Math 30, Thursday April 23, 2020

1pm class

Example/Application continued...

Determine distance traveled only using your speedometer.

If your speed is constant it's easy!

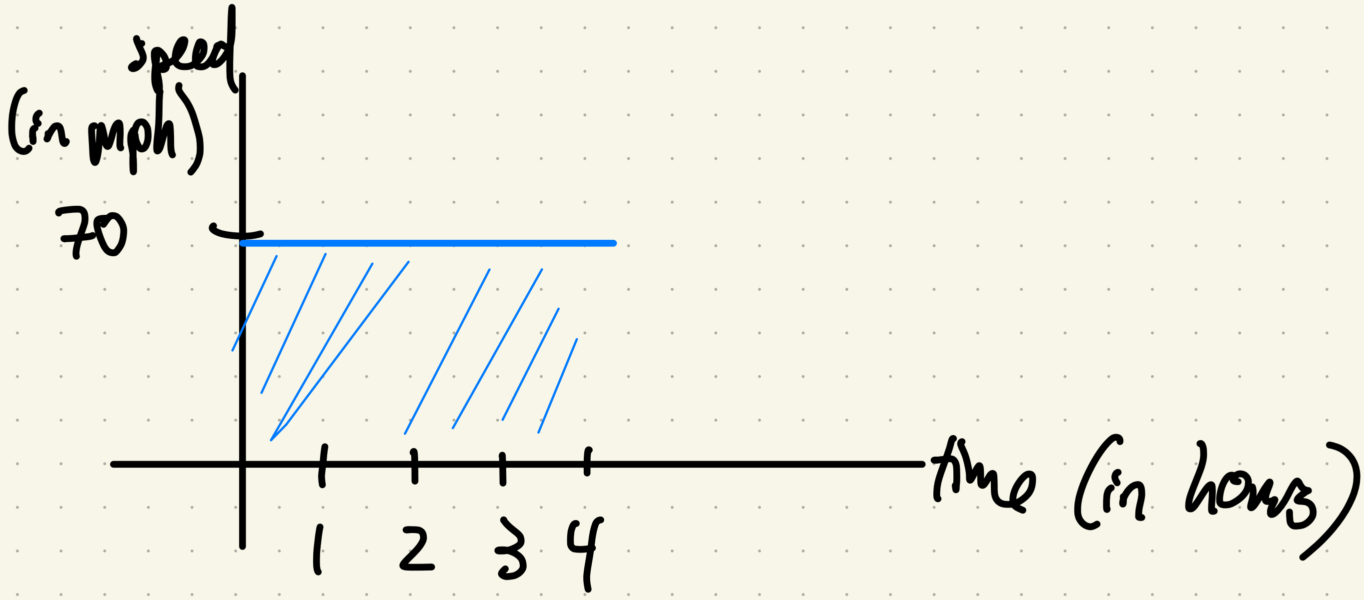
Say your speed is 70mph and are driving 4 hours.

Then distance traveled is

speed x time = distance

$$\left(70\,\frac{\text{miles}}{\text{hour}}\right)(4 \text{ hours}) = 280 \text{ miles}$$

right?

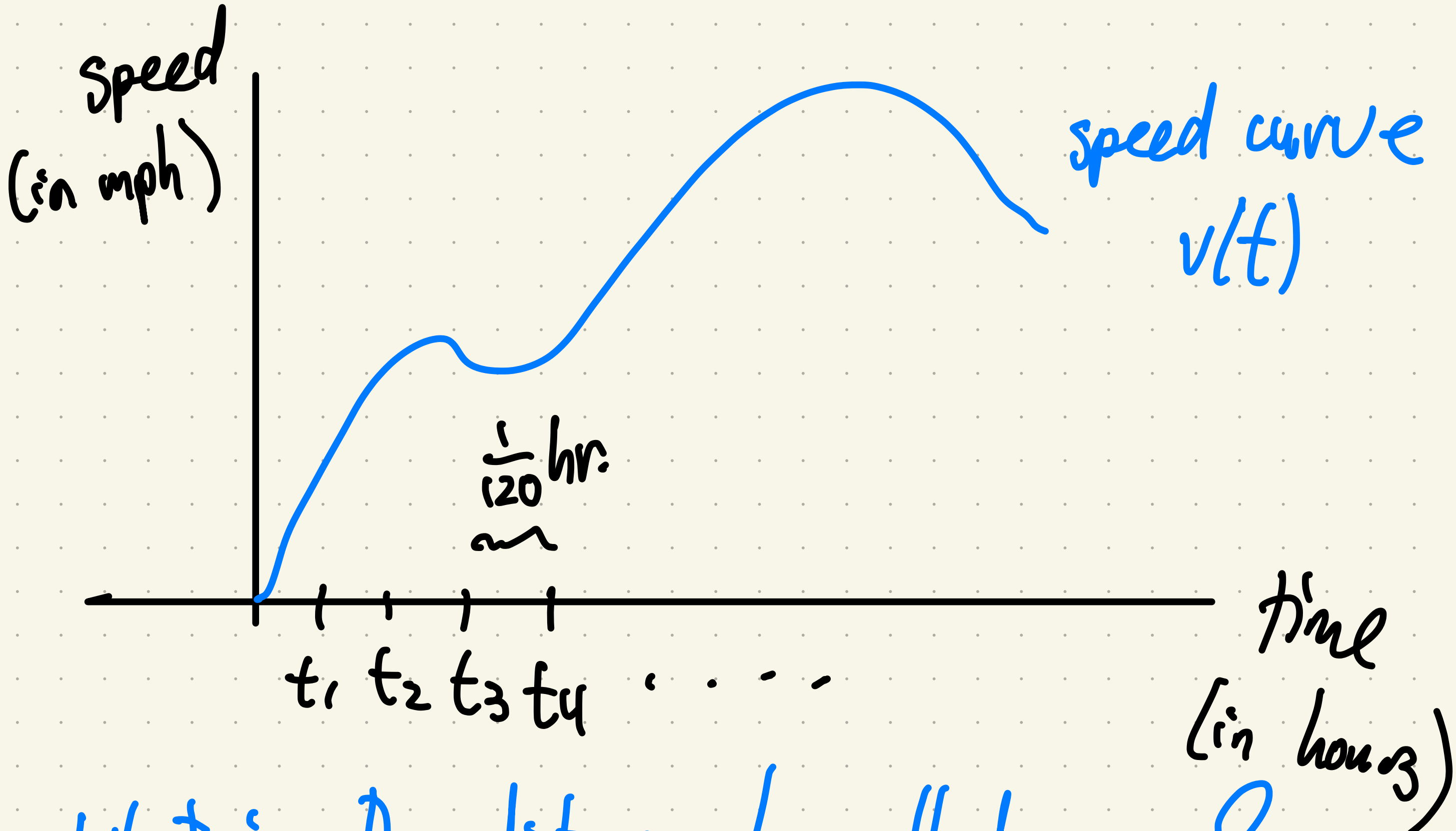


Area under "speed curve" is the distance traveled.

total here = 280 miles

Big <u>Question</u>: What if we are <u>not</u> going constant speed.

You've speeding up & slowing down..

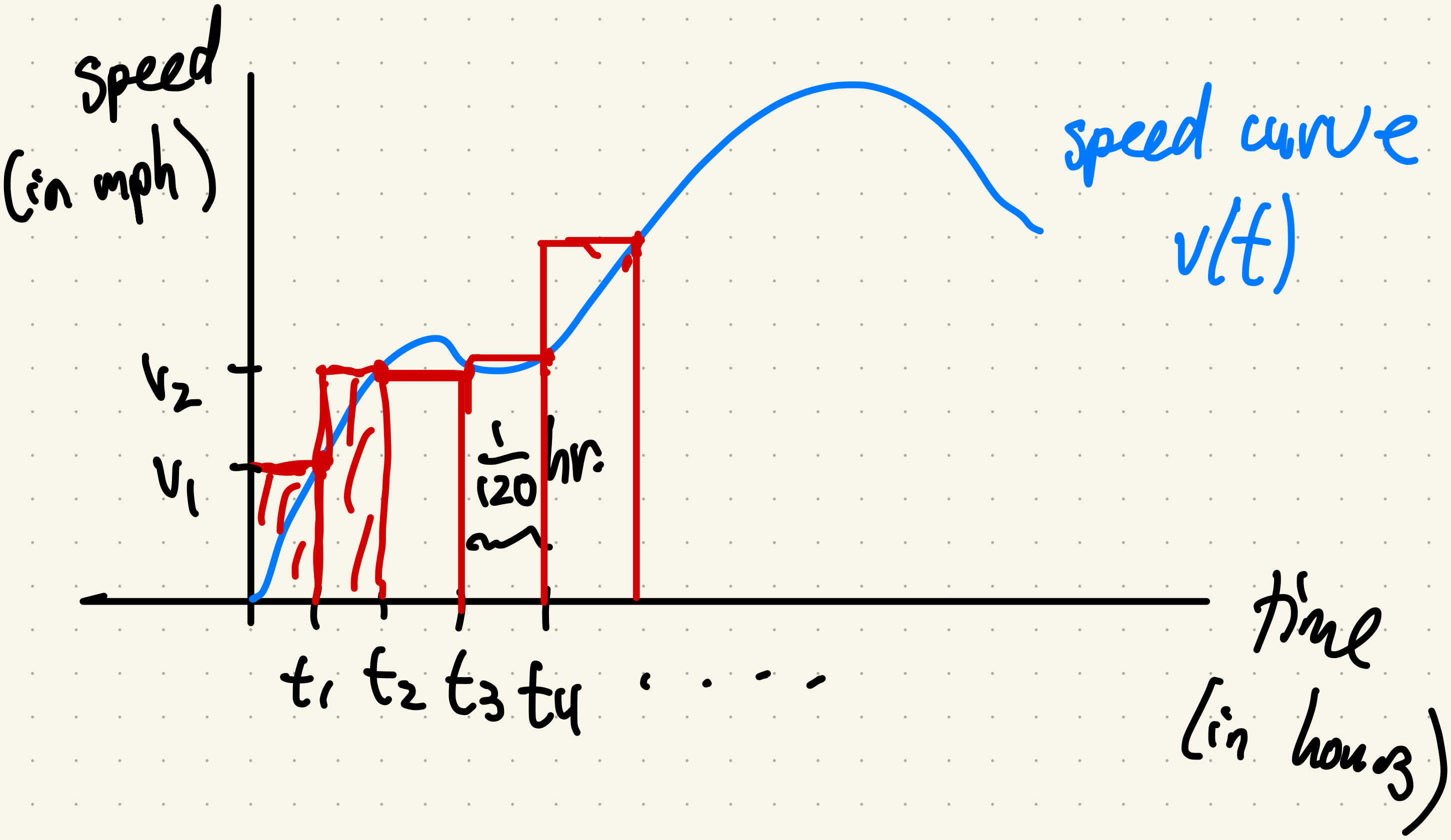


What is the distance travelled now?

Idea: approximate using rectangles.

Look at your speedometer every 30 seconds
($\frac{1}{120}$ hour)

say at time t_k hours
your speed is v_k miles per hour



Now approximate: if at time t_k going v_k mph,

in the prev. 30 sec $\left(= \frac{1}{120} \text{ hour} \right)$

you went approximately

$$\underbrace{v_k}_{\text{miles/hour}} \cdot \underbrace{\frac{1}{120}}_{\text{hours}} \text{ miles}$$

So the total dist. traveled is

speed at 30sec   speed at 60sec   etc.

$$\approx \underbrace{\overset{\text{speed at 30sec}}{v_1}\left(\frac{1}{120}\right)}_{\approx\text{ dist. trav. in first 30sec}} + \underbrace{\overset{\text{speed at 60sec}}{v_2}\left(\frac{1}{120}\right)}_{\approx\text{ dist. trav. in second 30 sec}} + v_3\left(\frac{1}{120}\right) + \cdots + v_n\left(\frac{1}{120}\right)$$

miles

To get a better approximation, use smaller time intervals —

eg. look at speedometer every 5 sec

or every 1 second ...

You'd be using more & skinnier rectangles...

To be more precise and write math formulas let me introduce some useful notation:

"Sigma notation":

Sigma is the Greek letter Σ

corresp. to English letter S

which stands for "sum"

Helps express sums that have patterns.

eg. a slick way to write

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$$

$$= 2^{-1} + 2^{-2} + 2^{-3} + 2^{-4} + 2^{-5} + \cdots$$

new notation: with $j=n$ term end

$$\sum_{j=1}^{n} a_j \quad \text{means} \quad = \underbrace{a_1}_{\substack{j=1 \\ \text{term}}} + \underbrace{a_2}_{\substack{j=2 \\ \text{term}}} + a_3 + \cdots + \underbrace{a_n}_{\substack{j=n \\ \text{term}}}$$

add them up

start with $j=1$ term

This Σ-notation helps us avoid writing "..."

Ex. $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}$

can be rewritten as:

$$\sum_{j=1}^{6} 2^{-j}$$

less writing ✓ & tells the pattern :)

Ex. $1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720}$

See the pattern?

$$= 1 + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3\cdot 4} + \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}$$

$$= \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!}$$

$$= \sum_{j=1}^{6} \frac{1}{j!}$$

less writing and shows the pattern

$\sum$ = "Sum"

Fancy example. $\sum_{j=1}^{100} j$ famous story: They say Gauss did This in his head as a child.

and famous!

Let me show you how:

$= 1+2+3+4+\cdots+98+99+100$

now do it again in The opposite order:

$100+99+98+97+\cdots+3+2+1$

trick add vertical pairs:

$101+101+101+101+\cdots+101+101+101$

$=(101)(100)$

But This is twice as much, so final answer is $\boxed{5050}$:)

Same trick works for

$$\sum_{j=1}^{n} j = 1+2+3+\cdots+n$$

$$= \frac{n(n+1)}{2}$$

(check)

General Facts about Sigma Notation:

1. If c is a constant (doesn't change)

Then $\sum_{j=1}^{n} c = \underbrace{c + c + c + \cdots + c}_{n \text{ times}}$

c doesn't depend on j

$= \boxed{nc}$

2. $\sum_{j=1}^{n} (a_j + b_j) = ?$

$= (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) +$

$\cdots + (a_n + b_n)$

regroup terms

$= (a_1 + a_2 + \cdots + a_n) + (b_1 + b_2 + \cdots + b_n)$

$= \sum_{j=1}^{n} a_j + \sum_{j=1}^{n} b_j$

3. If c is a constant, then

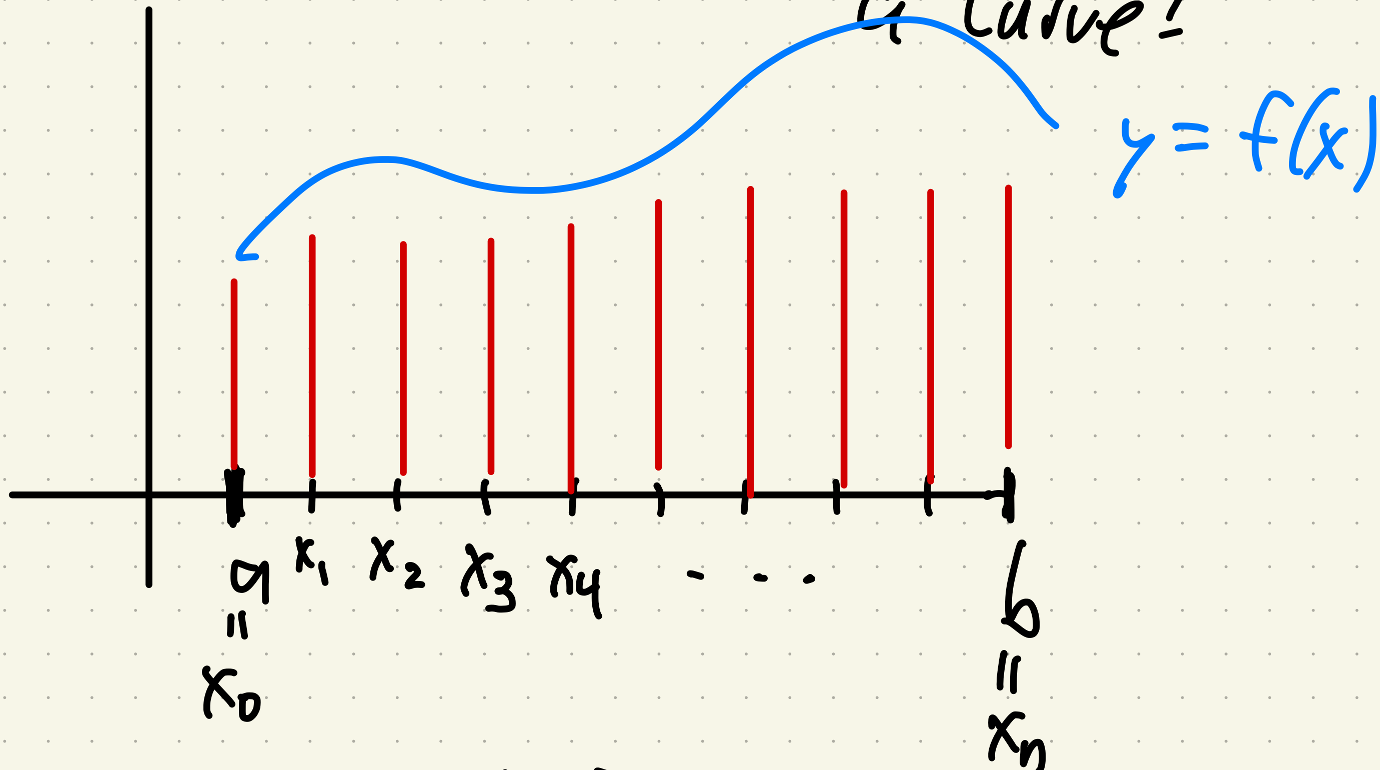
$$\sum_{j=1}^{n} c a_j = ca_1 + ca_2 + ca_3 + \cdots + ca_n$$

factor out the c:

$$= c\left(a_1 + a_2 + a_3 + \cdots + a_n\right)$$

$$= c\sum_{j=1}^{n} a_j$$

Q: How is this related to area under a curve?



Split interval $[a, b]$ into n equal parts,

each part has length

$$\frac{b-a}{n} = \underbrace{\Delta x}_{\text{"change in } x\text{"}}$$

How tall should each rectangle be?

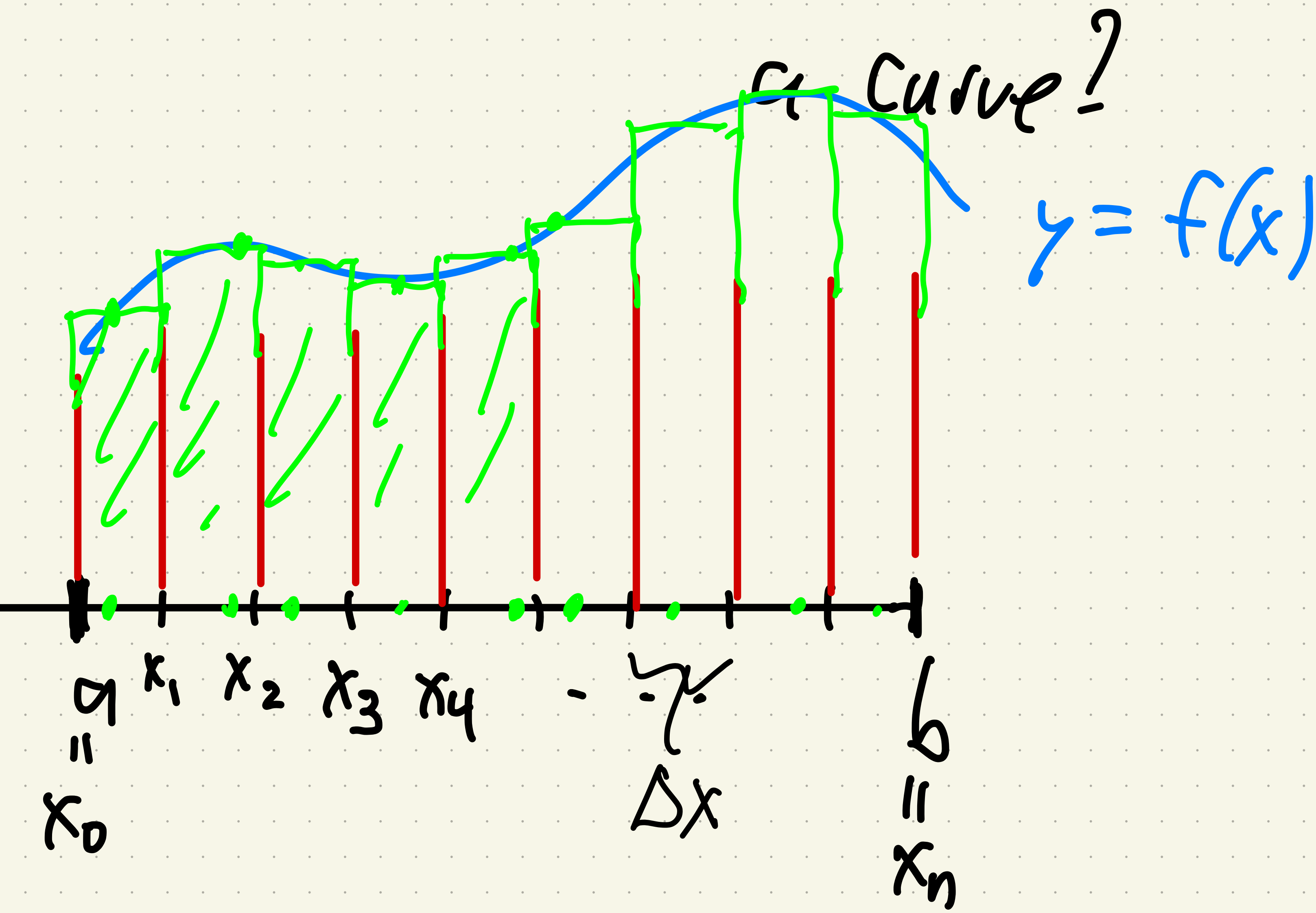
Lots of ways: can use

- left endpts
- right endpts
- midpts

etc.

Let x_j^* be a "sample point" in the j^{th} interval.

Let the height of the j^{th} rectangle be $f(x_j^*)$.



So the area under the curve is approximately:
blue

$$\underbrace{f(x_1^*)\Delta x}_{\text{area of } 1^{st} \text{ rectangle}} + \underbrace{f(x_2^*)\Delta x}_{\text{second}} + \cdots + \underbrace{f(x_n^*)\Delta x}_{n\text{th}}.$$

Use summation notation to rewrite it

as

$$\sum_{j=1}^{n} \underbrace{f(x_j^*)\Delta x}_{\text{area of } j^{th} \text{ rectangle.}}$$

out of time...

See you tomorrow!