Ejercicios sección 3.14

6. a) $\langle e^i | e_j \rangle = \delta^i_j$

Como $\langle e^i|$ es ortogonal a todo $|e_j\rangle$ entonces uní $\langle e^i|$ se puede escribir como $\langle e^i| = \alpha (e_j \times e_k)$

y $\langle e^i | e_i \rangle = 1$, entonces:

$$\alpha (e_j \times e_k) \cdot e_i = 1 \to \alpha = \frac{1}{(e_j \times e_k) \cdot e_i}$$

Entonces $$e^i = \frac{e_j \times e_k}{e_i \cdot (e_j \times e_k)}$$

$$\langle e^i | e_m \rangle = \frac{(e_j \times e_k) \cdot e_m}{e_i \cdot (e_j \times e_k)} \quad \text{Si } m = i$$

$$\frac{(e_j \times e_k) \cdot e_i}{e_i \cdot (e_j \times e_k)}$$, permutación cíclica, por tanto $\langle e^i | e_m \rangle = 1$

Si $m \neq i$ se repiten dos vectores,

por tanto $\langle e^i | e_m \rangle = 0$

lo cual cumple con la definición

$\langle e^i | e_m \rangle = \delta^i_m$

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b) $V\tilde{V} = (e_1 \cdot (e_2 \times e_3))(e^1 \cdot (e^2 \times e^3))$

$= e_1 \cdot (e_2 \times e_3)\left(\left(\frac{e_2 \times e_3}{V}\right) \cdot \left(\left(\frac{e_3 \times e_1}{V}\right) \times \left(\frac{e_1 \times e_2}{V}\right)\right)\right)$

$(e_3 \times e_1) \times (e_1 \times e_2) \Rightarrow$ Sea $c = e_3 \times e_1$

$c \times (e_1 \times e_2) = \varepsilon_{ijk} c_j \varepsilon_{kmn} e_1^m e_2^n$

$= \varepsilon_{ijk} \varepsilon_{mnk} c_j e_1^m e_2^n = (\delta_m^i \delta_n^j - \delta_n^i \delta_m^j) c_j e_1^m e_2^n$

$= \delta_m^i e_1^m \delta_n^j c_j e_2^n - \delta_m^j c_j e_1^m \delta_n^i e_2^n = e_1^i c_j e_2^j$

$- c_m e_1^m e_2^i$

$= e_1 (c \cdot e_2) - e_2 (c \cdot e_1)$

$= e_1((e_3 \times e_1) \cdot e_2) - e_2 \cancel{((e_3 \times e_1) \cdot e_1)}^{0}$

$= e_1((e_3 \times e_1) \cdot e_2) = e_1 V$

$\left(\frac{e_2 \times e_3}{V}\right) \cdot \left(\frac{(e_3 \times e_1) \times (e_1 \times e_2)}{V^2}\right) = \frac{1}{V^3} (e_2 \times e_3) \cdot e_1 V$

$= \frac{1}{V^2} (e_2 \times e_3) \cdot e_1 = \frac{1}{V^2}(V) = \frac{1}{V}$

$V\tilde{V} = e_1 \cdot (e_2 \times e_3)\left(\frac{1}{V}\right) = V\left(\frac{1}{V}\right) = \boxed{1}$

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c) $a \cdot e^i = 1$ $\quad a = a^j e_j$

$\langle a^j e_j | e^i \rangle = a^j \langle e_j | e^i \rangle = a^j \delta_j^i = a^i$

Entonces $a \cdot e^i = 1 = a^i$ por tanto $a^i = 1 \ \forall i$

Entonces $a = e_1 + e_2 + e_3$

Este es único porque la descomposición en la base es única

Además, consideremos entonces que existe otro $\vec{b}$ tal que $b \cdot e^i = 1$

entonces $d = a - b$ es la diferencia entre ambos

$d \cdot e^i = (a-b) \cdot e^i = a \cdot e^i - b \cdot e^i = 1 - 1 = 0$

Es decir $d \cdot e^i = 0 \ \forall i$

Sea $d = d^j e_j \to d \cdot e^i = d^j \langle e_j | e^i \rangle = d^j \delta_j^i = d^i = 0 \ \forall i$

Por tanto, $d^1 = d^2 = d^3 = 0$, entonces $d = a - b = 0 \to a = b$ $\square$

d)

I. $e^1 = \dfrac{(3\hat\imath + 3\hat\jmath) \times (2\hat k)}{(4\hat\imath + 2\hat\jmath + \hat k) \cdot [(3\hat\imath + 3\hat\jmath) \times (2\hat k)]} = \dfrac{(-6\hat\jmath + 6\hat\imath)}{(4\hat\imath + 2\hat\jmath + \hat k) \cdot (6\hat\imath - 6\hat\jmath)}$

$= \dfrac{6(\hat\imath - \hat\jmath)}{24 - 12} = \dfrac{6(\hat\imath - \hat\jmath)}{12} = \dfrac{(\hat\imath - \hat\jmath)}{2}$

$e^2 = \dfrac{(2\hat k) \times (4\hat\imath + 2\hat\jmath + 2\hat k)}{(3\hat\imath + 3\hat\jmath) \cdot [(2\hat k) \times (4\hat\imath + 2\hat\jmath + 2\hat k)]} = \dfrac{8\hat\jmath - 4\hat\imath}{(3\hat\imath + 3\hat\jmath) \cdot (-4\hat\imath + 8\hat\jmath)} = \dfrac{4(2\hat\jmath - \hat\imath)}{-12 + 24} =$
$\dfrac{(2\hat\jmath - \hat\imath)}{3}$

$e^3 = \dfrac{(4\hat\imath + 2\hat\jmath + \hat k) \times (3\hat\imath + 3\hat\jmath)}{(2\hat k) \cdot [(4\hat\imath + 2\hat\jmath + \hat k) \times (3\hat\imath + 3\hat\jmath)]} = \dfrac{12\hat k - 6\hat k + 3\hat\jmath - 3\hat\imath}{2\hat k \cdot (-3\hat\imath + 3\hat\jmath + 6\hat k)} = \dfrac{3(-\hat\imath + \hat\jmath + 2\hat k)}{12}$

$= \dfrac{(-\hat\imath + \hat\jmath + 2\hat k)}{4}$

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II Componente contravariante:

$a = a^i e_i$

$\hat{i} + 2\hat{j} + 3\hat{k} = a_1(4\hat{i} + 2\hat{j} + \hat{k}) + a_2(3\hat{i} + 3\hat{j}) + a_3(2\hat{k})$

$1 = 4a_1 + 3a_2 \rightarrow ①$ $① - ② \rightarrow -1 = 2a_1$

$2 = 2a_1 + 3a_2 \rightarrow ②$ $\boxed{a_1 = -\frac{1}{2}}$

$3 = a_1 + 2a_3 \rightarrow ③$ $1 = 4\left(-\frac{1}{2}\right) + 3a_2$

$1 = -2 + 3a_2$

$3 = -\frac{1}{2} + 2a_3$

$3 = 3a_2$

$\boxed{a_2 = 1}$

$6 = -1 + 4a_3$

$7 = 4a_3$

$\boxed{a_3 = 7/4}$

Componente covariante

$a = a_i e^i$

$\hat{i} + 2\hat{j} + 3\hat{k} = a_1\frac{(\hat{i} - \hat{j})}{2} + a_2\frac{(2\hat{j} - \hat{i})}{3} + a_3\frac{(-\hat{i} + \hat{j} + 2\hat{k})}{4}$

$1 = \frac{a_1}{2} + \left(-\frac{1}{3}\right)a_2 - \frac{1}{4}a_3 \rightarrow ①$ $① + ② \Rightarrow 3 = \frac{1}{3}a_2$

$2 = -\frac{a_1}{2} + \frac{2}{3}a_2 + \frac{1}{4}a_3 \rightarrow ②$ $\boxed{9 = a_2}$

$3 = \frac{2}{4}a_3$ $2 = -\frac{a_1}{2} + \frac{2}{3}(9) + \frac{1}{4}(6)$

$\boxed{6 = a_3}$ $2 = -\frac{a_1}{2} + 6 + \frac{3}{2}$

$-4 = \frac{3 - a_1}{2}$

$-8 = 3 - a_1 \rightarrow \boxed{a_1 = 3 + 8 = 11}$

7) $\langle \sigma^i | \sigma_j \rangle = \delta^i_j$

$\langle \sigma^0|\sigma_0\rangle = 1 \to \begin{pmatrix} \sigma_1^{0*} & \sigma_3^{0*} \\ \sigma_2^{0*} & \sigma_4^{0*}\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix} = \begin{pmatrix}\sigma_1^{0*} & \sigma_3^{0*}\\ \sigma_2^{0*} & \sigma_4^{0*}\end{pmatrix}$

$Tr(\sigma^{0\dagger}\sigma_0) = \sigma_1^{0*} + \sigma_4^{0*} = 1$

$\langle\sigma^0|\sigma_1\rangle = 0 \to \begin{pmatrix}\sigma_1^{0*} & \sigma_3^{0*}\\ \sigma_2^{0*} & \sigma_4^{0*}\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}\sigma_3^{0*} & \sigma_1^{0*}\\ \sigma_4^{0*} & \sigma_2^{0*}\end{pmatrix}$

$Tr(\sigma^{0\dagger}\sigma_1) = \sigma_3^{0*} + \sigma_2^{0*} = 0 \to \sigma_3^{0*} = -\sigma_2^{0*}$

$\langle\sigma^0|\sigma_2\rangle = 0 \to \begin{pmatrix}\sigma_1^{0*} & \sigma_3^{0*}\\ \sigma_2^{0*} & \sigma_4^{0*}\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix} = \begin{pmatrix} i\sigma_3^{0*} & -i\sigma_1^{0*}\\ i\sigma_4^{0*} & -i\sigma_2^{0*}\end{pmatrix}$

$Tr(\sigma^{0\dagger}\sigma_2) = i\sigma_3^{0*} - i\sigma_2^{0*} = 0 \to i\sigma_3^{0*} = i\sigma_2^{0*} \to \sigma_3^{0*} = \sigma_2^{0*}$

$\langle\sigma^0|\sigma_3\rangle = 0 \to \begin{pmatrix}\sigma_1^{0*} & \sigma_3^{0*}\\ \sigma_2^{0*} & \sigma_4^{0*}\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix} = \begin{pmatrix}\sigma_1^{0*} & -\sigma_3^{0*}\\ \sigma_2^{0*} & -\sigma_4^{0*}\end{pmatrix}$

$Tr(\sigma^{0\dagger}\sigma_3) = \sigma_1^{0*} - \sigma_4^{0*} = 0 \to \sigma_1^{0*} = \sigma_4^{0*}$

$\sigma_3^{0*} = \sigma_2^{0*} = 0$

$2\sigma_1^{0*} = 1$

$\sigma_1^{0*} = \frac{1}{2} = \sigma_4^{0*}$

$\sigma^0 = \begin{pmatrix}\frac{1}{2} & 0\\0&\frac{1}{2}\end{pmatrix}$

$\langle\sigma^1|\sigma_0\rangle = 0 \to \sigma_1^{1*} = -\sigma_4^{1*}$ $\qquad \sigma_1^1 = \sigma_4^1 = 0$

$\langle\sigma^1|\sigma_1\rangle = 1 \to \sigma_3^{1*} + \sigma_2^{1*} = 1,$ $\qquad 2\sigma_3^{1*} = 1$

$\langle\sigma^1|\sigma_2\rangle = 0 \to \sigma_3^{1*} = \sigma_2^{1*}$ $\qquad \sigma_3^{1*} = \frac{1}{2}$

$\langle\sigma^1|\sigma_3\rangle = 0 \to \sigma_1^{1*} = \sigma_4^{1*}$ $\qquad \sigma_3 = \frac{1}{2} = \sigma_2$

$$\sigma^1 = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$$

$\langle \sigma^2 | \sigma_0 \rangle = 0 \to \sigma_1^{2*} = -\sigma_4^{2*}$

$\langle \sigma^2 | \sigma_1 \rangle = 0 \to \sigma_3^{2*} = -\sigma_2^{2*}$

$\langle \sigma^2 | \sigma_2 \rangle = 1 \to (i\sigma_3^{2*} - i\sigma_2^{2*} = 1)\, i$

$\sigma_2^{2*} - \sigma_3^{2*} = i$

$\langle \sigma^2 | \sigma_3 \rangle = 0 \to \sigma_1^{2*} = \sigma_4^{2*}$

$\sigma_1^2 = \sigma_4^2 = 0$

$2\sigma_2^{2*} = i$

$\sigma_2^{2*} = \frac{i}{2} \to \sigma_2^2 = -\frac{i}{2}$

$\sigma_3^2 = i/2$

$$\sigma^2 = \begin{pmatrix} 0 & -i/2 \\ i/2 & 0 \end{pmatrix}$$

$$\left.\begin{array}{l} \langle \sigma^3 | \sigma_0 \rangle = 0 \to \sigma_1^{3*} = -\sigma_4^{3*} \\ \langle \sigma^3 | \sigma_1 \rangle = 0 \to \sigma_3^{3*} = -\sigma_2^{3*} \\ \langle \sigma^3 | \sigma_2 \rangle = 0 \to \sigma_3^{3*} = \sigma_2^{3*} \end{array}\right\} \sigma_3^3 = \sigma_2^3 = 0$$

$\langle \sigma^3 | \sigma_3 \rangle = 1 \to \sigma_1^{3*} - \sigma_4^{3*} = 1$

$2\sigma_1^{3*} = 1$

$\sigma_1^{3*} = 1/2 \to \sigma_1^3 = 1/2$

$\sigma_4^3 = -1/2$

$$\sigma^3 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$$

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Parte 2 punto 7.

$f_A[|\sigma_i\rangle] = \langle A|\sigma_i\rangle = \langle a_j \sigma^j|\sigma_i\rangle$
$= a_j \langle \sigma^j|\sigma_i\rangle = a_j \delta^j_i = a_i$ - Coef. covariantes

Pero: $f_A[|\sigma_i\rangle] = \langle A|\sigma_i\rangle = Tr(A^\dagger \sigma_i)$.

Sea entonces $A \in M_{2\times2}$ Hermíticas

$A = \begin{pmatrix} \alpha & \beta \\ \beta^* & \varphi \end{pmatrix}$ $\beta \in \mathbb{C}$, $\varphi, \alpha \in \mathbb{R}$

$a_0 = Tr(A^\dagger \sigma_0) = Tr\left[\begin{pmatrix} \alpha & \beta \\ \beta^* & \varphi \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] = \alpha + \varphi$

$a_1 = Tr\left[\begin{pmatrix} \alpha & \beta \\ \beta^* & \varphi \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] = Tr\begin{pmatrix} \beta & \alpha \\ \varphi & \beta^* \end{pmatrix} = \beta + \beta^* = 2Re\,\beta$

$a_2 = Tr\left[\begin{pmatrix} \alpha & \beta \\ \beta^* & \varphi \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right] = Tr\begin{pmatrix} \beta i & -\alpha i \\ \varphi i & -\beta^* i \end{pmatrix} = \beta i - \beta^* i$

Si $\beta = x + yi$, entonces $a_2 = (x+yi)i - (x-yi)i$
$= -y - y = -2y = -2 Im\,\beta$

$a_3 = Tr\left[\begin{pmatrix} \alpha & \beta \\ \beta^* & \varphi \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] = Tr\begin{pmatrix} \alpha & -\beta \\ \beta^* & -\varphi \end{pmatrix} = \alpha - \varphi$

Entonces $\langle A| = a_0\langle\sigma_0| + a_1\langle\sigma_1| + a_2\langle\sigma_2| + a_3\langle a_3|$
$= (\alpha+\varphi)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2x\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + (-2y)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + (\alpha-\varphi)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

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$$\langle A| = \begin{pmatrix} 2\alpha & 2x+2yi \\ 2x-2yi & 2\varphi \end{pmatrix} = 2\begin{pmatrix} \alpha & \beta \\ \beta^* & \varphi \end{pmatrix} = 2\,|A|$$