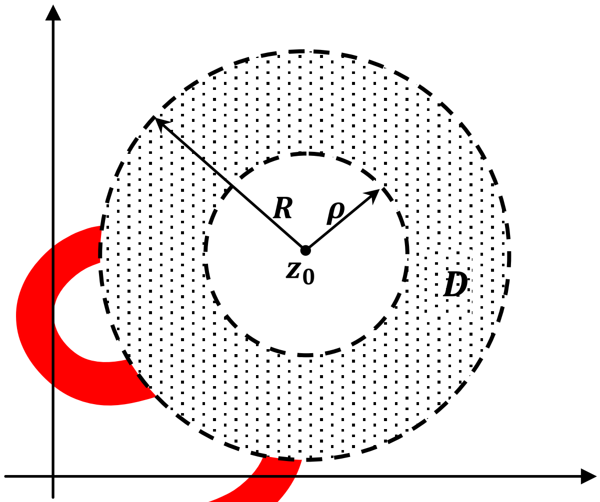
REPASO

SERIE DE LAURENT

Si f (función univaluada) es analítica en el anillo D: $\rho < |z - z_0| < R$, entonces en todo punto $z \in D$, f admite la representación en serie de potencias:

$$f(z) = \underbrace{\underbrace{\sum_{k=0}^{\infty} a_k (z - z_0)^k}_{\substack{Parte\ analítica\ o\\ de\ Taylor.\\ Converge\ para\ |z-z_0|<R.}} + \underbrace{\sum_{k=1}^{\infty} \frac{b_k}{(z - z_0)^k}}_{\substack{Parte\ principal.\\ Converge\ para\ |z-z_0|>\rho.}}}_{Converge\ para\ \rho<|z-z_0|<R}$$



Expanda $f(z)$ en una serie de Taylor o Laurent, según corresponda, válida para la región indicada.

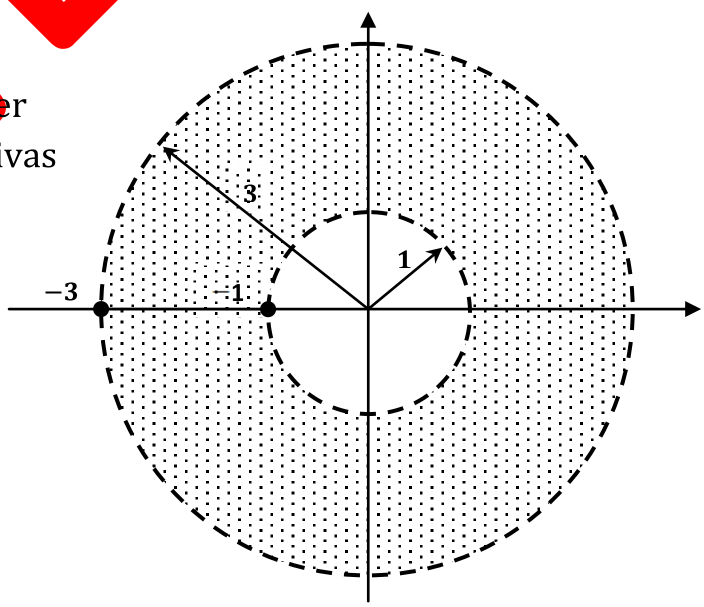
$$f(z) = \frac{1}{(z+1)(z+3)}$$

a) $1 < |z| < 3$

La serie centrada en $z_0 = 0$ va a tener potencias enteras negativas y no negativas de z.

$$\frac{1}{(z+1)(z+3)} = \frac{A}{z+1} + \frac{B}{z+3}$$

$$= \frac{\frac{1}{2}}{z+1} + \frac{-\frac{1}{2}}{z+3}$$



La serie para $\frac{1}{2}\left(\frac{1}{z+1}\right)$, $|z| > 1$ es:

$$\frac{1}{2}\left(\frac{1}{z+1}\right) = \frac{1}{2z}\left(\frac{1}{1+\frac{1}{z}}\right) = \frac{1}{2z}\left[1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right]$$

$$= \underbrace{\frac{1}{2z} - \frac{1}{2z^2} + \frac{1}{2z^3} - \frac{1}{2z^4} + \cdots}_{\sum_{k=1}^{\infty} \frac{\frac{1}{2}(-1)^{k+1}}{z^k}} \quad , |z| > 1 \; o \; \left|\frac{1}{z}\right| < 1$$

La serie para $-\frac{1}{2}\left(\frac{1}{z+3}\right)$, $|z| < 3$ es:

$$-\frac{1}{2}\left(\frac{1}{z+3}\right) = -\frac{1}{2}\left(\frac{1}{3}\right)\left(\frac{1}{1+\frac{z}{3}}\right) = \frac{1}{2}\left(-\frac{1}{3}\right)\left[1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 - \left(\frac{z}{3}\right)^3 + \cdots\right]$$

$$= \underbrace{\frac{1}{2}\left[-\frac{1}{3} + \left(\frac{1}{3}\right)^2 z + \left(-\frac{1}{3}\right)^3 z^2 + \left(\frac{1}{3}\right)^4 z^3 - \cdots\right]}_{\sum_{k=0}^{\infty}\left[\frac{1}{2}\left(-\frac{1}{3}\right)^{k+1}\right] z^k}, |z| < 3$$

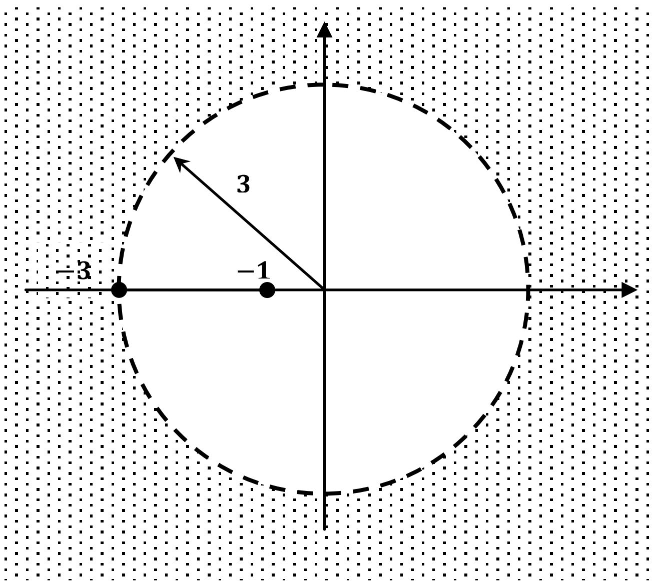
Luego la serie requerida, válida para $1 < |z| < 3$, es

$$\boxed{\frac{\mathbf{1}}{(\boldsymbol{z}+\mathbf{1})(\boldsymbol{z}+\mathbf{3})} = \overbrace{\sum_{k=0}^{\infty} \underbrace{\left[\frac{1}{2}\left(-\frac{1}{3}\right)^{k+1}\right]}_{a_k} z^k}^{\textit{Parte analítica o de Taylor}} + \overbrace{\sum_{k=1}^{\infty} \frac{\overbrace{\frac{1}{2}(-1)^{k+1}}^{b_k}}{z^k}}^{\textit{Parte principal}} \;;\; \mathbf{1} < |\boldsymbol{z}| < \mathbf{3}}$$

Serie de Laurent

b) $|z| > 3$

La serie centrada en $z_0 = 0$ va a tener sólo potencias enteras negativas de z.



$$\frac{1}{(z+1)(z+3)} = \frac{1}{2}\left(\frac{1}{z+1}\right) - \frac{1}{2}\left(\frac{1}{z+3}\right)$$

La serie para $\frac{1}{2}\left(\frac{1}{z+1}\right)$, $|z| > 1$ ya se obtuvo en a).

La serie de $-\frac{1}{2}\left(\frac{1}{z+3}\right)$ para $|z| > 3$ es

$$-\frac{1}{2}\left(\frac{1}{z+3}\right) = -\frac{1}{2z}\left(\frac{1}{1+\frac{3}{z}}\right) = -\frac{1}{2z}\left[\overbrace{1-\frac{3}{z}+\frac{3^2}{z^2}-\frac{3^3}{z^3}+\cdots}^{\text{válida para } \left|\frac{3}{z}\right|<1 \text{ o } |z|>3}\right]$$

$$= -\frac{1}{2z}+\frac{3}{2z^2}-\frac{3^2}{2z^3}+\frac{3^3}{2z^4}-\cdots$$

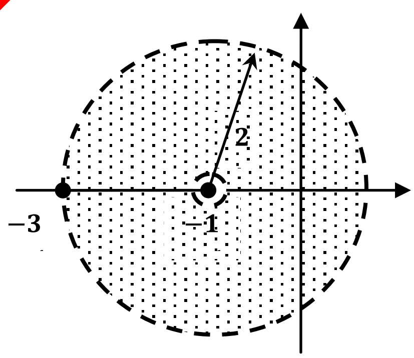
$$\frac{1}{(z+1)(z+3)} = \underbrace{\frac{1}{2z}-\frac{1}{2z^2}+\frac{1}{2z^3}-\frac{1}{2z^4}+\cdots}_{\sum_{k=1}^{\infty}\frac{\frac{1}{2}(-1)^{k+1}}{z^k}}\underbrace{-\frac{1}{2z}+\frac{3}{2z^2}-\frac{3^2}{2z^3}+\frac{3^3}{2z^4}-\cdots}_{\sum_{k=1}^{\infty}\frac{\frac{-1}{2}(-3)^{k-1}}{z^k}}$$

$$\boxed{\frac{\mathbf{1}}{(\boldsymbol{z}+\mathbf{1})(\boldsymbol{z}+\mathbf{3})} = \overbrace{\mathbf{0}}^{\textbf{Parte analítica o de Taylor}} + \overbrace{\sum_{k=1}^{\infty}\frac{\overbrace{\frac{1}{2}\left((-1)^{k+1}-(-3)^{k-1}\right)}^{b_k}}{z^k}}^{\textbf{Parte principal}} \;;\; |\boldsymbol{z}|>\mathbf{3}}$$

Serie de Laurent

c) $0<|z+1|<2$

La serie centrada en $z_0=-1$ va a tener potencias enteras negativas y no negativas de $z+1$.



Haciendo $u=z+1 \;\Rightarrow\; z=u-1$

$$\frac{1}{(z+1)(z+3)} = \frac{1}{u}\left(\frac{1}{u-1+3}\right) = \frac{1}{u}\left(\frac{1}{u+2}\right)$$

$$= \frac{1}{2u}\left(\frac{1}{1+\frac{u}{2}}\right)$$

$$= \frac{1}{2u}\left[\overbrace{1-\frac{u}{2}+\left(\frac{u}{2}\right)^2-\left(\frac{u}{2}\right)^3+\left(\frac{u}{2}\right)^4-\cdots}^{\text{válida para } \left|\frac{u}{2}\right|<1 \text{ o } \left|\overbrace{u}^{z+1}\right|<2}\right]$$

$$=\frac{1}{2u}-\frac{1}{2^2}+\frac{u}{2^3}-\frac{u^2}{2^4}+\frac{u^3}{2^5}-\cdots \quad ,0<|u|<2$$

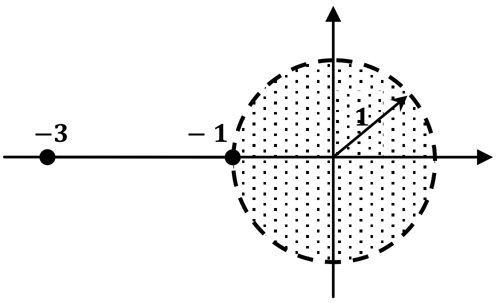
$$\frac{1}{(z+1)(z+3)}=\frac{1}{2(z+1)}+\underbrace{\frac{1}{2}\left[-\frac{1}{2}+\frac{z+1}{2^2}-\frac{(z+1)^2}{2^3}+\frac{(z+1)^3}{2^4}-\cdots\right]}_{\sum_{k=0}^{\infty}\left[\frac{1}{2}\left(-\frac{1}{2}\right)^{k+1}\right](z+1)^k} \quad ,0<|z+1|<2$$

$$\boxed{\frac{\mathbf{1}}{(\boldsymbol{z}+\mathbf{1})(\boldsymbol{z}+\mathbf{3})}=\overbrace{\sum_{k=0}^{\infty}\underbrace{\left[\frac{1}{2}\left(-\frac{1}{2}\right)^{k+1}\right]}_{a_k}(z+1)^k}^{\textit{Parte analítica o de Taylor}}+\overbrace{\frac{\overbrace{\frac{1}{2}}^{b_1}}{z+1}}^{\textit{Parte principal}}\;;\quad \mathbf{0}<|\boldsymbol{z}+\mathbf{1}|<\mathbf{2}}$$

Serie de Laurent

d) $|z|<1$

Como f es analítica en $|z|<1$, la serie requerida es la serie de Taylor.



$$\frac{1}{(z+1)(z+3)}=\frac{1}{2}\left(\underbrace{\frac{1}{1+z}}\right)-\underbrace{\frac{1}{2}\left(\frac{1}{z+3}\right)}$$

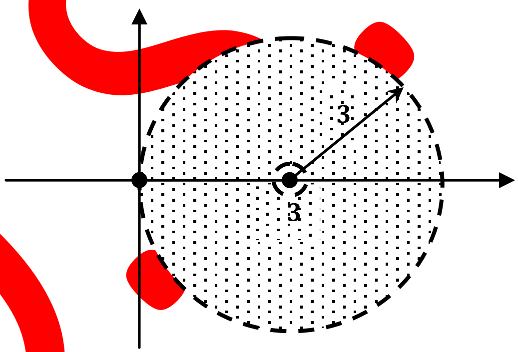
$$=\underbrace{\frac{1}{2}\left[\overbrace{1-z+z^2-z^3+\cdots}^{\text{válida para } |z|<1}\right]}+\overbrace{\underbrace{\frac{1}{2}\left(-\frac{1}{3}\right)\left[1-\frac{z}{3}+\left(\frac{z}{3}\right)^2-\left(\frac{z}{3}\right)^3+\cdots\right]}}^{\text{obtenida en a) (válida para } |z|<3)}$$

$$\frac{1}{(z+1)(z+3)}=\underbrace{\frac{1}{2}-\frac{1}{2}z+\frac{1}{2}z^2-\frac{1}{2}z^3+\cdots}_{\sum_{k=0}^{\infty}\left[\frac{1}{2}(-1)^k\right]z^k}+\underbrace{\frac{1}{2}\left[-\frac{1}{3}+\left(\frac{1}{3}\right)^2z+\left(-\frac{1}{3}\right)^3z^2+\left(\frac{1}{3}\right)^4z^3-\cdots\right]}_{\sum_{k=0}^{\infty}\left[\frac{1}{2}\left(-\frac{1}{3}\right)^{k+1}\right]z^k} \quad ,|z|<1$$

$$\boxed{\frac{1}{(z+1)(z+3)} = \sum_{k=0}^{\infty} \overbrace{\left[\frac{1}{2}\left((-1)^k + \left(-\frac{1}{3}\right)^{k+1}\right)\right]}^{a_k} z^k \ , \qquad |z| < 1 \quad \textit{Serie de Taylor}}$$

Para cada una de las siguientes funciones, obtenga la serie de Laurent alrededor de la singularidad indicada, clasifique el tipo de singularidad, dé la región de convergencia de la serie y determine el residuo.

209. $f(z) = \frac{1}{z(z-3)^2} \qquad ; \quad z_0 = 3$



Haciendo $u = z - 3 \;\Rightarrow\; z = 3 + u$

y

$$\frac{1}{z(z-3)^2} = \frac{1}{(3+u)u^2} = \frac{1}{3u^2}\left(\frac{1}{1+\frac{u}{3}}\right)$$

Llamando $v = \frac{u}{3} \;\Rightarrow\; \frac{1}{1+\frac{u}{3}} = \frac{1}{1+v} = \overbrace{\underbrace{1 - v + v^2 - v^3 + \cdots}_{\sum_{k=0}^{\infty}(-1)^k v^k}; \ \textit{válida para } |v| < 1}^{\textit{serie geométrica}}$

Luego

$$\frac{1}{1+\frac{u}{3}} = \frac{1}{1+v} = \sum_{k=0}^{\infty}(-1)^k v^k = \sum_{k=0}^{\infty}(-1)^k \left(\frac{u}{3}\right)^k \ ; \ \left|\frac{u}{3}\right| < 1 \ \ o \ |u| < 3$$

Entonces

$$\frac{1}{z(z-3)^2} = \frac{1}{3u^2}\overbrace{\left(\frac{1}{1+\frac{u}{3}}\right)} = \frac{1}{3u^2}\overbrace{\sum_{k=0}^{\infty}(-1)^k\left(\frac{u}{3}\right)^k} = \sum_{k=0}^{\infty}\frac{(-1)^k}{3^{k+1}}u^{k-2} \ , 0 < |u| < 3$$

Como $u = z - 3$ tenemos que:

$$\boxed{\frac{1}{z(z-3)^2}=\sum_{k=0}^{\infty}\frac{(-1)^k}{3^{k+1}}(z-3)^{k-2}=\frac{1}{3(z-3)^2}-\frac{1}{9(z-3)}+\frac{1}{27}-\frac{1}{81}(z-3)+\cdots\ ;\ 0<|z-3|<3}$$

$z_0=3$ es polo de segundo orden de f.

$$\boxed{Res(f(z),3)=b_1=-\frac{1}{9}}$$

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211. $f(z)=\frac{z-sen(z)}{z^3}$; $z_0=0$

$$\frac{z-sen(z)}{z^3}=\frac{z-\overbrace{\sum_{k=0}^{\infty}(-1)^k\frac{z^{2k+1}}{(2k+1)!}}^{Maclaurin\ \ de\ sen(z)}}{z^3}=\frac{z-\overbrace{\left(z+\sum_{k=1}^{\infty}(-1)^k\frac{z^{2k+1}}{(2k+1)!}\right)}^{\sum_{k=0}^{\infty}(-1)^k\frac{z^{2k+1}}{(2k+1)!}}}{z^3}$$

$$=\frac{-\sum_{k=1}^{\infty}(-1)^k\frac{z^{2k+1}}{(2k+1)!}}{z^3}=\frac{-\sum_{k=0}^{\infty}(-1)^{k+1}\frac{z^{2(k+1)+1}}{(2(k+1)+1)!}}{z^3}=\sum_{k=0}^{\infty}\frac{(-1)^{k+2}}{(2k+3)!}z^{2k},\qquad |z|>0$$

$$\boxed{\frac{z-sen(z)}{z^3}=\sum_{k=0}^{\infty}\frac{(-1)^{k+2}}{(2k+3)!}z^{2k}=\frac{1}{3!}-\frac{1}{5!}z^2+\frac{1}{7!}z^4-\cdots,\qquad |z|>0}$$

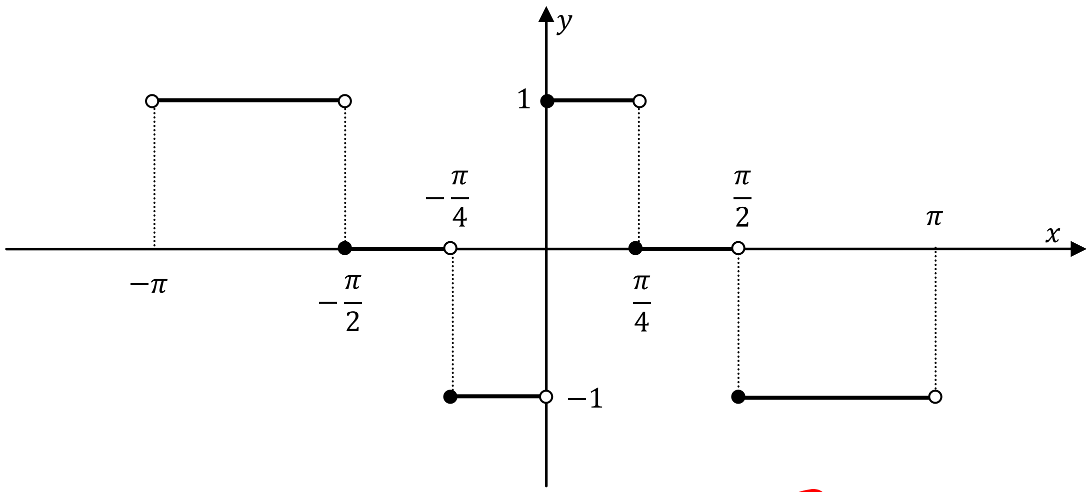
$z_0=0$ es punto singular evitable de f.

$$\boxed{Res(f(z),0)=b_1=0}$$

SERIES DE FOURIER

Obtenga la serie trigonométrica de Fourier para las siguientes funciones:

$$f(x)=\begin{cases}1\ , & si\quad -\pi<x<-\frac{\pi}{2}\\ 0\ , & si\quad -\frac{\pi}{2}\leq x<-\frac{\pi}{4}\\ -1\ , & si\quad -\frac{\pi}{4}\leq x<0\\ 1\ , & si\quad 0\leq x<\frac{\pi}{4}\\ 0\ , & si\quad \frac{\pi}{4}\leq x<\frac{\pi}{2}\\ -1\ , & si\quad \frac{\pi}{2}\leq x<\pi\end{cases}$$



$$(-L, L) = (-\pi, \pi) \quad , \boxed{L = \pi}$$

Como f es función **impar** su serie trigonométrica de Fourier es una serie de senos.

$$b_n = \frac{2}{L}\int_0^L f(x) sen\left(\frac{n\pi}{L}x\right) dx \quad , n = 1,2,3 \ldots$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{4}} 1 sen\left(\frac{n\pi}{\pi}x\right) dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 0 sen\left(\frac{n\pi}{\pi}x\right) dx + \int_{\frac{\pi}{2}}^{\pi} (-1) sen\left(\frac{n\pi}{\pi}x\right) dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{4}} sen(nx)dx - \int_{\frac{\pi}{2}}^{\pi} sen(nx)dx\right]$$

$$= \frac{2}{\pi}\left[-\frac{cos(nx)}{n}\Big|_0^{\frac{\pi}{4}} + \frac{cos(nx)}{n}\Big|_{\frac{\pi}{2}}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{1 - cos\left(n\frac{\pi}{4}\right) + \overbrace{cos(n\pi)}^{(-1)^n} - cos\left(n\frac{\pi}{2}\right)}{n}\right]$$

$$b_n = \frac{2}{\pi}\left[\frac{1 + (-1)^n - cos\left(n\frac{\pi}{4}\right) - cos\left(n\frac{\pi}{2}\right)}{n}\right] \quad , \quad n = 1,2,3 \ldots$$

$$\boxed{\boldsymbol{f(x) \sim \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{1 + (-1)^n - cos\left(\frac{n\pi}{4}\right) - cos\left(\frac{n\pi}{2}\right)}{n}\right) sen(nx)}}$$

297. $f(x) = x, \ -\pi < x < \pi$

$$(-L, L) = (-\pi, \pi) \quad , \boxed{L = \pi}$$

Como f es función **impar** su serie trigonométrica de Fourier es una serie de senos.

$$b_n = \frac{2}{L}\int_0^L f(x) sen\left(\frac{n\pi}{L}x\right)dx \ \ , n = 1,2,\cdots \qquad \boxed{\int xsen(nx)dx = \frac{sen(nx)}{n^2} - \frac{xcos(nx)}{n}}$$

$$b_n = \frac{2}{\pi}\int_0^\pi xsen\left(\frac{n\pi}{\pi}x\right)dx = \frac{2}{\pi}\int_0^\pi xsen(nx)dx = \frac{2}{\pi}\left(\frac{sen(nx)}{n^2} - \frac{xcos(nx)}{n}\right)\Bigg|_0^\pi$$

$$b_n = \frac{2}{\pi}\left(\frac{\overbrace{sen(n\pi)}^{0}}{n^2} - \frac{\pi \overbrace{cos(n\pi)}^{(-1)^n}}{n} - (0 - 0)\right) = -\frac{2}{\pi}\frac{\pi(-1)^n}{n} = \frac{2}{n}(-1)(-1)^n = \frac{2}{n}(-1)^{n+1} \ ; n = 1,2,\cdots$$

$$\boxed{f(x) \sim 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} sen(nx)}$$

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298. $f(x) = e^x, \ -\pi < x < \pi$

$$(-L, L) = (-\pi, \pi) \quad , \boxed{L = \pi}$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x dx = \frac{1}{\pi}e^x\Big|_{-\pi}^{\pi} = \frac{1}{\pi}(e^\pi - e^{-\pi}) = \frac{2}{\pi}\overbrace{\left(\frac{e^\pi - e^{-\pi}}{2}\right)}^{sen\,h(\pi)} = \frac{2senh(\pi)}{\pi}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)cos\left(\frac{n\pi}{L}x\right)dx \ , n = 1,2,\cdots \qquad \boxed{\int e^x cos(nx)dx = \frac{e^x cos(nx) + ne^x sen(nx)}{n^2 + 1}}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x cos\left(\frac{n\pi}{\pi}x\right)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x cos(nx)dx = \frac{1}{\pi}\left(\frac{e^x cos(nx) + ne^x sen(nx)}{n^2 + 1}\right)\Bigg|_{-\pi}^{\pi}$$

$$a_n = \frac{1}{\pi}\left(\frac{e^{\pi}\overbrace{\cos(n\pi)}^{(-1)^n} + ne^{\pi}\overbrace{sen(n\pi)}^{0} - e^{-\pi}\overbrace{\cos(-n\pi)}^{(-1)^n} - ne^{-\pi}\overbrace{sen(-n\pi)}^{0}}{n^2+1}\right)$$

$$a_n = \frac{2(-1)^n}{\pi(n^2+1)}\left(\frac{e^{\pi}-e^{-\pi}}{2}\right) = \frac{(-1)^n}{n^2+1}\frac{2senh(\pi)}{\pi} \quad , n = 1,2,\cdots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)sen\left(\frac{n\pi}{L}x\right)dx \quad , n = 1,2,\cdots$$

$$\boxed{\int e^x sen(nx)dx = \frac{e^x sen(nx) - ne^x\cos(nx)}{n^2+1}}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x sen\left(\frac{n\pi}{\pi}x\right)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x sen(nx)dx = \frac{1}{\pi}\left(\frac{e^x sen(nx) - ne^x\cos(nx)}{n^2+1}\right)\Bigg|_{-\pi}^{\pi}$$

$$b_n = \frac{1}{\pi}\left(\frac{e^{\pi}\overbrace{sen(n\pi)}^{0} - ne^{\pi}\overbrace{\cos(n\pi)}^{(-1)^n} - e^{-\pi}\overbrace{sen(-n\pi)}^{0} + ne^{-\pi}\overbrace{\cos(-n\pi)}^{(-1)^n}}{n^2+1}\right)$$

$$b_n = \frac{-2(-1)^n n}{\pi(n^2+1)}\left(\frac{e^{\pi}-e^{-\pi}}{2}\right) = -\frac{(-1)^n n}{n^2+1}\frac{2senh(\pi)}{\pi} \quad , n = 1,2,\cdots$$

$$\boxed{f(x) \sim \frac{2senh(\pi)}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}\left(\cos(nx) - nsen(nx)\right)\right]}$$

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299. $f(x) = \begin{cases} -2, & si \quad -3 < x < 0 \\ 4, & si \quad 0 \le x < 3 \end{cases}$

$$(-L, L) = (-3,3) \quad , \boxed{L = 3}$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx$$

$$a_0 = \frac{1}{3}\left[\int_{-3}^{0}(-2)dx + \int_{0}^{3}4dx\right] = \frac{1}{3}\left[-2x|_{-3}^{0} + 4x|_{0}^{3}\right] = \frac{1}{3}(-6+12) = 2$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)cos\left(\frac{n\pi}{L}x\right)dx \;\;, n = 1,2,\cdots$$

$$a_n = \frac{1}{3}\left[\int_{-3}^{0}(-2)cos\left(\frac{n\pi}{3}x\right)dx + \int_{0}^{3} 4cos\left(\frac{n\pi}{3}x\right)dx\right]$$

$$a_n = \frac{1}{3}\left[-2\left(\frac{3}{n\pi}\right)sen\left(\frac{n\pi}{3}x\right)\Big|_{-3}^{0} + 4\left(\frac{3}{n\pi}\right)sen\left(\frac{n\pi}{3}x\right)\Big|_{0}^{3}\right]$$

$$a_n = \frac{1}{3}\left[0 + 2\left(\frac{3}{n\pi}\right)\overbrace{sen(-n\pi)}^{0} + 4\left(\frac{3}{n\pi}\right)\overbrace{sen(n\pi)}^{0} - 0\right] = 0 \;\;, n = 1,2,\cdots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)sen\left(\frac{n\pi}{L}x\right)dx \;\;, n = 1,2,\cdots$$

$$b_n = \frac{1}{3}\left[\int_{-3}^{0}(-2)sen\left(\frac{n\pi}{3}x\right)dx + \int_{0}^{3} 4sen\left(\frac{n\pi}{3}x\right)dx\right]$$

$$b_n = \frac{1}{3}\left[2\left(\frac{3}{n\pi}\right)cos\left(\frac{n\pi}{3}x\right)\Big|_{-3}^{0} - 4\left(\frac{3}{n\pi}\right)cos\left(\frac{n\pi}{3}x\right)\Big|_{0}^{3}\right]$$

$$b_n = \frac{1}{3}\left[2\left(\frac{3}{n\pi}\right) - 2\left(\frac{3}{n\pi}\right)\overbrace{cos(-n\pi)}^{(-1)^n} - 4\left(\frac{3}{n\pi}\right)\overbrace{cos(n\pi)}^{(-1)^n} + 4\left(\frac{3}{n\pi}\right)\right]$$

$$b_n = \frac{1}{n\pi}[2 - 2(-1)^n - 4(-1)^n + 4] = \frac{1}{n\pi}[6 - 6(-1)^n] = \frac{6}{\pi}\left[\frac{1-(-1)^n}{n}\right] \;;n = 1,2,\cdots$$

$$\boxed{f(x) \sim 1 + \frac{6}{\pi}\sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n}\right)sen\left(\frac{n\pi}{3}x\right)}$$

<u>EJERCICIOS</u>

1. **Empleando el método de la transformada de Laplace obtenga la solución de la ecuación diferencial**

$$\boldsymbol{y'' + y = te^{-2t}}$$

que satisface las condiciones iniciales:

$$\boldsymbol{y(0) = 0 \;,\qquad y'(0) = 0}$$

Solución

$$\mathscr{L}\{y'' + y\} = \mathscr{L}\{te^{-2t}\}$$

$$\mathscr{L}\{y''\} + \mathscr{L}\{y\} = \mathscr{L}\{te^{-2t}\}$$

$$s^2Y(S) - s\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{0} + Y(s) = \frac{1}{(s+2)^2}$$

$$s^2Y(s) + Y(s) = \frac{1}{(s+2)^2}$$

$$Y(s)(s^2+1) = \frac{1}{(s+2)^2}$$

$$Y(s) = \frac{1}{(s+2)^2(s^2+1)} = \frac{1}{(s+2)^2(s+i)(s-i)}$$

$$y(t) = Res(Y(s)e^{st}, -2) + Res(Y(s)e^{st}, i) + Res(Y(s)e^{st}, -i)$$

$$Res(Y(s)e^{st}, -2) = \lim_{s\to -2}\left[\frac{d}{ds}\left((s+2)^2\frac{e^{st}}{(s+2)^2(s^2+1)}\right)\right]$$

$$= \lim_{s\to -2}\left[\frac{d}{ds}\left(\frac{e^{st}}{(s^2+1)}\right)\right] = \lim_{s\to -2}\frac{d}{ds}[(s^2+1)^{-1}e^{st}]$$

$$= \lim_{s\to -2}\left[\frac{-2se^{st}}{(s^2+1)^2} + \frac{te^{st}}{(s^2+1)}\right] = \frac{4}{25}e^{-2t} + \frac{1}{5}te^{-2t}$$

$$Res(Y(s)e^{st}, i) = \lim_{s\to i}\left[(s-i)\frac{e^{st}}{(s+2)^2(s+i)(s-i)}\right] = \lim_{s\to i}\left[\frac{e^{st}}{(s+2)^2(s+i)}\right]$$

$$= \frac{e^{it}}{(i+2)^2(i+i)} = \frac{e^{it}}{(-1+4i+4)(2i)} = \frac{e^{it}}{(3+4i)(2i)}$$

$$= \frac{e^{it}}{(-8+6i)}\frac{(-8-6i)}{(-8-6i)} = \frac{(-8-6i)e^{it}}{(-8)^2-(6i)^2} = \frac{(-8-6i)e^{it}}{100}$$

$$= -\frac{8}{100}e^{it} - \frac{6}{100}ie^{it}$$

$$= -\frac{2}{25}e^{it} - \frac{3}{50}ie^{it}$$

$$Res(Y(s)e^{st},-i) = \lim_{s\to i}\left[(s+i)\frac{e^{st}}{(s+2)^2(s+i)(s-i)}\right] = \lim_{s\to i}\left[\frac{e^{st}}{(s+2)^2(s-i)}\right]$$

$$= \cdots = -\frac{2}{25}e^{-it} + \frac{3}{50}ie^{-it}$$

$$y(t) = \overbrace{\frac{4}{25}e^{-2t} + \frac{1}{5}te^{-2t}}^{Res\left(Y(s)e^{st},-2\right)} \overbrace{-\frac{2}{25}e^{it} - \frac{3}{50}ie^{it}}^{Res\left(Y(s)e^{st},i\right)} \overbrace{-\frac{2}{25}e^{-it} + \frac{3}{50}ie^{-it}}^{Res\left(Y(s)e^{st},-i\right)}$$

$$y(t) = \frac{4}{25}e^{-2t} + \frac{1}{5}te^{-2t} - \frac{2}{25}\overbrace{\left(\frac{e^{it}+e^{-it}}{2}\right)}^{cos\,(t)}2 - \frac{3}{50}i\overbrace{\left(\frac{e^{it}-e^{-it}}{2i}\right)}^{sen\,(t)}2i$$

$$\boxed{y(t) = \frac{4}{25}e^{-2t} + \frac{1}{5}te^{-2t} - \frac{4}{25}cos(t) + \frac{3}{25}sen(t)}$$

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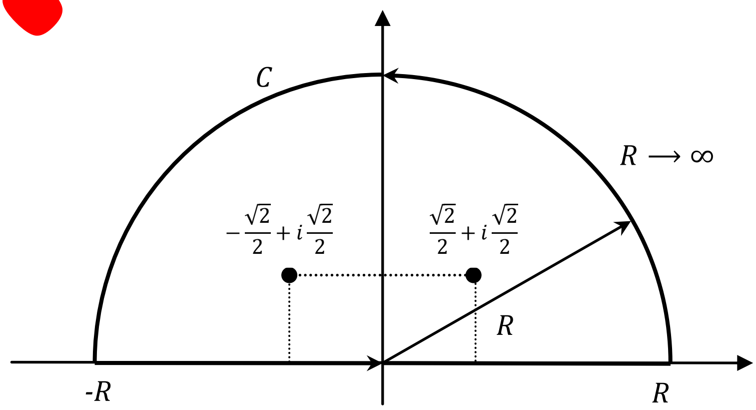
2. Empleando residuos obtenga el valor de la siguiente integral:

$$\int_{-\infty}^{\infty}\frac{1}{x^4+1}dx$$

Solución

$$\int_{-\infty}^{\infty}\frac{1}{x^4+1}dx = \oint_C \frac{1}{z^4+1}dz$$



$$f(z) = \frac{1}{z^4+1}$$

$$z^4+1=0 \quad\Rightarrow\quad z=(-1)^{\frac{1}{4}}$$

$$z_{k+1} = \left[1e^{i(\pi+2k\pi)}\right]^{\frac{1}{4}} \quad ; \quad k=0,1,2,3$$

$$z_{k+1} = 1^{\frac{1}{4}}e^{i\left(\frac{\pi}{4}+\frac{1}{2}k\pi\right)} = e^{i\left(\frac{\pi}{4}+\frac{1}{2}k\pi\right)} \quad ; \quad k=0,1,2,3$$

$$k=0,\quad \boxed{z_1 = e^{i\frac{\pi}{4}} = cos\left(\frac{\pi}{4}\right) + isen\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}}$$

$$k=1,\quad \boxed{z_2 = e^{i\frac{3\pi}{4}} = cos\left(\frac{3\pi}{4}\right) + isen\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}}$$

$$k=2,\quad z_2 = e^{i\frac{5\pi}{4}} = cos\left(\frac{5\pi}{4}\right) + isen\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$$

$$k=3,\quad z_2 = e^{i\frac{7\pi}{4}} = cos\left(\frac{7\pi}{4}\right) + isen\left(\frac{7\pi}{4}\right) = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$$

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1}dx = \oint_C \frac{1}{z^4+1}dz = 2\pi i\left[Res\left(f(z), e^{i\frac{\pi}{4}}\right) + Res\left(f(z), e^{i\frac{3\pi}{4}}\right)\right]$$

$$= 2\pi i\left[\left(\frac{1}{4z^3}\right)_{z=e^{i\frac{\pi}{4}}} + \left(\frac{1}{4z^3}\right)_{z=e^{i\frac{3\pi}{4}}}\right]$$

$$= 2\pi i\left[\frac{1}{4e^{i\frac{3\pi}{4}}} + \frac{1}{4e^{i\frac{9\pi}{4}}}\right]$$

$$= \frac{1}{2}\pi i\left[e^{-i\frac{3\pi}{4}} + e^{-i\frac{9\pi}{4}}\right]$$

$$= \frac{1}{2}\pi i\left[-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right]$$

$$= \frac{1}{2}\pi i\left[-i\sqrt{2}\right]$$

$$\boxed{\int_{-\infty}^{\infty} \frac{1}{x^4+1}dx = \frac{\sqrt{2}}{2}\pi}$$