1. Demostrar $\frac{d^2 f(x_i)}{dx^2} = \frac{f(x_{i+2}) - 2f(x_i) + f(x_{i-2})}{4h^2}$

Sabemos que $\frac{d^2 f(x_j)}{dx^2} = \frac{d}{dx}\left(\frac{d}{dx} f(x_j)\right)$ y que

$$\frac{d}{dx} f(x_j) = \frac{f(x_{j+1}) - f(x_{j-1})}{2h}$$

Entonces, $\frac{d^2 f(x_j)}{dx^2} = \frac{d}{dx}\left(\frac{f(x_{j+1}) - f(x_{j-1})}{2h}\right)$

Sea $g(x_j) = \frac{f(x_{j+1}) - f(x_{j-1})}{2h}$:

$$\frac{d^2 f(x_j)}{dx^2} = \frac{d}{dx} g(x_j) = \frac{g(x_{j+1}) - g(x_{j-1})}{2h}$$

Pero $g(x_{j+1}) = \frac{f(x_{j+2}) - f(x_j)}{2h}$ y $g(x_{j-1}) = \frac{f(x_j) - f(x_{j-2})}{2h}$

Entonces $\frac{d^2 f(x_j)}{dx^2} = \frac{\left(\frac{f(x_{j+2}) - f(x_j)}{2h}\right) - \left(\frac{f(x_j) - f(x_{j-2})}{2h}\right)}{2h}$

$$= \frac{\frac{f(x_{j+2}) - 2f(x_j) + f(x_{j-2})}{2h}}{2h} \boxed{= \frac{f(x_{j+2}) - 2f(x_j) + f(x_{j-2})}{4h^2}}$$

Q.E.D.

5. Demostrar $D^4(x_j)$

Hallamos la serie de Taylor de las siguientes funciones:

$$f(x+2h) = f(x) + 2hf'(x) + \frac{2^2h^2}{2}f''(x) + \frac{2^3h^3}{3!}f'''(x) + \frac{2^4}{4!}f^4(x) + \underbrace{O(h^5)}_{h^5(2^5/5!f^5(x)+\dots}\ 1$$

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^4(x) + \underbrace{O(h^5)}_{\frac{h^5}{5!}f^5(x)+\dots}\ 2$$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^4(x) + \underbrace{O(h^5)}_{-\frac{h^5}{5!}f^5(x)+\dots}\ 3$$

$$f(x-2h) = f(x) - 2hf'(x) + \frac{2^2h^2}{2}f''(x) - \frac{2^3h^3}{3!}f'''(x) + \frac{2^4f^4(x)}{4!} + \underbrace{O(h^5)}_{-h^5\frac{2^5}{5!}f^5(x)+\dots}\ 4$$

Sabiendo esto,

$$f(x+2h) - 4f(x+h) + 6f(x) - 4f(x-h) + f(x+2h)$$

$$= \left[f(x) + 2hf'(x) + \frac{2^2h^2}{2}f''(x) + \frac{2^3h^3}{3!}f'''(x) + \frac{2^4f^4(x)h^4}{4!} + O(h^5)^1\right]$$

$$-4\left[f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^4(x) + O(h^5)^2\right]$$

$$+6f(x)$$

$$-4\left[f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^4(x) + O(h^5)^3\right]$$

$$+\left[f(x) - 2hf'(x) + \frac{2^2h^2}{2}f''(x) - \frac{2^3h^3}{3!}f'''(x) + \frac{2^4f^4(x)h^4}{4!} + O(h^5)^4\right]$$

$$= [\underbrace{f(x) - 4f(x) + 6f(x) - 4f(x) + f(x)}_{=0}]$$

$$+ [\underbrace{2hf'(x) - 4hf'(x) + 4hf'(x) - 2hf'(x)}_{0}]$$

$$+ [\underbrace{4\frac{h^2}{2}f''(x) - 4\frac{h^2}{2}f''(x) - 4\frac{h^2}{2}f''(x) + 4\frac{h^2}{2}f''(x)}_{0}]$$

$$+ [\underbrace{8\frac{h^3}{3!}f'''(x) - 4\frac{h^3}{3!}f'''(x) + 4\frac{h^3}{3!}f'''(x) - 8\frac{h^3}{3!}f'''(x)}_{0}]$$

$$+ [\underbrace{16\frac{h^4}{4!}f^4(x) - 4\frac{h^4}{4!}f^4(x) - 4\frac{h^4}{4!}f^4(x) + 16\frac{h^4}{4!}f^4(x)}_{24 \cdot \frac{h^4}{4!}f^4(x) = h^4f^4(x)}]$$

$$+\underbrace{\left[32\frac{h^5}{5!}f^5(x)-4\frac{h^5}{5!}f^5(x)+4\frac{h^5}{5!}f^5(x)-32\frac{h^5}{5!}f^5(x)\right]}_{=0}$$

Toca ver el termino $f^{(6)}(x)$:

$$\underbrace{\left[2^6\frac{h^6}{6!}f^6(x)-4\frac{h^6}{6!}f^6(x)-4\frac{h^6}{6!}f^6(x)+2^6\frac{h^6}{6!}f^6(x)\right]}_{=\frac{120}{6!}h^6f^6(x)=\frac{1}{6}h^6f^6(x)}$$

$$\Rightarrow f(x+2h)-4f(x+h)+6f(x)-4f(x-h)+f(x-2h)=h^4f^4(x)+\frac{1}{6}h^6f^6(x)+\ldots$$

$$\Rightarrow f^4(x)+\frac{1}{6}h^2f^6(x)=\frac{f(x+2h)-4f(x+h)+6f(x)-4f(x-h)+f(x-2h)}{h^4}$$

$$\Rightarrow f^4(x)=\frac{f(x+2h)-4f(x+h)+6f(x)-4f(x-h)+f(x-2h)}{h^4}\underbrace{-\frac{1}{6}h^2f^6(x)}_{O(h^2)} \quad (1)$$

Discretizando para algún punto de la partición:

$$\Rightarrow \boxed{D^4f(x_j)\cong\frac{f(x_{j+2})-4f(x_{j+1})+6f(x_j)-4f(x_{j-1})+f(x_{j-2})}{h^4}} \quad Q.E.D.$$

$\Rightarrow$ Como se ve en (1), el orden de aproximación del operador $D^4f(x_j)$ es $O(h^2)$