APPENDIX A

CALCULATING k_e^*

In this appendix, we describe a method to calculate the k_e^* of a Boolean automaton (Boolean function).

A.1 CONCEPTS

The central concepts required to understand the formulation of k_e^* are defined in Chapters 1 and 3. Below, we introduce additional concepts necessary to understand the procedure for calculating k_e^*.

Cubes and subcubes. As described in Chapter 1, every k-input Boolean function can be represented as a k-dimensional hypercube or simply *cube*. Every such cube contains *subcubes* of every possible dimension from k down to 0 — the entire cube is the only subcube of dimension k, and every individual input vector (corner) is a subcube of dimension 0. Every subcube can be uniquely identified by an *identifier* that consists of input variables and their values that remain constant in the subcube. For example, consider the following 2-dimensional subcube consisting of the following input vectors in a 3-dimensional cube: $\{(0,1,0),(0,1,1),(1,1,0),(1,1,1)\}$. In this subcube, the only input variable whose value is a constant is i_2 with value 1. Thus, the identifier of this subcube is: $(i_2 = 1)$. A subcube identifier could involve multiple input variables, e.g., $(i_1 = 0, i_3 = 1, i_4 = 0)$. As a special case, the identifier of an input vector is that vector itself. A set of *parallel subcubes* consists of subcubes whose identifiers contain the same set of input variables; each subcube is distinguished by the values of the identifier variables. The following is a single set of parallel subcubes, for example: $\{(i_1 = 0, i_3 = 0), (i_1 = 0, i_3 = 1), (i_1 = 1, i_3 = 0), (i_1 = 1, i_3 = 1)\}$. The dimension of a subcube is equal to k minus the number of its identifier variables. In the parallel subcube above, $k = 4$ means that then dimension of every subcube in the set is equal to 2. For a cube of dimension k, the number of possible sets of parallel subcubes of dimension D is equal to $C(k, D)$. For example, in a $k = 3$ cube, the number of sets of parallel subcubes of dimension $D = 2$ is equal to $C(3, 2) = 3$ — the 3 pairs of parallel faces (left-right, front-back, top-bottom) of the cube.

Composite schemata. A *composite schema* is a symmetric schema with at least one fixed (non-permuting) '#' and at least one permuting permuting '#'; it is named so because it contains the characteristic features of both an ordinary schema and a two-symbol schema. All other symmetric schemata shall be referred to as *ordinary symmetric schemata*. For example, $(1, \#, \dot{0}, \dot{\#})$ is a composite schema, whereas $(1, 0, \dot{0}, \dot{\#})$ or $(1, 0, \dot{\#}, \dot{\#})$ are ordinary symmetric schemata. One way to identify a composite schema is by combining two or more symmetric schemata from a set of parallel subcubes. For example, $(\#, \dot{0}, \dot{1}, \dot{\#})$ can be obtained by combining $(0, \dot{0}, \dot{1}, \dot{\#})$ and $(1, \dot{0}, \dot{1}, \dot{\#})$, or by combining $(0, \dot{0}, \dot{\#}, \dot{\#})$ and $(1, \dot{1}, \dot{\#}, \dot{\#})$. Notice in the former that the composite schema is a full union of the combining schemata, whereas in the latter it is a union of portions of the combining schemata. In either case, the composite schema acts as "bridge" that unites ordinary parallel symmetric schemata — this is essentially why we consider parallel subcubes. Also, two symmetric schemata can combine to more than one composite schemata. For example, the symmetric schemata in the set $\{(0, 0, \dot{0}, \dot{\#}), (0, 1, \dot{0}, \dot{\#}), (1, 0, \dot{0}, \dot{\#})\}$ combine to produce the following set of composite schemata: $\{(0, \#, \dot{0}, \dot{\#}), (\#, 0, \dot{0}, \dot{\#})\}$. Naturally, if the original set was $\{(0, 1, \dot{0}, \dot{\#}), (1, 0, \dot{0}, \dot{\#})\}$, no composite schemata would be possible since the values of subcube identifier variables (i_1 and i_2)

can't combine.

A.2 METHOD

The procedure for calculating k_e^* involves the following steps:

Step 0: Initialize a "cover list" of length 2^k with all zeros — this list shall contain the dimensions of the largest covering symmetric schemata corresponding to each input vector, and will be updated throughout the procedure.

Repeat steps 1 to 4 below for every possible dimension D in decreasing order from k to 1, and for every possible set of parallel cubes of a given dimension.

Step 1: Consider a single set of parallel subcubes of a given dimension.

Step 2: Identify all ordinary symmetric schemata in each subcube in the set obtained in the previous step and for each output value.

Note that a subcube may contain more than one symmetric schema. Every symmetric schema in a subcube of dimension D must contain exactly D permuting symbols. One way to identify a symmetic schema is to compute the number of input vectors in the subcube that contain a certain number of 1s (n_1). If that number is equal to $C(D, n_1)$, then those input vectors may constitute a symmetric schema. Note that we identify only those symmetric schemata that cover input vectors with the associated number of 1s in some interval $[n_1, n_1 + w]$ where $w \geq 1$.

Step 3: Identify all composite schemata from the set of ordinary symmetric schemata obtained in the previous step and for each output value.

One way to identify a composite schema is to enumerate "signatures" of all possible ordinary symmetric schemata of dimension D and match them against the permuting symbols of the schemata obtained in the previous step. If more than one symmetric schema in the set matches a given signature, then it is an indication that they might combine to form a composite schema. An important point to note here is that a signature may only *partially* match the schemata and yet produce a composite schema. A representative example is: the signature $(\dot{0}, \dot{1}, \dot{\#})$ partially matches the permuting symbols of both the symmetric schemata $(0, \dot{0}, \dot{\#}, \dot{\#})$ and $(1, \dot{1}, \dot{\#}, \dot{\#})$, to produce the composite schema $(\#, \dot{0}, \dot{1}, \dot{\#})$.

Step 4: Record the dimension of the symmetric schema against every input vector it covers if and only if the current largest covering symmetric schema's dimension is smaller.

Finally, compute k_r^* by averaging over the covering dimensions of all 2^k input vectors. Compute $k_e^* = k - k_r^*$.

A.3 EXAMPLE

In this section, we apply the procedure described above to calculate the k_e^* of an example $k = 3$ function (Fig. A.1), and describe the steps involved in detail.

Initialize cover list: $(0, 0, 0, 0, 0, 0, 0, 0)$, where the order is the same as the numbers indicated in red in the figure.

D = 3:

Step 1. Since $D = k$, the only subcube in the set of parallel subcubes is the full cube itself.

Step 2. The only symmetric schema in this subcube is: $\{(\dot{1}, \dot{1}, \dot{\#})\}$, corresponding to output 1.

Step 3. No composite schemata exist since there is only symmetric schema.

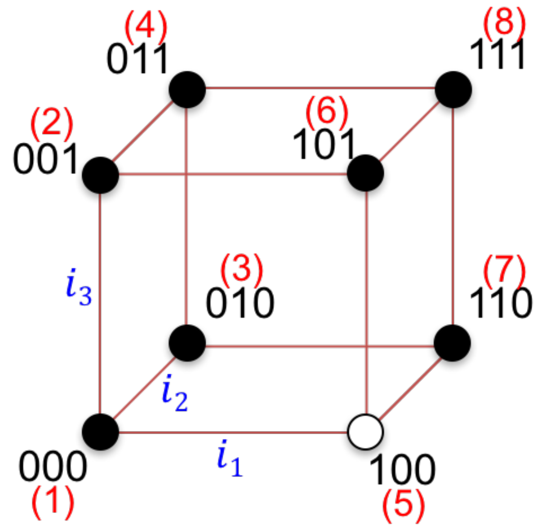

Figure A.1: An example $k = 3$ Boolean function $x = f(i_1, i_2, i_3)$. Each corner (input vector) is marked with a decimal number in red — to be used as an index to refer to the input vectors in the cover list.

Step 4. The set of all symmetric schemata now consists of $\{(\dot{1}, \dot{1}, \dot{\#})\}$, with dimension equal to $\log_2(4) = 2$, and covering the corners {(4,6,7,8)}. The updated cover list is: (0,0,0,2,0,2,2,2).

D = 2:

There are 3 sets of parallel subcubes: $\{(i_1 = 0), (i_1 = 1)\}$, $\{(i_2 = 0), (i_2 = 1)\}$ and $\{(i_3 = 0), (i_3 = 1)\}$.

Step 1a. Consider the set of parallel subcubes $\{(i_1 = 0), (i_1 = 1)\}$.

Step 2a. The set of symmetric schema in this set of parallel subcubes consists of: $\{(0, \dot{\#}, \dot{\#})$, $(1, \dot{1}, \dot{\#})\}$, corresponding to output 1.

Step 3a. The set of composite schemata obtained by combining the symmetric schemata in the set from above consists of: $\{(\#, \dot{1}, \dot{\#})\}$.

Step 4a. The set of all symmetric schemata now consists of $\{(0, \dot{\#}, \dot{\#}), (1, \dot{1}, \dot{\#}), (\#, \dot{1}, \dot{\#})\}$, with dimensions respectively equal to $\log_2(4) = 2$, $\log_2(3) \approx 1.58$ and $\log_2(6) \approx 2.58$, and covering the corners {(1,2,3,4),(6,7,8),(2,3,4,6,7,8)} respectively. The updated cover list is: (2,2.58,2.58,2.58,0,2.58,2.58,2.58).

Step 1b. Consider the set of parallel subcubes $\{(i_2 = 0), (i_2 = 1)\}$.

Step 2b. The set of symmetric schema in this set of parallel subcubes consists of: $\{(\dot{\#}, 1, \dot{\#})\}$, corresponding to output 1.

Step 3b. No composite schemata exist since there is only symmetric schema.

Step 4b. The set of all symmetric schemata now consists of $\{(\dot{\#}, 1, \dot{\#})\}$, with a dimensions equal to $\log_2(4) = 2$, and covering the corners {(3,4,7,8)}. The updated cover list is: (2,2.58,2.58,2.58,0,2.58,2.58,2.58) (no alteration).

Step 1c. Consider the set of parallel subcubes $\{(i_3 = 0), (i_3 = 1)\}$.

Step 2c. The set of symmetric schema in this set of parallel subcubes consists of: $\{(\dot{\#}, \dot{\#}, 1)\}$, corresponding to output 1.

Step 3c. No composite schemata exist since there is only symmetric schema.

Step 4c. The set of all symmetric schemata now consists of $\{(\dot{\#}, \dot{\#}, 1)\}$, with a dimension equal to $\log_2(4) = 2$, and covering the corners {(2,4,6,8)}. The updated cover list is: (2,2.58,2.58,2.58,0,2.58,2.58,2.58) (no alteration).

D = 1:

There are 3 sets of parallel subcubes: $\{(i_1 = 0, i_2 = 0), (i_1 = 0, i_2 = 1), (i_1 = 1, i_2 = 0), (i_1 = 1, i_2 = 1)\}$, $\{(i_1 = 0, i_3 = 0), (i_1 = 0, i_3 = 1), (i_1 = 1, i_3 = 0), (i_1 = 1, i_3 = 1)\}$ and $\{(i_2 = 0, i_3 = 0), (i_2 = 0, i_3 = 1), (i_2 = 1, i_3 = 0), (i_2 = 1, i_3 = 1)\}$.

Step 1a. Consider the set of parallel subcubes $\{(i_1 = 0, i_2 = 0), (i_1 = 0, i_2 = 1), (i_1 = 1, i_2 =$

$0), (i_1 = 1, i_2 = 1)\}$.

Step 2a. The set of symmetric schema in this set of parallel subcubes consists of: $\{(0, 0, \dot{\#})$, $(0, 1, \dot{\#}), (1, 1, \dot{\#}\}$, corresponding to output 1.

Step 3a. No composite schemata exist. Note that $(0, 0, \dot{\#})$ and $(0, 1, \dot{\#})$ can combine to form $(0, \dot{\#}, \dot{\#})$ but it is not a valid composite schema (see definition in Sec. A.1).

Step 4a. The set of all symmetric schemata now consists of $\{(0, 0, \dot{\#}), (0, 1, \dot{\#}), (1, 1, \dot{\#}\}$, with dimensions respectively equal to $\log_2(2) = 1$ each, and covering the corners {(1,2),(3,4),(7,8)} respectively. The updated cover list is: (2,2.58,2.58,2.58,0,2.58,2.58,2.58)s (no alteration).

Step 1b. Consider the set of parallel subcubes $\{(i_1 = 0, i_3 = 0), (i_1 = 0, i_3 = 1), (i_1 = 1, i_3 = 0), (i_1 = 1, i_3 = 1)\}$.

Step 2b. The set of symmetric schema in this set of parallel subcubes consists of: $\{(0, \dot{\#}, 0)$, $(0, \dot{\#}, 1), (1, \dot{\#}, 1)\}$, corresponding to output 1.

Step 3b. No composite schemata exist.

Step 4b. The set of all symmetric schemata now consists of $\{(0, \dot{\#}, 0), (0, \dot{\#}, 1), (1, \dot{\#}, 1\}$, with dimensions respectively equal to $\log_2(2) = 1$ each, and covering the corners {(1,3),(2,4),(6,8)} respectivelu. The updated cover list is: (2,2.58,2.58,2.58,0,2.58,2.58,2.58) (no alteration).

Step 1c. Consider the set of parallel subcubes $\{(i_2 = 0, i_3 = 0), (i_2 = 0, i_3 = 1), (i_2 = 1, i_3 = 0), (i_2 = 1, i_3 = 1)\}$.

Step 2c. The set of symmetric schema in this set of parallel subcubes consists of: $\{(\dot{\#}, 0, 1)$, $(\dot{\#}, 1, 1), (\dot{\#}, 1, 0)\}$, corresponding to output 1.

Step 3c. No composite schemata exist.

Step 4c. The set of all symmetric schemata now consists of $\{(\dot{\#}, 0, 1), (\dot{\#}, 1, 1), (\dot{\#}, 1, 0)\}$, with dimensions respectively equal to $\log_2(2) = 1$ each, and covering the corners {(2,6),(4,8),(3,7)} respectively. The updated cover list is: (2,2.58,2.58,2.58,0,2.58,2.58,2.58) (no alteration).

Finally, k_r^* is the mean of the values in the cover list: $k_r^* = 2.185 \implies k_e^* = k - k_r^* = 0.815$ (exact value is 0.8112781 if the log values above are not rounded).