Dynamics of First order Nonlinear Differential Equations
(Lotka-Volterra Model)

A Project By:

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Contents

Aim: To understand the relation, stability and other parameters of two quantities connected by first order Nonlinear Differential Equations.

Observed Quantities: The populations of Wolves (Predators) and Rabbits (Prey) in the wild, under few assumptions [1].

Introduction:

The Ecosystem is a delicately balanced system between various species and life forms, where the existence of one is necessary for the other. We shall take into account, this interdependence between them and try to relate the population, change in population of two species.

[1]We shall assume there are only two species that are varying with time and plants are sufficiently large, to make the case simpler for analysis. The Lotka-Volterra model also known as the Predator-Prey Model will be used for reference.

We shall consider Wolves and Rabbits as the Predator and Prey respectively. Wolves depend on rabbits for nutrition, which is responsible for their growth in population. The Rabbit's population is kept in control by the wolves, because of the food chain.

Equations and Solving:

Without losing generality, we can assume the growth rate can be divided into two parts, the increasing factor and the decreasing factor (responsible as the names suggest), the increasing factor of a species to depend on the food it has and the current population of the species, whereas the decreasing factor depends on the Predator it has and the competition among their species for food (prey). With increase in predator population, the competition for prey increases.

Hence the general equation can be framed as:

$$\dot{r} = \frac{dr}{dt} = (\alpha \cdot p \cdot r) - (\beta \cdot r \cdot w) - (\gamma \cdot r)$$

$$\dot{w} = \frac{dw}{dt} = (\delta \cdot r \cdot w) - (\tau \cdot w)$$

Figure 1: Generalized equations.

By assuming the plant's quantity as a constant, the factor "**αp**" can be assumed a constant. As there is no competition for rabbits for food (due to large availability of plants) and there are no predators for wolves, those factors are neglected.

$$\alpha p = a \quad | \text{ constant}$$
$$\gamma \approx 0 \quad | \text{ No competition}$$

Figure 2: Assumptions.

By above hypothesis and referring to the Lotka-Volterra model, we can conclude the following relation between the present population and growth.

$$\dot{r} = \frac{dr}{dt} = a \cdot r - b \cdot w \cdot r$$
$$\dot{w} = \frac{dw}{dt} = c \cdot w \cdot r - d \cdot w$$

Figure 3: Final expressions.

Parameters (all positives) **:**

"**a**" : The rate at which the Rabbits grow (*i.e.,* without wolf intervention).
"**b**" : The rate at which rabbits are eaten, dead.
"**c**" : The rate at which wolves grow (Depends on Rabbits, as they are the primary food).
"**d**" : The rate at which wolves die (Age and Competition to find food).
"**r**" : The Present population of Rabbits.
"**w**" : The Present population of Wolves.
"**p**" : Population of Plants(Assumed to be large, changes can be neglected).

We can observe the above system to be an **Autonomous dynamical system** due to the absence of explicit dependence on time. If the system follows the Hamiltonian mechanics, The Liouville's theorem holds true. We are approximating the system to follow Hamiltonian Laws and analyzing the same.

We can conclude from the above, that rabbit's population would exponentially grow, whereas in case of absence of rabbits, the wolf's population would exponentially fall to zero, which is logical.

Considering this as a system, the stability can be decided based on the values of the parameters and initial population. For the equilibrium to exist, the parameters **a, b, c and d** have to be carefully chosen, which are analyzed.

Hypothesis and Examples:

Using the Jacobi Matrix and The Liouville's Theorem to understand stability:

From the determinant of the Jacobi matrix, we can find out the stretch in the area before and after the transformation. In this case, after every time interval (*dt*) the transformation happens, the system acquires a new state. Now if [J] ≈ 1, which signifies the area is constant (nearly), the relative change in one variable is comparable to the relative change in another, hence there is no absurd change or impulsive blows to the population. For small changes in population of prey, the change in predator's population is also small and hence stays in equilibrium.

By setting up the Matrix:

$$r_1 = r(t) + r'\delta t + \sigma(\delta t^2) \quad | \text{ where } r' = dr/dt$$

$$w_1 = w(t) + w'\delta t + \sigma(\delta t^2) \quad | \text{ where } w' = dw/dt$$

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$$\frac{\partial r_1}{\partial r} = 1 + (a - bw)\,\delta t + \sigma(\delta t^2)$$

$$\frac{\partial r_1}{\partial w} = 0 + (-br)\,\delta t + \sigma(\delta t^2)$$

5(i)

$$\frac{\partial w_1}{\partial r} = 0 + (cw)\,\delta t + \sigma(\delta t^2)$$

$$\frac{\partial w_1}{\partial w} = 1 + (cr - d)\,\delta t + \sigma(\delta t^2)$$

5(ii)

$$\text{Hence } [\,J\,] = \begin{bmatrix} 1 + (a-bw)\,\delta t + \sigma(\delta t^2) & (-br)\,\delta t + \sigma(\delta t^2) \\ (cw)\,(\delta t) + \sigma(\delta t^2) & 1 + (cr - d)\,\delta t + \sigma(\delta t^2) \end{bmatrix}$$

$$\det(J) = 1 + (cr - d)\delta t + (a-bw)\delta t + \sigma(\delta t^2)$$

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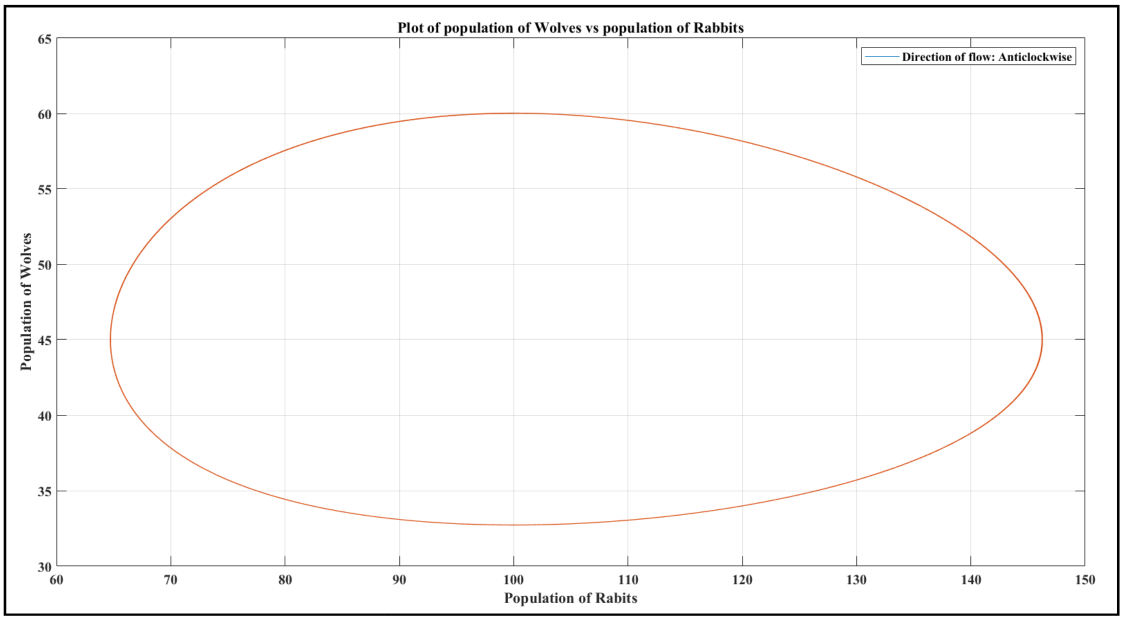
Figures- 4, 5(i), 5(ii) and 6 : Setting the components of the matrix.

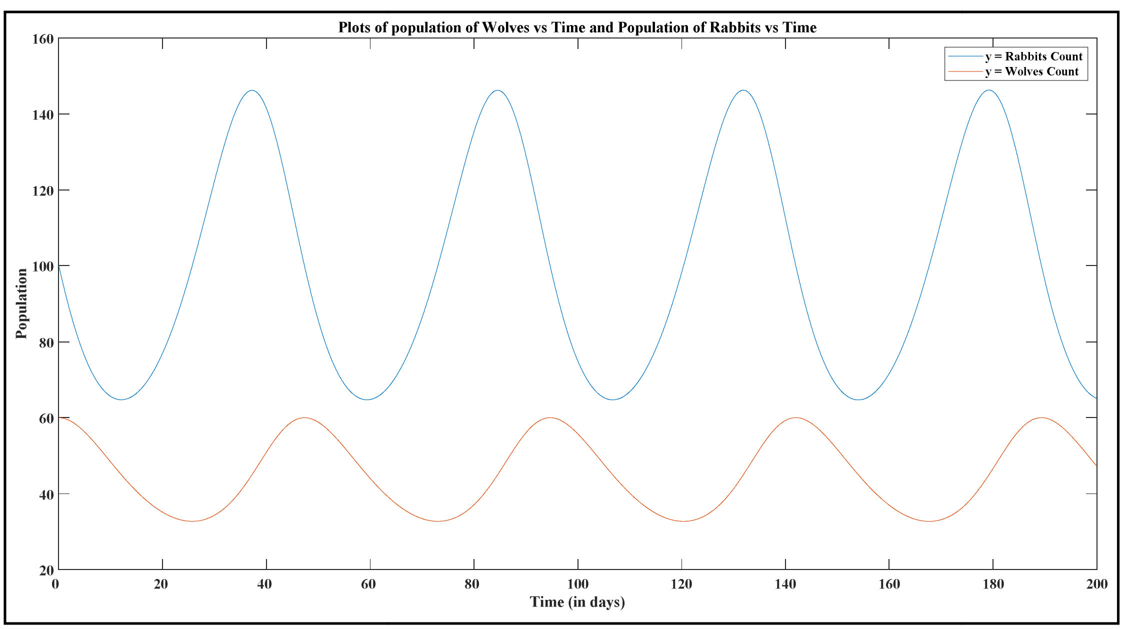
For the initial Conditions:

r = 100 ; w = 60 ; a = 0.18 ; b = 0.004 ; c = 0.001 ; d = 0.01 ……………..(1)

det(J) = 1+((0.001*100) - 0.01 + 0.018 - (0.004*60))dt (After neglecting Second order terms)
= 1 + (0.1 + 0.17 - 0.24)dt = 1 + (0.03)dt ≈ 1

For these conditions we achieve the following results:



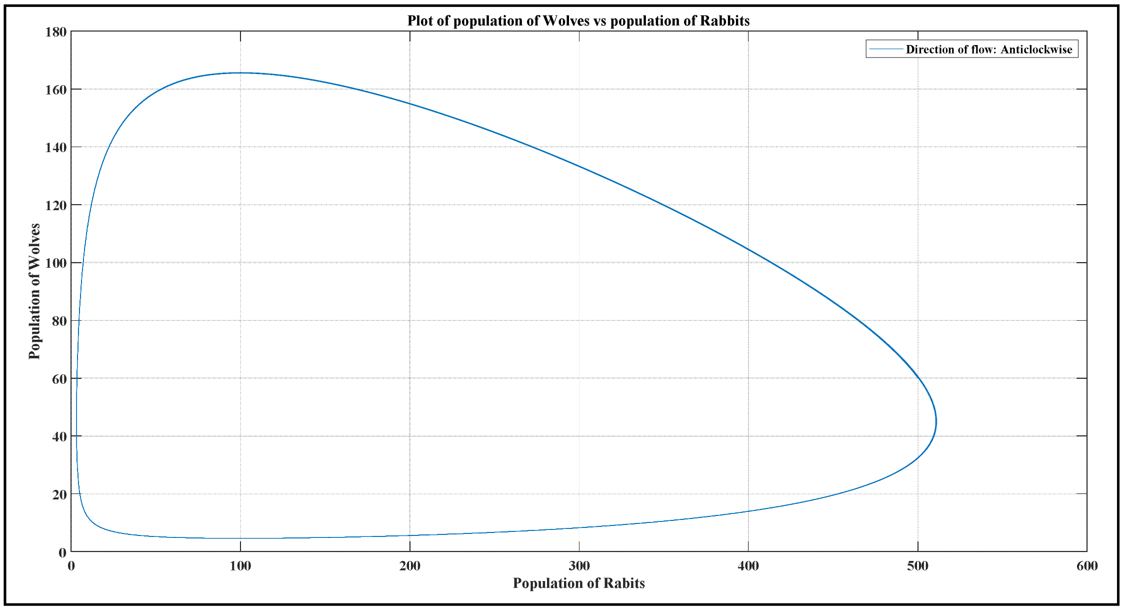


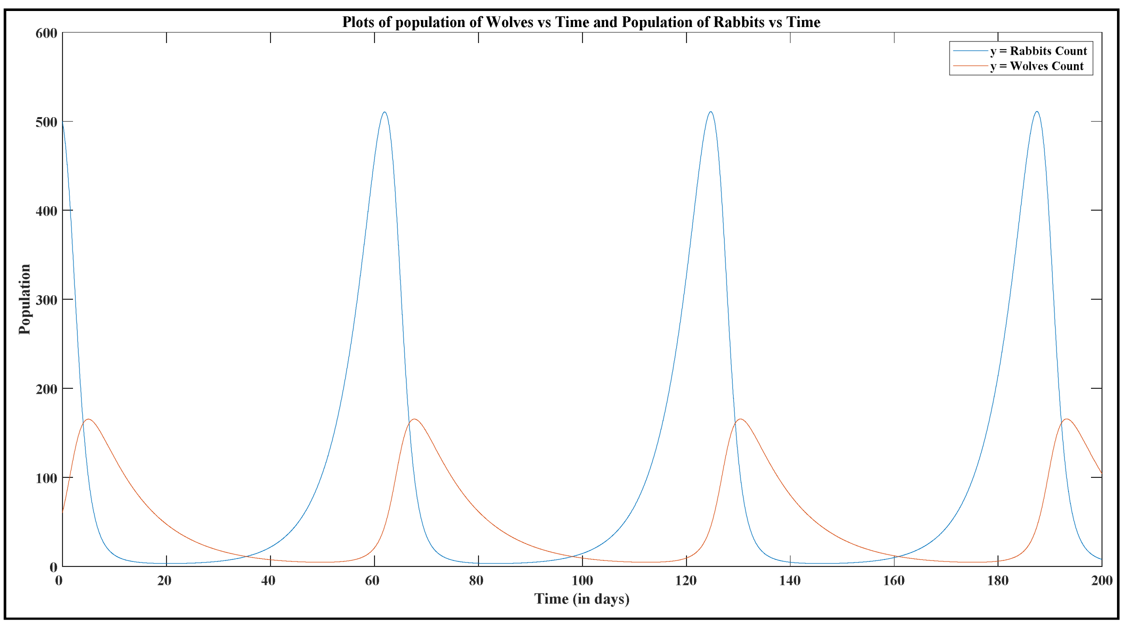
Figures - 7(i) Phase-space diagram, 7(ii) Variation of Population with respect to time.

Oscillations are clear and the system is stable. There is phase difference within the oscillation, which will be discussed further pages.

For Inputs

(i) r = 500 and rest all the same value, Det(J) = 1 + (0.43)dt There would be rapid changes and hence highly reduced stability.





Figures - 8(i) Phase-space diagram, 8(ii) Variation of Population with respect to time.

Impulsive changes are visible and the system is on the verge of collapse. (Animals going to be extinct *i.e.,* population approaching Zero).

For Inputs

(i) r = 1000 and rest all the same value, Det(J) = 1 + (0.93)dt There would be impulsive changes and hence high instability.

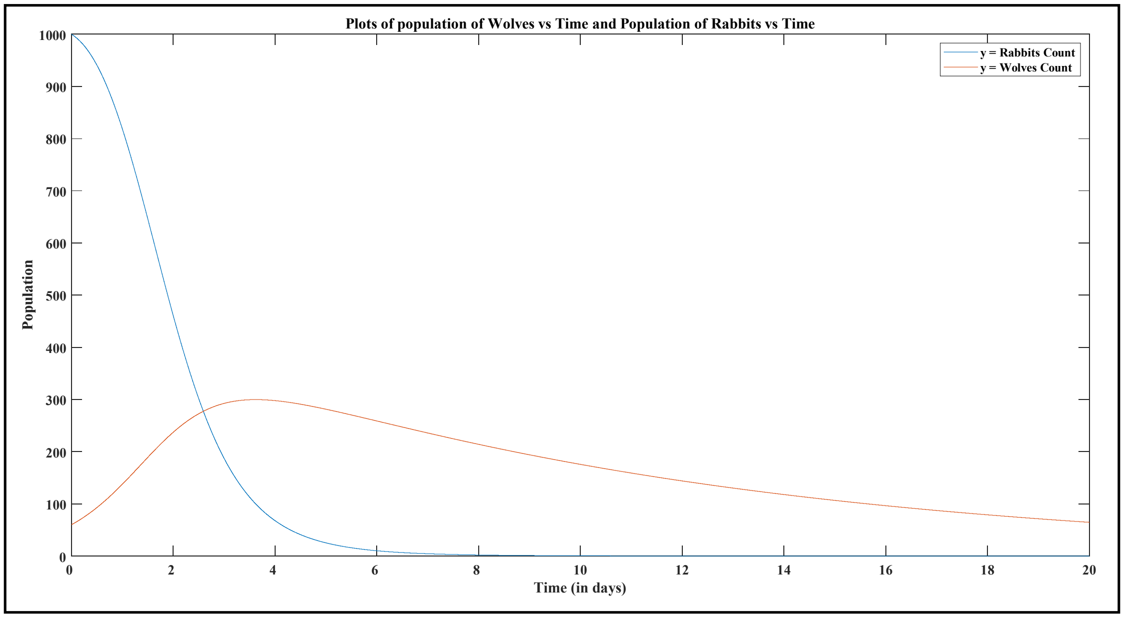


Figure - 9 Variation of Population with respect to time.

The Rabbit's population falls to zero very rapidly after which the wolf's population exponentially decays to zero as expected.

<u>Significant Points (States) and their features</u>:

Fixed points are those at which the system does not oscillate and remains constant. From equations in Figures - 3 and 6, we can understand we have two sets of Fixed points which are (0 , 0) and (a/b , d/c). At these points the values of first derivatives become Zero.

At (0 , 0) : By analyzing the Eigen values of Jacobi Matrix, we arrive at the following Solution $\boldsymbol{\lambda_1 = a; \lambda_2 = -d}$. As the signs are always opposite, we can conclude it to be a saddle point. The same can be explained by logical reasoning as follows: If wolf's population becomes Zero, the rabbit's population reaches infinity exponentially, whereas if Rabbit's population is zero, the wolf's population falls to zero exponentially. The nature of the graph is different in two directions hence it can be concluded to be a saddle point.

At (a/b , d/c): On solving for the Eigen values of Jacobi Matrix, we arrive at the equation $\boldsymbol{\lambda^2 + ad = 0}$; following which, we obtain the solutions as $\boldsymbol{\lambda_1}$= +i√(ad); $\boldsymbol{\lambda_2}$= -i√(ad). It is clear that the solutions are purely imaginary (real parts of the solutions are zero) and complex conjugates of each other. Hence the system is elliptic with solutions being periodic. The system oscillates on an ellipse around this particular point. The frequency of oscillations would be the geometric mean of the Eigen values. Hence the frequency of the oscillations **ω= √ad.**

From the following figure:

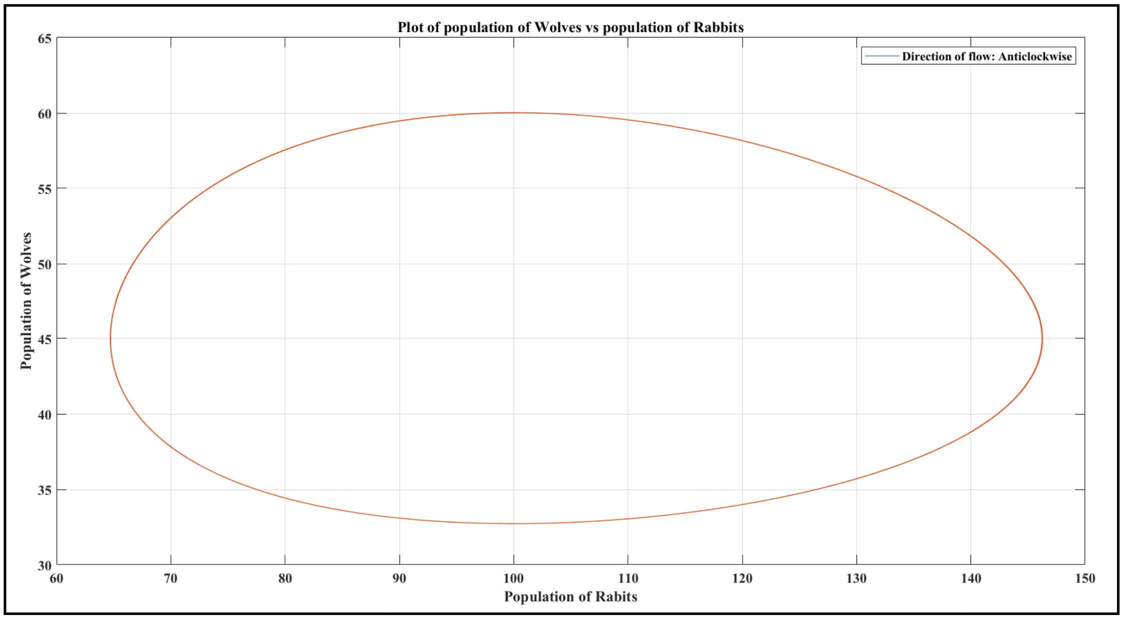


Figure – 10 Phase-space diagram.

We can observe that: when the population of rabbits or that of wolves reaches the specific value (*i.e.,* either r = d/c or w = a/b), there is a change of sign of the slope of the tangent. Those are the points that drive the system towards the stability. The elliptical trajectory explains the phase difference in oscillations and the phase difference will be related to the eccentricity of the ellipse.

<u>Conserved Quantity (Constant of the System)</u>:

On dividing the equations in Figure - 3, and splitting the variable, we attain a differential equation in two variables, which can be solved as follows:

$$\frac{dr}{dw} = \frac{\mathrm{ar-bwr}}{\mathrm{cwr-dw}}$$

$$(\mathrm{cwr} - \mathrm{dw})\,dr = (\mathrm{ar} - \mathrm{bwr})\,dw$$

divided by wr (if wr$\neq$0)

$$\left(\mathrm{c} - \frac{\mathrm{d}}{\mathrm{r}}\right) dr = \left(\frac{\mathrm{a}}{\mathrm{w}} - \mathrm{b}\right) dw$$

Upon integration on both sides

$$\int \left(\mathrm{c} - \frac{\mathrm{d}}{\mathrm{r}}\right) dr = \int \left(\frac{\mathrm{a}}{\mathrm{w}} - \mathrm{b}\right) dw$$

$$\mathrm{cr} - \mathrm{d}\cdot\ln(\mathrm{r}) = \mathrm{a}\cdot\ln(\mathrm{w}) - \mathrm{b}\cdot\mathrm{w} + \mathrm{K}$$

K => constant of integration

$$\Rightarrow \mathrm{K} = \mathrm{cr} + \mathrm{bw} - \ln(\mathrm{r^d}) - \ln(\mathrm{w^a})$$

$$\mathrm{K} = \mathrm{cr} + \mathrm{bw} - \ln(\mathrm{r^d w^a})$$

$$= \ln(\mathrm{e}^{(\mathrm{cr+bw})}) - \ln(\mathrm{r^d w^a}) = \ln\frac{\mathrm{e}^{(\mathrm{cr+bw})}}{\mathrm{r^d w^a}}$$

$$\frac{\mathrm{e}^{(\mathrm{cr+bw})}}{\mathrm{r^d w^a}} = \mathrm{e^K} = \mathrm{K}'$$

Hence the value $\frac{\mathrm{e}^{(\mathrm{cr+bw})}}{\mathrm{r^d w^a}}$ is constant

And is conserved throughout.

Verification of the above statement:

1) At r =100, w = 60, a = 0.18 ; b = 0.004 ; c = 0.001 ; d = 0.01

$$\frac{e^{0.001\times100+0.004\times60}}{100^{0.01}\times60^{0.18}}$$ is equal to 0.6420

2) r = 100, w = 33 is also a possible state:

$$\frac{e^{0.001\times100+0.004\times33}}{100^{0.01}\times33^{0.18}} \text{ is equal to } 0.6419$$

Hence proved, the small error is due to rounding off, but can be neglected.

Introducing Disease Factor:

Let us consider a case where : Due to overpopulation of rabbits, they have been infected by a disease causing their death. In simulation of the same, their chances of death are doubled when the population crosses a certain barrier.

The same has been plotted, which appears as following:

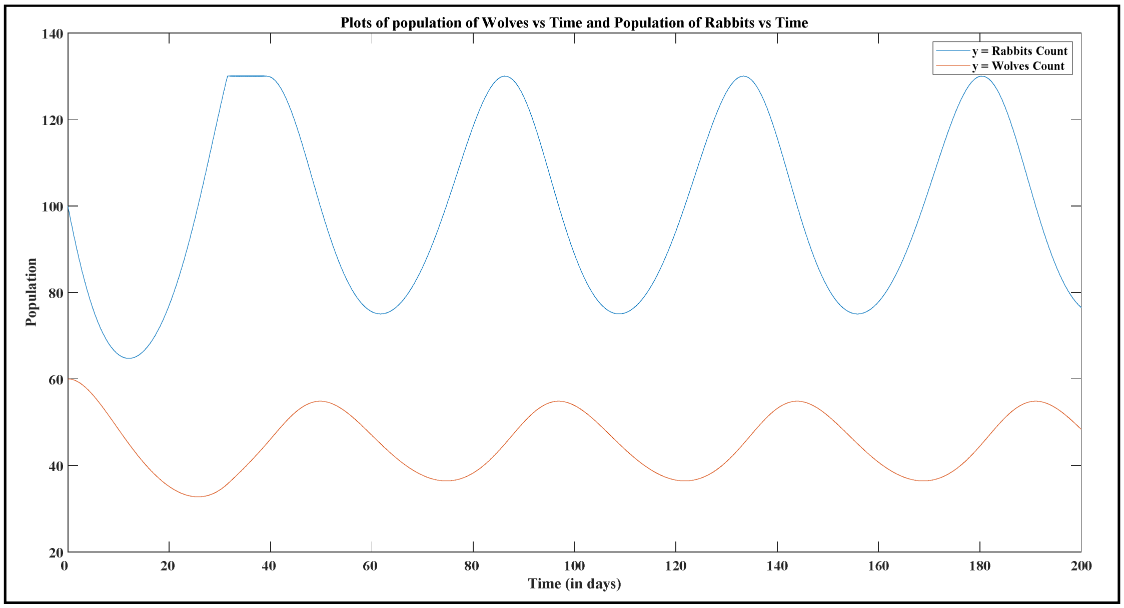


Figure - 11 Variation of Population with respect to time.

It is clear that as the rabbits start approaching their maximum population, the death rate becomes high enough to stagnate the population for some time and it is also clear that the amplitude of oscillation decreases after the disease and hence attains a new set of amplitudes and might also attain new equilibrium points.

Explanation of the Coding Part:

From equations in Figure - 3, when we consider small intervals of time, the changes can be assumed Linear, *i.e.,* second and higher order terms can be neglected. The equation changes to the following:

$$\frac{dx}{dt} = ax - bxy$$

$$dx = (ax - bxy)\,dt \quad | \text{ let } dt = \Delta t = h \ \& \ \frac{h}{T} \approx 0$$

$$\Delta x = x(a - by)\,dt \quad | \ dx = \Delta x = x(t + \Delta t) - x(t)$$

$$x_2 - x_1 = xh(a\text{-}by)$$

$$x_2 = x(t+\Delta t)$$

$$x_1 = x(t) = x$$

Hence

$$x(t+\Delta t) = x(1+h(a\text{-}by))$$

The above principle has been used for computing both the populations of wolves and rabbits, which were plotted using various MATLAB commands to visualize the data for attaining the conclusions.

<u>Conclusion</u>:

There is a delicate balance in the ecosystem and a very minor change can affect the whole system, as there are exponential terms being involved. The same is also known as the Butterfly effect.

Sampling down the above problem, we can mathematically predict the behavior with the help of Hamiltonian dynamics and other equations. A system can be called stable if the coefficient of ***"dt"*** in the determinant of Jacobi Matrix is nearly zero, (-0.05, 0.05) can be the considerable range for stability of the system. In this range, there wouldn't be impulsive changes when compared to the actual population and hence the system lies in the stability zone. Nature interferes when required and changes the parameter values to set the system back in the original track maintaining the delicate balance as seen when analyzing disease factor.