SERIES TRIGONOMÉTRICAS DE FOURIER

La serie trigonométrica de Fourier de una función f definida en el intervalo $(-L, L)$ viene dada por

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n cos\left(\frac{n\pi}{L}x\right) + b_n sen\left(\frac{n\pi}{L}x\right)\right]$$

donde

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx$$

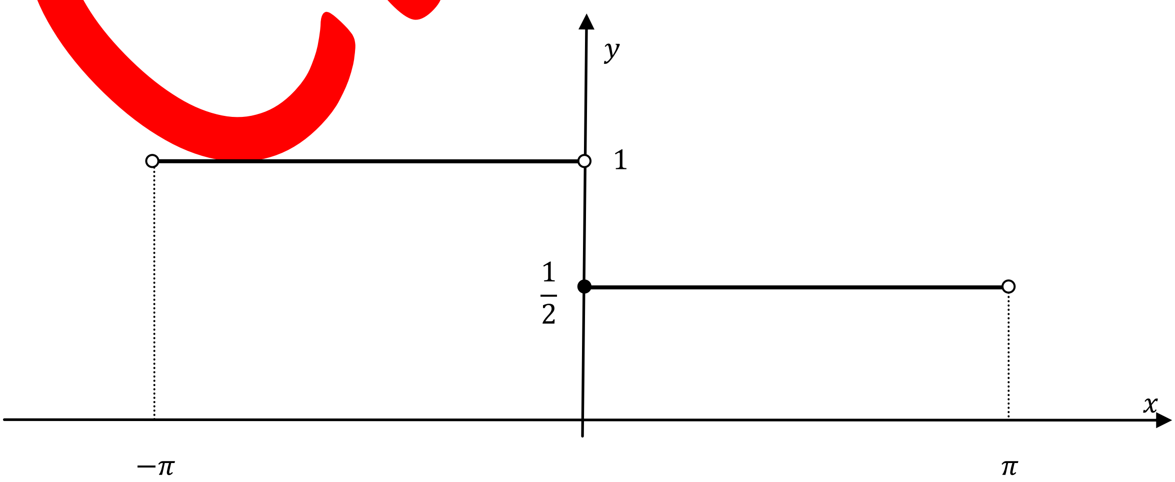
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)cos\left(\frac{n\pi}{L}x\right)dx$$

$$n = 1,2,3 \ldots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)sen\left(\frac{n\pi}{L}x\right)dx$$

Obtenga la serie trigonométrica de Fourier de las siguientes funciones:

293. $f(x) = \begin{cases} 1, & si \quad -\pi < x < 0 \\ \frac{1}{2}, & si \quad 0 \leq x < \pi \end{cases}$



$$(-L, L) = (-\pi, \pi) \quad , \boxed{L = \pi}$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx$$

$$a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0} 1dx + \int_{0}^{\pi} \frac{1}{2}dx\right] = \frac{1}{\pi}x|_{-\pi}^{0} + \frac{1}{2\pi}x|_{0}^{\pi} = 1 + \frac{1}{2}$$

$$\boxed{a_0 = \frac{3}{2}}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)cos\left(\frac{n\pi}{L}x\right)dx \ , \qquad n = 1,2,3 \ldots$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} 1cos\left(\frac{n\pi}{\pi}x\right)dx + \int_{0}^{\pi} \frac{1}{2}cos\left(\frac{n\pi}{\pi}x\right)dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 1cos(nx)dx + \int_{0}^{\pi} \frac{1}{2}cos(nx)dx\right]$$

$$\int cos(nx)dx = \int cos(u)\frac{du}{n} = \frac{sen(u)}{n} = \frac{sen(nx)}{n}$$

$$u = nx \ \Rightarrow \ du = ndx \ \Rightarrow \ dx = \frac{du}{n}$$

$$a_n = \frac{1}{\pi}\left[\frac{sen(nx)}{n}\Big|_{-\pi}^{0} + \frac{sen(nx)}{2n}\Big|_{0}^{\pi}\right] = \frac{1}{\pi}\left[-\frac{\overbrace{sen(-n\pi)}^{=0\ si\ n=1,2,3\ldots}}{n} + \frac{\overbrace{sen(n\pi)}^{=0\ si\ n=1,2,3\ldots}}{2n}\right]$$

$$\boxed{a_n = 0 \ , n = 1,2,3 \ldots}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)sen\left(\frac{n\pi}{L}x\right)dx \ , \quad n = 1,2,3 \ldots$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} 1sen(nx)dx + \int_{0}^{\pi} \frac{1}{2}sen(nx)dx\right]$$

$$= \frac{1}{\pi}\left[-\frac{cos(nx)}{n}\Big|_{-\pi}^{0} - \frac{cos(nx)}{2n}\Big|_{0}^{\pi}\right]$$

$$\int sen(nx)dx = \int sen(u)\frac{du}{n} = -\frac{cos(u)}{n} = -\frac{cos(nx)}{n}$$

$$u = nx \ \Rightarrow \ du = ndx \ \Rightarrow \ dx = \frac{du}{n}$$

$$= \frac{1}{\pi}\left[\frac{-1 + cos(-n\pi)}{n} - \left(\frac{cos(n\pi) - 1}{2n}\right)\right]$$

$$cos(-n\pi) = cos(n\pi) = (-1)^n \quad si\ n = 1,2,3 \ldots$$

$$= \frac{1}{\pi}\left[\frac{-2 + 2cos(-n\pi) - cos(n\pi) + 1}{2n}\right] = \frac{1}{\pi}\left[\frac{-2 + 2(-1)^n - (-1)^n + 1}{2n}\right]$$

$$\boxed{b_n = \frac{1}{2\pi}\left[\frac{(-1)^n - 1}{n}\right] \quad , \qquad n = 1,2,3\ldots}$$

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n cos\left(\frac{n\pi}{L}x\right) + b_n sen\left(\frac{n\pi}{L}x\right)\right]$$

$$\boxed{\boldsymbol{f(x) \sim \frac{3}{4} + \frac{1}{2\pi}\sum_{n=1}^{\infty}\left(\frac{(-1)^n - 1}{n}\right) sen(nx)}}$$

SERIES DE FOURIER DE COSENOS

Si f es una función **par** en el intervalo $(-L, L)$ su serie de Fourier es

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n cos\left(\frac{n\pi}{L}x\right)$$

donde

$$a_0 = \frac{2}{L}\int_0^L f(x)dx$$

$$a_n = \frac{2}{L}\int_0^L f(x) cos\left(\frac{n\pi}{L}x\right) dx \ , n = 1,2,3\ldots$$

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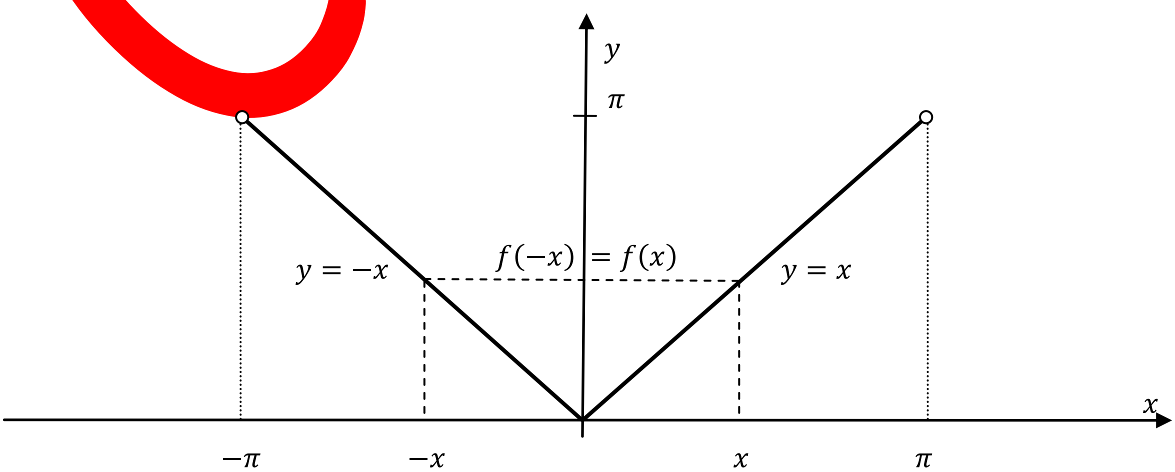
294. $f(x) = |x|, \ -\pi < x < \pi$



$$(-L, L) = (-\pi, \pi) \quad , \boxed{L = \pi}$$

Como f es función **par** su serie trigonométrica de Fourier es una serie de cosenos.

$$a_0 = \frac{2}{L}\int_0^L f(x)dx$$

$$= \frac{2}{\pi}\int_0^{\pi} xdx = \frac{2}{\pi}\frac{x^2}{2}\Big|_0^{\pi} \quad \Rightarrow \quad \boxed{a_0 = \pi}$$

$$a_n = \frac{2}{L}\int_0^L f(x)cos\left(\frac{n\pi}{L}x\right)dx \ , n = 1,2,3 \dots$$

$$\boxed{\int xcos(nx)dx = \frac{cos(nx)}{n^2} + \frac{xsen(nx)}{n}}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} xcos\left(\frac{n\pi}{\pi}x\right)dx$$

$$= \frac{2}{\pi}\int_0^{\pi} xcos(nx)dx = \frac{2}{\pi}\left(\frac{cos(nx)}{n^2} + \frac{xsen(nx)}{n}\Big|_0^{\pi}\right)$$

$$= \frac{2}{\pi}\left(\frac{\overbrace{cos(n\pi)}^{=(-1)^n}-1}{n^2} + \frac{\overbrace{\pi sen(n\pi)}^{=0}}{n}\right) \quad , \quad n = 1,2,3, \dots$$

$$\boxed{a_n = \frac{2}{\pi}\left(\frac{(-1)^n - 1}{n^2}\right) \quad , n = 1,2,3, \dots}$$

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n cos\left(\frac{n\pi}{L}x\right)$$

$$\boxed{\boldsymbol{f(x) \sim \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{(-1)^n - 1}{n^2}\right)cos(nx)}}$$

SERIES DE FOURIER DE SENOS

Si f es una función **impar** en el intervalo $(-L, L)$ su serie de Fourier es

$$f(x) \sim \sum_{n=1}^{\infty} b_n sen\left(\frac{n\pi}{L}x\right)$$

donde

$$b_n = \frac{2}{L}\int_0^L f(x)sen\left(\frac{n\pi}{L}x\right)dx \quad , n = 1,2,3 \dots$$

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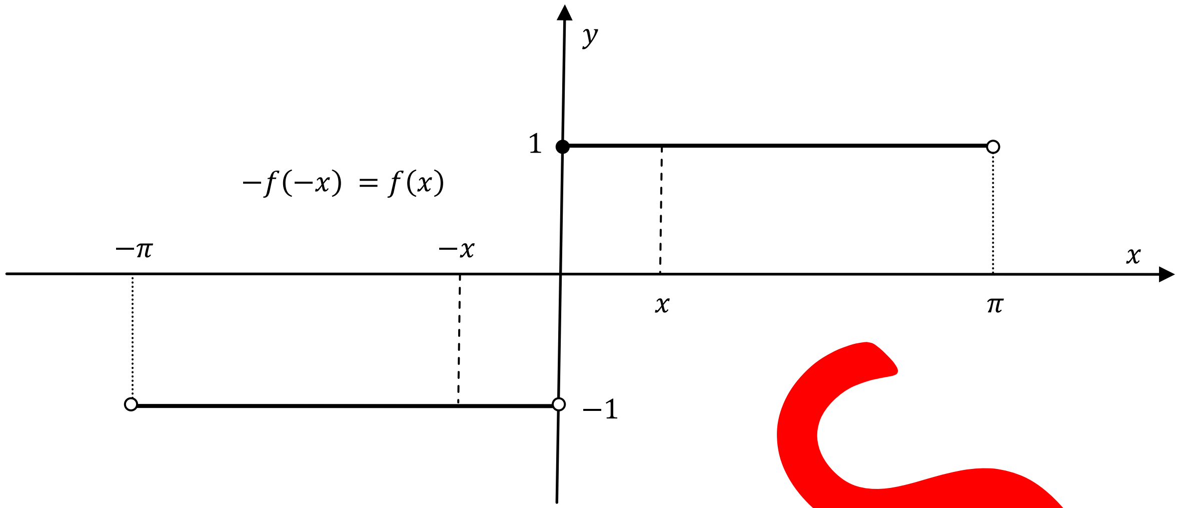
295. $f(x) = \begin{cases} -1, & si \quad -\pi < x < 0 \\ 1, & si \quad 0 \leq x < \pi \end{cases}$



$$(-L, L) = (-\pi, \pi) \quad , \boxed{L = \pi}$$

Como f es función **impar** su serie trigonométrica de Fourier es una serie de senos.

$$b_n = \frac{2}{L}\int_0^L f(x) sen\left(\frac{n\pi}{L}x\right) dx \quad , n = 1,2,3 \ldots$$

$$b_n = \frac{2}{\pi}\int_0^\pi 1 sen\left(\frac{n\pi}{\pi}x\right) dx = \frac{2}{\pi}\int_0^\pi sen(nx)dx = -\frac{2}{\pi}\frac{cos(nx)}{n}\Bigg|_0^\pi = -\frac{2}{\pi}\left(\frac{\overbrace{cos(n\pi)}^{(-1)^n} - 1}{n}\right)$$

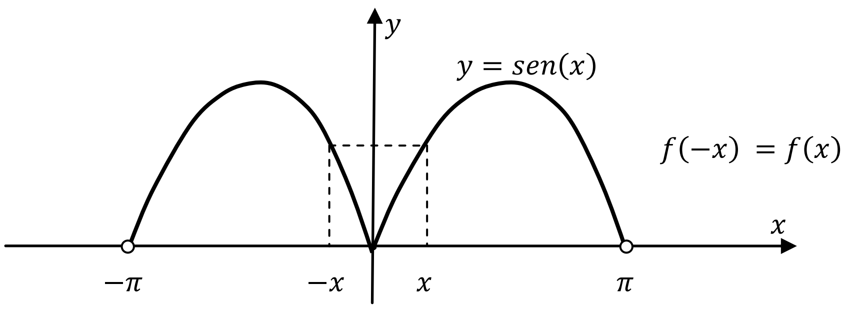
$$n = 1,2,3 \ldots$$

$$\boxed{b_n = \frac{2}{\pi}\left(\frac{1-(-1)^n}{n}\right) \quad , \quad n = 1,2,3 \ldots}$$

$$f(x) \sim \sum_{n=1}^{\infty} b_n sen\left(\frac{n\pi}{L}x\right)$$

$$\boxed{\boldsymbol{f(x) \sim \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n}\right) sen(nx)}}$$

296. $f(x) = |sen(x)|, \ -\pi < x < \pi$



$$(-L, L) = (-\pi, \pi) \quad , \ \boxed{L = \pi}$$

Como f es función **par** su serie trigonométrica de Fourier es una serie de cosenos.

$$a_0 = \frac{2}{L}\int_0^L f(x)dx$$

$$a_0 = \frac{2}{\pi}\int_0^\pi sen(x)dx = -\frac{2}{\pi}cos(x)|_0^\pi = -\frac{2}{\pi}(-1-1) \Rightarrow \boxed{a_0 = \frac{4}{\pi}}$$

$$\boxed{\int sen(x)cos(nx)dx = \frac{cos\big(x(n-1)\big)}{2(n-1)} - \frac{cos\big(x(n+1)\big)}{2(n+1)}}$$

$$a_n = \frac{2}{L}\int_0^L f(x)cos\left(\frac{n\pi}{L}x\right)dx$$

$$a_n = \frac{2}{\pi}\int_0^\pi sen(x)cos\left(\frac{n\pi}{\pi}x\right)dx = \frac{2}{\pi}\int_0^\pi sen(x)cos(nx)dx$$

$$a_n = \frac{2}{\pi}\left(\frac{cos\big(x(n-1)\big)}{2(n-1)} - \frac{cos\big(x(n+1)\big)}{2(n+1)}\right)\Bigg|_0^\pi$$

$$= \frac{2}{\pi}\left(\frac{\overbrace{cos\big(\pi(n-1)\big)}^{=-(-1)^n} - 1}{2(n-1)} - \frac{\left(\overbrace{cos\big(\pi(n+1)\big)}^{=-(-1)^n} - 1\right)}{2(n+1)}\right), \qquad n = 2,3\ldots$$

$$= \frac{1}{\pi}\left(\frac{-(-1)^n - 1}{n-1} - \frac{-(-1)^n - 1}{n+1}\right)$$

$$= \frac{1}{\pi}\left(\frac{(n+1)(-(-1)^n - 1) - (n-1)(-(-1)^n - 1)}{(n-1)(n+1)}\right)$$

$$= \frac{1}{\pi}\left(\frac{(-(-1)^n - 1)\left(\overbrace{n + 1 - (n-1)}^{=2}\right)}{n^2 - 1}\right) = \frac{-2}{\pi}\left(\frac{(-1)^n + 1}{n^2 - 1}\right)$$

$$\boxed{a_n = \frac{2}{\pi}\left(\frac{(-1)^n + 1}{1 - n^2}\right) \quad , \qquad n = 2,3,\ldots}$$

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right)$$

$$\boxed{\boldsymbol{f(x) \sim \frac{2}{\pi} + \frac{2}{\pi}\sum_{n=2}^{\infty}\left(\frac{(-1)^n + 1}{1 - n^2}\right)\cos(nx)}}$$

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297. $f(x) = x, \ -\pi < x < \pi$

$$Resp.: \ f(x) \sim 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} sen(nx)$$

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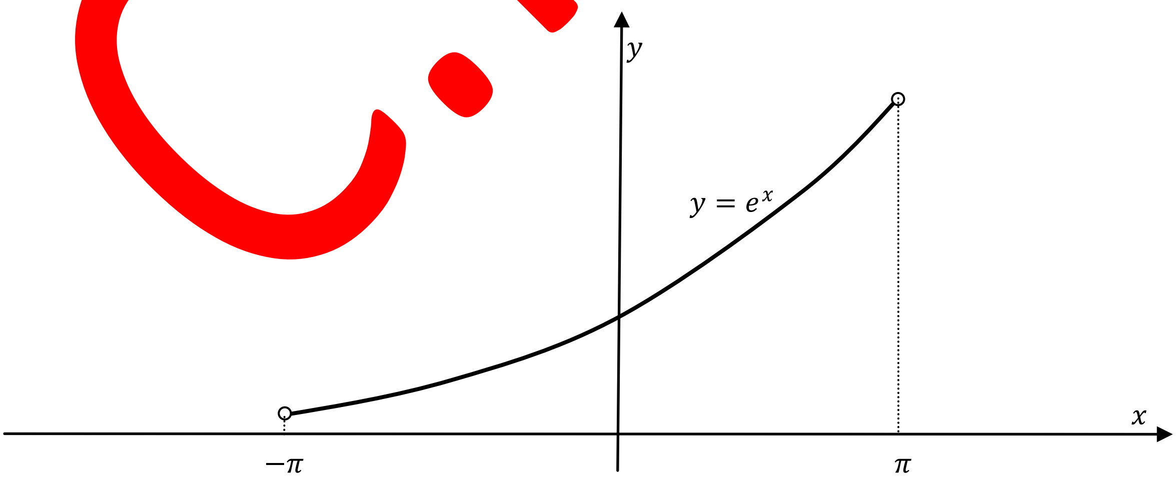
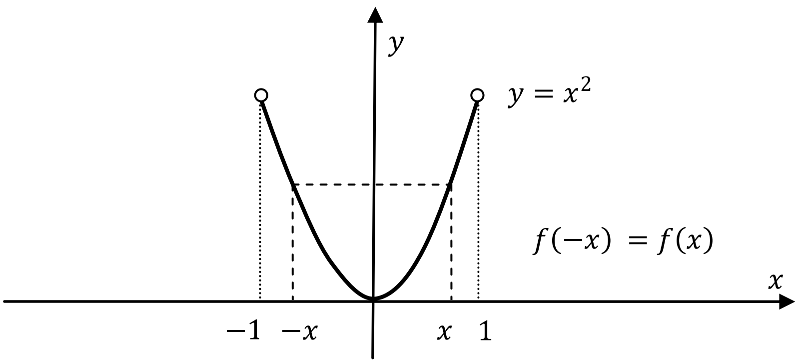
298. $f(x) = e^x, \ -\pi < x < \pi$



$$Resp.: \ f(x) \sim \frac{2senh(\pi)}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + 1}\left(\cos(nx) - nsen(nx)\right)\right]$$

308. $f(x) = x^2, \; -1 < x < 1$



$$(-L, L) = (-1,1) \quad , \; \boxed{L = 1}$$

Como f es función **par** su serie trigonométrica de Fourier es una serie de cosenos.

$$a_0 = \frac{2}{L}\int_0^L f(x)dx$$

$$a_0 = \frac{2}{1}\int_0^1 x^2 dx = \frac{2}{3}x^3|_0^1 \quad \Rightarrow \quad \boxed{a_0 = \frac{2}{3}}$$

$$a_n = \frac{2}{L}\int_0^L f(x)cos\left(\frac{n\pi}{L}x\right)dx \; , n = 1,2,3 \ldots$$

$$\boxed{\int x^2 cos(n\pi x)dx = \frac{2n\pi x cos(n\pi x) + (n^2\pi^2x^2 - 2)sen(n\pi x)}{n^3\pi^3}}$$

$$a_n = \frac{2}{1}\int_0^1 x^2 cos\left(\frac{n\pi}{1}x\right)dx = 2\left(\frac{2n\pi x cos(n\pi x) + (n^2\pi^2x^2 - 2)sen(n\pi x)}{n^3\pi^3}\right)\Bigg|_0^1$$

$$= 2\left(\frac{2n\pi \overbrace{cos(n\pi)}^{=(-1)^n} + (n^2\pi^2 - 2)\overbrace{sen(n\pi)}^{=0}}{n^3\pi^3}\right), n = 1,2,3 \ldots$$

$$\boxed{a_n = \frac{4}{n^2\pi^2}(-1)^n \; , \quad n = 1,2,3 \ldots}$$

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n cos\left(\frac{n\pi}{L}x\right)$$

$$\boxed{\boldsymbol{f(x) \sim \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}cos(n\pi x)}}$$

309. $f(x) = \begin{cases} -2, & si \quad -3 < x < 0 \\ 4, & si \quad 0 \leq x < 3 \end{cases}$

$$Resp.: \ f(x) \sim 1 + \frac{6}{\pi} \sum_{n=1}^{\infty} \left(\frac{1-(-1)^n}{n} \right) sen\left(\frac{n\pi}{3} x\right)$$

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301. $f(x) = \begin{cases} 5, \ si & -1 < x < -\frac{1}{4} \\ -3, \ si & -\frac{1}{4} \leq x < \frac{1}{4} \\ 5, \ si & \frac{1}{4} \leq x < 1 \end{cases}$

$$Resp.: \ f(x) \sim 3 - \frac{16}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} sen\left(\frac{n\pi}{4}\right) cos(n\pi x)$$

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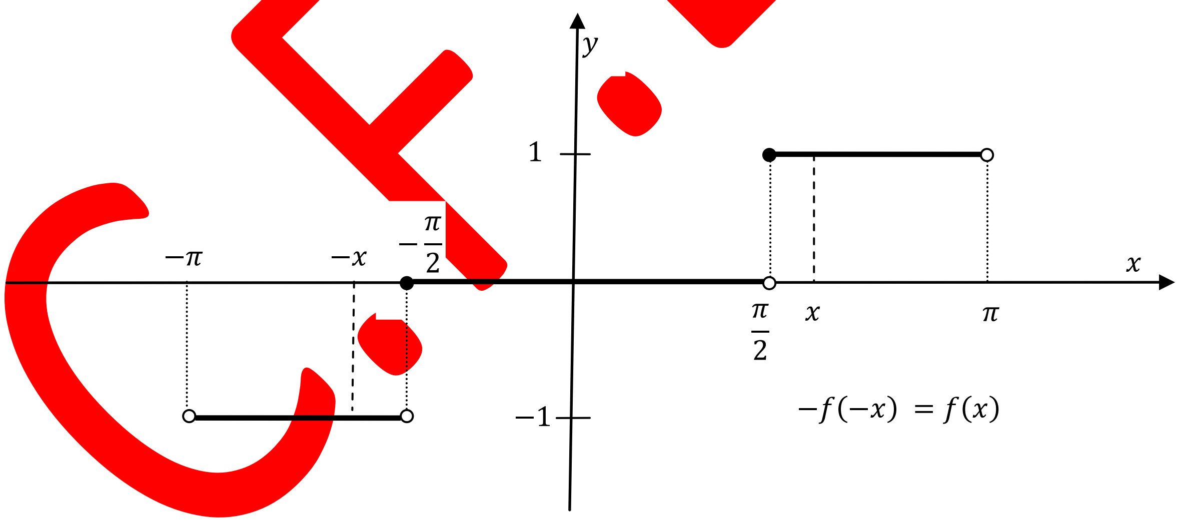
302. $f(x) = \begin{cases} -1, \ si & -\pi < x < -\frac{\pi}{2} \\ 0, \ si & -\frac{\pi}{2} \leq x < \frac{\pi}{2} \\ 1, \ si & \frac{\pi}{2} \leq x < \pi \end{cases}$



$$(-L, L) = (-\pi, \pi) \quad , \ \boxed{L = \pi}$$

Como f es función **impar** su serie trigonométrica de Fourier es una serie de senos.

$$b_n = \frac{2}{L} \int_0^L f(x) sen\left(\frac{n\pi}{L} x\right) dx \quad , n = 1,2,3 \ldots$$

$$b_n = \frac{2}{\pi} \left[\int_0^{\frac{\pi}{2}} 0 sen\left(\frac{n\pi}{\pi} x\right) dx + \int_{\frac{\pi}{2}}^{\pi} 1 sen\left(\frac{n\pi}{\pi} x\right) dx \right]$$

$$= \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} sen(nx)dx = -\frac{2}{\pi}\frac{cos(nx)}{n}\Bigg|_{\frac{\pi}{2}}^{\pi} = -\frac{2}{\pi n}\left(\overbrace{cos(n\pi)}^{(-1)^n} - cos\left(\frac{n\pi}{2}\right)\right), n = 1,2,3 \ldots$$

$$\boxed{b_n = \frac{2}{\pi n}\ \left(cos\left(\frac{n\pi}{2}\right) - (-1)^n\right), \qquad n = 1,2,3 \ldots}$$

$$f(x) \sim \sum_{n=1}^{\infty} b_n sen\left(\frac{n\pi}{L}x\right)$$

$$\boxed{\boldsymbol{f(x) \sim \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(cos\left(\frac{n\pi}{2}\right) - (-1)^n\right) sen(nx)}}$$

C.F.D.S