1. Let $A = PDP^{-1}$ and compute A^4. $P = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$.

1. $A = PDP^{-1}$, compute A^4. $P = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

For [illegible]:

$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ $\quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$P^{-1} = \frac{1}{1\cdot 3 - 2\cdot 2}\begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix} \Rightarrow \frac{1}{-1}\begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix} \to \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$

$3 - 4$

$A^k = PD^kP^{-1} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1^k & 0 \\ 0 & 3^k \end{bmatrix}\begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$

$A^4 = PD^4P = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1^4 & 0 \\ 0 & 3^4 \end{bmatrix}\begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$

$\left(\begin{bmatrix} 1\cdot 1 + 2\cdot 0 & 1\cdot 0 + 2\cdot 81 \\ 2\cdot 1 + 3\cdot 0 & 2\cdot 0 + 3\cdot 81 \end{bmatrix}\right)\begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$

$\begin{bmatrix} 1 & 162 \\ 2 & 243 \end{bmatrix}\begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$ $\quad \begin{bmatrix} A & B \\ C & D \end{bmatrix}\cdot\begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} AE+BG & AF+BH \\ CE+DG & CF+DH \end{bmatrix}$

$\begin{bmatrix} 1\cdot -3 + 162\cdot 2 & 1\cdot 2 + 162\cdot -1 \\ 2\cdot -3 + 243\cdot 2 & 2\cdot 2 + 243\cdot -1 \end{bmatrix}$ $\quad \begin{matrix} I & J \\ K & L \end{matrix}$

$\boxed{\begin{bmatrix} 321 & -160 \\ 480 & -239 \end{bmatrix}}$

2. Diagonalize the following matrices, if possible. The real eigenvalues for (*b*) are given.

(a) $\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$

2. ① Find eigenvalues: $\det(A-\lambda I)=0$

a) $\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1-\lambda & 3 \\ 4 & 2-\lambda \end{bmatrix}$ $(1-\lambda)(2-\lambda) - 4(3)$

add? mult?

$\lambda^2 - 3\lambda - 10 \leftarrow 2-\lambda-2\lambda+\lambda^2-12$

$(\lambda+2)(\lambda-5) \rightarrow$ Eigenvalues: $-2, 5$

$\lambda+2=0 \rightarrow \lambda=-2$

$\lambda-5=0 \rightarrow \lambda=5$

② Find eigenvectors

Basis for -2: $A-(-2)I$ $\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xleftarrow{\text{takes to}} \begin{bmatrix} 3 & 3 & 0 \\ 4 & 4 & 0 \end{bmatrix} \leftarrow \begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix}$

$x_1 + x_2 = 0$ $\quad x_1 = -x_2$

x_2 is free $\quad x_2$ is free

Basis for -2: $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Cont'd Next

cont'd

Basis for 5? $A - 5I$ $\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$

$\begin{bmatrix} -4 & 3 \\ 4 & -3 \end{bmatrix}$

$\begin{bmatrix} -4 & 3 & 0 \\ 4 & -3 & 0 \end{bmatrix}$

$\downarrow R_2 + R_1 \to R_2$

$\begin{bmatrix} -4 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{-1 \cdot R_1} \begin{bmatrix} 4 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\downarrow \frac{1}{4} R_1$

$\begin{bmatrix} 1 & -3/4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$x_1 - 3/4 x_2 = 0$
x_2 is free

$\downarrow$

$x_1 = 3/4\, x_2$ $\quad x_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3/4\, x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 3/4 \\ 1 \end{bmatrix}$
x_2 is free

Basis for 5 could be $\begin{bmatrix} 3/4 \\ 1 \end{bmatrix}$, but we'll use $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ to avoid fractions

③ Construct P from vectors

$\begin{bmatrix} -1 & 3 \\ 1 & 4 \end{bmatrix}$

cont'd

Cont'd

④ Construct D

Our eigenvalues were -2 and 5, so we use these eigenvalues in the order chosen for the cols of P!

$$\begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix}$$

Because we were able to find 2 L.I. eigenvectors of A for a 2x2 matrix, we can say this is diagonalizable!

To see if P and D works, check:

$$\begin{array}{cc} -1 \quad 3 & 3 \quad 12 \end{array}$$

$$AP = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1\cdot -1 + 3\cdot 1 & 1\cdot 3 + 3\cdot 4 \\ 4\cdot -1 + 2\cdot 1 & 4\cdot 3 + 2\cdot 4 \end{bmatrix}$$

$$\begin{array}{cc} -4 \quad 2 & 12 \quad 8 \end{array}$$

$$\downarrow$$

$$\begin{bmatrix} 2 & 15 \\ -2 & 20 \end{bmatrix}$$

$$\begin{array}{cc} 2 \quad 0 & 0 \quad 15 \end{array}$$

$$PD = \begin{bmatrix} -1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} -1\cdot -2 + 3\cdot 0 & -1\cdot 0 + 3\cdot 5 \\ 1\cdot -2 + 4\cdot 0 & 1\cdot 0 + 4\cdot 5 \end{bmatrix}$$

$$\begin{array}{cc} -2 \quad 0 & 0 \quad 20 \end{array}$$

$$\downarrow$$

AP = PD? ✓

$$\begin{bmatrix} 2 & 15 \\ -2 & 20 \end{bmatrix}$$

So,

$$\boxed{P = \begin{bmatrix} -1 & 3 \\ 1 & 4 \end{bmatrix}, \quad D = \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix}}$$

Cont'd

(b) $\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$, and $\lambda = 2, 5$

cont'd

(b) $\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ and $\lambda = 2, 5$

① Find eigenvalues: → We can skip since they are given!

② Find eigenvectors

Basis for 2? $A-(2)I$ $\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

reduces to

$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Basis for 2: $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$1x_1 + 1x_2 + 1x_3 = 0$ ⇒ $x_1 = -x_2 - x_3$
x_2 is free x_2 is free
x_3 is free x_3 is free

$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

cont'd

Cont'd

Basis for 5? $A-(5)I$ $\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{2}R_1} \begin{bmatrix} 1 & -1/2 & -1/2 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{bmatrix} \xrightarrow{R_2 - R_1 \to R_2} \begin{bmatrix} 1 & -1/2 & -1/2 & 0 \\ 0 & -3/2 & 3/2 & 0 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

$$\xrightarrow{R_3 - R_1 \to R_3} \begin{bmatrix} 1 & -1/2 & -1/2 & 0 \\ 0 & -3/2 & 3/2 & 0 \\ 0 & 3/2 & -3/2 & 0 \end{bmatrix} \xrightarrow{-\frac{2}{3}\cdot R_2} \begin{bmatrix} 1 & -1/2 & -1/2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 3/2 & -3/2 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 + \frac{1}{2}R_2 \to R_1} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 3/2 & -3/2 & 0 \end{bmatrix} \xrightarrow{R_3 - \frac{3}{2}\cdot R_2 \to R_3} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 - 1x_3 = 0 \rightarrow x_1 = x_3$

$x_2 - 1x_3 = 0 \rightarrow x_2 = x_3$

x_3 is free $\quad x_3$ is free

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Basis for 5: $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Cont'd 3*

Cont 13 (b)

③ Construct P

$$\begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

④ Construct D

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

So, $\boxed{P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix},\ D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}}$

3. Show that if A is both diagonalizable and invertible, then so is A^{-1}.

A square matrix A is said to be **diagonalizable** if A is similar to a diagonal matrix, that is, if $A = PDP^{-1}$ for some invertible matrix P and some diagonal matrix D. The next theorem gives a characterization of diagonalizable matrices and tells how to construct a suitable factorization.

THEOREM 5 **The Diagonalization Theorem**

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

In fact, $A = PDP^{-1}$, with D a diagonal matrix, if and only if the columns of P are n linearly independent eigenvectors of A. In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.

EXAMPLE 5 Let $A = \begin{bmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & 3 & 4 \end{bmatrix}$. The eigenvalues of A are 3, 0, and 2. The eigenvalues of B are 4 and 1. ■

What does it mean for a matrix A to have an eigenvalue of 0, such as in Example 5? This happens if and only if the equation

$$A\mathbf{x} = 0\mathbf{x} \tag{4}$$

has a nontrivial solution. But (4) is equivalent to $A\mathbf{x} = \mathbf{0}$, which has a nontrivial solution if and only if A is not invertible. Thus 0 *is an eigenvalue of A if and only if A is not invertible*. This fact will be added to the Invertible Matrix Theorem in Section 5.2.

The following important theorem will be needed later. Its proof illustrates a typical calculation with eigenvectors.

THEOREM 2

If $\mathbf{v}_1, \dots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \dots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ is linearly independent.

THEOREM

The Invertible Matrix Theorem (continued)

Let A be an $n \times n$ matrix. Then A is invertible if and only if:

s. The number 0 is *not* an eigenvalue of A.

t. The determinant of A is *not* zero.

THEOREM 1

The eigenvalues of a triangular matrix are the entries on its main diagonal.

3. Show if A is both diagonalizable and invertible, then so is A^{-1}.

A is diagonalizable, so $A = PDP^{-1}$ for some invertible matrix P and diagonal matrix D. A is invertible, so 0 is not an eigenvalue of A. Since 0 is not an eigenvalue, the diagonal entries of D won't be zero, and D is invertible as well. Otherwise, D having an eigenvalue of 0 would be a contradiction to the fact that A is invertible.

Cont'd

(cont'd)

So, we have that:

$A^{-1} = (PDP^{-1})^{-1} = P^{-1}D^{-1}(P^{-1})^{-1} = P^{-1}D^{-1}P$

Seeing how D^{-1} is a diagonal matrix, A^{-1} would be diagonalizable as well.

*I know my handwriting is bad, so I am going to type this out for your convenience. So, using the excerpts above, I answered the following:

A is diagonalizable, so A = PDP^{-1} for some invertible matrix P and diagonal matrix D. A is invertible, so 0 is not an eigenvalue of A. Since 0 is not an eigenvalue, the diagonal entries of D won't be 0, and D is invertible as well. Otherwise, D having an eigenvalue of 0 could be a contradiction to the fact that A is invertible. So, we have that:

$A^{-1} = (PDP^{-1})^{-1} = P^{-1}D^{-1}(P^{-1})^{-1} = P^{-1}D^{-1}P$

Seeing how $D^{-1}$is a diagonal matrix, A^{-1} would be diagonalizable as well.

4. Find a unit vector in the direction of the given vector: $\begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix}$.

4. Find unit vec. in direction of given vector!

$$\begin{bmatrix}-6\\4\\-3\end{bmatrix}$$

① Compute length of vector!

$$\|v\|^2 = v \cdot v = (-6)^2 + (4)^2 + (-3)^2 = 61$$

$$\|v\| = \sqrt{61}$$

② Mult v by $1/\|v\|$

$$u = \frac{1}{\|v\|} v = \frac{1}{\|\sqrt{61}\|}\begin{bmatrix}-6\\4\\-3\end{bmatrix} = \boxed{\begin{bmatrix}-6/\sqrt{61}\\4/\sqrt{61}\\-3/\sqrt{61}\end{bmatrix}}$$

To check that $\|u\|=1$, show that $\|u\|^2=1$

$$\|u\|^2 \; u \cdot u = \left(\frac{-6}{\sqrt{61}}\right)^2 + \left(\frac{4}{\sqrt{61}}\right)^2 + \left(\frac{-3}{\sqrt{61}}\right)^2$$

$$= \frac{36}{61} + \frac{16}{61} + \frac{9}{61} = \frac{61}{61} = 1$$

5. Find the distance between $\mathbf{u} = \begin{bmatrix}0\\-5\\2\end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix}-4\\-1\\8\end{bmatrix}$.

5. Find dist. between

$$u = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix} \text{ and } v = \begin{bmatrix} -4 \\ -1 \\ 8 \end{bmatrix}$$

$$\text{dist}(u,v) = \|u - v\| = \sqrt{(u-v)\cdot(u-v)}$$

$$= \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2}$$

$$u - v = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix} - \begin{bmatrix} -4 \\ -1 \\ 8 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ -6 \end{bmatrix}$$

$$= \sqrt{(4)^2 + (-4)^2 + (-6)^2}$$

$$= \sqrt{16 + 16 + 36}$$

$$= \sqrt{68}$$

$$68/4 = 17$$

$$\boxed{2\sqrt{17}}$$

6. Suppose a vector **y** is orthogonal to vectors **u** and **v**. Show that **y** is orthogonal to the vector $\mathbf{u} + \mathbf{v}$.

DEFINITION Two vectors **u** and **v** in $\mathbb{R}^n$ are **orthogonal** (to each other) if $\mathbf{u} \cdot \mathbf{v} = 0$.

The Inner Product

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, then we regard $\mathbf{u}$ and $\mathbf{v}$ as $n \times 1$ matrices. The transpose $\mathbf{u}^T$ is a $1 \times n$ matrix, and the matrix product $\mathbf{u}^T\mathbf{v}$ is a 1×1 matrix, which we write as a single real number (a scalar) without brackets. The number $\mathbf{u}^T\mathbf{v}$ is called the **inner product** of $\mathbf{u}$ and $\mathbf{v}$, and often it is written as $\mathbf{u}\cdot\mathbf{v}$. This inner product, mentioned in the exercises for Section 2.1, is also referred to as a **dot product**. If

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

then the inner product of $\mathbf{u}$ and $\mathbf{v}$ is

$$\begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

THEOREM 1

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^n$, and let c be a scalar. Then

a. $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$

b. $(\mathbf{u} + \mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot\mathbf{w} + \mathbf{v}\cdot\mathbf{w}$

c. $(c\mathbf{u})\cdot\mathbf{v} = c(\mathbf{u}\cdot\mathbf{v}) = \mathbf{u}\cdot(c\mathbf{v})$

d. $\mathbf{u}\cdot\mathbf{u} \geq 0$, and $\mathbf{u}\cdot\mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$

6. Suppose vector y is orthogonal to vectors u and v. Show y is orthogonal to the vector u+v.

If y is orthogonal to u and v, then:

$y \cdot u = 0$ and $y \cdot v = 0$

From a given property from Thm. 1, we get that:

$$y \cdot (u+v) = \overset{0}{y \cdot u} + \overset{0}{y \cdot v}$$
$$= 0 + 0 = 0$$

Thus, y is orthogonal to the vector u+v!

*Using the excerpts above, for the answer I wrote:

If y is orthogonal to u and v, then:

y*u = 0 and y*v = 0

From a given property from Theorem 1, we get that:

y*(u+v) = y*u + y*v

= 0 + 0 = 0

Thus, y is orthogonal to the vector u + y!

7. Let $W = \text{Span}\{\mathbf{v}_1, ..., \mathbf{v}_p\}$. Show that if $\mathbf{x}$ is orthogonal to each $\mathbf{v}_j$, for $1 \leq j \leq p$, then $\mathbf{x}$ is orthogonal to every vector in W.

DEFINITION

If $\mathbf{v}_1, \dots, \mathbf{v}_p$ are in $\mathbb{R}^n$, then the set of all linear combinations of $\mathbf{v}_1, \dots, \mathbf{v}_p$ is denoted by Span$\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ and is called the **subset of $\mathbb{R}^n$ spanned** (or **generated**) **by** $\mathbf{v}_1, \dots, \mathbf{v}_p$. That is, Span$\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is the collection of all vectors that can be written in the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

with $c_1, \dots, c_p$ scalars.

Asking whether a vector $\mathbf{b}$ is in Span$\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ amounts to asking whether the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{b}$$

has a solution, or, equivalently, asking whether the linear system with augmented matrix $[\,\mathbf{v}_1 \ \cdots \ \mathbf{v}_p \ \ \mathbf{b}\,]$ has a solution.

Note that Span$\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ contains every scalar multiple of $\mathbf{v}_1$ (for example), since $c\mathbf{v}_1 = c\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_p$. In particular, the zero vector must be in Span$\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$.

THEOREM 1

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^n$, and let c be a scalar. Then

a. $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$

b. $(\mathbf{u}+\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot\mathbf{w} + \mathbf{v}\cdot\mathbf{w}$

c. $(c\mathbf{u})\cdot\mathbf{v} = c(\mathbf{u}\cdot\mathbf{v}) = \mathbf{u}\cdot(c\mathbf{v})$

d. $\mathbf{u}\cdot\mathbf{u} \geq 0$, and $\mathbf{u}\cdot\mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$

Properties (b) and (c) can be combined several times to produce the following useful rule:

$$(c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p)\cdot\mathbf{w} = c_1(\mathbf{u}_1\cdot\mathbf{w}) + \cdots + c_p(\mathbf{u}_p\cdot\mathbf{w})$$

DEFINITION

Two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are **orthogonal** (to each other) if $\mathbf{u}\cdot\mathbf{v} = 0$.

7. Let $W = \text{Span}\{v_1, \dots, v_p\}$. Show that if x is orthogonal to each v_i, for $1 \le i \le p$, then x is orthogonal to any vector in W.

For any vector w in the set W, there are scalars $c_1, \dots, c_p$, such that

$w = c_1 v_1 + \dots c_p v_p$

If x is orthogonal to each v_i, then by theorem b we get that:

cont'd

(cont'd)

$W = C_1V_1 + \dots C_pV_p$

$W \cdot X = (C_1V_1 + \dots C_pV_p)X$

$= (C_1V_1)X + \dots (C_pV_p)X$

$= C_1(V_1 \cdot X) + \dots C_p(V_p \cdot X)$

$= C_1(0) + \dots C_p(0)$

$= 0$

So, X is orthogonal to every vector in W!

*Using the excerpts above, for the answer I wrote:

For any vector w in the set W, there are scalars c1,...cp such that:

w $= c_1 v_1 + \dots c_p v_p$

If x is orthogonal to each v_j, then by theorem 1, we get that:

w $= c_1 v_1 + \dots c_p v_p$

w*x $= (c_1 v_1 + \dots c_p v_p)x$

$= c_1(v_1 * x) + \dots c_p(v_p * x)$

$= c_1(0) + \dots c_p(0)$

=0

So, x is orthogonal to every vector in W!

8. Show that $\{\mathbf{u}_1, \mathbf{u}_2\}$, $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal basis for $\mathbb{R}^2$ or $\mathbb{R}^3$, respectively. Then express $\mathbf{x}$ as a linear combination of the $\mathbf{u}$'s.

(a) $\mathbf{u}_1 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} 9 \\ -7 \end{bmatrix}$

THEOREM 4

If $S = \{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then S is linearly independent and hence is a basis for the subspace spanned by S.

PROOF If $\mathbf{0} = c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p$ for some scalars $c_1, \dots, c_p$, then

$$\begin{aligned} 0 = \mathbf{0}\cdot\mathbf{u}_1 &= (c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_p\mathbf{u}_p)\cdot\mathbf{u}_1 \\ &= (c_1\mathbf{u}_1)\cdot\mathbf{u}_1 + (c_2\mathbf{u}_2)\cdot\mathbf{u}_1 + \cdots + (c_p\mathbf{u}_p)\cdot\mathbf{u}_1 \\ &= c_1(\mathbf{u}_1\cdot\mathbf{u}_1) + c_2(\mathbf{u}_2\cdot\mathbf{u}_1) + \cdots + c_p(\mathbf{u}_p\cdot\mathbf{u}_1) \\ &= c_1(\mathbf{u}_1\cdot\mathbf{u}_1) \end{aligned}$$

because $\mathbf{u}_1$ is orthogonal to $\mathbf{u}_2, \dots, \mathbf{u}_p$. Since $\mathbf{u}_1$ is nonzero, $\mathbf{u}_1\cdot\mathbf{u}_1$ is not zero and so $c_1 = 0$. Similarly, $c_2, \dots, c_p$ must be zero. Thus S is linearly independent. ■

The next theorem suggests why an orthogonal basis is much nicer than other bases. The weights in a linear combination can be computed easily.

THEOREM 5

Let $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$. For each $\mathbf{y}$ in W, the weights in the linear combination

$$\mathbf{y} = c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p$$

are given by

$$c_j = \frac{\mathbf{y}\cdot\mathbf{u}_j}{\mathbf{u}_j\cdot\mathbf{u}_j} \qquad (j = 1, \dots, p)$$

PROOF As in the preceding proof, the orthogonality of $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ shows that

$$\mathbf{y}\cdot\mathbf{u}_1 = (c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_p\mathbf{u}_p)\cdot\mathbf{u}_1 = c_1(\mathbf{u}_1\cdot\mathbf{u}_1)$$

Since $\mathbf{u}_1\cdot\mathbf{u}_1$ is not zero, the equation above can be solved for c_1. To find c_j for $j = 2, \dots, p$, compute $\mathbf{y}\cdot\mathbf{u}_j$ and solve for c_j. ■

8. a) $u_1 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$, $u_2 = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$, and $x = \begin{bmatrix} 9 \\ -7 \end{bmatrix}$

Consider the only possible pair of distinct vectors, namely $\{u_1, u_2\}$.

$u_1 \cdot u_2 = 2(6) + (-3)(4) = 12 - 12 = 0$

$u_1 \cdot u_2 = 0$, so orthogonal!

Since the vectors are nonzero, they are linearly independent from Thm 4!

Vectors form a basis for $\mathbb{R}^2$ → $\{u_1, u_2\}$ is orthogonal basis for $\mathbb{R}^2$!

By Thm 5, → show x is linear comb of u's

$y = c_1 u_1 + \dots + c_p u_p$

$c_j = \frac{y \cdot u_j}{u_j \cdot u_j}$ $(j = 1, \dots, p)$

$x = \frac{x \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{x \cdot u_2}{u_2 \cdot u_2} u_2$

$x \cdot u_1 = 9(2) + (-7)(-3) = 39$

$x \cdot u_2 = 9(6) + (-7)(4) = 26$

$u_1 \cdot u_1 = 2(2) + (-3)(-3) = 13$

$u_2 \cdot u_2 = 6(6) + (4)(4) = 52$

$= \frac{39}{13} \cdot u_1 + \frac{26}{52} u_2$

$x = 3 \cdot u_1 + \frac{1}{2} u_2$

Cont'd

~~cont'd~~

$$\boxed{y = 3\begin{bmatrix} 2 \\ -3 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 6 \\ 4 \end{bmatrix}}$$

(b) $\mathbf{u}_1 = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$

b) $u_1 = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, u_3 = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$ and $y = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$

$$u_1 \cdot u_2 = \overset{6}{3(2)} + \overset{-6}{(-3)(2)} + \overset{0}{0(-1)} = 0$$

$$u_1 \cdot u_3 = \overset{3}{3(1)} + \overset{-3}{(-3)(1)} + \overset{0}{0(4)} = 0$$

$$u_2 \cdot u_3 = \overset{2}{2(1)} + \overset{2}{(2)(1)} + \overset{-4}{(-1)(4)} = 0$$

$\rightarrow$ so all orthogonal

Vectors are nonzero, so L.I from Thm 4

$\downarrow$

Vectors form a basis for R^3.

$\downarrow$

$\{u_1, u_2, u_3\}$ is an orthogonal basis for R^3!

Using Thm 5,

$$y = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2 + \frac{y \cdot u_3}{u \cdot u_3} u_3$$

Cont'd 5*

(Cont'd 5*)

$x \cdot u_1 = 5(3) + (-3)(-3) + (1)(0) = 24$

$x \cdot u_2 = (5)(2) + (-3)(2) + (1)(-1) = 3$

$x \cdot u_3 = 5(1) + (-3)(1) + (1)(4) = 6$

$u_1 \cdot u_1 = 3(3) + (-3)(-3) + (0)(0) = 18$

$u_2 \cdot u_2 = 2(2) + 2(2) + (-1)(-1) = 9$

$u_3 \cdot u_3 = 1(1) + 1(1) + (4)(4) = 18$

$x = \frac{24}{18} u_1 + \frac{3}{9} u_2 + \frac{6}{18} u_3$

$$\boxed{x = \frac{4}{3}\begin{bmatrix}3\\-3\\0\end{bmatrix} + \frac{1}{4}\begin{bmatrix}2\\2\\-1\end{bmatrix} + \frac{1}{3}\begin{bmatrix}1\\1\\4\end{bmatrix}}$$

9. Compute the orthogonal projection of $\begin{bmatrix}1\\7\end{bmatrix}$ onto the line through $\begin{bmatrix}-4\\2\end{bmatrix}$ and the origin.

9. Compute orthogonal projection of $\begin{bmatrix}1\\7\end{bmatrix}$ onto line through $\begin{bmatrix}-4\\2\end{bmatrix}$ and origin.

Let's say

$y=\begin{bmatrix}1\\7\end{bmatrix}, u=\begin{bmatrix}-4\\2\end{bmatrix}$ $\quad y\cdot u=\begin{bmatrix}1\\7\end{bmatrix}\cdot\begin{bmatrix}-4\\2\end{bmatrix}$

$= \underbrace{1(-4)}_{-4}+\underbrace{7(2)}_{14}=10$

$u\cdot u=\begin{bmatrix}-4\\2\end{bmatrix}\cdot\begin{bmatrix}-4\\2\end{bmatrix}=\underbrace{(-4)(-4)}_{16}+\underbrace{(2)(2)}_{4}=20$

$\hat{y}=\frac{y\cdot u}{u\cdot u}u=\frac{10}{20}u=\frac{1}{2}\begin{bmatrix}-4\\2\end{bmatrix}=\boxed{\begin{bmatrix}-2\\1\end{bmatrix}}$

10. Let $\mathbf{y}=\begin{bmatrix}2\\3\end{bmatrix}$ and $\mathbf{u}=\begin{bmatrix}4\\-7\end{bmatrix}$. Write $\mathbf{y}$ as the sum of two orthogonal vectors, one in Span$\{\mathbf{u}\}$ and one orthogonal to $\mathbf{u}$.

10. Let $y=\begin{bmatrix}2\\3\end{bmatrix}$ and $u=\begin{bmatrix}4\\-7\end{bmatrix}$ Sum of 2 orth. vectors:
↳ 1 in Span$\{u\}$ and 1 orth. to u.

① Find orth. projection

$y\cdot u=\begin{bmatrix}2\\3\end{bmatrix}\cdot\begin{bmatrix}4\\-7\end{bmatrix}=\overbrace{2(4)}^{8}+\overbrace{3(-7)}^{-21}=-13$

$u\cdot u=\begin{bmatrix}4\\-7\end{bmatrix}\cdot\begin{bmatrix}4\\-7\end{bmatrix}=\overbrace{4(4)}^{16}+\overbrace{(-7)(-7)}^{49}=65$

$\hat{y}=\frac{y\cdot u}{u\cdot u}u=\frac{-13}{65}u=\frac{-13}{65}\begin{bmatrix}4\\-7\end{bmatrix}=\begin{bmatrix}-52/65\\91/65\end{bmatrix}$

Cont'd 6★

cont'd 6

$$y - \hat{y} = \begin{bmatrix}2\\3\end{bmatrix} - \begin{bmatrix}-52/65\\91/65\end{bmatrix} = \begin{bmatrix}130/65\\195/65\end{bmatrix} - \begin{bmatrix}-52/65\\91/65\end{bmatrix}$$

$$= \begin{bmatrix}182/65\\104/65\end{bmatrix}$$

The sum of these two vectors is y.

$$\boxed{\underbrace{\begin{bmatrix}2\\3\end{bmatrix}}_{y} = \underbrace{\begin{bmatrix}-52/65\\91/65\end{bmatrix}}_{\hat{y}} + \underbrace{\begin{bmatrix}182/65\\104/65\end{bmatrix}}_{(y-\hat{y})}}$$

11. Let $\mathbf{y} = \begin{bmatrix}-3\\9\end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix}1\\2\end{bmatrix}$. Compute the distance from $\mathbf{y}$ to the line through $\mathbf{u}$ and the origin.

11. $v = \begin{bmatrix} -3 \\ 9 \end{bmatrix}, u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ Compute dist. from v to line through u and origin.

Dist is $\|v - p\|$

$p = \frac{v \cdot u}{u \cdot u} u$ $\quad v \cdot u = \underset{}{\overset{-3}{(-3)}}(1) + \overset{18}{(9)}(2) = 15$

$u \cdot u = \underset{1}{(1)(1)} + \underset{4}{(2)(2)} = 5$

$\frac{15}{5}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

$v - p = \begin{bmatrix} -3 \\ 9 \end{bmatrix} - \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$

Cont'd

Cont'd

$\|v - p\| = \sqrt{(-6)^2 + (3)^2}$

$\begin{bmatrix} -6 \\ 3 \end{bmatrix}$ $\quad = \sqrt{36 + 9}$ $\qquad 45/9 = 5$

$= \sqrt{45}$

$= \boxed{3\sqrt{5}}$ $\quad \sqrt{9} = 3$