MATH 30, 3/23/2020: INTRO TO MAXIMA AND MINIMA

Another Related Rates Review Problem (from Stewart's **Calculus**, #39): A plane flied horizontally at an altitude of 5 kilometers and passes directly over a tracking telescope on the ground. When the angle of elevation is $\pi/3$, this angle is decreasing at a rate of $\pi/6$ radians per minute. How fast is the plane traveling at that time?

Answer. $\frac{10\pi}{9}$ kilometers per minute.

Linear Approximation Review Problem. Use linear approximation to approximate the value of $\sqrt{3.9}$.

Answer. The approximate value is $79/40 = 1.975$. The true value is $1.97484\ldots$ Pretty good, right?

Because of coronavirus, we will skip the material on "differentials." It's the same as "linear approximation," but in different notation.

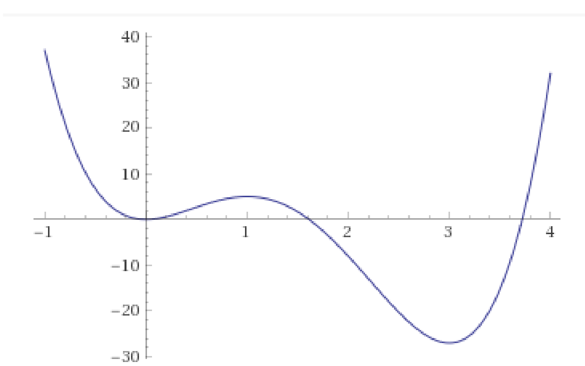
New topic: Maxima and Minima.

"Optimization Problems": What is the "best" way to do something? When is a quantity *maximized* or *minimized*? This shows up *a lot* in physics.

Definitions.
a function f has an **absolute maximum** (aka global maximum) at c if $f(x) \leq f(c)$ for all x.
a function f has an **absolute minimum** (aka global minimum) at c if $f(x) \geq f(c)$ for all x.

Definitions.
Let c be a point in the interior of the domain of f (not on the boundary of the domain).
a function f has an **local maximum** at c if $f(x) \leq f(c)$ for all x near c.
a function f has an **local minimum** at c if $f(x) \geq f(c)$ for all x near c.

Example. Consider $f(x) = 3x^4 - 16x^3 + 18x^2$ for $-1 \leq x \leq 4$.



There is a global maximum at $x = -1$, a local minimum at $x = 0$, a local maximum at $x = 1$, and a local *and* global minimum at $x = 3$. There is not a local max at $x = 4$ because it is a boundary point.

Definition. a **critical points** of f is a number c in the interior of the domain of f where
(1) $f'(c) = 0$ or
(2) $f'(c)$ does not exist.

In the above example, $x = -1$ and $x = 4$ are on the boundary of the domain, not in the interior.

"Fermat's Theorem." If f has a local max or local min at c, and if $f'(c)$ exists, then $f'(c) = 0$.

That is, if you want to find local max and min, the critical points are the possible candidates.

In the above example, $f(x) = 3x^4 - 16x^3 + 18x^2$ for $-1 \leq x \leq 4$, we have

$$f'(x) = 12x(x-1)(x-3),$$

so the critical points of f are $x = 0, 1, 3$. These are the only possible places where we can have local max or min. (Of course, we knew that, based on the picture.)

Here is a common min/max problem: what is the point on a line that is closest to a given point?

Example. Find the point on the line $y = 9 - 6x$ that is closest to the point $(-3, 1)$.

First of all, the distance from $(-3, 1)$ to a general point $(x, 9 - 6x)$ on the line is:

$$d(x) = \sqrt{(x+3)^2 + (8-6x)^2}.$$

We would like to find the value of x that *minimizes* this distance.

But this value of x is the *same* as the x that minimizes

$$f(x) = (x+3)^2 + (8-6x)^2,$$

a simpler function.

Using Fermat's theorem, we would like to find the critical points of f: that is, the points x where $f'(x) = 0$.

After some calculation, we find $x = \frac{45}{37}$, which corresponds to the point on the line $(\frac{45}{37}, \frac{63}{37})$.

Find the point on the line $y = 9 - 6x$ that is closest to the point $(-3, 1)$.