Punto 25 (Teórico) Parcial 2:

* Vamos a hallar a mano la regla de cuadratura de Laguerre de 2 puntos

(a) Usemos la formula de Rodrigues para los polinomios de Laguerre para hallar el de orden 2.

▸ $\mathcal{L}_n(x) = \dfrac{e^x}{n!} \cdot \dfrac{d^n}{dx^n}\left(x^n e^{-x}\right)$ ... (Formula de Rodrigues)

Para orden 2: $\mathcal{L}_2(x) = \dfrac{e^x}{2!} \cdot \dfrac{d^2}{dx^2}\left(x^2 e^{-x}\right)$

$$\hookrightarrow \mathcal{L}_2(x) = \frac{e^x}{2!} \cdot \left(e^{-x}x^2 - 4e^{-x}x + 2e^{-x}\right)$$

$$= \frac{\cancel{e^x}}{2!}\left(\cancel{e^{-x}}\left(x^2 - 4x + 2\right)\right)$$

$$\mathcal{L}_2(x) = \frac{x^2}{2} - 2x + 1 = \frac{1}{2}\left(x^2 - 4x + 2\right)$$

(b) Hallar las raices de $\mathcal{L}_2(x)$:

$\mathcal{L}_2(x) = 0 \Rightarrow \frac{1}{2}(x^2 - 4x + 2) = 0$ $\left\{\begin{array}{l} a = 1 \\ b = -4 \\ c = 2 \end{array}\right.$

$$x_0 = \frac{+4 - \sqrt{16-8}}{2} = \frac{4 - 2\sqrt{2}}{2} = 2 - \sqrt{2}$$

$$x_1 = \frac{+4 + \sqrt{16-8}}{2} = \frac{4 + 2\sqrt{2}}{2} = 2 + \sqrt{2}$$

(c) Vamos a integrar para hallar los pesos de la cuadratura

$\sigma(x) = e^{-x}$, $x_0 = 2-\sqrt{2}$, $x_1 = 2+\sqrt{2}$

Por Partes $\begin{pmatrix} u = x-2-\sqrt{2} \\ dv = e^{-x} \end{pmatrix}$

Peso 1 $$W_0 = \int_0^\infty e^{-x}\left(\frac{x-2-\sqrt{2}}{2-\sqrt{2}-2-\sqrt{2}}\right)dx = \frac{1}{-2\sqrt{2}}\int_0^\infty e^{-x}(x-2-\sqrt{2})\,dx$$

$$= \frac{1}{2\sqrt{2}}\left(e^{-x}(x-2-\sqrt{2}) - \int e^{-x}\,dx\right)$$

$$= \frac{1}{2\sqrt{2}}\left(e^{-x}(x-2-\sqrt{2}+1)\right)\Big|_0^\infty \quad \left[\text{Integral sin evaluar}\right]$$

$$= \frac{1}{2\sqrt{2}}\left(\lim_{t\to\infty} e^{-t}(t-1-\sqrt{2}) - \left(e^{-0}(0-1-\sqrt{2})\right)\right)$$

$$= \frac{1}{2\sqrt{2}}\left(\left(\lim_{t\to\infty}\frac{t-1-\sqrt{2}}{\frac{1}{e^x}}\right) + \left(1+\sqrt{2}\right)\right)$$

$$\frac{1}{1}$$

$$= \frac{1}{2\sqrt{2}}\left(\lim_{t\to\infty} e^{-t} + 1 + \sqrt{2}\right) = \underbrace{\frac{1+\sqrt{2}}{2\sqrt{2}} \approx 0.853553}_{\text{Primer Peso ¡¡}}$$

Por Partes $\begin{pmatrix} u = x-2+\sqrt{2} \\ dv = e^{-x} \end{pmatrix}$

Peso 2 $$W_1 = \int_0^\infty e^{-x}\left(\frac{x-2+\sqrt{2}}{2+\sqrt{2}-2+\sqrt{2}}\right)dx = \frac{1}{2\sqrt{2}}\int_0^\infty e^{-x}(x-2+\sqrt{2})\,dx$$

$$= \frac{1}{2\sqrt{2}}\left(-e^{-x}(x-2+\sqrt{2}) + \int e^{-x}\,dx\right) = \frac{1}{2\sqrt{2}}\left(e^{-x}(-x+2-\sqrt{2}-1)\right)\Big|_0^\infty$$

$$= \frac{1}{2\sqrt{2}}\left[\left(\lim_{t\to\infty}\frac{-t+\cancel{1-\sqrt{2}}}{\frac{1}{e^t}}\right) + \left(e^0(-0+1-\sqrt{2})\right)\right]$$

$$= \frac{-1}{2\sqrt{2}}\left[\left(\lim_{t\to\infty} e^{-t}\right) + 1 - \sqrt{2}\right] = \underbrace{-\frac{1+\sqrt{2}}{2\sqrt{2}} \approx 0.146446}_{\text{Segundo peso !!}}$$

$W_0 = 0.853553$

$W_1 = 0.146446$ Corresponden a los pesos de orden 2

(d) Mostrar que la regla para el polinomio de grado 3 es:

$$\int_0^{\infty} e^{-x} x^3 \, dx = \sum_{i=0}^{1} \omega_i f(x_i) = 6$$

▶ Usemos primero la funcion Gamma para resolver la integral

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{n-1} \, dt = (n-1)!$$

(remplazo x por n para no confundir)

☆ Notemos que $t^{n-1} = x^3 \rightarrow n = 4$, entonces: ya que ambos acompañan a la e^{-x}.

$$(n-1)! = (4-1)! = 3! = 6 = \Gamma(4)$$

Usemos la regla de cuadratura:

$$\int_0^\infty e^{-x} x^3 dx = w_0 f(x_0) + w_1 f(x_1) = \frac{1+\sqrt{2}}{2\sqrt{2}}(2-\sqrt{2})^3 + \frac{-1+\sqrt{2}}{2\sqrt{2}}(2+\sqrt{2})^3$$

$$= \frac{(1+\sqrt{2})(2-\sqrt{2})^3 + (-1+\sqrt{2})(2+\sqrt{2})^3}{2\sqrt{2}}$$

$$= \frac{(1+\sqrt{2})(8-4\sqrt{2}+4-2\sqrt{2}) + (-1+\sqrt{2})(8+4\sqrt{2}+4+2\sqrt{2})}{2\sqrt{2}}$$

$$= \frac{(1+\sqrt{2})(12-6\sqrt{2}) + (-1+\sqrt{2})(12+6\sqrt{2})}{2\sqrt{2}}$$

$$= \frac{\cancel{12} - 6\sqrt{2} + 12\sqrt{2} - \cancel{12} - \cancel{12} - 6\sqrt{2} + 12\sqrt{2} + \cancel{12}}{2\sqrt{2}}$$

$$= \frac{24\sqrt{2} - 12\sqrt{2}}{2\sqrt{2}} = \frac{12\cancel{\sqrt{2}}}{2\cancel{\sqrt{2}}} \boxed{= 6}$$

Vemos que entonces $\Gamma(4) = \sum_{i=0}^{1} w_i f(x_i) = \int_0^\infty e^{-x} x^3 dx = 6$

$\boxed{\text{Q.E.D.}}$