FUNCIONES ARMÓNICAS

Una función real ϕ de dos variables reales x e y se dice armónica en un dominio D del plano xy si sobre D tiene derivadas parciales de segundo orden continuas y satisfacen la ecuación de Laplace:

$$\phi_{xx}(x,y) + \phi_{yy}(x,y) = 0$$

TEOREMA

Si $f(z) = u(x,y) + iv(x,y)$ es analítica en un dominio *D,* entonces u y v son armónicas en *D.*

¿Para qué valores de z se satisface la ecuación de Laplace?:

108. $\boldsymbol{\phi(x,y) = x^4 + y^3}$ **¿Por qué no es armónica?**

$$\phi_x = 4x^3 \qquad \phi_{xx} = 12x^2$$

$$\phi_y = 3y^2 \qquad \phi_{yy} = 6y$$

La ecuación de Laplace: $\overbrace{12x^2}^{\phi_{xx}} + \overbrace{6y}^{\phi_{yy}} = 0$ se satisface cuando $y = -2x^2$.

Las derivadas parciales de segundo orden de ϕ son continuas en todo el plano pero la ecuación de Laplace se satisface sólo en el conjunto: $\{z \in \mathbb{C} \mid Im(z) = -2Re^2(z)\}$, el cual no constituye un domino, por lo tanto $\phi(x,y)$ no es armónica.

Determine si las siguientes funciones son armónicas y en qué dominio:

109. $\boldsymbol{\phi(x,y) = x + y}$

$$\phi_x = 1 \qquad \phi_{xx} = 0$$

$$\phi_y = 1 \qquad \phi_{yy} = 0$$

$\phi(x,y)$ es armónica en todo el plano ya que sus derivadas parciales de segundo orden son continuas en todo el plano y satisfacen la ecuación de Laplace:

$$\overbrace{0}^{\phi_{xx}} + \overbrace{0}^{\phi_{yy}} = 0 \text{ en todo el plano.}$$

110. $\boldsymbol{\phi(x,y) = xy}$

$$\phi_x = y \qquad\qquad \phi_{xx} = 0$$

$$\phi_y = x \qquad\qquad \phi_{yy} = 0$$

$\phi(x,y)$ es armónica en todo el plano ya que sus derivadas parciales de segundo orden son continuas en todo el plano y satisfacen la ecuación de Laplace:

$$\overbrace{0}^{\phi_{xx}} + \overbrace{0}^{\phi_{yy}} = 0 \text{ en todo el plano.}$$

ARMÓNICA CONJUGADA

Si 2 funciones $u(x,y)$ y $v(x,y)$ son armónicas en un dominio D y satisfacen las condiciones de C-R en D se dice que v es armónica conjugada de u.

TEOREMA

$f(z) = u(x,y) + iv(x,y)$ es analítica en un dominio D si y sólo si v es armónica conjugada de u.

Compruebe si $\boldsymbol{u(x,y)}$ es armónica en todo el plano. Encuentre (si es posible) su conjugada armónica $\boldsymbol{v(x,y)}$ tal que $\boldsymbol{w = u + iv}$ sea analítica en todo el plano:

111. $\boldsymbol{u(x,y) = 2x(1-y)}$

$$u_x = 2(1-y) \qquad\qquad u_{xx} = 0$$

$$u_y = -2x \qquad\qquad u_{yy} = 0$$

$u(x,y)$ es armónica en todo el plano ya sus derivadas parciales de segundo orden son continuas en todo el plano y satisfacen la ecuación de Laplace:

$$\overbrace{0}^{u_{xx}} + \overbrace{0}^{u_{yy}} = 0 \text{ en todo el plano.}$$

Se busca su conjugada armónica utilizando la fórmula resolvente:

$$\boxed{\boldsymbol{v(x,y) = -\int_{x_0}^{x} u_y\,(x,y)dx + \int_{y_0}^{y} u_x\,(x_0,y)dy + c}}$$

Si se elige $(x_0, y_0) = (0,0)$, se tiene

$$v(x,y) = -\int_0^x (-2x)\,dx + \int_0^y 2(1-y)\,dy + c$$

$$= 2\int_0^x x\,dx + 2\int_0^y (1-y)\,dy + c$$

$$= [x^2]_0^x + [2y - y^2]_0^y + c$$

$$\boxed{v(x,y) = x^2 + 2y - y^2 + c} \text{ armónica conjugada de } u(x,y).$$

$$w = u + iv = 2x(1-y) + i(x^2 + 2y - y^2 + c) \text{ es entera.}$$

112. $\boldsymbol{u(x,y) = senh(x)sen(y)}$

$u_x = cosh(x)sen(y)$ $\qquad$ $u_{xx} = senh(x)sen(y)$

$u_y = senh(x)cos(y)$ $\qquad$ $u_{yy} = -senh(x)sen(y)$

$u(x,y)$ es armónica en todo el plano ya que sus derivadas parciales de segundo orden son continuas en todo el plano y satisfacen la ecuación de Laplace:

$$\overbrace{senh(x)sen(y)}^{u_{xx}} + \overbrace{(-senh(x)sen(y))}^{u_{yy}} = 0 \text{ en todo el plano.}$$

Se busca su conjugada armónica utilizando la fórmula resolvente:

$$v(x,y) = -\int_{x_0}^{x} u_y\,(x,y)dx + \int_{y_0}^{y} u_x\,(x_0,y)dy + \tilde{c}$$

Si se elige $(x_0, y_0) = (0,0)$, se tiene

$$v(x,y) = -\int_0^x senh(x)cos(y)\,dx + \int_0^y \overbrace{cosh(0)}^{=1} sen(y)\,dy + \tilde{c}$$

$$= -cos(y)\int_0^x senh(x)\,dx + \int_0^y sen(y)\,dy + \tilde{c}$$

$$= -cos(y)[cosh(x)]_0^x + [-cos(y)]_0^y + \tilde{c}$$

$$= -cos(y)cosh(x) + cos(y) - cos(y) + \underbrace{1 + \tilde{c}}_{c}$$

$\boxed{v(x,y) = -cosh(x)cos(y) + c}$ armónica conjugada de $u(x,y)$.

$w = u + iv = senh(x)sen(y) + i(-cosh(x)cos(y) + c)$ es entera.

FUNCIONES ELEMENTALES

Función exponencial

Se define para todo $z = x + iy$ como:

$$e^z = e^{x+iy} = e^x\big(cos(y) + isen(y)\big) \;;\; \text{con } y \text{ en radianes}$$

Módulo

$$|e^z| = \sqrt{\big(e^x cos(y)\big)^2 + \big(e^x sen(y)\big)^2} = e^x > 0 \;\; \forall x$$

$$\boxed{|e^z| = e^x}$$

Argumento

Como $e^z = e^{x+iy} = \underbrace{e^x}_{módulo}\;\; e^{iy}$

el argumento es:

$$\boxed{arg(e^z) = y + 2k\pi \;,\quad k \in \mathbb{Z}}$$

$e^z \neq 0 \quad \forall z,$ "la exponencial compleja nunca se anula"

Periodicidad

$$e^{z+2\pi i} = e^z\, \overbrace{e^{i2\pi}} = e^z\left(\overbrace{cos(2\pi) + isen(2\pi)}\right)$$

$\boxed{e^{z+2\pi i} = e^z}$, e^z es periódica de periodo imaginario puro $2\pi i$

Exprese como $w = u + iv$:

117. $w = e^{3+4i}$

Fórmula de Euler

$$w = e^{3}\,\overbrace{e^{i4}} = e^{3}\left(\overbrace{\cos(4) + i sen(4)}\right)$$

$$\boxed{w = \underbrace{e^{3}\cos(4)}_{u} + i\,\underbrace{e^{3} sen(4)}_{v}}$$

120. $w = e^{i\pi}$

Fórmula de Euler

$$\boxed{w = \overbrace{e^{i\pi}} = \overbrace{cos(\pi) + isen(\pi)} = \underbrace{-1}_{u} + i\ \underbrace{0}_{v}}$$

124. $w = e^{\frac{2+3\pi i}{4}}$

Fórmula de Euler

$$w = e^{\frac{1}{2}}\,\overbrace{e^{i\frac{3\pi}{4}}} = e^{\frac{1}{2}}\left(\overbrace{cos\left(\frac{3\pi}{4}\right) + isen\left(\frac{3\pi}{4}\right)}\right) = \sqrt{e}\left(-\frac{\sqrt{2}}{2} + i\ \frac{\sqrt{2}}{2}\right)$$

$$\boxed{w = \underbrace{-\frac{\sqrt{2e}}{2}}_{u} + i\,\underbrace{\frac{\sqrt{2e}}{2}}_{v}}$$

125. $w = e^{\frac{1}{1-i}}$

Fórmula de Euler

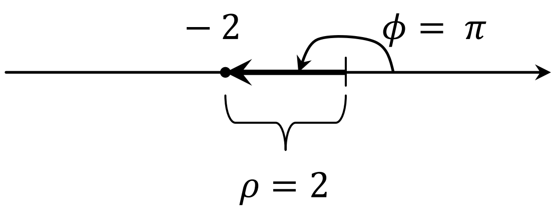
$$w = e^{\left(\frac{1}{1-i}\right)\left(\frac{1+i}{1+i}\right)} = e^{\frac{1+i}{2}} = e^{\frac{1}{2}}\,\overbrace{e^{i\frac{1}{2}}} = \sqrt{e}\left(\overbrace{cos\left(\frac{1}{2}\right) + i\ sen\left(\frac{1}{2}\right)}\right)$$

$$\boxed{w = \sqrt{e}\ cos\left(\frac{1}{2}\right) + i\sqrt{e}\ sen\left(\frac{1}{2}\right)}$$

Determine todos los valores de z tales que:

126. $e^z = -2$

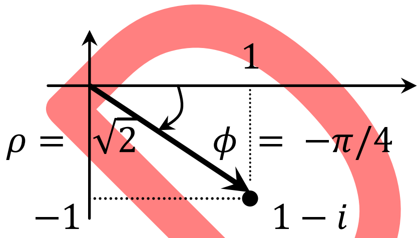
$$e^{x+iy} = \underbrace{2e^{i(\pi+2k\pi)}}_{-2}$$



$$\overbrace{e^{x}}\, e^{i\underbrace{y}} = \overbrace{2}\, e^{i\underbrace{(\pi+2k\pi)}} \quad \Leftrightarrow \quad \begin{cases} e^x = 2 \quad \Rightarrow \quad x = ln(2) \\ \qquad y = \pi + 2k\pi \end{cases}$$

$$\boxed{z = \underbrace{ln(2)}_{x} + i\left(\underbrace{\pi + 2k\pi}_{y}\right) \quad ; \quad k \in \mathbb{Z}}$$

127. $e^z = 1 - i$

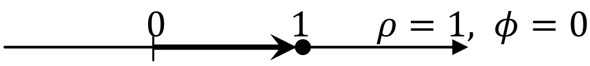


$$e^{x+iy} = \underbrace{\sqrt{2}e^{i\left(-\frac{\pi}{4}+2k\pi\right)}}_{1-i}$$

$$\overbrace{e^{x}}\, e^{i\underbrace{y}} = \overbrace{\sqrt{2}}\, e^{i\underbrace{\left(-\frac{\pi}{4}+2k\pi\right)}} \quad \Leftrightarrow \quad \begin{cases} e^x = \sqrt{2} \quad \Rightarrow \quad x = ln\left(\sqrt{2}\right) \\ \qquad y = -\frac{\pi}{4} + 2k\pi \end{cases}$$

$$\boxed{z = \underbrace{ln\left(\sqrt{2}\right)}_{x} + i\left(\underbrace{-\frac{\pi}{4} + 2k\pi}_{y}\right) \quad ; \quad k \in \mathbb{Z}}$$

128. $e^{2z-1} = 1$

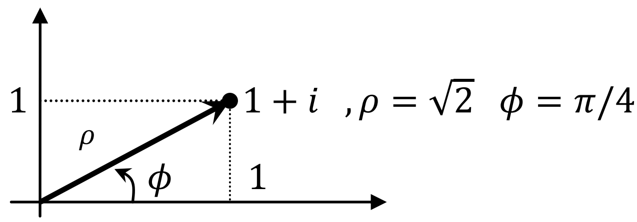


$$e^{2x-1+i2y} = \overbrace{1e^{i(0+2k\pi)}}^{1}$$

$$\overbrace{e^{2x-1}}\, e^{i\underbrace{2y}} = \overbrace{1}\, e^{i\underbrace{(0+2k\pi)}} \quad \Leftrightarrow \quad \begin{cases} e^{2x-1} = 1 \quad \Rightarrow 2x - 1 = \overbrace{ln(1)}^{=0} \quad \Rightarrow x = \frac{1}{2} \\ 2y = 2k\pi \qquad\qquad\qquad\qquad \Rightarrow \quad y = k\pi \end{cases}$$

$$\boxed{z = \underbrace{\frac{1}{2}}_{x} + i\,\underbrace{k\pi}_{y} \quad ; \quad k \in \mathbb{Z}}$$

133. $\boldsymbol{e^{iz} = 1 + i}$



$$e^{i(x+iy)} = \overbrace{\sqrt{2}e^{i\left(\frac{\pi}{4}+2k\pi\right)}}^{1+i}$$

$$\overbrace{e^{-y}}\, e^{i\underbrace{x}} = \overbrace{\sqrt{2}}\, e^{i\underbrace{\left(\frac{\pi}{4}+2k\pi\right)}} \quad \Leftrightarrow \quad \begin{cases} e^{-y} = \sqrt{2} \quad \Rightarrow \; y = -ln(\sqrt{2}) \\ \qquad x = \frac{\pi}{4} + 2k\pi \end{cases}$$

$$\boxed{z = \underbrace{\frac{\pi}{4} + 2k\pi}_{x} + i\underbrace{\left(-ln(\sqrt{2})\right)}_{y} \quad ; \quad k \in \mathbb{Z}}$$

<u>Funciones trigonométricas e hiperbólicas</u>

$$cos(z) = \frac{e^{iz} + e^{-iz}}{2} \qquad sen(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

$$cosh(z) = \frac{e^{z} + e^{-z}}{2} \qquad senh(z) = \frac{e^{z} - e^{-z}}{2}$$

$$sen(z) = sen(x)cosh(y) + icos(x)senh(y)$$

$$cos(z) = cos(x)cosh(y) - isen(x)senh(y)$$

$$cosh(z) = cosh(x)cos(y) + isenh(x)sen(y)$$

$$senh(z) = senh(x)cos(y) + icosh(x)sen(y)$$

Halle todos los valores de *z* que satisfacen:

135. $\boldsymbol{cos(z) = 2}$

$$\frac{\overbrace{e^{iz} + e^{-iz}}^{cos(z)}}{2} = 2 \quad \Rightarrow \quad e^{iz} + e^{-iz} = 4$$

Multiplicando por e^{iz}:

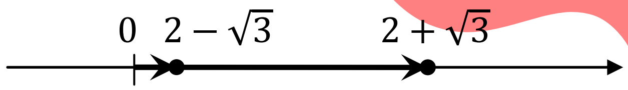
$$e^{iz}e^{iz} + e^{-iz}e^{iz} = 4e^{iz}$$

$$e^{2iz} + 1 = 4e^{iz}$$

$$\left(e^{iz}\right)^2 - 4e^{iz} + 1 = 0 \qquad ; \qquad a = 1, \qquad b = -4, \quad c = 1$$

$$e^{iz} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{4 \pm \sqrt{(-4)^2 - 4.(1).(1)}}{2.(1)} = \frac{4 \pm \sqrt{12}}{2}$$

$$e^{iz} = 2 \pm \sqrt{\frac{12}{4}} = 2 \pm \sqrt{3}$$



$$e^{i(x+iy)} = \overbrace{\left(2 \pm \sqrt{3}\right)e^{i(0+2k\pi)}}^{2\pm\sqrt{3}}$$

$$\overbrace{e^{-y}}\, e^{i\underbrace{x}} = \overbrace{\left(2 \pm \sqrt{3}\right)}\, e^{i\underbrace{2k\pi}} \quad \Leftrightarrow \quad \begin{cases} e^{-y} = 2 \pm \sqrt{3} \;\Rightarrow\; -y = ln\left(2 \pm \sqrt{3}\right) \\ x = 2k\pi \end{cases}$$

$$\boxed{z = \underbrace{2k\pi}_{x} + i\underbrace{\left(-ln\left(2 \pm \sqrt{3}\right)\right)}_{y} = \underbrace{2k\pi}_{x} + i\underbrace{\left(\pm ln\left(2 - \sqrt{3}\right)\right)}_{y} \;;\quad k \in \mathbb{Z}}$$

136. $\boldsymbol{cosh(z) = \frac{1}{2}}$

$$\frac{\overbrace{e^{z} + e^{-z}}^{cos\,h(z)}}{2} = \frac{1}{2} \quad \Rightarrow \quad e^{z} + e^{-z} = 1$$

Multiplicando por e^{z}:

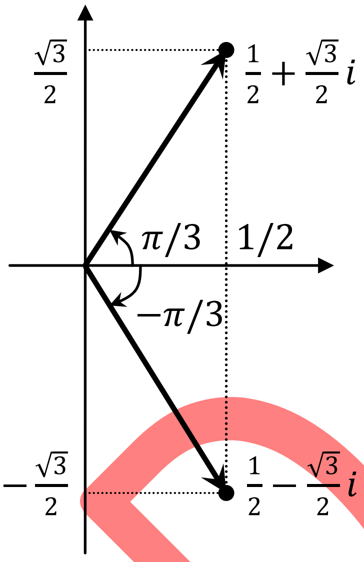
$$e^{z}e^{z} + e^{-z}e^{z} = 1e^{z}$$

$$e^{2z} + 1 = e^{z}$$

$$(e^{z})^2 - e^{z} + 1 = 0 \qquad ; \qquad a = 1, \qquad b = -1, \quad c = 1$$

$$e^z = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{1 \pm \sqrt{(-1)^2 - 4.(1).(1)}}{2.(1)} = \frac{1 \pm \sqrt{-3}}{2}$$

$$e^z = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$



$$e^{x+iy} = \overbrace{1e^{i\left(\pm\frac{\pi}{3}+2k\pi\right)}}^{\frac{1}{2}\pm\frac{\sqrt{3}}{2}i}$$

$$\overbrace{e^x}\, e^{\underbrace{iy}} = \overbrace{1}\, e^{i\underbrace{\left(\pm\frac{\pi}{3}+2k\pi\right)}} \Leftrightarrow \begin{cases} e^x = 1 \Rightarrow x = ln(1) = 0 \\ y = \pm\frac{\pi}{3} + 2k\pi \end{cases}$$

$$\boxed{z = \underbrace{0}_{x} + i\underbrace{\left(\pm\frac{\pi}{3} + 2k\pi\right)}_{y} \quad ; \quad k \in \mathbb{Z}}$$

139. $\boldsymbol{cosh(z) = i}$

$$\overbrace{\frac{e^z + e^{-z}}{2}}^{cosh(z)} = i \Rightarrow \quad e^z + e^{-z} = 2i$$

Multiplicando por e^z:

$$e^z e^z + e^{-z} e^z = 2ie^z$$

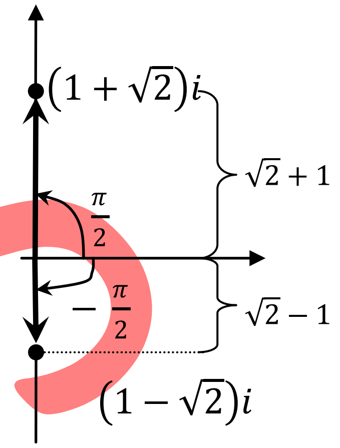
$$e^{2z} + 1 = 2ie^z$$

$$(e^z)^2 - 2ie^z + 1 = 0 \;; \qquad a = 1, \qquad b = -2i, \quad c = 1$$

$$e^z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2i \pm \sqrt{(-2i)^2 - 4.(1).(1)}}{2.(1)} = \frac{2i \pm \sqrt{-8}}{2}$$

$$e^z = \frac{2i \pm 2\sqrt{2}\, i}{2} = \underbrace{\left(1 \pm \sqrt{2}\right) i}$$

$$e^z = \overbrace{\left(\sqrt{2} \pm 1\right) e^{i\left(\pm\frac{\pi}{2} + 2k\pi\right)}}$$



$$\overbrace{e^x}\, e^{i\underset{\smile}{y}} = \overbrace{\left(\sqrt{2} \pm 1\right)}\, e^{i\underbrace{\left(\pm\frac{\pi}{2} + 2k\pi\right)}} \quad \Leftrightarrow \quad \begin{cases} e^x = \sqrt{2} \pm 1 \Rightarrow x = ln\left(\sqrt{2} \pm 1\right) = \pm ln\left(\sqrt{2} + 1\right) \\ y = \pm\frac{\pi}{2} + 2k\pi \end{cases}$$

$$\boxed{z = \underbrace{\pm ln\left(\sqrt{2} + 1\right)}_{x} + i\left(\underbrace{\pm\frac{\pi}{2} + 2k\pi}_{y}\right) \quad ; \quad k \in \mathbb{Z}}$$

141. $\boldsymbol{sen(z) = cosh(4)}$

$$\frac{\overbrace{e^{iz} - e^{-iz}}^{sen\,(z)}}{2i} = cos\,h(4) \quad \Rightarrow \quad e^{iz} - e^{-iz} = 2icos\,h(4)$$

Multiplicando por e^{iz}:

$$e^{iz} e^{iz} - e^{-iz} e^{iz} = 2icos\,h(4)\, e^{iz}$$

$$\left(e^{iz}\right)^2 - 2icos\,h(4)\, e^{iz} - 1 = 0 \qquad ; \qquad a = 1, \qquad b = -2icos\,h(4)\,, \quad c = -1$$

$$e^{iz} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2icos\,h(4) \pm \sqrt{(-2icos\,h(4))^2 - 4.(1).(-1)}}{2.(1)}$$

$$e^{iz} = \frac{2icos\,h(4) \pm \sqrt{-4cos\,h^{\,2}(4) + 4}}{2} = \frac{2icos\,h(4) \pm \sqrt{-4(cos\,h^{\,2}(4) - 1)}}{2}$$

Como: $\cos h^{\,2}(x) - \operatorname{sen} h^{\,2}(x) = 1 \Rightarrow \cos h^{\,2}(x) - 1 = \operatorname{sen} h^{\,2}(x)$

Luego, $\cos h^{\,2}(4) - 1 = \operatorname{sen} h^{\,2}(4)$

$$e^{iz} = \frac{2i\cos h(4) \pm \sqrt{-4\underbrace{\operatorname{sen} h^{\,2}(4)}_{>0}}}{2}$$

$$e^{iz} = \frac{2i\cos h(4) \pm 2i\operatorname{sen} h(4)}{2}$$

$$e^{iz} = \left(\cos h(4) \pm \operatorname{sen} h(4)\right) i$$

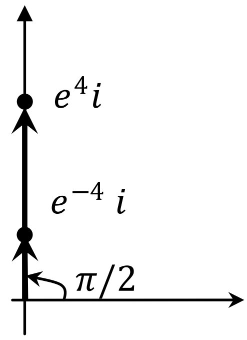
$$e^{iz} = \left(\underbrace{\cos h(4)}_{\frac{e^{4}+e^{-4}}{2}} \pm \underbrace{\operatorname{sen} h(4)}_{\frac{e^{4}-e^{-4}}{2}}\right) i = \left(\frac{e^{4}+e^{-4} \pm (e^{4}-e^{-4})}{2}\right) i$$

$$e^{iz} = \left(\frac{e^{4}+e^{-4} \pm e^{4} \mp e^{-4}}{2}\right) i = e^{\pm 4} i$$

$$e^{iz} = \underbrace{e^{\mp 4} i}$$

$$e^{i(x+iy)} = \overbrace{e^{\mp 4} e^{i\left(\frac{\pi}{2}+2k\pi\right)}}$$

$$\underbrace{e^{-y}}\, e^{\underbrace{ix}} = \overbrace{e^{\mp 4}}\, e^{\underbrace{i\left(\frac{\pi}{2}+2k\pi\right)}}$$



$$\begin{cases} e^{-y} = e^{\mp 4} \Rightarrow -y = \mp 4 ln(e) \Rightarrow y = \pm 4 \\ x = \dfrac{\pi}{2} + 2k\pi \end{cases}$$

$$\boxed{z = \underbrace{\frac{\pi}{2} + 2k\pi}_{x} + i\underbrace{(\pm 4)}_{y} \quad ; \quad k \in \mathbb{Z}}$$

Otra forma

$$\underbrace{\operatorname{sen}(z)} = \cosh(4)$$

$$\overbrace{\operatorname{sen}(x)\cosh(y) + i\cos(x)\operatorname{senh}(y)} = \cosh(4) + i0$$

$$\begin{cases} \operatorname{sen}(x)\cosh(y) = \cosh(4) & (1) \\ \cos(x)\operatorname{senh}(y) = 0 & (2) \end{cases}$$

De (2) $\underbrace{cos(x) = 0}_{\Downarrow}$ o $senh(y) = 0 \Rightarrow y = 0$

$$\overbrace{x = \pm\frac{\pi}{2} + 2k\pi,\ k \in \mathbb{Z}}$$

Si $y = 0$ en (1) $\underbrace{sen(x)}_{-1 \leq sen\,(x) \leq 1} \cdot \overbrace{1}^{cos\,h\,(0)} = \overbrace{cosh(4)}^{\approx 27{,}32}$ Imposible

Luego, se descarta $y = 0$

Si $x = \pm\frac{\pi}{2} + 2k\pi,\ k \in \mathbb{Z}$ en (1) $\underbrace{sen\left(\pm\frac{\pi}{2} + 2k\pi\right)}_{\pm 1} \underbrace{cosh(y)}_{\geq 1} = \overbrace{cosh(4)}^{\approx 27{,}32}$

Luego: $\underbrace{sen\left(\frac{\pi}{2} + 2k\pi\right)}_{1} \underbrace{cosh(y)}_{\geq 1} = \overbrace{cosh(4)}^{\approx 27{,}32} \Rightarrow y = \pm 4$ $(cosh\ es\ función\ par)$

Por lo tanto

$$\boxed{z = \underbrace{\frac{\pi}{2} + 2k\pi}_{x} + i\underbrace{(\pm 4)}_{y} \quad ; \quad k \in \mathbb{Z}}$$

142. $\boldsymbol{cos(z) = isenh(4)}$

$$\overbrace{\frac{e^{iz} + e^{-iz}}{2}}^{cos\,(z)} = isen\,h(4) \quad \Rightarrow \quad e^{iz} + e^{-iz} = 2isen\,h(4)$$

Multiplicando por e^{iz}:

$$e^{iz}e^{iz} + e^{-iz}e^{iz} = 2isen\,h(4)\,e^{iz}$$

$$\left(e^{iz}\right)^2 - 2isen\,h(4)\,e^{iz} + 1 = 0 \qquad ; \qquad a = 1, \qquad b = -2isen\,h(4), \quad c = 1$$

$$e^{iz} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2isen\,h(4) \pm \sqrt{(-2isen\,h(4))^2 - 4.(1).(1)}}{2.(1)}$$

$$e^{iz} = \frac{2isen\,h(4) \pm \sqrt{-4(sen\,h^{\,2}(4) + 1)}}{2}$$

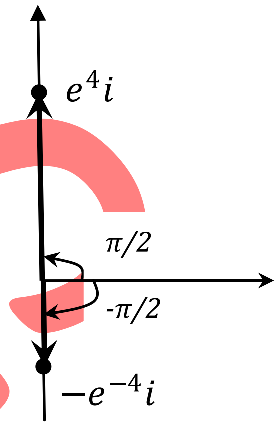
Como: $cos\,h^{\,2}(x) - sen\,h^{\,2}(x) = 1 \Rightarrow sen\,h^{\,2}(x) + 1 = cos\,h^{\,2}(x)$

Luego, $sen\,h^{\,2}(4) + 1 = cos\,h^{\,2}(4)$

$$e^{iz} = \frac{2i\,sen\,h(4) \pm \sqrt{\underbrace{-4\cos h^{\,2}(4)}_{>0}}}{2} = \frac{2i\,sen\,h(4) \pm 2i\cos h(4)}{2}$$

$$e^{iz} = \left(\underbrace{sen\ h(4)}_{\frac{e^{4}-e^{-4}}{2}} \pm \underbrace{cos\ h(4)}_{\frac{e^{4}+e^{-4}}{2}}\right) i$$

$$e^{iz} = \left(\frac{e^{4} - e^{-4} \pm e^{4} \pm e^{-4}}{2}\right) i$$



$$e^{iz} = \underbrace{\pm e^{\pm 4} i}$$

$$e^{i(x+iy)} = \overbrace{e^{\pm 4} e^{i\left(\pm\frac{\pi}{2}+2k\pi\right)}}$$

$$\overbrace{e^{-y}}\, \underbrace{e^{ix}} = \overbrace{e^{\pm 4}}\, e^{\underbrace{i\left(\pm\frac{\pi}{2}+2k\pi\right)}}$$

$$\begin{cases} e^{-y} = e^{\pm 4} \quad \Rightarrow \quad -y = \pm 4ln(e) \Rightarrow y = \mp 4 \\ \qquad\qquad x = \pm\dfrac{\pi}{2} + 2k\pi \end{cases}$$

$$\boxed{z = \underbrace{\pm\frac{\pi}{2} + 2k\pi}_{x} + i\,\underbrace{(\mp 4)}_{y} \quad ; \quad k \in \mathbb{Z}}$$

Otra forma

$$\underbrace{cos(z)} = isenh(4)$$

$$\overbrace{cos(x)cosh(y) - isen(x)senh(y)} = 0 + isenh(4)$$

$$\begin{cases} cos(x)cosh(y) = 0 & (1) \\ -sen(x)senh(y) = senh(4) & (2) \end{cases}$$

De (1) $x = \pm\frac{\pi}{2} + 2k\pi,\ \ k \in \mathbb{Z}$. Reemplazando en (2)

$$-sen\left(\pm\frac{\pi}{2} + 2k\pi\right) senh(y) = senh(4)$$

$$-(\pm 1)senh(y) = senh(4)$$

$$\mp senh(y) = senh(4) \quad \Rightarrow \quad y = \mp 4 \quad (senh\ es\ función\ impar)$$

$$\boxed{z = \underbrace{\pm\frac{\pi}{2} + 2k\pi}_{x} + i\,\underbrace{(\mp 4)}_{y} \quad ; \quad k \in \mathbb{Z}}$$

Función logaritmo

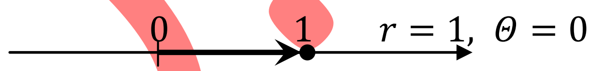
$$log(z) = ln(r) + i(\Theta + 2k\pi)\,, z \neq 0\,, k \in \mathbb{Z}$$

$$Log(z) = ln(r) + i\Theta\ ,\ z \neq 0\ \text{ Valor principal}$$

Halle todos los valores de:

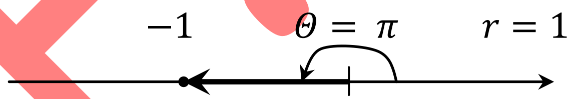
144. $\boldsymbol{log(1)}$

$$log(1) = ln(1) + i(0 + 2k\pi)$$



$$\boxed{log(1) = i2k\pi\ ;\ k \in \mathbb{Z}}$$

145. $\boldsymbol{Log(-1)}$



$$\boxed{Log(-1) = ln(1) + i\pi = i\pi}$$

Halle todos los valores $\boldsymbol{z}$ tales que:

150. $\boldsymbol{log(z) = 1 + i\frac{2\pi}{3}}$

$$e^{log\,(z)} = e^{1+i\frac{2\pi}{3}}$$

$$z = e^{1+i\frac{2\pi}{3}} = ee^{i\frac{2\pi}{3}} = e\left[cos\left(\frac{2\pi}{3}\right) + isen\left(\frac{2\pi}{3}\right)\right] = e\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$

$$\boxed{z = -\frac{e}{2} + \frac{e\sqrt{3}}{3}i}$$

Usando logaritmo obtenga todos los valores de z tales que:

153. $\boldsymbol{e^{z} = e^{2+i}}$

$$\log(e^{z}) = \log\left(e^{2+i}\right)$$

$$z + i2m\pi = \log\left(e^{2}e^{i1}\right) \quad ; m \in \mathbb{Z}$$

$$z + i2m\pi = \ln(e^{2}) + i(1 + 2n\pi) \quad ; n \in \mathbb{Z}$$

$$z = 2\ln(e) + i[1 + 2(n - m)\pi]; \;\; m, n \in \mathbb{Z}$$

$$\boxed{z = 2 + i(1 + 2k\pi) \quad ; k \in \mathbb{Z}}$$

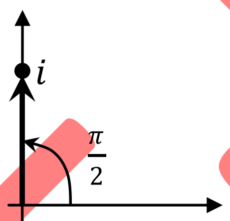
EXPONENTES COMPLEJOS

$$z^{c} = e^{c\log(z)} \; ; \; z \neq 0$$

Halle todos los valores de:

156. $\boldsymbol{i^{i}}$

$$z = i \;, \qquad c = i$$



$$i^{i} = e^{i\log(i)} = e^{i\left[\ln(1) + i\left(\frac{\pi}{2} + 2k\pi\right)\right]}$$

$$\boxed{i^{i} = e^{-\left(\frac{1}{2} + 2k\right)\pi} + i0, \quad k \in \mathbb{Z}}$$

157. $\boldsymbol{2^{1+i}}$

$$z = 2 \;,\; c = 1 + i$$

$r = 2, \; \theta = 0$

$$2^{1+i} = e^{(1+i)\log(2)} = e^{(1+i)[\ln(2) + i(0 + 2k\pi)]}$$

$$= e^{\ln(2) + i2k\pi + i\ln(2) - 2k\pi}$$

$$= e^{\ln(2) - 2k\pi + i(\ln(2) + 2k\pi)}$$

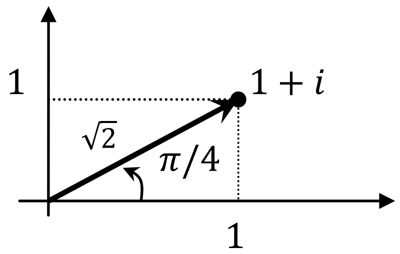
$$= e^{\ln(2) - 2k\pi} \;\; \underbrace{e^{i(\ln(2) + 2k\pi)}}$$

$$= e^{\ln(2)} \; e^{-2k\pi} \left[\overbrace{\cos(\ln(2) + 2k\pi) + i\,sen(\ln(2) + 2k\pi)}\right]$$

$$\boxed{2^{1+i} = 2e^{-2k\pi}\cos\big(ln(2)\big) + i2e^{-2k\pi}sen\big(ln(2)\big)\ ,\quad k \in \mathbb{Z}}$$

158. $\boldsymbol{(1+i)^i}$

$z = 1 + i\ ;\ c = i$



$$(1+i)^i = e^{i\,log\,(1+i)} = e^{i\left[ln(\sqrt{2})+i\left(\frac{\pi}{4}+2k\pi\right)\right]}$$

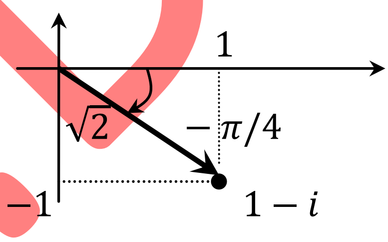
$$(1+i)^i = e^{-\left(\frac{\pi}{4}+2k\pi\right)}e^{i\,ln(\sqrt{2})}$$

$$= e^{-\left(\frac{\pi}{4}+2k\pi\right)}\left[cos\left(ln(\sqrt{2})\right) + isen\left(ln(\sqrt{2})\right)\right]$$

$$\boxed{(1+i)^i = e^{-\left(\frac{1}{4}+2k\right)\pi}cos\left(ln(\sqrt{2})\right) + ie^{-\left(\frac{1}{4}+2k\right)\pi}sen\left(ln(\sqrt{2})\right),\qquad k \in \mathbb{Z}}$$

159. $\boldsymbol{(1-i)^{4i}}$

$z = 1 - i\ ,\ \ c = 4i$

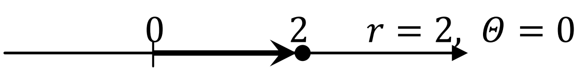


$$(1-i)^{4i} = e^{4i\,log\,(1-i)} = e^{4i\left[ln(\sqrt{2})+i\left(-\frac{\pi}{4}+2k\pi\right)\right]} = e^{i\,4ln(\sqrt{2})}e^{\pi-8k\pi}$$

$$\boxed{(1-i)^{4i} = e^{(1-8k)\pi}cos\big(ln(4)\big) + ie^{(1-8k)\pi}sen\big(ln(4)\big)\ ,k \in \mathbb{Z}}$$

160. $\boldsymbol{2^{\pi}}$

$z = 2\ ,\ c = \pi$



$$2^{\pi} = e^{\pi log\,(2)} = e^{\pi[ln\,(2)+i(0+2k\pi)]}$$

$$= e^{\pi ln\,(2)}e^{i\,2k\pi^2}$$

$$= e^{ln\,(2^{\pi})}[cos(2k\pi^2) + isen(2k\pi^2)]$$

$$\boxed{2^{\pi} = 2^{\pi}\ cos(2k\pi^2) + i2^{\pi}sen(2k\pi^2)\ ,\ k \in \mathbb{Z}}$$