TRANSFORMADA DE LAPLACE

Sea

- $f(t)$ una función real o compleja de la variable real t, definida para $t \geq 0$.
- $s = \sigma + i\omega$ un parámetro complejo.

Definimos la **transformada unilateral de Laplace** de f como

$$\mathscr{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt = F(s)$$

siempre que esta integral converja.

Notación

A las funciones del tiempo las denotamos con letras minúsculas. Por ejemplo:

$$f(t)\,, \qquad g(t), \qquad y(t)$$

Y a sus respectivas transformadas de Laplace con letras mayúsculas:

$$F(s)\,, \qquad G(s), \qquad Y(s)$$

Es decir

$$\mathscr{L}\{f(t)\} = F(s), \qquad \mathscr{L}\{g(t)\} = G(s), \qquad \mathscr{L}\{y(t)\} = Y(s)$$

Algunas propiedades de la transformada de Laplace

Linealidad

Si

- c_1 y c_2 son constantes complejas.
- $\mathscr{L}\{f(t)\} = F(s)$
- $\mathscr{L}\{g(t)\} = G(s)$

Entonces

$$\mathscr{L}\{c_1 f(t) + c_2 g(t)\} = c_1 \mathscr{L}\{f(t)\} + c_2 \mathscr{L}\{g(t)\} = c_1 F(s) + c_2 G(s)$$

Transformadas de las derivadas

Si

- $\mathscr{L}\{f(t)\} = F(s)$

Entonces

$$\mathscr{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$$

Por ejemplo si $\mathscr{L}\{y(t)\} = Y(s)$ entonces

$$\mathscr{L}\{y'(t)\} = sY(s) - y(0)$$

$$\mathscr{L}\{y''(t)\} = s^2 Y(s) - sy(0) - y'(0)$$

Tabla de algunas transformadas de Laplace

$f(t)$	$F(s)$
1	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$cos(bt)$	$\frac{s}{s^2+b^2}$
$sen(bt)$	$\frac{b}{s^2+b^2}$
$e^{at} cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
$e^{at} sen(bt)$	$\frac{b}{(s-a)^2+b^2}$

Aplicación de la transformada de Laplace a las ecuaciones diferenciales

Supongamos que se quiere resolver la siguiente ecuación diferencial lineal ordinaria de segundo orden con coeficientes constantes:

$$ay'' + by' + cy = f(t)$$

sujeta a las condiciones iniciales: $y(0) = k_1 \quad , y'(0) = k_2$.

Es decir, se busca determinar la expresión de la función incógnita $y(t)$.

Aplicando la transformada de Laplace a la ecuación diferencial tenemos

$$\mathscr{L}\{ay'' + by' + cy\} = \mathscr{L}\{f(t)\}$$

Por propiedad de linealidad (en el primer miembro)

$$a\mathscr{L}\{y''\} + b\mathscr{L}\{y'\} + c\mathscr{L}\{y\} = \mathscr{L}\{f(t)\}$$

$$a\left[s^2Y(s) - s\underbrace{y(0)}_{k_1} - \underbrace{y'(0)}_{k_2}\right] + b\left[sY(s) - \underbrace{y(0)}_{k_1}\right] + cY(s) = F(s)$$

$$as^2Y(s) - ask_1 - ak_2 + bsY(s) - bk_1 + cY(s) = F(s)$$

Sacando $Y(s)$ factor común en el primer miembro

$$Y(s)[as^2 + bs + c] = F(s) + (as + b)k_1 + ak_2$$

Despejando la transformada de la función incógnita $Y(s)$ se obtiene

$$Y(s) = \frac{F(s) + \overbrace{(as + b)k_1 + ak_2}^{\substack{\text{términos de} \\ \text{condiciones iniciales}}}}{as^2 + bs + c}$$

Ocurre con frecuencia que $Y(s) = \frac{P(s)}{Q(s)}$ suele ser un **cociente de polinomios** irreducible (P y Q no tienen factores comunes) **con *grado Q* $>$ *grado P***, entonces los puntos singulares de $Y(s)$ son **polos**, y (se demuestra que) la función buscada $y(t)$ se puede obtener por residuos:

$$y(t) = \overbrace{\mathscr{L}^{-1}\{Y(s)\}}^{\textbf{transformada inversa de Laplace de } Y(s)}$$

$$\boxed{y(t) = \sum_{k=1}^{N} Res\,(Y(s)e^{st}, s_k) \quad , s_k : polos\ de\ Y(s)e^{st}}$$

Para el cálculo de los residuos en los polos de $Y(s)e^{st}$ (= polos de $Y(s)$) se usan las fórmulas:

$$\boxed{Res(Y(s)e^{st}, s_0) = \lim_{s \to s_0} [(s - s_0)Y(s)e^{st}]}$$

si s_0 es **polo simple** de $Y(s)e^{st}$ y

$$\boxed{Res(Y(s)e^{st}, s_0) = \frac{1}{(n-1)!} \lim_{s \to s_0} \left\{ \frac{d^{n-1}}{ds^{n-1}} [(s - s_0)^n Y(s)e^{st}] \right\}}$$

si s_0 es **polo de orden *n*** de $Y(s)e^{st}$.

Aplicando el método de la transformada de Laplace obtenga la solución de las siguientes ecuaciones diferenciales, con las condiciones iniciales fijadas:

1) $y'' - 4y = 0 \quad ; \quad y(0) = 0 \ , \ y'(0) = -6$

$Resp.: y(t) = -3senh(2t)$

2) $y'' + y = 2 \quad ; \quad y(0) = 0 \ , \ y'(0) = 3$

$Resp.: y(t) = 2 - 2cos(t) + 3sen(t)$

3) $y'' + 2y' + y = e^t \quad ; \quad y(0) = -1 \ , \ y'(0) = 1$

$Resp.: y(t) = \left(-\frac{1}{2}t - \frac{5}{4}\right)e^{-t} + \frac{1}{4}e^t$

4) $9y'' - 6y' + y = 0$; $y(0) = 3$, $y'(0) = 1$

Resp.: $y(t) = 3e^{\frac{1}{3}t}$

5) $y'' - y = sen(3t)$; $y(0) = 0$, $y'(0) = 0$

Resp.: $y(t) = \frac{3}{10}senh(t) - \frac{1}{10}sen(3t)$

6) $y'' - y = cos(2t)$; $y(0) = 0$, $y'(0) = 0$

Resp.: $y(t) = \frac{1}{5}cosh(t) - \frac{1}{5}cos(2t)$

7) $y'' - 4y = e^{3t} + 3e^{-t}$; $y(0) = 0$, $y'(0) = 0$

Resp.: $y(t) = \frac{1}{5}e^{3t} - e^{-t} + \frac{4}{5}e^{-2t}$

8) $y'' + y = t + 1$; $y(0) = 0$, $y'(0) = 0$

Resp.: $y(t) = 1 + t - cos(t) - sen(t)$

9) $y'' + y = 4te^{t}$; $y(0) = 0$, $y'(0) = 0$

Resp.: $y(t) = 2cos(t) + 2(t-1)e^{t}$

10) $y'' - 6y' + 9y = t$; $y(0) = 0$, $y'(0) = 1$

Resp.: $y(t) = \left(\frac{10}{9}t - \frac{2}{27}\right)e^{3t} + \frac{1}{9}t + \frac{2}{27}$

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1) $y'' - 4y = 0$; $y(0) = 0$, $y'(0) = -6$

$$\mathscr{L}\{y'' - 4y\} = \mathscr{L}\{0\}$$

$$\mathscr{L}\{y''\} - 4\,\mathscr{L}\{y\} = 0$$

$$s^2Y(S) - s\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{-6} - 4Y(s) = 0$$

$$s^2Y(s) + 6 - 4Y(s) = 0$$

$$Y(s)(s^2 - 4) = -6$$

$$Y(s) = \frac{-6}{s^2 - 4} = \frac{-6}{(s-2)(s+2)}$$

$$y(t)=\mathscr{L}^{-1}\left\{\overbrace{\frac{-6}{(s-2)(s+2)}}^{Y(s)}\right\}$$

$$y(t)=\sum_{k=1}^{2} Res\left(Y(s)e^{st},s_k\right)\ , \qquad con\ s_1=2\ y\ s_2=-2\ polos\ simples\ de\ Y(s)e^{st}$$

$$y(t)=\underbrace{Res(Y(s)e^{st},2)}+\underbrace{Res(Y(s)e^{st},-2)}$$

$$y(t)=\overbrace{\lim_{s\to 2}\left[(s-2)\left(\frac{-6}{(s-2)(s+2)}\right)e^{st}\right]}+\overbrace{\lim_{s\to -2}\left[(s+2)\left(\frac{-6}{(s-2)(s+2)}\right)e^{st}\right]}$$

$$y(t)=\lim_{s\to 2}\left[\left(\frac{-6}{s+2}\right)e^{st}\right]+\lim_{s\to -2}\left[\left(\frac{-6}{s-2}\right)e^{st}\right]$$

$$y(t)=-\frac{6}{4}e^{2t}+\frac{6}{4}e^{-2t}$$

$$y(t)=-\frac{3}{2}e^{2t}+\frac{3}{2}e^{-2t}$$

$$y(t)=-\frac{3}{2}e^{2t}+\frac{3}{2}e^{-2t}$$

$$\boxed{\boldsymbol{y(t)=-3\left(\frac{e^{2t}-e^{-2t}}{2}\right)=-3senh(2t)}}$$

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2) $\boldsymbol{y''+y=2}$; $\boldsymbol{y(0)=0}$, $\boldsymbol{y'(0)=3}$

$$\mathscr{L}\{y''+y\}=\mathscr{L}\{2\}$$

$$\mathscr{L}\{y''\}+\mathscr{L}\{y\}=2\,\mathscr{L}\{1\}$$

$$s^2Y(s)-s\underbrace{y(0)}_{0}-\underbrace{y'(0)}_{3}+Y(s)=2\frac{1}{s}$$

$$s^2Y(s)-3+Y(s)=\frac{2}{s}$$

$$Y(s)(s^2+1)=\frac{2}{s}+3$$

$$Y(s)(s^2+1)=\frac{2+3s}{s}$$

$$Y(s)=\frac{2+3s}{s(s^2+1)}=\frac{2+3s}{s(s-i)(s+i)}$$

$$y(t)=\mathscr{L}^{-1}\left\{\overbrace{\frac{2+3s}{s(s-i)(s+i)}}^{Y(s)}\right\}$$

$$y(t)=\sum_{k=1}^{3} Res\,(Y(s)e^{st},s_k),\quad con\ s_1=0, s_2=i, y\ s_3=-i\quad polos\ simples\ de\ Y(s)e^{st}$$

$$y(t)=Res(Y(s)e^{st},0)+Res(Y(s)e^{st},i)+Res(Y(s)e^{st},-i)$$

$$y(t)=\lim_{s\to 0}\left[s\left(\underbrace{\frac{2+3s}{s(s^2+1)}}_{Y(s)}\right)e^{st}\right]+\lim_{s\to i}\left[(s-i)\left(\underbrace{\frac{2+3s}{s(s-i)(s+i)}}_{Y(s)}\right)e^{st}\right]$$

$$+\lim_{s\to -i}\left[(s+i)\left(\underbrace{\frac{2+3s}{s(s-i)(s+i)}}_{Y(s)}\right)e^{st}\right]$$

$$y(t)=\lim_{s\to 0}\left[\left(\frac{2+3s}{s^2+1}\right)e^{st}\right]+\lim_{s\to i}\left[\left(\frac{2+3s}{s(s+i)}\right)e^{st}\right]+\lim_{s\to -i}\left[\left(\frac{2+3s}{s(s-i)}\right)e^{st}\right]$$

$$y(t)=2+\left(\frac{2+3i}{i(2i)}\right)e^{it}+\left(\frac{2-3i}{-i(-2i)}\right)e^{-it}$$

$$y(t)=2+\left(\frac{2+3i}{-2}\right)e^{it}+\left(\frac{2-3i}{-2}\right)e^{-it}$$

$$y(t)=2-e^{it}-\frac{3}{2}ie^{it}-e^{-it}+\frac{3}{2}ie^{-it}$$

$$y(t)=2-2\left(\frac{e^{it}+e^{-it}}{2}\right)-(3i)i\left(\frac{e^{it}-e^{-it}}{2i}\right)$$

$$y(t)=2-2\underbrace{\left(\frac{e^{it}+e^{-it}}{2}\right)}+3\underbrace{\left(\frac{e^{it}-e^{-it}}{2i}\right)}$$

$$\boxed{\boldsymbol{y(t)=2-2\,\overbrace{cos(t)}+3\,\overbrace{sen(t)}}}$$

3) $\boldsymbol{y'' + 2y' + y = e^t}$; $\boldsymbol{y(0) = -1}$, $\boldsymbol{y'(0) = 1}$

$$\mathscr{L}\{y'' + 2y' + y\} = \mathscr{L}\{e^t\}$$

$$\mathscr{L}\{y''\} + 2\mathscr{L}\{y'\} + \mathscr{L}\{y\} = \mathscr{L}\{e^t\}$$

$$s^2Y(s) - s\,\underbrace{y(0)}_{-1} - \underbrace{y'(0)}_{1} + 2\left[sY(s) - \underbrace{y(0)}_{-1}\right] + Y(s) = \frac{1}{s-1}$$

$$s^2Y(s) + s - 1 + 2sY(s) + 2 + Y(s) = \frac{1}{s-1}$$

$$Y(s)[s^2 + 2s + 1] = \frac{1}{s-1} - (s+1)$$

$$Y(s)[s^2 + 2s + 1] = \frac{1 - (s+1)(s-1)}{s-1}$$

$$Y(s) = \frac{1 - (s^2 - 1)}{(s^2 + 2s + 1)(s-1)} = \frac{2 - s^2}{(s+1)^2(s-1)}$$

$$y(t) = \sum_{k=1}^{2} Res\,(Y(s)e^{st}, s_k),\ con\ s_1 = -1\ polo\ de\ orden\ 2\ y\ s_2 = 1\ polo\ simple$$

$$y(t) = Res(Y(s)e^{st}, -1) + Res(Y(s)e^{st}, 1)$$

$$y(t) = \frac{1}{1!}\lim_{s\to -1}\left\{\frac{d}{ds}\left[(s+1)^2\left(\frac{2-s^2}{(s+1)^2(s-1)}\right)e^{st}\right]\right\} + \lim_{s\to 1}\left[(s-1)\left(\frac{2-s^2}{(s+1)^2(s-1)}\right)e^{st}\right]$$

$$y(t) = \lim_{s\to -1}\left\{\frac{d}{ds}\left[\left(\frac{2-s^2}{s-1}\right)e^{st}\right]\right\} + \lim_{s\to 1}\left[\left(\frac{2-s^2}{(s+1)^2}\right)e^{st}\right]$$

$$y(t) = \lim_{s\to -1}\left\{\frac{d}{ds}[(2-s^2)(s-1)^{-1}e^{st}]\right\} + \lim_{s\to 1}\left[\left(\frac{2-s^2}{(s+1)^2}\right)e^{st}\right]$$

$$y(t) = \lim_{s\to -1}\{-2s(s-1)^{-1}e^{st} - (2-s^2)(s-1)^{-2}e^{st} + (2-s^2)(s-1)^{-1}te^{st}\} + \frac{1}{4}e^t$$

$$y(t) = -e^{-t} - \frac{1}{4}e^{-t} - \frac{1}{2}te^{-t} + \frac{1}{4}e^t$$

$$\boxed{\boldsymbol{y(t) = \left(-\frac{1}{2}t - \frac{5}{4}\right)e^{-t} + \frac{1}{4}e^t}}$$

6) $\boldsymbol{y'' - y = cos(2t)}$; $\boldsymbol{y(0) = 0 \;,\; y'(0) = 0}$

$$\mathscr{L}\{y'' - y\} = \mathscr{L}\{cos(2t)\}$$

$$\mathscr{L}\{y''\} - \mathscr{L}\{y\} = \mathscr{L}\{cos(2t)\}$$

$$s^2Y(s) - s\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{0} - Y(s) = \frac{s}{s^2 + 2^2}$$

$$Y(s)(s^2 - 1) = \frac{s}{s^2 + 4}$$

$$Y(s) = \frac{s}{(s^2 - 1)(s^2 + 4)} = \frac{s}{(s-1)(s+1)(s-2i)(s+2i)}$$

$$y(t) = Res(Y(s)e^{st}, 1) + Res(Y(s)e^{st}, -1) + Res(Y(s)e^{st}, 2i) + Res(Y(s)e^{st}, -2i)$$

$$\begin{aligned} y(t) = &\lim_{s\to 1}\left[(s-1)\frac{s}{(s-1)(s+1)(s^2+4)}e^{st}\right] \\ &+ \lim_{s\to -1}\left[(s+1)\frac{s}{(s-1)(s+1)(s^2+4)}e^{st}\right] \\ &+ \lim_{s\to 2i}\left[(s-2i)\frac{s}{(s^2-1)(s-2i)(s+2i)}e^{st}\right] \\ &+ \lim_{s\to -2i}\left[(s+2i)\frac{s}{(s^2-1)(s-2i)(s+2i)}e^{st}\right] \end{aligned}$$

$$\begin{aligned} y(t) = &\lim_{s\to 1}\left[\frac{s}{(s+1)(s^2+4)}e^{st}\right] + \lim_{s\to -1}\left[\frac{s}{(s-1)(s^2+4)}e^{st}\right] \\ &+ \lim_{s\to 2i}\left[\frac{s}{(s^2-1)(s+2i)}e^{st}\right] + \lim_{s\to -2i}\left[\frac{s}{(s^2-1)(s-2i)}e^{st}\right] \end{aligned}$$

$$y(t) = \frac{1}{10}e^{t} + \frac{1}{10}e^{-t} + \frac{2i}{(-5)(4i)}e^{i2t} - \frac{2i}{(-5)(-4i)}e^{-i2t}$$

$$y(t) = \frac{1}{5}\left(\underbrace{\frac{e^{t} + e^{-t}}{2}}\right) - \frac{1}{5}\left(\underbrace{\frac{e^{i2t} + e^{-i2t}}{2}}\right)$$

$$\boxed{\boldsymbol{y(t) = \frac{1}{5}\overbrace{cosh(t)} - \frac{1}{5}\overbrace{cos(2t)}}}$$

7) $\boldsymbol{y'' - 4y = e^{3t} + 3e^{-t}}$; $\boldsymbol{y(0) = 0\ ,\ y'(0) = 0}$

$$\mathscr{L}\{y'' - 4y\} = \mathscr{L}\{e^{3t} + 3e^{-t}\}$$

$$\mathscr{L}\{y''\} - 4\mathscr{L}\{y\} = \mathscr{L}\{e^{3t}\} + 3\mathscr{L}\{e^{-t}\}$$

$$s^2Y(s) - s\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{0} - 4Y(s) = \frac{1}{s-3} + \frac{3}{s+1}$$

$$Y(s)(s^2-4) = \frac{s+1+3(s-3)}{(s-3)(s+1)}$$

$$Y(s)(s^2-4) = \frac{s+1+3s-9}{(s-3)(s+1)}$$

$$Y(s)(s^2-4) = \frac{4s-8}{(s-3)(s+1)}$$

$$Y(s) = \frac{4s-8}{(s-3)(s+1)(s^2-4)}$$

$$Y(s) = \frac{4s-8}{(s-3)(s+1)(s-2)(s+2)}$$

$$y(t) = Res(Y(s)e^{st},3) + Res(Y(s)e^{st},-1) + Res(Y(s)e^{st},2) + Res(Y(s)e^{st},-2)$$

$$\begin{aligned} y(t) = \lim_{s\to 3}&\left[(s-3)\frac{(4s-8)e^{st}}{(s-3)(s+1)(s-2)(s+2)}\right] \\ &+ \lim_{s\to -1}\left[(s+1)\frac{(4s-8)e^{st}}{(s-3)(s+1)(s-2)(s+2)}\right] \\ &+ \lim_{s\to 2}\left[(s-2)\frac{(4s-8)e^{st}}{(s-3)(s+1)(s-2)(s+2)}\right] \\ &+ \lim_{s\to -2}\left[(s+2)\frac{(4s-8)e^{st}}{(s-3)(s+1)(s-2)(s+2)}\right] \end{aligned}$$

$$\begin{aligned} y(t) = \lim_{s\to 3}&\left[\frac{(4s-8)e^{st}}{(s+1)(s-2)(s+2)}\right] + \lim_{s\to -1}\left[\frac{(4s-8)e^{st}}{(s-3)(s-2)(s+2)}\right] \\ &+ \lim_{s\to 2}\left[\frac{(4s-8)e^{st}}{(s-3)(s+1)(s+2)}\right] + \lim_{s\to -2}\left[\frac{(4s-8)e^{st}}{(s-3)(s+1)(s-2)}\right] \end{aligned}$$

$$y(t)=\frac{(4.3-8)e^{3t}}{(3+1)(3-2)(3+2)}+\frac{(4.(-1)-8)e^{-t}}{(-1-3)(-1-2)(-1+2)}$$
$$+\frac{(4.2-8)e^{2t}}{(2-3)(2+1)(2+2)}+\frac{(4.(-2)-8)e^{-2t}}{(-2-3)(-2+1)(-2-2)}$$

$$\boxed{\boldsymbol{y(t)=\frac{1}{5}e^{3t}-e^{-t}+\frac{4}{5}e^{-2t}}}$$