Mathematical Pendulum

Group 8
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Abstract

The report presents the calculation of the acceleration due to gravity by two different methods and represents the functional relationship between the time period and the angle of deflection. The acceleration due to gravity in Bremen North is observed to be $(9.824 \pm 0.062)ms^{-2}$ using the stop-watch and $(9.8239 \pm 0.0073)ms^{-2}$ using the light barrier, which agrees within the error range of the theoretical value. The relation between the time period and the angle of deflection is observed from the best fit line of the graph between $\ln \theta_{max}$ and $\ln\left(\frac{T}{T_0} - 1\right)$. The slope of the best fit line is 2.1654, which agrees with the literature value with some errors, and the value of the correlation coefficient is 0.987, which shows that the time period and the angle of deflection are related.

1 Introduction and Theory

A simple pendulum is a system of mass suspended by a string from rigid support, which oscillates about its equilibrium position under the influence of gravity. The interval of time in which the system completes one oscillation is called the time period of the pendulum.

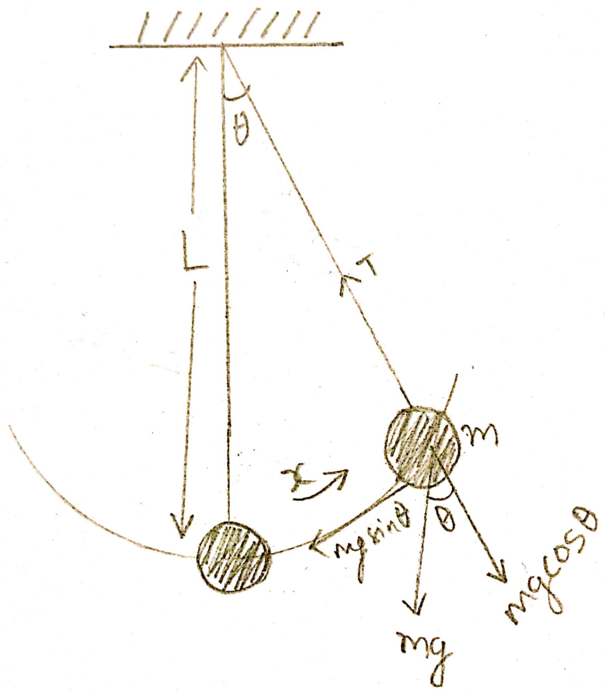

Figure 1: Forces acting on the mass of a pendulum

In the figure, a mass m is connected to one end of the string of length L, fixed at the pivot. The path of the mass of a pendulum describes an arc of a circle with radius L. Let ω be the angular velocity

$$\omega = \frac{d\theta}{dt} = \frac{v}{L}$$

where θ is the angular displacement and v is the tangential velocity. The angular acceleration is given by:

$$\alpha = \frac{d^2\theta}{dt^2}$$

The tangential restoring force pulling the mass back to its equilibrium position is

$$F_\theta = -mg\sin\theta \tag{1}$$

Using Newton's Law, $F = m \cdot a$ for the tangential force gives:

$$\begin{aligned} F_\theta &= m \cdot a_{tang} = m \cdot \frac{dv_{tang}}{dt} \\ &= mL\frac{d\omega}{dt} = mL\frac{d^2\theta}{dt^2} \end{aligned} \tag{2}$$

Equating 1 and 2,

$$\begin{aligned} -mg\sin\theta &= mL\frac{d^2\theta}{dt^2} \\ \ddot{\theta} + \frac{g}{L}\sin\theta &= 0 \end{aligned} \tag{3}$$

For small $\theta, \sin\theta \approx \theta$
Equation (3) then writes,

$$\ddot{\theta} + \frac{g}{L}\theta = 0$$

This gives, time period

$$T = 2\pi\sqrt{\frac{L}{g}} \tag{4}$$

The period of a pendulum with large deflections is given by

$$T = T_0\left[1 + \frac{1}{16}\theta_{max}^2\right] \tag{5}$$

where T is the "real" and T_0 the "ideal" period (for small amplitudes).
The equation 5 can be written as

$$\ln\left(\frac{T}{T_0} - 1\right) = -4\ln 2 + 2\ln\theta_{max} \tag{6}$$

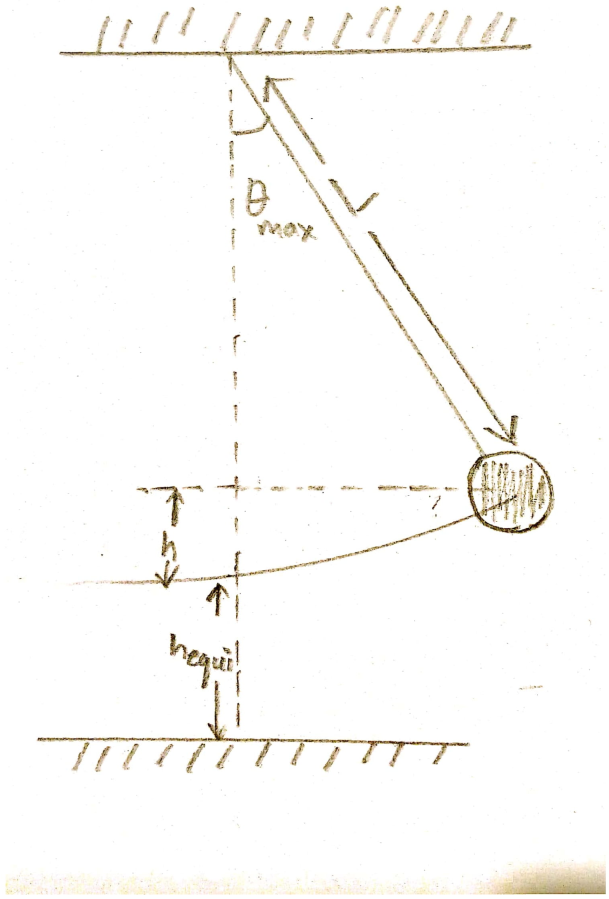

Figure 2: Relation between height h and angle of deflection θ_{max}

If h is the difference between the lowest position of the sphere and the height of the sphere at θ_{max}, θ_{max} is given by:

$$h = L \cdot (1 - \cos(\theta_{max})) \tag{7}$$

where L is the effective length of the pendulum.

2 Experimental and Procedure

2.1 Set-up and Procedure

In the following experiments, the pendulum with a steel ball tied to the wire was mounted on the ceiling. The time period of the pendulum was measured using the stopwatch and light barrier connected to Cassy. The diameter of the sphere was measured using the vernier calipers three times, and the average value was calculated.

2.2 Measuring gravitational acceleration using a stopwatch

The pendulum was oscillated within the small deflections ($\theta < 10°$). The time period for ten complete oscillations was measured using the stopwatch. The same procedure was repeated ten times.

2.3 Measuring gravitational acceleration using a light barrier

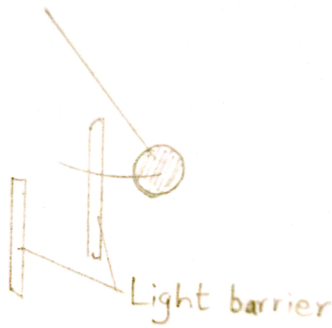

Figure 3: Setup for Light Barrier

In this experiment, the light barrier was used to measure the time period of the pendulum. The light barrier, connected to Cassy, was placed below the equilibrium position of the pendulum. The pendulum was oscillated through the light barrier with small deflections. The time period for each oscillation was displayed on the laptop. The results were compared with the observed acceleration due to gravity using stopwatch and the literature value.

2.4 Functional relationship between period and angle of deflection

In the following experiment, the light barrier was used to determine the time period of the pendulum with higher angles of deflection. The height of the ball in its equilibrium position was measured. Further using the mathematical calculations, the height of the ball for different deflection angles including 40 degrees was determined. The pendulum was oscillated for different deflections between 10 and 40 degrees including. The time period displayed for each oscillation on the laptop was noted down, and the graphs of T vs θ_{max} and $\ln(\theta_{max})$ vs $\ln\left(\frac{T}{T_0} - 1\right)$ were plotted.

3 Results and Data Analysis

3.1 Measuring gravitational acceleration: Preparation

The measurements for the diameter of the steel ball is tabulated below:

Diameter $[10^{-3}m]$	Average $[10^{-3}m]$	Error of the mean $[10^{-3}m]$
59.35 59.45 59.40	59.40	0.03

Table 1: Diameter of the Steel Ball

See Error Analysis section for the calculation of the error of the mean

The average value $\bar{d}$ is found to be $59.4 \times 10^{-3} m$.
The length l of the wire used to suspend the steel ball is $(2.508 \pm 0.001)m$. Thus, the effective length of the pendulum is

$$L = l + \frac{d}{2} = 2.508 + 29.7 \times 10^{-3} = 2.537m$$

. From equation 4,

$$g = \frac{4\pi^2 L}{T^2} \tag{8}$$

3.2 Measuring gravitational acceleration using a stopwatch

From observations, the time period for ten oscillations t_{10} is tabulated as:

Time period for ten oscillations $[s]$ t_{10}	Average $[s]$ $\bar{t_{10}}$	Error of the mean $[s]$ $\Delta\bar{t}_{10}$	$\bar{T} = \frac{\bar{t_{10}}}{10}[s]$	$\Delta\bar{T} = \frac{\Delta t_{10}}{10}[s]$
32.00 31.90 31.75 31.81 31.87 31.82 31.75 31.78 31.75 32.06	31.850	0.034	3.1850	0.0034

Table 2: Time period using Stop Watch

From Table 2, the period T for one oscillation is $3.19s$ with uncertainty $0.01s$ [See Error Analysis Section].
From equation 8,

$$\begin{aligned} g &= \frac{4\pi^2 L}{T^2} \\ &= \frac{4\pi^2 L}{T^2} \\ &= 4\pi^2 \times \frac{2.537}{3.19^2} \\ &= 9.842ms^{-2} \end{aligned}$$

The acceleration due to gravity using the stop watch is observed to be $(9.842 \pm 0.062)ms^{-2}$. [See Error Analysis section].

3.3 Measuring gravitational acceleration using a light barrier

The measured time period using light barrier is noted in the following table:

Time period [s] T	Average [s] $\bar{T}$	Error of the mean [s] $\bar{\Delta T}$
3.191		
3.193		
3.196		
3.193		
3.193		
3.193	3.19320	0.00053
3.191		
3.193		
3.196		
3.193		

Table 3: Time period using Light Barrier

The time period T is calculated to be $(3.193 \pm 0.001)s$.[See Error Analysis Section]
From equation 8,

$$\begin{aligned} g &= \frac{4\pi^2 L}{T^2} \\ &= \frac{4\pi^2 L}{T^2} \\ &= 4\pi^2 \frac{2.537}{3.193^2} \\ &= 9.8239 ms^{-2} \end{aligned}$$

The acceleration due to gravity using the light barrier is observed to be $(9.8239 \pm 0.0073)ms^{-2}$.
See Error Analysis section.

The measurements for the acceleration due to gravity from using the stop watch and light barrier are tabulated below and compared with the literature value g_0 in Bremen North.

g_{stop}	$(9.824 \pm 0.062)ms^{-2}$
g_{cassy}	$(9.8239 \pm 0.0073)ms^{-2}$
g_0	$(9.813310 \pm 0.000041)ms^{-2}$

Table 4: Comparison of Results

3.4 Functional relationship between period and angle of deflection

The height h_{equi} of the ball in its equilibrium position is measured to be $(21.5 \pm 0.1)cm$.
The calculated height of the ball from the equilibrium position for a maximum deflection of $40°$ is

$$\begin{aligned} h &= L \cdot (1 - \cos\theta_{max}) \\ &= 253.7(1 - cos40) \\ &= 59.4mm \end{aligned} \tag{9}$$

Time period [s] T	$h_{start}[cm]$ $h_{equi} + h$	h [cm] $h = L(1 - cos\theta)$	$\theta_{max}[deg]$	$\theta_{max}[rad]$	$\ln \theta_{max}$	$\left(\frac{T}{T_0} - 1\right)$ [s]	$\ln\left(\frac{T}{T_0} - 1\right)$
3.197	25.3	3.9	10	0.175	-1.746	1.25	-6.502
3.200	30.1	8.6	15	0.262	-1.340	2.19	-6.124
3.211	36.8	15.3	20	0.350	-1.052	5.64	-5.178
3.225	45.3	23.8	25	0.436	-0.830	10.02	-4.603
3.234	55.5	34.0	30	0.524	-0.647	12.84	-4.355
3.277	80.9	59.4	40	0.698	-0.360	26.30	-3.638

Table 5: Time period for large angle deflections

The measured time periods using the light barrier for different deflections is tabulated below:

The graph of T vs θ_{max} is shown in the figure, which is in the form of parabola.

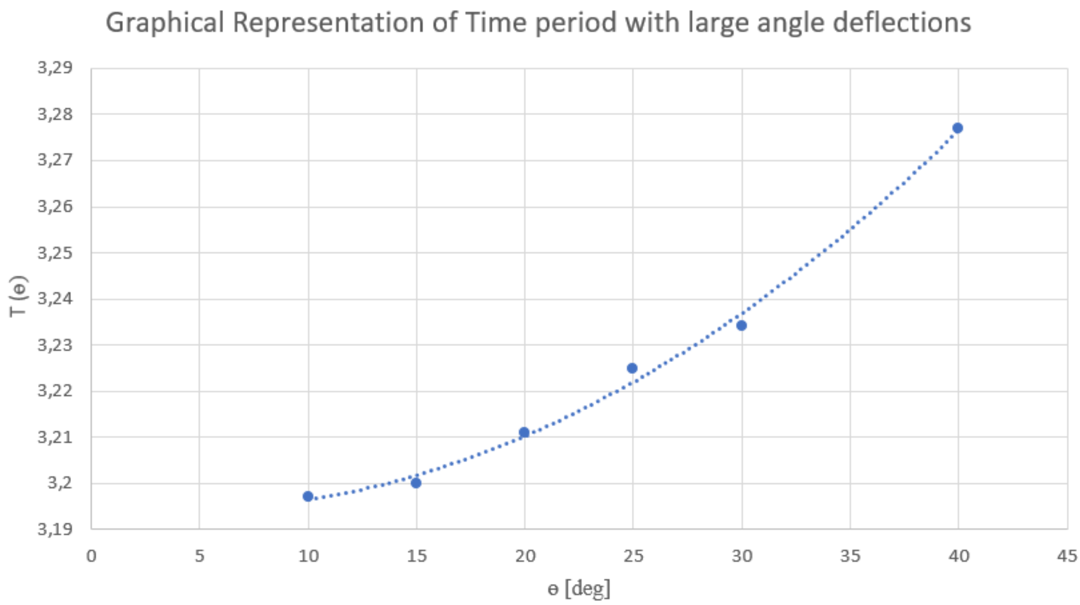

Figure 4: Graph of T vs θ_{max}

Also, the graph of $\ln(\theta_{max})$ and $\ln\left(\frac{T}{T_0} - 1\right)$ is plotted, and the best fit line of slope 2.1654 is sketched.

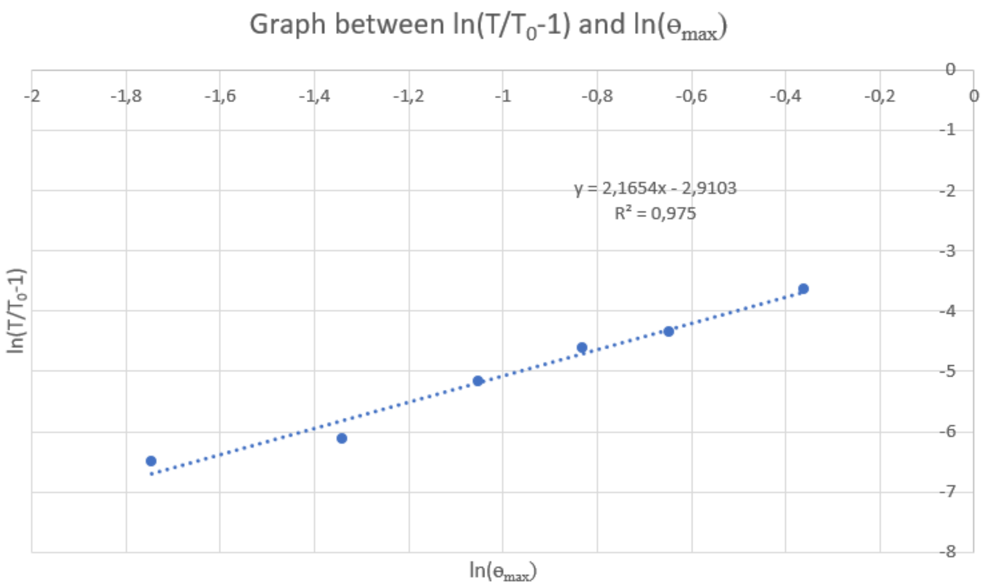

Figure 5: Graph of $\ln(\theta_{max})$ vs $\ln\left(\frac{T}{T_0} - 1\right)$

The measurement of T for the largest angle i.e. 40° is noted down.

Time period [s] T	Average [s] $\bar{T}$	Error of the mean[s] $\Delta\bar{T}$
3.286 3.285 3.283 3.280 3.278	3.2824	0.0015

Table 6: Time period for maximum angle 40°

4 Error Analysis

The least count of the vernier calipers used to measure the diameter of the steel ball is $5\times10^{-5}m$, and the least count of the meter scale used to measure the vertical height of the ball is $0.001m$. Similarly, the stopwatch used to measure the time has least count of $0.01s$, and the Cassy has least count of $0.001s$.

4.1 Statistical Treatment

From Table 1, the error of the mean value $\Delta\bar{d}$ is calculated as:

$$\Delta\bar{d} = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(d_i - \bar{d})^2}$$
$$= 0.03 \times 10^{-3}m$$

The instrumental error being higher than the statistical error, instrumental error is taken in account.

Therefore, $\Delta d = 5 \times 10^{-5} m$.
From Table 2, the error of the mean value $\Delta \bar{t_{10}}$ is calculated as

$$\begin{aligned} \Delta \bar{t_{10}} &= \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (t_{10_i} - \bar{t_{10}})^2} \\ &= 0.034s \end{aligned} \tag{10}$$

And,

$$\Delta \bar{T} = \frac{\Delta \bar{t_{10}}}{10} = \frac{0.034}{10} = 0.0034s$$

which is lower than the instrumental error i.e, $0.01s$.
Thus, the instrumental error is taken in account.

From Table 3,

$$\begin{aligned} \Delta \bar{T} &= \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (T_i - \bar{T})^2} \\ &= 0.00053s \end{aligned}$$

The error of the mean, $\Delta \bar{T}$ is $0.00053s$, which is lower than the least count of the Cassy i.e, $0.001s$.
Hence, the dominating error i.e, instrumental error is taken.

From Table 6,

$$\begin{aligned} \Delta \bar{T} &= \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (T_i - \bar{T})^2} \\ &= 0.0015s \end{aligned} \tag{11}$$

The error of the mean, $\Delta \bar{T}$ is $0.0015s$.

4.2 Propagated Error

If ΔL be the uncertainty in measuring the effective length of the pendulum,

$$\Delta L = \Delta l + \Delta d = 0.001 + 5 \times 10^{-5} = 0.001m$$

The instrumental error produces uncertainty in measuring the effective length and time period of the pendulum. Hence, there would be uncertainty in measuring the acceleration due to gravity. Let Δg be the uncertainty in g.Thus, the propagated error in g using stop watch is

$$\begin{aligned} \Delta g_{avg} &= \sqrt{\left[\left(\frac{\partial g}{\partial T}\right)_L \Delta T\right]^2 + \left[\left(\frac{\partial g}{\partial L}\right)_T \Delta L\right]^2} \\ &= \sqrt{\left[\frac{-8\pi^2 \cdot L \cdot \Delta T}{T^3}\right]^2 + \left[\frac{4\pi^2 \cdot \Delta L}{T^2}\right]^2} \\ &= 0.062m/s^2 \end{aligned} \tag{12}$$

Similarly, propagated error in g using the light barrier

$$\begin{aligned} \Delta g_{avg} &= \sqrt{\left[\left(\frac{\partial g}{\partial T}\right)_L \Delta T\right]^2 + \left[\left(\frac{\partial g}{\partial L}\right)_T \Delta L\right]^2} \\ &= 0.0073 ms^{-2} \end{aligned} \tag{13}$$

5 Discussion and Conclusion

The value for the acceleration due to gravity using the stopwatch and the light barrier is $(9.824 \pm 0.062)ms^{-2}$ and $(9.8239 \pm 0.0073)ms^{-2}$ respectively. Both values agree within the error range of each other and the literature value, $g_0 = (9.813310 \pm 0.000041)ms^{-2}$. The large error range for the value of g calculated using the stopwatch shows that the instruments of high least count are taken. In the experiment to measure g using the light barrier, the problem of high instrumental error is avoided. Thus, the observed value of g using light barrier is much reliable.

The plot of $T(\theta_{max})$ is in the form of equation 5, which should be a parabola. Also, the graph of $\ln(\theta_{max})$ vs $\ln\left(\frac{T}{T_0} - 1\right)$ has the slope in approximate to the theoretical value of the slope in equation 6. The observed slope is 2.1654 with correlation coefficient 0.987. The small offset in the slope should be because the system is not ideal. As the table 6 shows that there is gradual decrease in the time period for the same deflection. This should be because of damping due to air presence in the experimental room and non frictionless pivot. In addition, the measurements of vertical height of the ball have uncertainty due to instrumental error. Nevertheless, the value of correlation coefficient is ca. 1, showing that there is positive relationship between the time period and the angle of deflections, which agrees with the data presented in Table 5.

References

[1] Prof. Dr. Jürgen Fritz and Faezeh Mohaghegh, Classical Physics Lab (CH-140-B) Fall 2019