Ejercicio 3.14.8.3

$$\int_a^b \frac{(x-b)(x-x_m)}{(a-b)(a-x_m)} f(a) + \frac{(x-a)(x-b)}{(x_m-a)(x_m-b)} f(x_m) + \frac{(x-a)(x-x_m)}{(b-a)(b-x_m)} f(b) \, dx \quad , \text{ con } x_m = \frac{a+b}{2}$$

$$= \frac{f(a)}{(a-b)(a-x_m)} \int_a^b x^2 - x_m x - bx + x_m b \, dx + \frac{f(x_m)}{(x_m-a)(x_m-b)} \int_a^b x^2 - ax - bx + ab \, dx + \frac{f(b)}{(b-a)(b-x_m)} \int_a^b x^2 - ax - x_m x + x_m a \, dx$$

$$= \frac{f(a)}{(a-b)(a-x_m)} \left[\frac{b^3}{3} - \frac{x_m b^2}{2} - \frac{b^3}{2} + x_m b^2 - \frac{a^3}{3} + \frac{x_m a^2}{2} + \frac{ba^2}{2} - x_m ba \right] + \frac{f(x_m)}{(x_m-a)(x_m-b)} \left[\frac{b^3}{3} - \frac{ab^2}{2} - \frac{b^3}{2} + ab^2 - \frac{a^3}{3} + \frac{a^3}{2} + \frac{ba^2}{2} - a^2 b \right] + \frac{f(b)}{(b-a)(b-x_m)} \left[\frac{b^3}{3} - \frac{ab^2}{2} - \frac{x_m b^2}{2} + x_m ab - \frac{a^3}{3} + \frac{a^3}{2} + \frac{x_m a^2}{2} - x_m a^2 \right]$$

$$= \frac{-f(a)}{6(a-b)(a-x_m)} \cdot \left(\frac{-b^3}{2} + \frac{3ab^2}{2} - \frac{3a^2 b}{2} + \frac{a^3}{2} \right) + \frac{f(x_m)}{6(x_m-a)(x_m-b)} \cdot (a^3 - 3a^2 b + 3ab^2 - b^3)$$

$$- \frac{f(b)}{6(b-a)(b-x_m)} \cdot \left(\frac{-b^3}{2} + \frac{3ab^2}{2} - 3a^2 b + \frac{a^3}{2} \right)$$

$$= \frac{f(a)}{12(a-b)(x_m-a)} \cdot (a-b)^3 + \frac{f(x_m)}{6(x_m-a)(x_m-b)} \cdot (a-b)^3 + \frac{f(b)}{12(a-b)(b-x_m)} \cdot (a-b)^3$$

$$= \frac{(a-b)^2}{6} \left[\frac{f(a)}{2(x_m-a)} + \frac{f(x_m)}{(x_m-a)(x_m-b)} \cdot (a-b) + \frac{f(b)}{2(b-x_m)} \right]$$

Si $x_m - a = b - x_m = h$, entonces:

$$= \frac{(2h)^2}{6} \left(\frac{f(a)}{2h} - \frac{f(x_m)}{h^2} \cdot (-2h) + \frac{f(b)}{2h} \right) = \frac{2h}{3} \left(\frac{f(a)}{2} + 2f(x_m) + \frac{f(b)}{2} \right)$$

$$= \frac{h}{3} \left(f(a) + 4f(x_m) + f(b) \right)$$