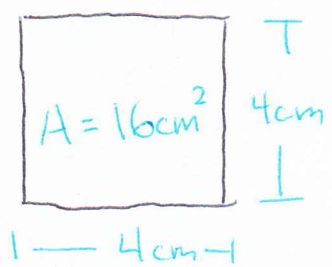
# 1 Applications of Derivatives

1. Each side of a square is increasing at a rate of 6 cm/s. At what rate is the area of the square increasing when the area of the square is 16 cm$^2$.




$A = s^2, \quad s = 4\text{cm}, \quad \frac{ds}{dt} = 6\text{cm/s}$

$\frac{dA}{dt} = 2s \cdot \frac{ds}{dt} \rightarrow \frac{dA}{dt} = 2(4)(6)$

$\boxed{\frac{dA}{dt} = 48\ \text{cm}^2/\text{s}}$

2. If a snowball melts so that its surface area decreases at a rate of 1 cm$^2$/min, find the rate at which the diameter decreases when the diameter is 10 cm.


$A = 4\pi r^2, \quad r = 5\text{cm}, \quad \frac{dA}{dt} = -1\ \text{cm}^2/\text{min}$

$\frac{dA}{dt} = 8\pi r \cdot \frac{dr}{dt} \rightarrow -1 = 8\pi(5) \cdot \frac{dr}{dt}$

$\boxed{\frac{dr}{dt} = -\frac{1}{40\pi}\ \text{cm/min}}$

3. Find the critical numbers of the function $g(y) = \frac{y-1}{y^2-y+1}$.

$g'(y) = -\frac{y(y-2)}{(y^2-y+1)^2} = 0 \rightarrow \boxed{y = 0, 2}$

Relative Minimum at $y=0$

Relative Maximum at $y=2$

| Interval | Test | Sign | Conclusion |
|---|---|---|---|
| $(-\infty, 0)$ | $x=-1$ | $-$ | Decreasing |
| $(0, 2)$ | $x=1$ | $+$ | Increasing |
| $(2, \infty)$ | $x=3$ | $-$ | Decreasing |

4. An object with weight $W$ is dragged along a horizontal plane by a force acting along a rope attached to the object. If the rope makes an angle $\theta$ with the plane, then the magnitude of force is

$$F = \frac{\mu W}{\mu \sin\theta + \cos\theta}$$

where $\mu$ is a positive constant called the coefficient of friction and where $0 \leq \theta \leq \pi/2$. Show that $F$ is minimized when $\tan\theta = \mu$.

$F'(\theta) = \frac{0(\mu\sin\theta + \cos\theta) - \mu W(\mu\cos\theta - \sin\theta)}{(\mu\sin\theta + \cos\theta)^2} = \frac{\mu W(\sin\theta - \mu\cos\theta)}{(\mu\sin\theta + \cos\theta)^2}$

$F'(\theta) = 0 \rightarrow \frac{\mu W(\sin\theta - \mu\cos\theta)}{(\mu\sin\theta + \cos\theta)^2} = 0 \rightarrow \sin\theta - \mu\cos\theta = 0$

$\rightarrow \sin\theta = \mu\cos\theta$



This is a critical #, where $F$ has max/min $\rightarrow \boxed{\tan\theta = \mu}$

5. Consider the function $f(x) = \sin x + \cos x$, $0 \leq x \leq 2\pi$. $\rightarrow f'(x) = \cos x - \sin x$
(a) Find the intervals on which $f$ is increasing or decreasing. $\rightarrow \cos x - \sin x = 0$
(b) Find the local maximum and minimum values of $f$.
(c) Find the intervals of concavity and the inflection points.

$\rightarrow \cos x = \sin x$

$\rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4}$

$f''(x) = -\sin x - \cos x$

$\rightarrow -\sin x - \cos x = 0$

$\rightarrow -\sin x = \cos x$

$\rightarrow x = \frac{3\pi}{4}, \frac{7\pi}{4}$

(a)

| Interval | Test | Sign | Conclusion |
|---|---|---|---|
| $[0, \pi/4)$ | $\pi/6$ | + | Increasing |
| $(\pi/4, 5\pi/4)$ | $\pi/2$ | − | Decreasing |
| $(5\pi/4, 2\pi]$ | $11\pi/6$ | + | Increasing |

(b) Maximum @ $x = \pi/4$
Minimum @ $x = 5\pi/4$

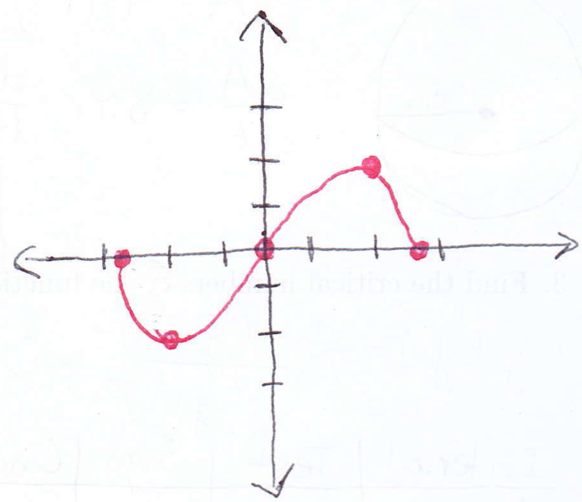
(c) Inflection @ $x = 3\pi/4, 7\pi/4$

6. Sketch a graph of the function $y = x\sqrt{2 - x^2}$.

x-intercepts: $(0,0)$ $(-\sqrt{2},0)$ $(\sqrt{2},0)$ domain: $[-\sqrt{2}, \sqrt{2}]$

First Derivative Test
relative extrema: $(-1,-1)$ min $(1,1)$ max
decreasing: $(-\sqrt{2},-1) \cup (1,\sqrt{2})$
increasing: $(-1,1)$

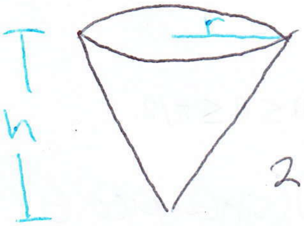
Second Derivative Test
inflection point: $(0,0)$
concave down: $(0,\sqrt{2})$
concave up: $(-\sqrt{2},0)$



$$y' = \sqrt{2-x^2} - \frac{x^2}{\sqrt{2-x^2}}$$

$$y'' = \frac{x(3x^2-10)}{2(2-x^2)\sqrt{2-x^2}}$$

7. A cone shaped paper drinking cup is to be made to hold 27 $cm^3$ of water. Find the height and radius of the cup that will use the smallest amount of paper.




$V = \frac{1}{3}\pi r^2 h$ $\quad V = 27 cm^3$

$A = \pi r\sqrt{r^2+h^2} \rightarrow A^2 = \pi^2 r^2 (r^2+h^2) = S$

$27 = \frac{1}{3}\pi r^2 h$

$81 = \pi r^2 h$

$r^2 = \frac{81}{\pi h}$

$S = \pi^2\left(\frac{81}{\pi h}\right)\left(\frac{81}{\pi h} + h^2\right) = \frac{81^2}{h^2} + 81\pi h$

$S' = -2 \cdot 81^2 \cdot h^{-3} + 81\pi = 0$

$81\pi = \frac{2 \cdot 81^2}{h^3} \rightarrow \boxed{h = \sqrt[3]{\frac{162}{\pi}} \approx 3.72 cm}$

$\boxed{r = \sqrt{\frac{81}{\pi \sqrt[3]{\frac{162}{\pi}}}} \approx 2.63 cm}$

## 2 Integration

1. Find the general antiderivative of $f(x) = x^{3.4} - 2x^{\sqrt{2}-1}$.

$$F(x) = \frac{1}{3.4+1} x^{3.4+1} - 2\left(\frac{1}{\sqrt{2}-1+1}\right) x^{\sqrt{2}-1+1}$$

$$\boxed{F(x) = 0.227 x^{4.4} - \sqrt{2}\, x^{\sqrt{2}}}$$

2. Find the function $f$ if $f'''(x) = \cos x$ and $f(0) = 1, f'(0) = 2, f''(0) = 3$.

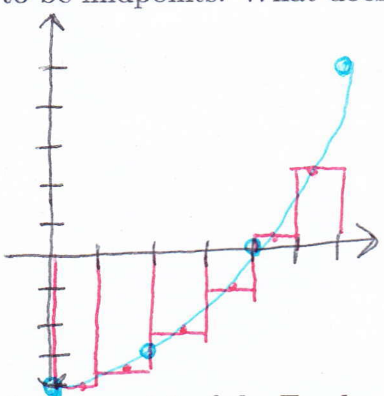
$$f'''(x) = \cos x$$

$$f''(x) = \sin x + a \rightarrow f''(x) = \sin x + 3$$

$$f'(x) = -\cos x + ax + b \rightarrow f'(x) = -\cos x + 3x + 3$$

$$f(x) = -\sin x + \frac{1}{2} a x^2 + bx + c \rightarrow \boxed{f(x) = -\sin x + \frac{3}{2} x^2 + 3x + 1}$$

3. If $f(x) = x^2 - 4$, $0 \leq x \leq 3$, find the Riemann sum with $n = 6$, taking the sample points to be midpoints. What does the Riemann sum represent? Illustrate with a diagram.



$$M_6 = \frac{1}{2}\left[\left(\tfrac{1}{4}\right)^2 - 4 + \left(\tfrac{3}{4}\right)^2 - 4 + \left(\tfrac{5}{4}\right)^2 - 4 + \left(\tfrac{7}{4}\right)^2 - 4 + \left(\tfrac{9}{4}\right)^2 - 4 + \left(\tfrac{11}{4}\right)^2 - 4\right]$$

$$M_6 = \frac{1}{2}(-3.9375 - 3.4375 - 2.4375 - 0.9375 + 1.0625 + 3.5625)$$

$$\boxed{M_6 = -3.0625}$$

4. Use part 1 of the Fundamental Theorem of Calculus to find the derivative of the function

$$g(x) = \int_1^x \ln(1 + t^2)dt$$

$$\boxed{g'(x) = \ln(1 + t^2)}$$

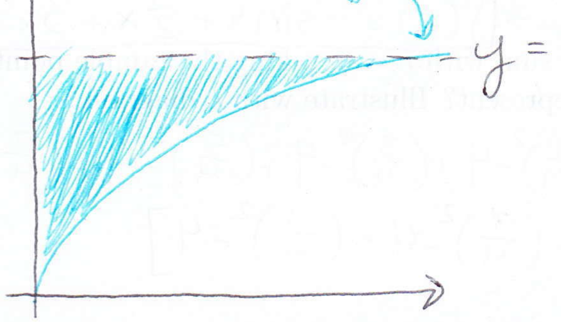
5. Evaluate the integral $\int_{-1}^{2}(3u-2)(u+1)du$

$$\int_{-1}^{2}(3u-2)(u+1)\,du = \int_{-1}^{2}(3u^2+u-2)\,du$$
$$= u^3+\frac{1}{2}u^2-2u\Big|_{-1}^{2}$$
$$= \boxed{4.5}$$

6. Find the area of a shaded region that is bounded by the y-axis, the line $y = 1$, and the curve $y = \sqrt[4]{x}$. Find the area by writing $x$ as a function of $y$ and integrating with respect to $y$.

$y=\sqrt[4]{x} \rightarrow x=y^4$

$$\int_0^1 y^4\,dy = \frac{1}{5}y^5\Big|_0^1 = \boxed{\frac{1}{5}}$$




7. Evaluate the indefinite integral.

$$\int \frac{dt}{\cos^2 t\sqrt{1-\tan t}}$$

$$\int\frac{dt}{\cos^2 t\sqrt{1-\tan t}} = \int\frac{\sec^2 t\,dt}{\sqrt{1-\tan t}} = -\int\frac{-\sec^2 t\,dt}{\sqrt{1-\tan t}}$$

Let $u=1-\tan t$, $du=-\sec^2 t\,dt$

$$= -\int\frac{du}{\sqrt{u}} = -\int u^{-1/2}\,du = -2u^{1/2}+C$$

$$= \boxed{-2\sqrt{1-\tan t}+C}$$