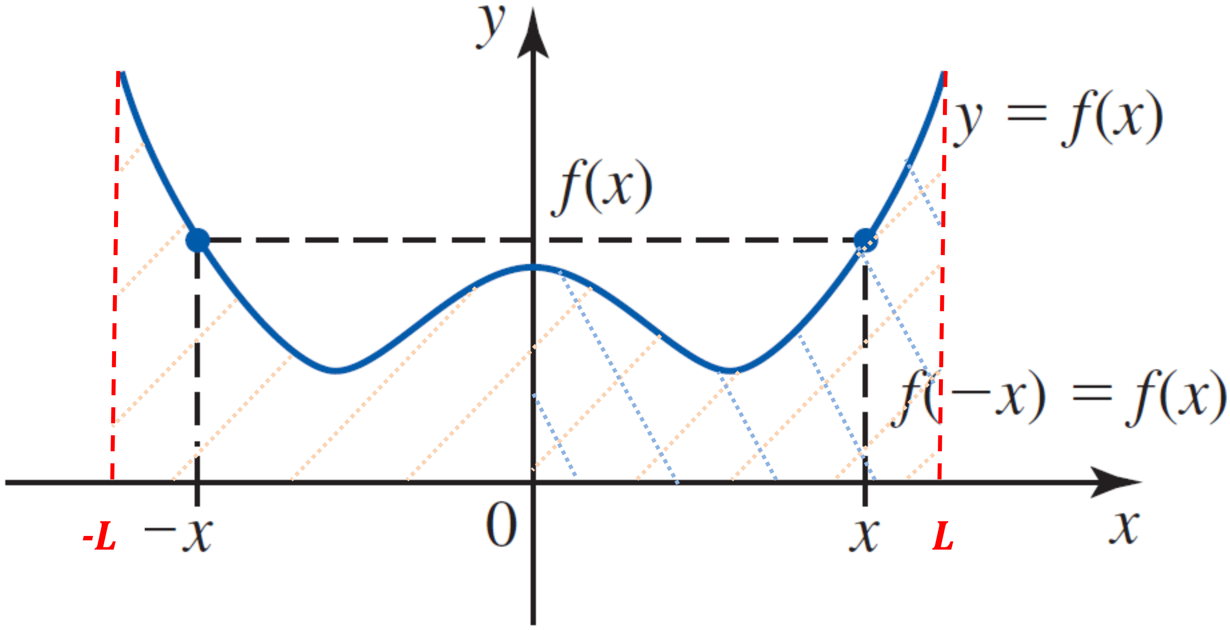
REPASO: FUNCIÓN PAR E IMPAR. PROPIEDADES

Función par

f es **par** en $[-L, L]$ si $f(-x) = f(x)$ para $-L \leq x \leq L$

Geométricamente, una función f es par si su gráfica es simétrica respecto del eje y.



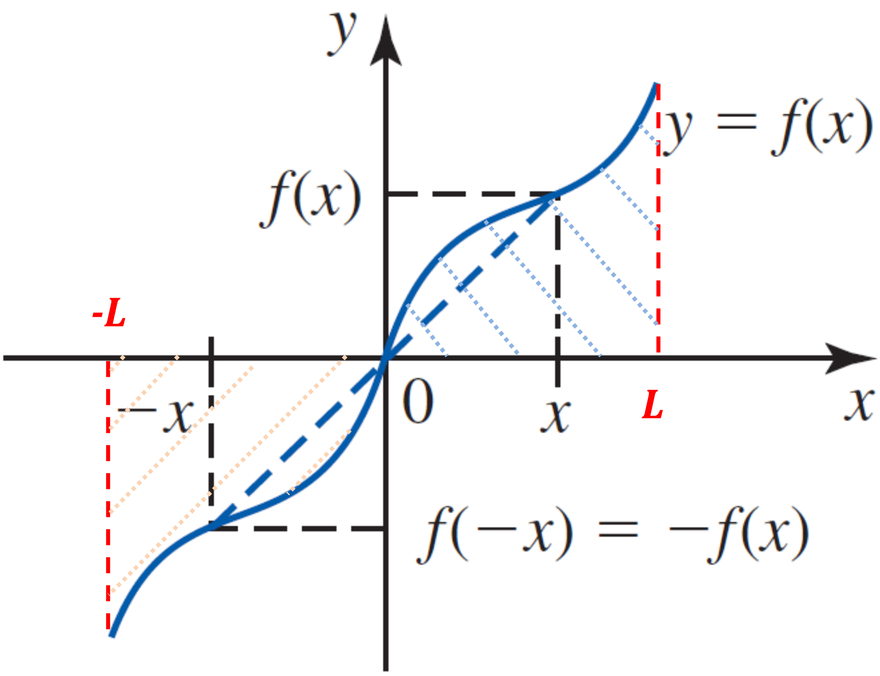
Por lo tanto tiene la propiedad de que:

$$\int_{-L}^{L} f(x)\,dx = 2\int_{0}^{L} f(x)\,dx$$

Función impar

f es **impar** en $[-L, L]$ si $f(-x) = -f(x)$ para $-L \leq x \leq L$

Geométricamente, una función f es impar si su gráfica es simétrica respecto del origen.



Por lo tanto tiene la propiedad de que:

$$\int_{-L}^{L} f(x)\,dx = 0$$

Si

$$f_p: función\ par$$

$$f_i: función\ impar$$

se tienen las siguientes propiedades:

- $f_{p/i} \;\pm\; f_{p/i} = f_{p/i}$
- $f_{p/i} \;\overset{\times}{\div}\; f_{p/i} = f_p$
- $f_{p/i} \;\overset{\times}{\div}\; f_{i/p} = f_i$

CÁLCULO DE INTEGRALES REALES UTILIZANDO RESIDUOS

A) INTEGRALES DEFINIDAS DE FUNCIONES TRIGONOMÉTRICAS

El método de los residuos es útil en el cálculo de integrales reales definidas del tipo

$$\int_0^{2\pi} F(cos\theta, sen\theta)\, d\theta$$

donde

F: cociente de polinomios en $cos\theta \ y \ sen\theta$.

El hecho de que θ varíe de $0 \, a \, 2\pi$ sugiere que si consideramos a θ como el argumento de z sobre la circunferencia unitaria

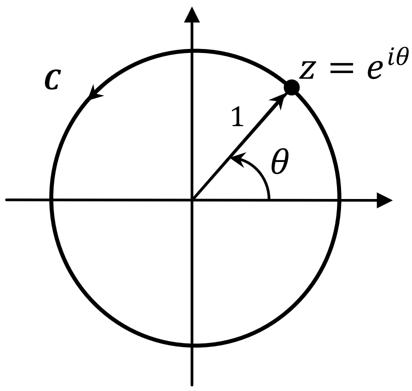
$$z = e^{i\theta} \quad , \qquad 0 \leq \theta \leq 2\pi$$

podemos escribir:

$$cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{1}{2}\left(e^{i\theta} + \frac{1}{e^{i\theta}}\right)$$

$$= \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$\boxed{cos(\theta) = \frac{z^2 + 1}{2z}}$$



$$sen(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{1}{2i}\left(e^{i\theta} - \frac{1}{e^{i\theta}}\right)$$

$$= \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

$$\boxed{sen(\theta) = \frac{z^2 - 1}{i2z}}$$

Y como $z = e^{i\theta} \quad \Rightarrow \quad dz = ie^{i\theta} d\theta \quad \Rightarrow \ dz = izd\theta \quad \Rightarrow \quad \boxed{d\theta = \frac{dz}{iz}}$

Entonces la integral original se convierte en una integral compleja a lo largo de un contorno C que es una circunferencia unitaria (orientada positivamente) centrada en el origen. Es decir:

$$\int_0^{2\pi} F(cos\theta, sen\theta)\, d\theta = \oint_C \overbrace{\underbrace{F\left(\frac{z^2+1}{2z}, \frac{z^2-1}{i2z}\right)\frac{1}{iz}}}^{cociente\ \ de\ polinomios\ \ en\ z}\, dz$$

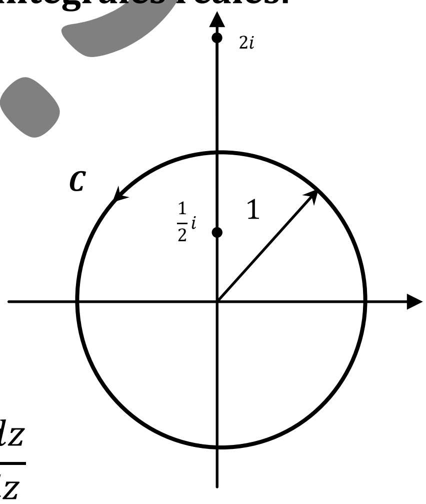
$$= \underbrace{\oint_C \overbrace{f(z)}\, dz}_{\substack{Por\ el\ teorema\\ de\ los\ residuos}}$$

$$= \overbrace{2\pi i \sum_{k=1}^{N} Res(f(z), z_k)} \quad , \quad |z_k| < 1$$

donde z_k $(k = 1,2, \dots, N)$ polos de f interiores a *C.*

Empleando residuos obtenga el valor de las siguientes integrales reales:

263. $\int_0^{2\pi} \frac{1}{5-4sen(\theta)} d\theta$



$$sen(\theta) = \frac{z^2-1}{i2z} \quad ; \quad d\theta = \frac{dz}{iz} \quad ; \quad C: |z| = 1$$

$$\int_0^{2\pi} \frac{1}{5-4sen(\theta)} d\theta = \oint_C \frac{1}{5-4\left(\frac{z^2-1}{i2z}\right)}\frac{dz}{iz}$$

$$= \oint_C \frac{1}{5-4\left(\frac{z^2-1}{i2z}\right)}\frac{dz}{iz}$$

$$= \oint_C \frac{1}{5-2\left(\frac{z^2-1}{iz}\right)}\frac{dz}{iz}$$

$$= \oint_C \frac{1}{\left(\frac{i5z-2z^2+2}{iz}\right)}\frac{dz}{iz}$$

$$= -\oint_C \frac{1}{2z^2-i5z-2} dz \ ; \quad f(z) = \frac{\overbrace{1}^{h(z)}}{\underbrace{2z^2-i5z-2}_{g(z)}}$$

$$2z^2 - i5z - 2 = 0$$

$$z = \frac{5i \pm \sqrt{(-i5)^2 - 4(2)(-2)}}{4}$$

$$= \frac{5i \pm \sqrt{-25+16}}{4} = \frac{5i \pm 3i}{4}$$

$$z_1 = 2i \ , \qquad z_2 = \frac{1}{2}i$$

$$= -2\pi i \ \underbrace{Res\left(f(z), \overbrace{\tfrac{1}{2}i}^{z_0}\right)} \quad ; z_0 = \tfrac{1}{2}i \ \textit{polo simple de } f$$

$$= -2\pi i \ \frac{\overbrace{h(z_0)}}{g'(z_0)}$$

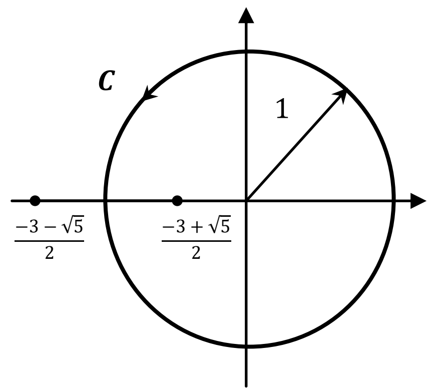
$$= -2\pi i \left[\frac{1}{4z - 5i}\right]_{z=\frac{1}{2}i}$$

$$= -2\pi i \left(\frac{1}{-3i}\right)$$

$$\boxed{\int_0^{2\pi} \frac{1}{5 - 4sen(\theta)} d\theta = \frac{2}{3}\pi}$$

262. $\int_0^{\pi} \frac{1}{3+2cos(\theta)} d\theta$

$$\int_0^{\pi} \frac{\overbrace{1}^{fp}}{\underbrace{\underbrace{3}_{fp} + \underbrace{2cos(\theta)}_{fp}}_{\substack{\textit{el integrando es} \\ \textit{funci ón par } (fp)}}} d\theta = \frac{1}{2}\int_{-\pi}^{\pi} \frac{1}{3+2cos(\theta)} d\theta$$



$$C: |z| = 1$$

$$z = e^{i\theta} \quad ; \ -\pi \le \theta \le \pi$$

$$\int_0^{\pi} \frac{1}{3+2cos\,(\theta)} d\theta = \frac{1}{2}\int_{-\pi}^{\pi} \frac{1}{3+2cos\,(\theta)} d\theta = \frac{1}{2}\oint_C \ \frac{1}{3+2\left(\frac{z^2+1}{2z}\right)} \frac{dz}{iz}$$

$$\boxed{\begin{gathered} z^2 + 3z + 1 = 0 \\ z = \frac{-3 \pm \sqrt{9-4}}{2} \\ z = \frac{-3 \pm \sqrt{5}}{2} \end{gathered}}$$

$$= \frac{1}{2}\oint_C \ \frac{1}{\frac{3z + z^2 + 1}{z}} \frac{dz}{iz}$$

$$= \frac{1}{2i} \underbrace{\oint_C \frac{1}{z^2+3z+1} dz} \quad ; \; f(z) = \frac{1}{z^2+3z+1}$$

$$= \frac{1}{2i} \, 2\pi i \; \overbrace{\underbrace{Res\left(f(z), \frac{-3+\sqrt{5}}{2}\right)}}$$

$$= \pi \overbrace{\left[\frac{1}{2z+3}\right]_{z=\frac{-3+\sqrt{5}}{2}}}$$

$$\boxed{\int_0^{\pi} \frac{\mathbf{1}}{\mathbf{3+2}\boldsymbol{cos(\theta)}} \boldsymbol{d\theta} = \frac{\boldsymbol{\pi}}{\sqrt{\mathbf{5}}}}$$

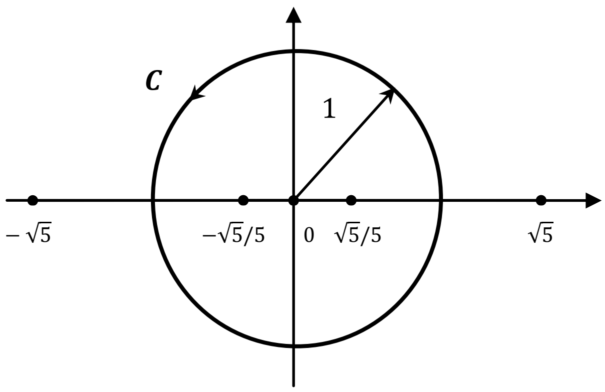
264. $\int_0^{\pi} \frac{\cos^2(\theta)}{13-5\cos(2\theta)} d\theta$

$$z = e^{i\theta} \quad , \quad d\theta = \frac{dz}{iz}$$

$$cos(\theta) = \frac{z^2+1}{2z} \quad \Rightarrow \quad cos^2(\theta) = \left(\frac{z^2+1}{2z}\right)^2 = \frac{(z^2+1)^2}{4z^2}$$

$$cos(\theta) = \frac{e^{i\theta}+e^{-i\theta}}{2} \quad \Rightarrow \quad cos(2\theta) = \frac{\left(e^{i\theta}\right)^2 + \left(e^{-i\theta}\right)^2}{2} = \frac{z^2+z^{-2}}{2} = \frac{z^4+1}{2z^2}$$

$$\int_0^{\pi} \underbrace{\frac{cos^2(\theta)}{13-5cos(2\theta)}}_{funci\acute{o}n\ par} d\theta = \frac{1}{2}\oint_C \frac{\frac{(z^2+1)^2}{4z^2}}{\left[13-5\left(\frac{z^4+1}{2z^2}\right)\right]} \frac{dz}{iz}$$



$$= \frac{1}{2}\oint_C \frac{\frac{(z^2+1)^2}{4z^2}}{\left(\frac{26z^2-5z^4-5}{2z^2}\right)} \frac{dz}{iz}$$

$$= -\frac{1}{4}\oint_C \frac{(z^2+1)^2}{(5z^4-26z^2+5)} \frac{dz}{iz}$$

$$z^2=\frac{26\pm\sqrt{(-26)^2-4(5)(5)}}{10}=\frac{26\pm 24}{10}=\frac{13}{5}\pm\frac{12}{5}$$

$$z^2=5 \quad \Rightarrow \quad z=\pm\sqrt{5}$$

$$z^2=\frac{1}{5} \quad \Rightarrow \; z=\pm\frac{\sqrt{5}}{5}$$

$$f(z)=\frac{(z^2+1)^2}{(5z^5-26z^3+5z)i}$$

$$Res\left(f(z),\pm\frac{\sqrt{5}}{5}\right)=\left[\frac{\left(z^2+1\right)^2}{(25z^4-78z^2+5)i}\right]_{z=\pm\frac{\sqrt{5}}{5}}=\frac{\left(\frac{1}{5}+1\right)^2}{\left(25\frac{1}{25}-78\frac{1}{5}+5\right)i}=\frac{\frac{36}{25}}{-\frac{48}{5}i}=-\frac{3}{20i}$$

$$Res(f(z),0)=\left[\frac{\left(z^2+1\right)^2}{(25z^4-78z^2+5)i}\right]_{z=0}=\frac{1}{5i}$$

$$\boxed{\int_0^\pi \frac{\boldsymbol{cos^2(\theta)}}{\boldsymbol{13-5cos(2\theta)}}\boldsymbol{d\theta}=-\frac{1}{4}(2\pi i)\left(\underbrace{-\frac{3}{20i}-\frac{3}{20i}+\frac{1}{5i}}_{\sum residuos}\right)=-\frac{\pi}{2}i\left(-\frac{2}{20i}\right)=\frac{\pi}{20}}$$

B) <u>INTEGRALES IMPROPIAS</u>

Las siguientes integrales:

(1) $\int_{-\infty}^{\infty} f(x)\,dx$ (2) $\int_{-\infty}^{\infty} f(x)cos(\alpha x)\,dx$ (3) $\int_{-\infty}^{\infty} f(x)sen(\alpha x)\,dx$

$$\alpha>0$$

donde

$$f(x)=\frac{p(x)}{q(x)} \quad (cociente\ de\ polinomios)$$

con

- p y q polinomios con coeficientes reales sin factores comunes.
- q no tiene ceros reales (sobre el eje x), es decir que f no tiene polos sobre el eje x.

se pueden convertir en integrales complejas y resolverse utilizando residuos.

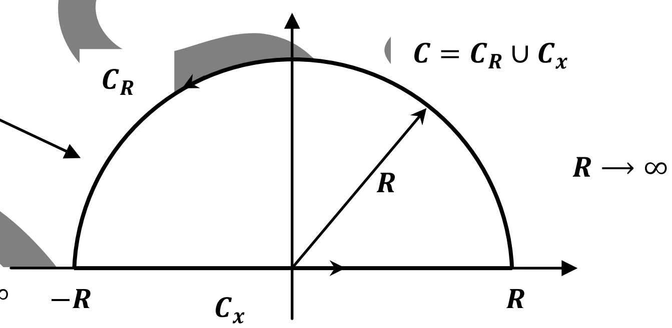
Cálculo de (1)

$$\int_{-\infty}^{\infty} \boldsymbol{f(x)\,dx} \qquad \textbf{donde } \boldsymbol{f(x) = \frac{p(x)}{q(x)}} \textbf{ con grado } \boldsymbol{q \geq} \textbf{ grado de } \boldsymbol{p + 2}$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \overbrace{\oint_C f(z)dz = 2\pi i \sum_{k=1}^{N} Res(f(z), z_k)}^{Por\ el\ teorema\ de\ los\ residuos} \quad , \quad Im\,(z_k) > 0$$

donde

- $f(z) = \frac{p(z)}{q(z)}$ $\quad (se\ cambia\ x\ por\ z)$
- $C = C_R \cup C_x$ es el siguiente contorno



$$\oint_C f(z)dz = \underbrace{\int_{C_x} f(z)dz}_{\int_{-R}^{R} f(x)dx} + \underbrace{\int_{C_R} f(z)dz}_{\to 0\ cuando\ \ R \to \infty}$$

- z_k $(k = 1,2, \dots, N)$ polos de $f(z)$ interiores a C (polos de $f(\,z)$ del semiplano superior).

Cálculo de (2) y (3)

$$\textbf{(2)} \int_{-\infty}^{\infty} \boldsymbol{f(x) cos(\alpha x)\,dx} \qquad\qquad \textbf{(3)} \int_{-\infty}^{\infty} \boldsymbol{f(x) sen(\alpha x)\,dx}$$

$$\boldsymbol{\alpha} > 0$$

$$\textbf{donde } \boldsymbol{f(x) = \frac{p(x)}{q(x)}} \textbf{ con grado } \boldsymbol{q \geq} \textbf{ grado de } \boldsymbol{p + 1}$$

Usando la fórmula de Euler $e^{i\alpha x} = cos(\alpha x) + isen(\alpha x)$ tenemos que:

$$\int_{-\infty}^{\infty} f(x) e^{i\alpha x}\,dx = \int_{-\infty}^{\infty} f(x) cos(\alpha x)\,dx + i \int_{-\infty}^{\infty} f(x) sen(\alpha x)\,dx$$

donde

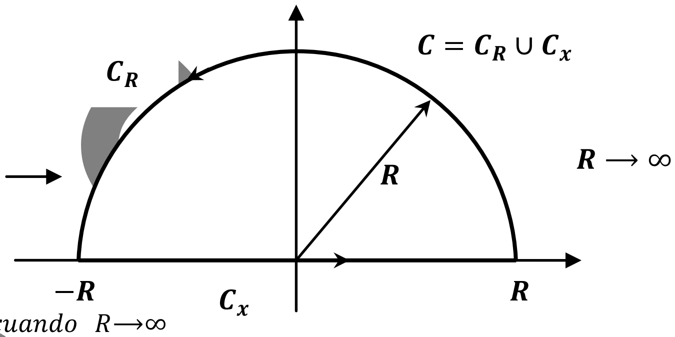
$$\int_{-\infty}^{\infty} f(x) \begin{Bmatrix} cos(\alpha x) \\ sen(\alpha x) \end{Bmatrix} dx = \begin{Bmatrix} Re \\ Im \end{Bmatrix} \int_{-\infty}^{\infty} f(x)\, e^{i\alpha x}\,dx$$

$$= \begin{Bmatrix} Re \\ Im \end{Bmatrix} \underbrace{\oint_C \overbrace{f(z)e^{i\alpha z}}^{F(z)} dz}$$

$$= \begin{Bmatrix} Re \\ Im \end{Bmatrix} \overbrace{2\pi i \sum_{k=1}^{N} Res\left(\underbrace{f(z)e^{i\alpha z}}_{F(z)}, z_k\right)}^{Por\ teorema\ de\ los\ residuos} \ ; \ Im(z_k) > 0$$

donde

- $F(z) = \dfrac{\overbrace{p(z)}^{f(z)}}{q(z)}\ e^{i\alpha z}$
- $C = C_R \cup C_x$ es el siguiente contorno $\longrightarrow$

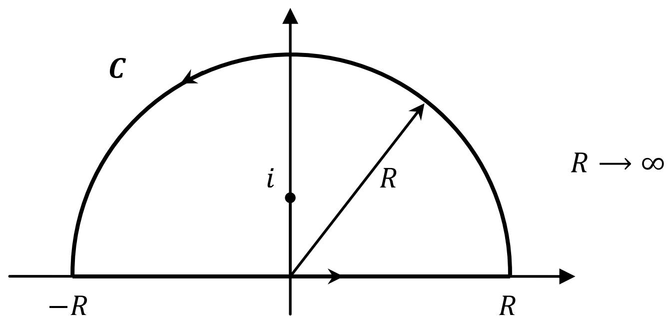


$$\oint_C \overbrace{f(z)e^{i\alpha z}}^{F(z)} dz = \overbrace{\int_{C_x} f(z)e^{i\alpha z} dx}^{\int_{-R}^{R} f(x)e^{i\alpha x}\, dx} + \overbrace{\int_{C_R} f(z)e^{i\alpha z} dz}^{\rightarrow 0\ cuando\ \ R\rightarrow\infty}$$

- z_k $(k = 1,2,\dots,N)$ polos de $F(z)$ interiores a C (polos de $F(\ z)$ del semiplano superior).

Empleando residuos obtenga el valor de las siguientes integrales reales:

273. $\int_0^\infty \frac{1}{(1+x^2)^2} dx$



$$\int_0^\infty \overbrace{\frac{1}{(1+x^2)^2}}^{fp} dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^2} dx$$

$$=\frac{1}{2}\underbrace{\oint_C \frac{1}{\left(1+z^2\right)^2}dz}\ ;\quad f(z)=\frac{1}{\left(1+z^2\right)^2}\ ;\quad z^2+1=0\Rightarrow \underbrace{z=\pm i}_{\substack{polos\\ de\ orden\ 2\ de\ f}}$$

$$\downarrow$$

$$=\frac{1}{2}\left[\overbrace{2\pi i\ \underbrace{Res(f(z),i)}}\right]\ ;\qquad f(z)=\frac{1}{\left(1+z^2\right)^2}=\frac{1}{(z-i)^2(z+i)^2}$$

$$\searrow$$

$$=\pi i\ \overbrace{\lim_{z\to i}\left\{\frac{d}{dz}\left[(z-i)^2\frac{1}{(z-i)^2(z+i)^2}\right]\right\}}$$

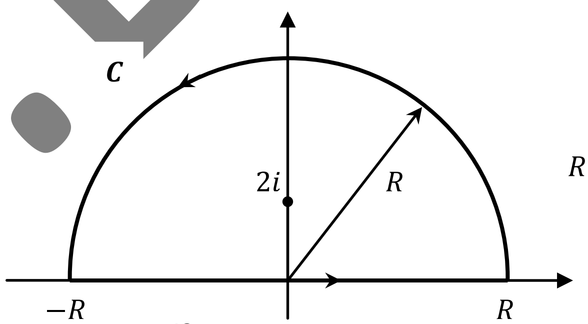
$$=\pi i\lim_{z\to i}\left\{\frac{d}{dz}[(z+i)^{-2}]\right\}$$

$$=\pi i\lim_{z\to i}[(-2)(z+i)^{-3}(1)]$$

$$=\pi i\left[\frac{-2}{(2i)^3}\right]$$

$$\boxed{\int_0^\infty \frac{1}{(1+x^2)^2}dx=\frac{\pi}{4}}$$

281. $\int_{-\infty}^{\infty}\frac{\cos(3x)}{x^2+4}dx$



$R\longrightarrow\infty$

$\alpha=3$

$$F(z)=\overbrace{\frac{1}{z^2+4}}^{f(z)}e^{i3z}=\frac{h(z)}{g(z)}=\frac{e^{i3z}}{z^2+4}=\frac{e^{i3z}}{(z-2i)(z+2i)}$$

$$z^2+4=0\ \Rightarrow\ z^2=-4\ \Rightarrow z=\pm 2i$$

$$\int_{-\infty}^{\infty}\frac{\cos(3x)}{x^2+4}dx=Re\left\{\oint_C \frac{\overbrace{e^{i3z}}^{F(z)}}{z^2+4}dz\right\}=Re\left\{2\pi i\ \underbrace{Res\left(F(z),\overbrace{\underset{polo\ simple\ de\ F}{2i}}^{z_0}\right)}\right\}$$

$$\swarrow$$

$$=Re\left\{2\pi i\ \overbrace{\frac{h(z_0)}{g'(z_0)}}\right\}$$

$$= Re\left\{2\pi i\ \left[\frac{e^{i3z}}{2z}\right]_{z=2i}\right\}$$

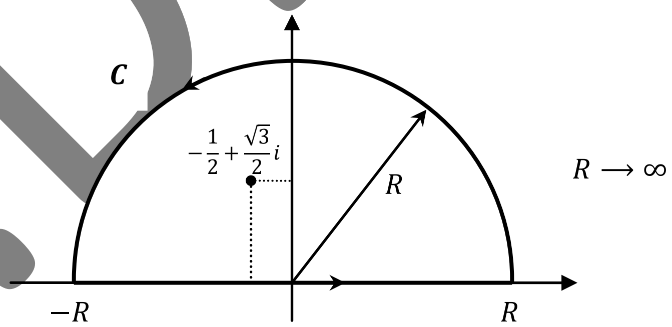
$$= Re\left\{2\pi i\ \left[\frac{e^{i3(2i)}}{2(2i)}\right]\right\}$$

$$= Re\left\{2\pi i\left[\frac{e^{-6}}{4i}\right]\right\}$$

$$= Re\left\{\frac{\pi}{2}e^{-6}\right\}$$

$$\boxed{\int_{-\infty}^{\infty}\frac{\boldsymbol{cos(3x)}}{\boldsymbol{x^2+4}}\boldsymbol{dx}=\frac{\boldsymbol{\pi}}{\boldsymbol{2}}\boldsymbol{e^{-6}}}$$

285. $\int_{-\infty}^{\infty}\frac{\boldsymbol{sen(x)}}{\boldsymbol{x^2+x+1}}\boldsymbol{dx}$



$\alpha = 1$

$$F(z)=\overbrace{\frac{1}{z^2+z+1}}^{f(z)}e^{iz}=\frac{h(z)}{g(z)}=\frac{e^{iz}}{z^2+z+1}$$

$$z^2+z+1=0$$

$$z=\frac{-1\pm\sqrt{1-4}}{2}=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$$

$$\int_{-\infty}^{\infty}\frac{sen(x)}{x^2+x+1}dx=Im\left\{\oint_c \overbrace{\frac{e^{iz}}{z^2+z+1}}^{F(z)}dz\right\}=Im\left\{\underbrace{2\pi i\ Res\left(F(z),\ \overbrace{\underbrace{-\frac{1}{2}+\frac{\sqrt{3}}{2}i}_{polo\ simple\ de\ F}}^{z_0}\right)}\right\}$$

$$= Im\left\{2\pi i\ \frac{\overbrace{h(z_0)}}{g'(z_0)}\right\}$$

$$= Im\left\{2\pi i\ \left[\frac{e^{iz}}{2z+1}\right]_{z=-\frac{1}{2}+\frac{\sqrt{3}}{2}i}\right\}$$

$$= Im\left\{2\pi i\left[\frac{e^{i\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)}}{2\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)+1}\right]\right\}$$

$$= Im\left\{2\pi i\left[\frac{e^{i\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)}}{\sqrt{3}i}\right]\right\}$$

$$= Im\left\{\frac{2\pi}{\sqrt{3}}e^{i\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)}\right\}$$

$$= Im\left\{\frac{2\pi}{\sqrt{3}}e^{-\sqrt{3}/2}e^{-i(1/2)}\right\}$$

$$= Im\left\{\frac{2\pi}{\sqrt{3}}e^{-\sqrt{3}/2}\left[cos\left(\frac{1}{2}\right)-isen\left(\frac{1}{2}\right)\right]\right\}$$

$$\boxed{\int_{-\infty}^{\infty}\frac{\boldsymbol{sen(x)}}{\boldsymbol{x^2+x+1}}\boldsymbol{dx}=-\frac{\boldsymbol{2\pi}}{\boldsymbol{\sqrt{3}}}\boldsymbol{e^{-\sqrt{3}/2}sen\left(\frac{1}{2}\right)}}$$