y^{2/3}}{y^{2/3}}\right)^{1/2}dy$$



$$S = 2\pi\int_0^1 (1-y^{2/3})^{3/2}\left(\frac{1}{y^{2/3}}\right)^{1/2}dy$$

Let $u = 1-y^{2/3}$

$du = -\frac{2}{3}y^{-1/3}dy$

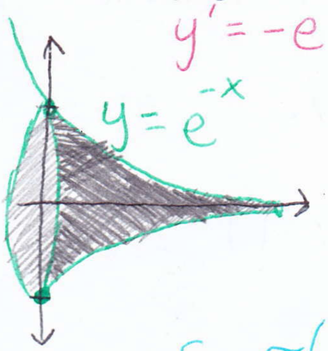
$$S = 2\pi\int_0^1 (1-y^{2/3})^{3/2}\left(\frac{1}{y^{1/3}}\right)dy$$

$$S = -3\pi\int_1^0 u^{3/2}\,du = 3\pi\int_0^1 u^{3/2}\,du$$

$$S = 3\pi\left(\frac{2}{5}u^{5/2}\Big|_0^1\right) = \frac{6\pi}{5} \approx \boxed{3.77}$$

5. If the infinite curve $y = e^{-x}$ for $x \ge 0$ is rotated about the x-axis, find the area of the resulting figure.

$y' = -e^{-x}$



$$S = \int_0^{\infty} 2\pi e^{-x}\sqrt{1+(-e^{-x})^2}\,dx = 2\pi\int_0^{\infty} e^{-x}\sqrt{1+e^{-2x}}\,dx$$

Let $u = e^{-x}$

$du = -e^{-x}dx$

As $x \to \infty$, $u \to 0$

As $x \to 0$, $u \to 1$

$$S = -2\pi\int_1^0 \sqrt{1+u^2}\,du \quad \text{(see \#3)}$$

$$S = 2\pi\int_0^1 \sqrt{1+u^2}\,du = 2\pi\left[\frac{1}{2}\left(u\sqrt{u^2+1} + \ln\left|\sqrt{u^2+1}+u\right|\right)\right]_0^1$$

$$S = \pi\left(\sqrt{2} + \ln(\sqrt{2}+1)\right) \approx \boxed{7.212}$$