Assignment 2

Instructions: Please answer the following legibly, logically, and **show all work** on a separate sheet of paper. No credit will be given for unjustified or unclear work. Please clearly label every problem and work them in order. When you are finished, please scan your work (or take pictures) and submit via the Assignments link on Canvas if you are an online student. Otherwise, you can turn in a separate sheet of paper in-person.

1. Determine if $\mathbf{b}$ is a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$.
   (Use row notation to show your steps)

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{a}_2 = \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix} \quad \mathbf{a}_3 = \begin{bmatrix} -6 \\ 7 \\ 5 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 11 \\ -5 \\ 9 \end{bmatrix}$$

2. Determine if $\mathbf{b}$ is a linear combination of the vectors formed from the columns of the matrix A.

$$A = \begin{bmatrix} 1 & -4 & 2 \\ 0 & 3 & 5 \\ -2 & 8 & -4 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 3 \\ -7 \\ -3 \end{bmatrix}$$

3. For what value(s) of h is $\mathbf{b}$ in the plane generated by $\mathbf{a}_1$ and $\mathbf{a}_2$?

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \qquad \mathbf{a}_2 = \begin{bmatrix} -5 \\ -8 \\ 2 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 3 \\ -5 \\ h \end{bmatrix}$$

4. Let $A = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 3 & -2 \\ -2 & 6 & 3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ -4 \end{bmatrix}$.

   Denote the columns of A by $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, and let $W = \text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3.\}$

   (a) Is $\mathbf{b}$ in $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$?
   (b) How many vectors are in $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$?
   (c) Is $\mathbf{b}$ in W? (Show your work)
   (d) How many vectors are in W?
   (e) Show that $\mathbf{a}_1$ is in W. [*Hint* Row operations are unnecessary.]

5. Use the definition of $A\mathbf{x}$ to write the following matrix equation as a vector equation.

$$\begin{bmatrix} 1 & 2 & -3 & 1 \\ -2 & -3 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

6. Use the definition of $A\mathbf{x}$ to write the following vector equation as a matrix equation. Then write this vector equation as a system of equations.

   Note: *A matrix equation is not the same as an augmented matrix.*

$$z_1 \begin{bmatrix} 2 \\ -4 \end{bmatrix} + z_2 \begin{bmatrix} -1 \\ 5 \end{bmatrix} + z_3 \begin{bmatrix} -4 \\ 3 \end{bmatrix} + z_4 \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$$

7. Given A and $\mathbf{b}$, write the augmented matrix for the linear system that corresponds to the matrix equation $A\mathbf{x} = \mathbf{b}$. Then solve the system and write the solution as a vector.

$$A = \begin{bmatrix} 1 & 3 & -4 \\ 1 & 5 & 2 \\ -3 & -7 & 6 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} -2 \\ 4 \\ 12 \end{bmatrix}$$

8. Let $A = \begin{bmatrix} 1 & -2 & -1 \\ -2 & 2 & 0 \\ 4 & -1 & 3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

   (a) Show that the equation $A\mathbf{x} = \mathbf{b}$ does not have a solution for all possible $\mathbf{b}$. (Meaning there is not a solution for every possible combination of b_1, b_2, b_3.)

   (b) Based on (a), do the columns of A span $\mathbb{R}^3$? In other words, can every vector in $\mathbb{R}^3$ be written as a linear combination of the columns of A? Why or why not?

   (c) Describe the set of all $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ *does* have a solution.

9. For the following statements determine whether they or true or false. If false, then explain why it is false. If true, then point to a Theorem or definition that supports the statement.

   (a) The solution set of the linear system whose augmented matrix is $\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{b} \end{bmatrix}$ is the same as the solution set for $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}$.

   (b) Asking whether the linear system corresponding to an augmented matrix $\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{b} \end{bmatrix}$ has a solution amounts to asking whether $\mathbf{b}$ is in $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$.

   (c) The equation $A\mathbf{x} = \mathbf{b}$ is consistent if the augmented matrix $\begin{bmatrix} A & \mathbf{b} \end{bmatrix}$ has a pivot position in every row.

   (d) If the columns of an $m \times n$ matrix A span $\mathbb{R}^m$, then the equation $A\mathbf{x} = \mathbf{b}$ is consistent for each $\mathbf{b}$ in $\mathbb{R}^m$.

   (e) If the equation $A\mathbf{x} = \mathbf{b}$ is consistent, then $\mathbf{b}$ is in the set spanned by the columns of A.

   (f) Any linear combination of vectors can always be written in the form $A\mathbf{x}$ for a suitable matrix A and vector $\mathbf{x}$.