HW 7

① a) $5\begin{bmatrix}3\\-5\end{bmatrix}+3\begin{bmatrix}-4\\6\end{bmatrix}=\begin{bmatrix}15-12\\-25+18\end{bmatrix}=\begin{bmatrix}3\\-7\end{bmatrix}$

b) $1\begin{bmatrix}1\\-2\\3\end{bmatrix}+0\begin{bmatrix}5\\0\\-2\end{bmatrix}-2\begin{bmatrix}4\\-3\\0\end{bmatrix}=\begin{bmatrix}1+0-8\\-2+0+6\\3+0+0\end{bmatrix}=\begin{bmatrix}-7\\4\\3\end{bmatrix}$

② a) $\left[\begin{array}{cc|c}1&2&-1\\-4&-3&-6\end{array}\right]\xrightarrow{4R_1+R_2\to R_2}\left[\begin{array}{cc|c}1&2&-1\\0&5&-10\end{array}\right]\xrightarrow{\frac{1}{5}R_2\to R_2}\left[\begin{array}{cc|c}1&2&-1\\0&1&-2\end{array}\right]\xrightarrow{-2R_2+R_1\to R_1}\left[\begin{array}{cc|c}1&0&3\\0&1&-2\end{array}\right]\to[x]_\beta=\begin{bmatrix}3\\-2\end{bmatrix}$

b) $\left[\begin{array}{ccc|c}1&2&1&0\\1&0&-1&0\\3&8&3&-2\end{array}\right]\xrightarrow[-3R_1+R_3\to R_3]{-R_2+R_1\to R_1}\left[\begin{array}{ccc|c}0&2&2&0\\1&0&-1&0\\0&2&0&-2\end{array}\right]\xrightarrow{-R_3+R_1\to R_1}\left[\begin{array}{ccc|c}0&0&2&2\\1&0&-1&0\\0&2&0&-2\end{array}\right]\xrightarrow[\frac{1}{2}R_3\to R_3]{\frac{1}{2}R_1\to R_1}\left[\begin{array}{ccc|c}0&0&1&1\\1&0&-1&0\\0&1&0&-1\end{array}\right]\xrightarrow{R_1+R_2\to R_2}\left[\begin{array}{ccc|c}0&0&1&1\\1&0&0&1\\0&1&0&-1\end{array}\right]$

(Switch Rows) $\to\left[\begin{array}{ccc|c}1&0&0&1\\0&1&0&-1\\0&0&1&1\end{array}\right]\to[x]_\beta=\begin{bmatrix}1\\-1\\1\end{bmatrix}.$

③ a) $\underset{A}{\begin{bmatrix}1&-3\\-2&5\end{bmatrix}}\to \det=5-(6)=-1\to -1\begin{bmatrix}5&3\\2&1\end{bmatrix}=\underset{A^{-1}}{\begin{bmatrix}-5&-3\\-2&-1\end{bmatrix}}\to\begin{bmatrix}-5&-3\\-2&-1\end{bmatrix}\underset{\vec{x}}{\begin{bmatrix}2\\-5\end{bmatrix}}=\begin{bmatrix}5\\1\end{bmatrix}=[x]_\beta$

b) $\underset{A}{\begin{bmatrix}1&2\\-1&-1\end{bmatrix}}\to \det=-1-(-2)=1\to 1\begin{bmatrix}-1&-2\\1&1\end{bmatrix}=\underset{A^{-1}}{\begin{bmatrix}-1&-2\\1&1\end{bmatrix}}\to\begin{bmatrix}-1&-2\\1&1\end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix}=\begin{bmatrix}-8\\5\end{bmatrix}=[x]_\beta$

④ $1-t^2=1+0t-1t^2\to\begin{bmatrix}1\\0\\-1\end{bmatrix}$, $t-t^2=0+t-t^2\to\begin{bmatrix}0\\1\\-1\end{bmatrix}$, $2-t+t^2\to\begin{bmatrix}2\\-1\\1\end{bmatrix}$, $\vec{p}(t)=1+3t-6t^2\to\begin{bmatrix}1\\3\\-6\end{bmatrix}$

Thus $\left[\begin{array}{ccc|c}1&0&2&1\\0&1&-1&3\\-1&-1&1&-6\end{array}\right]\xrightarrow{R_1+R_3\to R_3}\left[\begin{array}{ccc|c}1&0&2&1\\0&1&-1&3\\0&-1&3&-5\end{array}\right]\xrightarrow{R_2+R_3\to R_3}\left[\begin{array}{ccc|c}1&0&2&1\\0&1&-1&3\\0&0&2&-2\end{array}\right]\xrightarrow{\frac{1}{2}R_3\to R_3}\left[\begin{array}{ccc|c}1&0&2&1\\0&1&-1&3\\0&0&1&-1\end{array}\right]\xrightarrow[-2R_3+R_1\to R_1]{R_3+R_2\to R_2}\left[\begin{array}{ccc|c}1&0&0&3\\0&1&0&2\\0&0&1&-1\end{array}\right]$

So, $[\vec{p}(t)]_\beta=\begin{bmatrix}3\\2\\-1\end{bmatrix}.$

(5)

a) $\begin{bmatrix} 2c \\ a-b \\ b-3c \\ a+2b \end{bmatrix} = a\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0 \\ -1 \\ 1 \\ 2 \end{bmatrix} + c\begin{bmatrix} 2 \\ 0 \\ -3 \\ 0 \end{bmatrix}$ i) A basis for $H \Rightarrow \mathcal{B}_H = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -3 \\ 0 \end{bmatrix} \right\}$

ii) 3 vectors in $\mathcal{B}_H$ thus $\dim H = 3$.

b) As discussed in class $H = \{(0,0,0)\}$ thus i) Basis for $H \to \mathcal{B}_H = \{(0,0,0)\}$

ii) Since a basis is only the zero-vector then $\dim H = 0$.

(6) Let $A = \begin{bmatrix} 1 & -3 & -2 & -3 \\ -2 & -6 & 3 & 5 \\ 0 & 0 & 5 & 5 \end{bmatrix} \xrightarrow{2R_1+R_2\to R_2} \begin{bmatrix} 1 & -3 & -2 & -3 \\ 0 & -12 & -1 & -1 \\ 0 & 0 & 5 & 5 \end{bmatrix} \xrightarrow{5R_2+R_3\to R_3} \begin{bmatrix} 1 & -3 & -2 & -3 \\ 0 & -12 & -1 & -1 \\ 0 & -60 & 0 & 0 \end{bmatrix} \xrightarrow{-\frac{1}{60}R_3\to R_3} \begin{bmatrix} & '' & & \\ & '' & & \\ 0 & 1 & 0 & 0 \end{bmatrix}$

$\xrightarrow[3R_3+R_1\to R_1]{12R_3+R_2\to R_2} \begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ pivots

Since there are 3 pivots then $\dim \operatorname{Col} A = 3$, thus the dimension of the subspace spanned by these vectors is 3.

(7) $A = \begin{bmatrix} 1 & -6 & 9 & 0 & -2 \\ 0 & 1 & 2 & -4 & 5 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to$ there's 3 pivots, so $\boxed{\dim \operatorname{Col} A = 3}$ Note that, $\dim \operatorname{Col} A = \operatorname{rank} A = 3$ and since there are 5 columns, by the Rank-Theorem (Thm 14) $\operatorname{rank} A + \dim \operatorname{Nul} A = 5$

$\to 3 + \dim \operatorname{Nul} A = 5$

$\to \boxed{\dim \operatorname{Nul} A = 2}$

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⑧ $1 \to \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$ $(1+0t+0t^2+0t^3)$, $2t \to \begin{bmatrix}0\\2\\0\\0\end{bmatrix}$ $(0+2t+0t^2+0t^3)$, $-2+4t^2 \to \begin{bmatrix}-2\\0\\4\\0\end{bmatrix}$ $(-2+0t+4t^2+0t^3)$, $-12t+8t^3 \to \begin{bmatrix}0\\-12\\0\\8\end{bmatrix}$ $(0-12t+0t^2+8t^3)$

So, $A = \begin{bmatrix}1&0&-2&0\\0&2&0&-12\\0&0&4&0\\0&0&0&8\end{bmatrix} \to \det A$ (down 1st column) $= 1 \cdot \begin{vmatrix}2&0&-12\\0&4&0\\0&0&8\end{vmatrix}$ (across 2nd row) $= 1 \cdot 4 \begin{vmatrix}2&-12\\0&8\end{vmatrix} = 1 \cdot 4 \cdot 16 = 64 \neq 0$

Thus the columns of A are L.I., and since $\dim \mathbb{P}_3 = 4$ and this is 4 L.I. vectors from $\mathbb{P}_3$ then by the Basis Theorem (Thm 12) these polynomials are a basis for $\mathbb{P}_3$.

⑨ $\vec{p}(t) = -1 + 0t + 8t^2 + 8t^3 \to \begin{bmatrix}-1\\0\\8\\8\end{bmatrix}$ then

$$\left[\begin{array}{cccc|c}1&0&-2&0&-1\\0&2&0&-12&0\\0&0&4&0&8\\0&0&0&8&8\end{array}\right] \xrightarrow[\frac{1}{8}R_4 \to R_4]{\frac{1}{4}R_3 \to R_3} \left[\begin{array}{cccc|c}1&0&-2&0&-1\\0&2&0&-12&0\\0&0&1&0&2\\0&0&0&1&1\end{array}\right] \xrightarrow[12R_4+R_2 \to R_2]{2R_3+R_1 \to R_1} \left[\begin{array}{cccc|c}1&0&0&0&3\\0&2&0&0&12\\0&0&1&0&2\\0&0&0&1&1\end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_2 \to R_2} \left[\begin{array}{cccc|c}1&0&0&0&3\\0&1&0&0&12\\0&0&1&0&2\\0&0&0&1&1\end{array}\right] \quad [\vec{p}(t)]_{\mathcal{B}} = \begin{bmatrix}3\\12\\2\\1\end{bmatrix}$$

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(10) $A \sim B = \begin{bmatrix} 1 & 3 & 4 & -1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ these are the pivots in the 1^{st}, 3^{rd}, and 5^{th} columns.

(Rank Thm)

So, $\underline{\text{rank } A = 3}$ (3 pivots) and $3 + \dim Nul A = 5 \rightarrow \underline{\dim Nul A = 2}$

✓ For the basis of Col A look at column 1, 3, and 5 of matrix A,

thus $\mathcal{B}_{Col A} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -3 \\ 0 \end{bmatrix} \right\}$.

✓ For basis of row A, look at the non-zero rows of B.

thus $\mathcal{B}_{Row A} = \{(1, 3, 4, -1, 2), (0, 0, 1, -1, 1), (0, 0, 0, 0, -5)\}$.

✓ For basis of Nul A, we need to reduce B further:

$$\left[\begin{array}{ccccc|c} 1 & 3 & 4 & -1 & 2 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[-4R_2+R_1 \to R_1]{-\frac{1}{5}R_3 \to R_3} \left[\begin{array}{ccccc|c} 1 & 3 & 0 & 3 & -2 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & & & & & 0 \end{array}\right] \xrightarrow[2R_3+R_1 \to R_1]{-R_3+R_2 \to R_2} \left[\begin{array}{ccccc|c} 1 & 3 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & & & & & 0 \end{array}\right]$$

So, $x_1 = -3x_2 - 3x_4$
$x_3 = x_4$
$x_5 = 0$

$$\rightarrow \vec{x} = \begin{bmatrix} -3x_2 - 3x_4 \\ x_2 \\ x_4 \\ x_4 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

Thus, $\mathcal{B}_{Nul A} = \left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$

(11) 'A is 7×5 and rank $A = 2$.

✓ rank A + dim Nul A = n → $2 + \dim \text{Nul} A = 5$ → $\dim \text{Nul} A = 3$.

✓ dim Row A = dim Col A = rank A = 2.

✓ rank A^T = dim Col A^T = dim Row A = 2.

Since the transpose switches columns and rows

(12) 'A is 4×6 and dim Nul $A = 3$

columns of A.

✓ Once again we have rank A + dim Nul A = n

$\text{rank} A + 3 = 6$

$\text{rank} A = 3$

✓ rank A = dim Col A = 3. Since there are 4 rows in A then the columns of A have 4 entries thus,

Col A is a 3-dimensional subspace of $\mathbb{R}^4$, and cannot equal $\mathbb{R}^3$.