Exploring Imaginary Coordinates: Disparity in the Shape of Quantum State Space in Even and Odd Dimensions

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The state of a finite-dimensional quantum system is described by a density matrix that can be decomposed into a real diagonal, a real off-diagonal and and an imaginary off-diagonal part. The latter plays a peculiar role. While it is intuitively clear that some of the imaginary coordinates cannot have the same extension as their real counterparts the precise relation is not obvious. We give a complete characterization of the constraints in terms of tight inequalities for real and imaginary Bloch-type coordinates. Our description entails a three-dimensional Bloch ball-type model for the state space. We uncover a surprising qualitative difference for the state-space boundaries in even and odd dimensions.

I. INTRODUCTION

From the very beginning, quantum mechanics was naturally represented in terms of complex numbers. The state of a quantum system was identified with an element of a complex Hilbert space [1, 2]. Whether this complex formulation is actually necessary or merely more elegant than a hypothetical real version was an open question until very recently. Assuming the dimension of the system to be bounded, it is not difficult to design experiments that show that complex numbers are needed to describe the observed data. If the dimension is bounded by two, a sequential Stern-Gerlach experiment will suffice to come to this conclusion [3]. However, if no constraints on the dimension are assumed, it is always possible to explain the measurement data of a single system by a real system of higher dimension. This is even possible for two parties that are restricted to local operations [4, 5]. Only recently it was shown that real quantum theory can be experimentally refuted by designing an experiment in a quantum network involving three parties [6].

Thus, while complex numbers are intrinsically needed to describe quantum mechanics, the role of 'imaginarity' has yet to be fully understood, cf., e.g., Refs. [7–11]. One might ask 'how real' or 'how imaginary' a quantum state can be. The resource theory of imaginarity [12–14] measures imaginarity as a quantity that cannot be increased under the set of free operations, real quantum channels. We approach this fundamental question from a different perspective. In analogy with the qubit Bloch ball description based on the three Pauli matrices [15] we define a diagonal, a real off-diagonal and an purely imaginary coordinate and ask what weight can be put on these coordinates to be compatible with a positive operator. We then completely characterize the set of attainable values for these coordinates and obtain a Bloch ball-type model [16] of the quantum state space.

The Bloch representation of a qubit suggests the quantum space state to be rotation invariant with respect to these three coordinates. As it turns out, this only holds for the real diagonal and real off-diagonal coordinate in arbitrary dimensions, while the imaginary coordinate has to be treated separately. In this article we give a precise quantitative bound on the imaginary Bloch length of a quantum state compared to both its real diagonal and real off-diagonal Bloch components. Surprisingly, we thereby find a qualitative difference between systems of even dimensions and systems of odd dimensions. This difference is most pronounced for small system size and vanishes in the asymptotic limit as one might expect. It is noteworthy that the generic form for systems of odd dimension appears only from dimension 5 onwards.

II. NOTATION

Every quantum state can be written in the following way,

$$\rho \;=\; \frac{1}{d}(\mathbb{1}_d + D + X + I) \quad , \tag{1}$$

where D is a diagonal matrix, X a real off-diagonal matrix and I a purely imaginary (off-diagonal) and skew-symmetric matrix. The operators D, X and I are all Hermitian, traceless, and orthogonal to each other, that is $\mathrm{Tr}(DX) = \mathrm{Tr}(DI) = \mathrm{Tr}(XI) = 0$. For dimension $d = 2$ we recover the Bloch representation in terms of

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the Pauli-matrices, where $D = \langle\sigma_Z\rangle\sigma_Z$, $X = \langle\sigma_X\rangle\sigma_X$ and $I = \langle\sigma_Y\rangle\sigma_Y$. Let us introduce the quantities

$$S_{\mathrm{D}} = \sqrt{\frac{\operatorname{Tr} D^2}{d}}, \qquad S_{\mathrm{X}} = \sqrt{\frac{\operatorname{Tr} X^2}{d}} \qquad S_{\mathrm{I}} = \sqrt{\frac{\operatorname{Tr} I^2}{d}}$$

as weights of the diagonal, real off-diagonal and the imaginary part of the state ρ. For a qubit we recover $S_{\mathrm{D}} = \langle\sigma_Z\rangle$, $S_{\mathrm{X}} = \langle\sigma_X\rangle$ and $S_{\mathrm{I}} = \langle\sigma_Y\rangle$. It holds that

$$\begin{aligned}\operatorname{Tr}\rho^2 &= \frac{1}{d^2}(d + \operatorname{Tr} D^2 + \operatorname{Tr} X^2 + \operatorname{Tr} I^2) \\ &= \frac{1}{d}\left(1 + S_{\mathrm{D}}^2 + S_{\mathrm{X}}^2 + S_{\mathrm{I}}^2\right) \quad . \end{aligned} \tag{2}$$

In dimensions $d > 2$ the quantities S_{D}, S_{X} and S_{I} can be seen as a coarse-graining of the Bloch vector of ρ. Every quantum state ρ can be expanded in an orthogonal matrix basis as

$$\rho = \frac{1}{d}\left(\mathbb{1}_d + \sum_{k=1}^{d^2-1} v_k \mu_k\right) \quad , \tag{3}$$

where we call a basis $\{\mu_k\}_{i=0}^{d^2-1}$ satisfying $\mu_0 = \mathbb{1}_d$ and $\operatorname{Tr}(\mu_k \mu_l^\dagger) = d\delta_{kl}$ a Bloch basis [17, 18]. The vector $[\mathbf{v}]_k = v_k = \langle\mu_k\rangle$ is called the Bloch vector of the state ρ, where $v_0 = 1$ is not included. Choosing a matrix basis that includes only diagonal elements μ_k^{d}, strictly real off-diagonal elements μ_k^{x} and strictly imaginary elements μ_k^{i}, e.g., the generalized Gell-Mann matrices, the state can be expanded as

$$\rho = \frac{1}{d}\left(\mathbb{1}_d + \sum_{k=1}^{d-1} v_k^{\mathrm{d}}\mu_k^{\mathrm{d}} + \sum_{k=1}^{d(d-1)/2} v_k^{\mathrm{x}}\mu_k^{\mathrm{x}} + \sum_{k=1}^{d(d-1)/2} v_k^{\mathrm{i}}\mu_k^{\mathrm{i}}\right), \tag{4}$$

where

$$D = \sum_{k=1}^{d-1} v_k^{\mathrm{d}}\mu_k^{\mathrm{d}} \ , \quad X = \sum_{k=1}^{d(d-1)/2} v_k^{\mathrm{x}}\mu_k^{\mathrm{x}} \ , \quad I = \sum_{k=1}^{d(d-1)/2} v_k^{\mathrm{i}}\mu_k^{\mathrm{i}} \ .$$

It follows immediately that

$$S_{\mathrm{D}}^2 = \sum_{k=1}^{d-1} |v_k^{\mathrm{d}}|^2 \ , \quad S_{\mathrm{X}}^2 = \sum_{k=1}^{d(d-1)/2} |v_k^{\mathrm{x}}|^2 \ , \quad S_{\mathrm{I}}^2 = \sum_{k=1}^{d(d-1)/2} |v_k^{\mathrm{i}}|^2 \ .$$

III. BOUNDS

We now investigate which values of S_{D}, S_{X} and S_{I} are compatible with a quantum state. The purity $\operatorname{Tr}\rho^2$ of a quantum state is bounded by one and therefore from Eq. (2) it follows that,

$$S_{\mathrm{D}}^2 + S_{\mathrm{X}}^2 + S_{\mathrm{I}}^2 \leq d - 1 \quad . \tag{5}$$

For qubits this is the only restriction and the quantum set, that is, the set $\{(S_{\mathrm{D}}(\rho), S_{\mathrm{X}}(\rho), S_{\mathrm{I}}(\rho) : \rho \geq 0,\ \operatorname{Tr}\rho = 1\}$, is rotation invariant. We will see that the same applies in higher dimension only for the values S_{D} and S_{X}, while the coordinate S_{I} plays a special role. We therefore start our investigation by bunching the two real coordinates together to $S_{\mathrm{R}} = \sqrt{S_{\mathrm{D}}^2 + S_{\mathrm{X}}^2}$. It turns out that the constraints we get for the S_{R} and S_{I} values are the only ones that exist, and that the real coordinates are subject to rotational symmetry.

Every quantum state ρ can be split into its real and imaginary parts,

$$\rho = \frac{1}{d}(R + I) \quad , \tag{6}$$

with $R = d(\rho + \rho^T)/2$ and $I = d(\rho - \rho^T)/2$ [12]. We introduce the operator $R = \mathbb{1} + D + X$, a real positive, symmetric operator with $\operatorname{Tr} R = d$ which is orthogonal to the imaginary part, I. Note that, while R includes the identity $\mathbb{1}$, the length S_{R} does not contain its contribution. That is, $S_{\mathrm{R}}^2 = \operatorname{Tr} R^2/d - 1$.

Proposition 1. *For every finite-dimensional quantum state it holds that the quantities S_R and S_I satisfy*

$$S_I^2 \leq 1 + S_R^2 \quad . \tag{7}$$

Proof. It follows from the positivity of ρ and ρ^T that

$$\begin{aligned} 0 \leq d\operatorname{Tr}\left(\rho\rho^T\right) &= \frac{1}{d}\operatorname{Tr}[(R+I)(R-I)] \\ &= \frac{1}{d}(\operatorname{Tr} R^2 - \operatorname{Tr} I^2) = 1 + S_{\mathrm{R}}^2 - S_{\mathrm{I}}^2 \quad . \end{aligned} \tag{8}$$

□

The bound is saturated by states of the form

$$\rho = \begin{pmatrix} \alpha_2 & -i\alpha_2 & & & & & \\ i\alpha_2 i & \alpha_2 & & & & & \\ & & \alpha_4 & -i\alpha_4 & & & \\ & & i\alpha_4 & \alpha_4 & & & \\ & & & & \ddots & & \\ & & & & & \alpha_d & -i\alpha_d \\ & & & & & i\alpha_d & \alpha_d \end{pmatrix} \tag{9}$$

for even dimension and states of the form

$$\rho = \begin{pmatrix} \alpha_2 & -i\alpha_2 & & & & & & \\ i\alpha_2 & \alpha_2 & & & & & & \\ & & \alpha_4 & -i\alpha_4 & & & & \\ & & i\alpha_4 & \alpha_4 & & & & \\ & & & & \ddots & & & \\ & & & & & \alpha_{d-1} & -i\alpha_{d-1} & \\ & & & & & i\alpha_{d-1} & \alpha_{d-1} & \\ & & & & & & & 0 \end{pmatrix} \tag{10}$$

for odd dimensions if $S_{\mathrm{R}} \geq 1/\sqrt{d-1}$. This is easy to see by noting that in both cases $\operatorname{Tr} R^2 = \operatorname{Tr} I^2 = d^2 \sum \alpha_k^2$. This bound is shown in blue in Figs. 1–3.

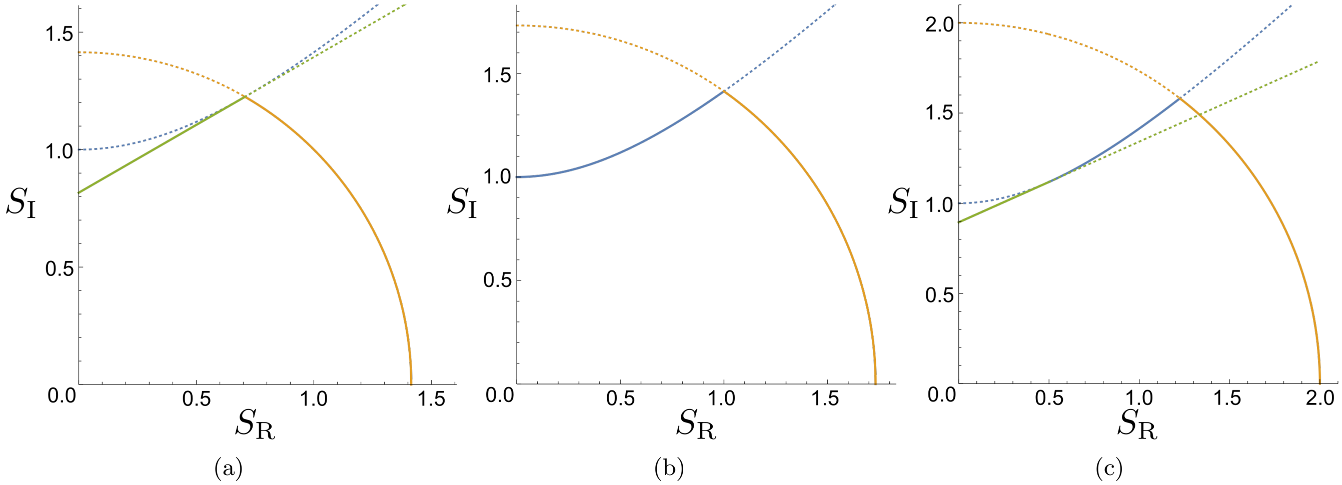

FIG. 1. The three figures display the bound of Eq. (5) in orange, the bound of Eq. (7) in blue and the bound of Eq. (11) in green, (a) $d = 3$, (b) $d = 4$, (c) $d = 5$. Continuous lines show the actual bound, dotted lines are either not valid or not tight.

Proposition 2. *For a quantum state in odd dimension it holds that the quantities S_R and S_I satisfy*

$$\sqrt{d}\; S_I \;\leq\; \sqrt{d-1} \;+\; S_R \quad , \tag{11}$$

whenever $S_R \leq 1/\sqrt{d-1}$.

Before proving this bound in full generality, we first show the case $S_{\mathrm{R}} = 0$. Every state satisfying this can be written as

$$\rho \;=\; \frac{1}{d}(\mathbb{1} + I) \quad . \tag{12}$$

The operator $\mathbb{1} + I$ needs to be positive, that is, its eigenvalues need to be positive. Let U be a unitary diagonalizing I. Since unitary transformations do not change the eigenvalues we can write

$$U(\mathbb{1} + I)U^\dagger \;=\; \mathbb{1} + UIU^\dagger \;=\; \mathbb{1} + I_{\mathrm{diag}} \quad . \tag{13}$$

Therefore it must hold for the eigenvalues of I that $|\lambda_k(I)| \leq 1$. Since I is traceless it further holds that $\sum \lambda_k(I) = 0$. The quantity $\operatorname{Tr} I^2 = \sum \lambda_k(I)^2$ is convex, hence maximizing it gives a solution on the boundary. Under the constraints $\sum \lambda_k(I) = 0$ $|\lambda_k(I)| \leq 1$ we find the optimal solution to be $\lambda_{2k-1}(I) = -\lambda_{2k}(I) = 1$ with $\lambda_d(I) = 0$ for odd dimension. Therefore,

$$S_{\mathrm{I}}^2 \;=\; \frac{1}{d}\operatorname{Tr} I^2 \;\leq\; \frac{2}{d}\left\lfloor\frac{d}{2}\right\rfloor \quad , \tag{14}$$

so that $S_{\mathrm{I}}^2 \leq 1$ for even dimension and $S_{\mathrm{I}}^2 \leq (d-1)/d$ otherwise.

Subsequently we prove Proposition 2 in the general case.

Proof. We know that I is a skew-symmetric matrix and that it has the same eigenvalues as its transpose. Therefore the eigenvalues of I come in pairs with opposite sign, with an additional 0 for odd dimensions. Let $\lambda_k(I)$ be ordered such that $\lambda_{2k}(I) = -\lambda_{2k-1}(I)$. We also know that both $R + I$ and $R - I$ are positive matrices. Let U be a unitary diagonalizing $I = UI_{\mathrm{diag}}U^\dagger$. It then holds that $U^\dagger RU \pm I_{\mathrm{diag}} \geq 0$.

Since R and I are orthogonal to each other, also $\tilde{R} = U^\dagger RU$ and I_{diag} are orthogonal,

$$\begin{aligned} 0 &= \operatorname{Tr}(RI) = \operatorname{Tr}(RUI_{\mathrm{diag}}U^\dagger) = \operatorname{Tr}(U^\dagger RUI_{\mathrm{diag}}) = \\ &= \operatorname{Tr}(\tilde{R}I_{\mathrm{diag}}) = \sum_{k=1}^{d} \tilde{R}_{kk}(I_{\mathrm{diag}})_{kk} = \sum_{k=1}^{d} \tilde{R}_{kk}\lambda_k(I) \ . \end{aligned} \tag{15}$$

It holds that $\tilde{R}_{kk} \geq |\lambda_k(I)|$ and it is known that the diagonal $\tilde{R}_{kk}$ is majorized by the eigenvalues $\lambda_k(\tilde{R}) = \lambda_k(R)$. It then holds that $\sum_{k=1}^{d} \tilde{R}_{kk}^2 \leq \sum_{k=1}^{d} \lambda_k(R)^2$ and we conclude

$$\operatorname{Tr} I^2 = \sum \lambda_k(I)^2 \leq \sum_{k=1}^{d-1} \tilde{R}_{kk}^2 \leq \sum_{k=1}^{d-1} \lambda_k(R)^2 \leq \operatorname{Tr} R^2 \ . \tag{16}$$

With this we recover the previous result. The first and the second inequality are saturated if all eigenvalues satisfy $\lambda_{2k-1}(I) = -\lambda_{2k}(I) = \lambda_{2k-1}(R) = \lambda_{2k}(R)$. For the third inequality to be saturated we have to choose $\lambda_d(R) = 0$ in odd dimensions. But this implies $S_{\mathrm{R}} \geq 1/\sqrt{d-1}$ so that the bound cannot be tight for $S_{\mathrm{R}} < 1/\sqrt{d-1}$. Set now the smallest eigenvalue of R $\lambda_d(R) = t \leq 1$, it immediately follows that $\operatorname{Tr} R^2 \geq d(d - 2t + t^2)/(d-1)$ where the minimum is attained for $\lambda_k(R) = (d-t)/(d-1)$. For a given $\operatorname{Tr} R^2$ it must therefore hold that

$$t \;\geq\; 1 - \frac{\sqrt{(d-1)\operatorname{Tr} R^2 - d^2 + d}}{\sqrt{d}} \;=\; 1 - \sqrt{d-1}S_{\mathrm{R}} \ . \tag{17}$$

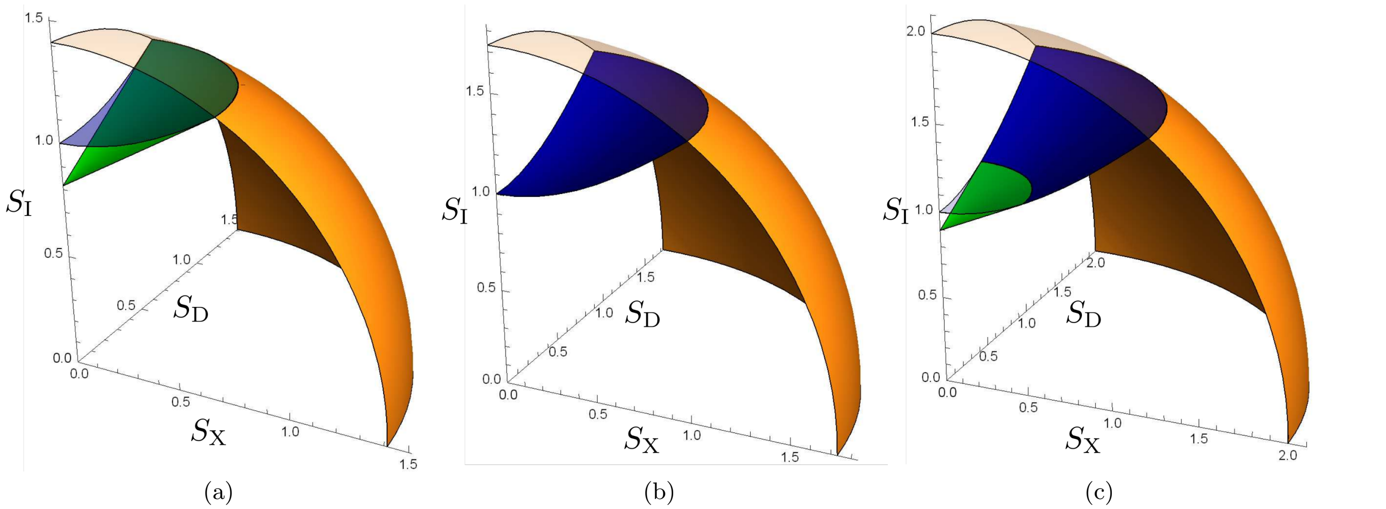

FIG. 2. The parametrization of qudit states in Eqs. (1)–(4) defines a three-dimensional model of the state space. The pure states saturate the bound in Eq. (5). They are located on the surface of the model (shown in orange) and have a minimum real coordinate $S_{\mathrm{R}} \geq \sqrt{(d-2)/2}$. This is because there is no pure-state projector corresponding to a superposition with (nearly) purely imaginary off-diagonal elements. The shape of the mixed-state surface part close to the imaginary axis varies as a function of the dimension. (a) $d = 3$: The boundary is described by the linear bound Eq. (11) (green), the quadratic bound Eq. (7) (green) is not tight for any mixed state. (b) $d = 4$: For even dimensions the surface is given by the quadratic bound (blue). (c) $d = 5$: The generic case for odd dimensions where for small S_{R} the surface is conic (green) and for $S_{\mathrm{R}} \geq 1/\sqrt{d-1}$ saturates the quadratic bound (blue).

It now holds in odd dimensions for $S_{\mathrm{R}} \leq 1/\sqrt{d-1}$,

$$
\begin{aligned}
dS_{\mathrm{I}}^2 &= \mathrm{Tr}\, I^2 = \sum \lambda_k(I)^2 \leq \sum_{k=1}^{d-1} \lambda_k(R)^2 = \mathrm{Tr}\, R^2 - t^2 \leq \\
&\leq d + dS_{\mathrm{R}}^2 - (1-\sqrt{d-1}S_{\mathrm{R}})^2 = (\sqrt{d-1}+S_{\mathrm{R}})^2,
\end{aligned}
$$

which proves the result. □

The inequality (11) for odd dimensions and $S_{\mathrm{R}} \leq 1/\sqrt{d-1}$ is saturated by states of the form

$$
\rho = \begin{pmatrix}
\alpha & -i\alpha & & & & & & \\
i\alpha & \alpha & & & & & & \\
& & \alpha & -i\alpha & & & & \\
& & i\alpha & \alpha & & & & \\
& & & & \ddots & & & \\
& & & & & \alpha & -i\alpha & \\
& & & & & i\alpha & \alpha & \\
& & & & & & & 1-(d-1)\alpha
\end{pmatrix} \tag{18}
$$

with $1/(d-1) \geq \alpha \geq 1/d$. To see this it suffices to note that $\mathrm{Tr}\, R^2 = d^2(d(d-1)\alpha^2 - 2(d-1)\alpha + 1)$ and $\mathrm{Tr}\, I^2 = d^2(d-1)\alpha^2$. Figures 1–3 show this bound in green.

For even dimension $d > 2$, Eqs. (5) and (7) describe the boundary, they intersect at $S_{\mathrm{R}} = \sqrt{(d-2)/2}$. For odd dimension the boundary (for $S_{\mathrm{R}} \leq 1/\sqrt{d-1}$) is described by Eq. (11), it tangentially touches the bound in Eq. (7) at $S_{\mathrm{R}} = 1/\sqrt{d-1}$.

Going back to our original coordinates S_{D}, S_{X} and S_{I}, we immediately find the following result.

Corollary 1. *The three quantities S_D, S_X and S_I satisfy the bounds*

$$
S_I^2 \leq 1 + S_D^2 + S_X^2 \tag{19a}
$$

$$
\sqrt{d} S_I \leq \sqrt{d-1} + \sqrt{S_D^2 + S_X^2}\ , \tag{19b}
$$

where the last inequality holds only for $S_D^2 + S_X^2 \leq 1/(d-1)$.

Proof. The bounds follows trivially from Eq. (7) and (11) by replacing $S_{\mathrm{R}}^2 = S_{\mathrm{D}}^2 + S_{\mathrm{X}}^2$. □

Proposition 3. *The bounds in Corollary 1 define a tight upper bound. Together with Eq. (5) they are optimal, in the sense that there exist no further constraints on the quantities S_D, S_X and S_I that do not trivially follow from them.*

Proof. The states in Eq. (9),(10) and (18) have no real off-diagonal contribution and thus saturate the bounds in Corollary 1 for $S_{\mathrm{X}} = 0$, together with the pure states $\beta|0\rangle + i\sqrt{1-\beta^2}|1\rangle$, where $0 \leq \beta \leq 1$.

We will now define a continuous orthogonal transformation that shifts weight from the diagonal component to the real off-diagonal component as long as $S_{\mathrm{D}} > 0$. Note that any orthogonal (i.e., real) transformation leaves the coordinates S_{R} and S_{I} unchanged. Consider an orthogonal transformation $O_{kl}(\theta) = \cos\theta(|k\rangle\langle k| + |l\rangle\langle l|) + \sin\theta(|k\rangle\langle l| - |l\rangle\langle k|)$ acting on the subspace spanned by the

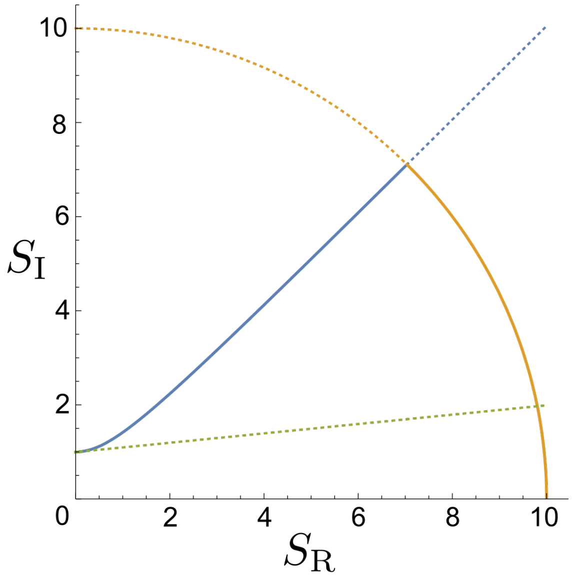

FIG. 3. The figure shows the bound of Eq. (5) in orange, the bound of Eq. (7) in blue and the bound of Eq. (11) in green for dimension $d = 101$.

states $\{|k\rangle, |l\rangle\}$. Start acting with this transformation on the first and the last entry until either one of the new diagonal entries becomes $1/d$. If it is the last entry we perform a second orthogonal transformation from $|0\rangle$ to $|d-2\rangle$ until one of them reaches $1/d$. If it is the first entry we apply the transformation from $|1\rangle$ to $|d-1\rangle$. We repeat this process until all diagonal entries are equal to $1/d$. This procedure continuously shifts weight from the diagonal coordinate $S_{\rm D}$ onto the real off-diagonal coordinate $S_{\rm X}$ until the diagonal coordinate vanishes. The orbits of the states in Eqs. (9), (10) and (18) form the boundary. This concludes the proof.

□

IV. BLOCH BALL FOR A QUDIT

We now discuss the geometric implications of our investigation. In recent years considerable effort was devoted to the question of constructing models for the quantum state space of dimension d, e.g., Refs. [16, 19–26]. That is, one asks for the shape of this (d^2-1)-dimensional object if it is represented in a lower dimension, thereby maintaining as many geometric properties of the original state space as possible. Because of the reduction in the number of parameters for dimensions $d \geq 3$ it is clear that any model, while correctly displaying some of the properties of the state space, will suffer from a loss of information and inadequately represent other features. Consequently, any model is the result of a judicious choice of the coordinates, depending on the set of properties that are to be maintained. The corresponding objects may have rather different shapes.

Our work provides a striking illustration for these considerations. As is evident from Fig. 2, our parametrization of the qudit state Eqs. (1), (2) in imaginary, real diagonal and off-diagonal coordinates leads to a three-dimensional model for the qudit state space for all $d \geq 2$. While, e.g., the model in Ref. [19] emphasizes the simplex of diagonal states (in $d = 3$) the present model focuses on the bounds for the imaginary coordinates and, for example, is not convex for $d \geq 3$. Note that convexity is restored for $d = 3$ if one uses just one real coordinate S_R (instead of S_X and S_D), as can be observed in Fig. 1 (a).

Another salient feature of the present model is its rotation invariance about the imaginary axis, cf. Fig. 2. It is a direct consequence of the unitary invariance of the purity, Eq. (2): Any transformation that leaves S_I unchanged, necessarily preserves also the radius $S_{\rm R}$ in the real plane.

Intriguingly, the bounds in Eqs. (7), (11) reveal that the geometry of the state space boundary is different in even vs. odd dimensions: While for odd dimensions there is always a flat surface part (of mixed states) close to the most distant point on the imaginary axis, the surface is curved in the corresponding region for even dimensions, cf. Fig. 2 (b). We note that the concave part of the model in $d = 3$ is completely described by the linear bound Eq. (11). Hence the first dimension for which the generic odd-d behavior is observed is $d = 5$: In the vicinity of $S_{\rm I} = \sqrt{(d-1)/d}$ the linear bound describes the surface before the quadratic bound, Eq. (7), takes effect for $S_{\rm R} = 1/\sqrt{d-1}$. The linear surface part remains present for all odd dimensions but becomes negligibly small for large d, see Fig. 3. At the same time the restriction imposed by Eq. (7) becomes more stringent. The respective part in $(S_{\rm R}, S_{\rm I})$ coordinates that remains compatible with a quantum state decreases, until asymptotically only half of the original circular area defined by the purity bound in Eq. (2) remains, see Fig. 3.

V. RELATION TO RESOURCE THEORY

Recently, the resource theory of imaginarity has received considerable attention [12–14]. A measure of imaginarity is a quantity that is non-increasing under real quantum channels, i.e., completely positive and trace-preserving (CPTP) maps that allow a description with purely real Kraus operators. The robustness of imaginarity is defined as $\mathcal{I}(\rho) = \|I/d\|_1$ and the geometric measure of imaginarity is defined as $\mathcal{I}_G(\psi) = 1 - \sup_{\phi\in\mathcal{R}} |\langle\phi|\psi\rangle|^2$, with the convex roof extension for mixed states. While these quantities are imaginarity monotones, our imaginary coordinate $\|\rho_{\rm I}\|_2$ is not an imaginary measure [9]. This stems from the fact that the Hilbert-Schmidt norm is generally not contractive under CPTP maps for dimension larger than two [27], in contrast to the trace norm [28]. Our goal is not the study of imaginarity as a resource, rather we characterize the constraints on coordinates describing the quantum state space. Nonetheless, our analysis establishes a relation between the real weight and the robustness of imaginarity $\mathcal{I}$ for odd dimensions. For $S_R \geq 1/\sqrt{d-1}$ there exist states that achieve $\mathcal{I} = 1$ (cf. the states saturating the bound in Proposition 1), while for $S_R = 0$ no state can reach full imaginarity. The question remains open for the region $0 < S_R < 1/\sqrt{d-1}$, as none of our presented states reaches maximum imaginarity. On the other hand, for even dimensions, no restriction exists concerning real weight and imaginarity, showing again remarkable differences between even and odd dimensions.

VI. CONCLUSION

We generalize the Bloch coordinates of a qubit describing the diagonal, real off-diagonal and imaginary part of a quantum state to arbitrary finite dimensions. We completely characterize the set of coordinates compatible with a quantum state. While for the simplest quantum system of a qubit these coordinates are interchangeable, this property is lost in higher dimensions. The imaginary coordinate plays a distinctive role and is bounded with respect to the other coordinates. Surprisingly, this bound exhibits a qualitative difference between even and odd dimensions, where the general bound for odd dimensions is first observed in dimension 5. With these findings we obtain a three-dimensional model of the generally higher-dimensional quantum state space that joins the already established models. It further illustrates the peculiar properties of finite-dimensional quantum state spaces and represents an important step towards their understanding.

VII. ACKNOWLEDGEMENTS

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