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Teaching and learning angles in elementary school: physical *versus* paper-and-pencil sequences

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ABSTRACT

This paper discusses the relevance of using physical situations to introduce the concept of angles at elementary school. We compare the effectiveness of two geometry teaching sequences. In the first one (physical sequence), the pupils learned the angle concept by experimenting on the playground (i.e. mesospace) and then modelling the situation. In the second one (paper-and-pencil sequence), the pupils worked solely in the space of a sheet of paper (i.e. microspace). In both sequences, pupils compared areas of space delineated by an angle between two directions. Pupils in two Grade 3 classes were exposed to one of the two teaching sequences. The unfolding of these sequences was videotaped and analyzed, and the pupils were tested individually, before and after teaching, to measure each sequence's effectiveness. Results showed that both sequences are effective to grasp the angle concept: Most pupils overcame the common erroneous conception of comparing angles' sides' lengths instead of angle openness. The comparison of areas of space delineated by an angle between two infinite directions, which is the two sequences' common core, seems to be the key factor underlying angle conceptualization. This paper ends with a discussion of these results' teaching implications and the merits of each sequence.

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KEYWORDS
Angle; elementary school; geometry; physics; teaching sequences

1. Introduction

The question of the relationship between mathematics and physics has been raised in many countries' teaching programmes since at least the early twentieth century. For example, France's 2007 official teaching instructions for elementary school emphasize that 'to the greatest possible extent, the experimental sciences, technology, and mathematics must be interconnected in school curricula' (MEN, 2007). Similarly, the National Council of Teachers of Mathematics (NCTM, 2000) standards point to the necessity to interrelate mathematics and science. In our research, we study the links between mathematics and physics in situations involving the concept of angles in elementary school. This paper extends a previous study (Devichi & Munier, 2013) showing that a physically-based sequence founded on the introduction of angles through the notion of the visual field leads

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to a good grasp of this concept in Grades 3 and 4. The aim of the present study is to better understand the reasons for these teaching sequences' effectiveness: Was it actually (or solely) the use of a physical framework to introduce angles that made the lesson effective?

2. Background

2.1. Geometry teaching

Several theories have been developed to model the way in which children construct geometry concepts. The analysis of the way children construct representative space was given by Piaget and Inhelder (1956) in their seminal work. The scholars singled out two phases in the process: topological relationships acquired by children (inside-outside, next to, open–closed) followed by the acquisition of projective and Euclidean relations. Their developmental approach placed greater emphasis on angles, which mark the transition from topological to Euclidean relationships.

Whereas Piaget's theory does not take into consideration instruction, other theories tackle the ways geometry is taught to children. For example, White and Mitchelmore (2010) advocated a teaching method wherein young learners go through four distinct phases: familiarity, similarity, reification, and application. In this method called 'teaching by abstraction', familiarity comes as a result of 'informal' knowledge gained in early childhood, the stage when children associate various situations with surface similarities. The second phase, that of similarity, is observable by the time children complete primary school. In this phase, children start to gain a better insight into similarities between different situations that can be grouped together under the same geometric shape. In this way, children start to build diverse contexts. In the third phase, that of reification, children begin to observe and recognize similarities in different contextual situations. This necessarily entails engaging in mental or physical action, referred to as 'reflective abstraction' by Piaget. At the last step, children must be confronted with new situations in which they can use the concept (application).

Teaching geometry in the elementary classroom (from First to Fifth Grade) has received sustained critical attention among scholars. French Berthelot and Salin (1998) put forward theoretical discussions in geometry teaching to distinguish between geometric knowledge on the one hand and spatial knowledge, which is indispensable to controlling the relationships to space on a regular basis, on the other. The two authors proceeded to demonstrate that children would struggle to draw upon spatial knowledge to deal with practical problems. Consequently, they proposed activities wherein students would model the real world, thus having them acquire both spatial and geometric knowledge.

Their approach is in accordance with NCTM standards (2000), which, too, emphasize the relationships between spatial and geometric knowledge.

2.2. About the concept of angles

Many authors have highlighted the complexity of the concept of angles (Mitchelmore & White, 1998). Keiser (2004) went on to draw parallels between the ways sixth-graders develop the notion of angles on the one hand and the difficulties mathematicians have in defining this complex concept on the other. There are three possible definitions of an angle:

a rotation angle (an amount of turning), a sector angle (the quantity shared by the set of all superimposable angular sectors), and an angle regarded as two half-lines extending from a same point (openness or inclination). The construction of this 'multifaceted concept' (Mitchelmore & White, 2000) is slow and progressive (Lehrer et al., 1998; Mitchelmore, 1997, 1998) and numerous obstacles occur frequently. Many authors have pinpointed an array of difficulties. One of the examples reported by Mitchelmore (1998) is that many young learners consider that it is the radius of the arc marking an angle that determines the size of the angle. Sometimes, the students believe that an angle must necessarily have one horizontal arm (for a review, see Bütüner & Filiz, 2017). Almost all authors report that the main obstacle in the construction of the concept of angles is that children believe that the length of the angle's arms determine its size (Aubert et al., 2008; Baldy et al., 2005; Clements, 2003; Fyhn, 2008; Mitchelmore, 1998; Piaget et al., 1960). This misconception may persist for an extended period of time (Lehrer et al., 1998), and for this reason, it is a particular focus in our research.

2.3. How should the concept of angles be taught?

According to Mitchelmore and White (1998), teaching geometric concepts requires drawing upon the informal knowledge of students. Similarly, Berthelot and Salin (1998) underline the necessity to use problem-solving activities in mesospace (Brousseau, 1997). Given that mesospace represents a space which allows the subject's movement and observation of objects (e.g. the playground), it therefore differs from the usual space used in geometry teaching, microspace, which is smaller in size and where the subject has a complete and almost immediate perception of the spatial domain within which they are interacting. According to these authors, pupils encounter difficulties because the working space at school is essentially the sheet of paper. Indeed, the concept of angles involves the notion of half-lines, which presupposes an unlimited space. For Berthelot and Salin (1995), the quasi-exclusive use of paper can explain why pupils comparing angles consider the lengths of the sides rather than the angle's openness. Although experimenting with geometry in mesospace seems to be heuristic, few studies have tested this idea directly.

A number of authors such as Mitchelmore (1998) and Masuda (2009) have proposed a different means of emphasizing openness and, consequently, undermined the importance of side length in measuring angle size: students should take part in classroom activities including both static and dynamic angles. As Browning et al. (2007) state, 'trying to make sense of a dynamic view of angles requires more than paper-and-pencil tasks' (p. 284). Like many authors, they proposed using technology, for example, LOGO, simulations programmes, or dynamic geometry environments (Clements & Burns, 2000; Clements & Sarama, 1995; Healy et al., 2002; Vadcard, 2002; and more recently Crompton, 2015; Cullen et al., 2018 and Kaur, 2020). Another approach to improving the dynamic view of angles is to use body movements (Fyhn, 2006, 2008; Kim et al., 2011; Smith et al., 2014; Wilson & Adams, 1992). These authors proposed 'mathematizing' physical activities involving angles (e.g. skating, snowboarding, climbing) to show that a dynamic approach facilitates the conceptualization of angles.

In a previous paper, we developed a design-based experiment involving angles in which pupils are confronted by problem-solving activities (Devichi & Munier, 2013). The specificity of our approach was to start with a physical situation to allow the pupils to grasp

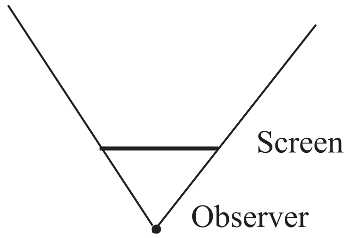


Figure 1. Boundaries of the visual field for an observer located in front of a screen.

the concept of angles. Sequences started with a practical problem corresponding to a real-life situation that the pupils must model. The problem was to delineate the hidden space for an observer in the presence of an obstacle, in this case, a screen. This visual field[1] is delineated by an angle, the vertex of which is located at the observer and the sides of which are two (infinite) half-lines that pass through the screen's extremities (Figure 1).

To solve the problem, pupils had to experiment in the mesospace of the playground and then model the initial situation in the microspace of the sheet of paper. They were then confronted with situations in which the angle varied (when the observer moved away from the screen) and were instructed to use body movements to help them apprehend these variations. The angle that had to be constructed was then defined between two infinite directions. In this previous paper, the sequence's effectiveness was studied through qualitative and quantitative approaches. The first one was based on a fine-grained analysis of the unfolding of the teaching sequence, while the second one compared the effectiveness of this experimental sequence and a standard teaching sequence (control group). The results showed the greater effectiveness of the proposed teaching sequence. We concluded that introducing this concept by modelling a physical situation in the mesospace, in which the angles appear between two directions, rather than using two objects with finite dimensions (e.g. the hands of a clock, as in the control group), is efficient and that it allows the pupils to understand that the length of the sides is irrelevant. We also concluded that using models as a bridge between abstraction and reality thus seems efficient at helping pupils to learn geometry.

In the present paper, we question the key aspects of the situation that were responsible for its effectiveness: Was it really the work in mesospace during the initial situation and the modelling of this situation and/or the fact that the concept of angles was introduced between two directions in space? Could the sequences' effectiveness also lie in the tasks proposed to the pupils?

To answer these questions, we implemented and compared two sequences. The first one was replicated from the 2013 study, while the second one confronted pupils directly with a geometric model of the situation, without having them experience or model the first situation. This second situation was solely paper-and-pencil, that is, limited to the microspace of the sheet of paper and designed so that the tasks proposed to the pupils would be as similar as possible: In both sequences, they had to find the edges of an area in space bounded by an angle between two spatial directions.[2] In both situations, pupils had to compare plan areas delineated by angles, either in the playground or on a sheet of paper.

3. Method

3.1. Participants

The sequences were used with two Grade 3 classes and parental consent was obtained prior to this study. 48 third-grade students (whose mean age was 8 years and 6 months) took part in the experiment. The students were mainly from middle-class families and attended schools from the Montpellier metropolitan area. The classes were heterogeneous, i.e. consisting of students with various levels of academic achievement. The concept of angles had not been taught to any of the students before, apart from right angles (which are introduced in France in Grade 1 or 2 as a means to distinguish geometric shapes, for example, rhombus versus square, using a set square).

3.2. Procedure

To limit any potential effects due to teacher or achievement differences between the classes, each class was split into two equivalent groups on the basis of a pretest addressing informal knowledge about angles (angle drawing, comparison of angle pairs). Half of the pupils in each class were given the physical sequence, and the other half were given the paper-and-pencil sequence. The experimental teaching sequences were administered to each half classes by the pupils' usual teachers after a training session led by our research team. Thus, two half-classes followed the physical sequence, and two half-classes followed the paper-and-pencil sequence. All of the sequences were videotaped. As in the 2013 study, we combined qualitative and quantitative approaches. Quantitative analysis was based on pre- and posttests as well as on written evaluations taken in class after the sequences. Qualitative analysis, which aimed to better understand the quantitative results, was based on fine-grained analysis of the unfolding of the teaching sequences and on the pupils' intermediate written productions.

3.3. Pre- and posttests

The preassessment data were collected one month before the teaching sequences via individual interviews. The pupils were asked to compare four pairs of angles. This task was designed to verify whether the pupils considered side length or angle openness. The pairs were presented in a counterbalanced order. The post-assessment data were collected three weeks after the teaching sequences using the same procedure. These data were used to establish equivalent groups before the sequences and compare the pupils' progress after the sequences.

4. Results

In this section, we first present the qualitative analysis, and then we report the results of the pre- and posttests as well as the written evaluations.

4.1. Description and analysis of the teaching sequences

Analysis of the unfolding of the sequences aimed to describe in detail pupils' activities in order to understand each sequence's effects and identify the key aspects of the situations

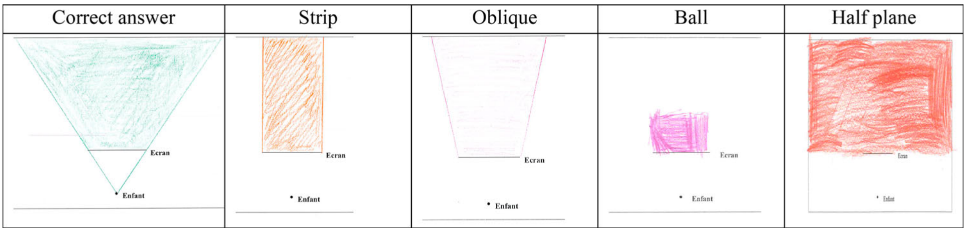


Figure 2. Examples of pupils' hypotheses.

that help pupils grasp the angle concept. Both sequences (physical/paper-and-pencil) unfolded very similarly in the two half-classes (in terms of pupils' reactions and activities, and duration), so we present the results grouped. Each session lasted approximately one hour Table 1.

4.2. Physical sequence

As explained above, this teaching sequence is a replication of one of the sequences Devichi and Munier (2013) described.

4.2.1. Session 1: elaborating the problem, setting hypotheses, and experimenting on the playground

The first session aims at leading pupils to inquire about the concept of the visual field. To introduce this point, the teacher first makes use of a drawing illustrating the following situation: a number of children are on the point of crossing a street, but they are in danger because of a vehicle moving past a parked bus. The driver of the vehicle is unable to see some children and vice versa. The teacher asks the students to determine which children in the drawing are in danger and to provide explanation. Following a class discussion, the question is asked: 'If there is an obstacle, what can we see?' In order to simplify this situation concerning road safety, the teacher uses a screen serving as the obstacle in the remaining sequences. The students are then asked to fill in a diagram representing the situation in question seen from above on a two-dimensional surface. In the diagram, they must apply colour to the area that the children in the drawing are unable to see. This model of the situation from an overhead view was easily understood by all of the pupils, who offered several hypotheses (see examples in Figure 2).

There were some correct answers in each half-class. However, most students got the answers wrong, either saying that the observer could not see the strip perpendicular to the screen ('the strip hypothesis') or drawing oblique lines going from the end points of the screen and not through the observer ('the oblique hypothesis'). After a class debate, the students suggested verifying their hypotheses on the playground. This experiment occurred in mesospace (Brousseau, 1997), both in terms of the size of the space (school playground) and the possibilities for control (possibility of acting upon the observer and screen positions, possibility of attaining an overview of the situation).

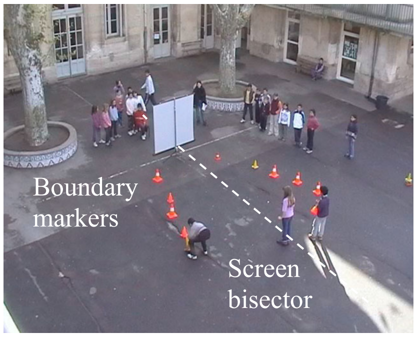
The teacher carefully placed a chair on the playground on the right bisector of the screen.[3] The student-observer sat on the chair without being able to see the other students who were standing behind the screen. Holding boundary markers, the students initially

Table 1. Overview of the research design.

	Physical Sequence	Paper-and-pencil Sequence
Pre-test	Individual interview: each pupil was asked to compare four pairs of angles in a paper-pencil task.	
Session 1	This session begins with a real-life situation involving the notion of visual field. The following question arises: 'What can we see when there's an obstacle in front of us?' The teacher proposes a diagram modelling this concrete situation and asks the children to make 'paper-and-pencil' hypotheses. After a collective debate, they propose playground experimentation to test their hypotheses. Back in the classroom, pupils are given individual paper-and-pencil exercises to model the playground experiment.	This session begins with a geometry exercise without reference to visual field or real-life situations. The teacher proposes individual 'paper-and-pencil' exercises analogous to those involved in the end of physical session 1: a schema representing two points called O and M on either side of an [AB] segment. The pupils must find all of the points that could be connected to point O while crossing segment [AB] (see Figure 6). In this sequence, the playground experiment is replaced by paper-and-pencil experimentation.
Session 2	Emergence of the concept of angle Finding techniques to compare and reproduce angles	
Session 3	Mathematics session: decontextualized session on angles with application exercises	
Written tests in class	Mathematics exercises (production and comparison of angles) and exercises about the situation experienced (physical or paper-and-pencil)	
Posttest	Individual interview: each pupil was asked to compare four pairs of angles in a paper-pencil task.	

lined up near the right bisector (see Photos 1–3). Then, they started walking one at a time in either direction parallel to the screen until the student-observer was able to spot them on either the left or the right of the screen. The moment a moving student entered the visual field of the observer, the latter gave a sign and the moving student laid his or her boundary marker on the ground.

Photos illustrating the playground experiment.[4]



When observing the boundary markers' positions, the children noticed that a straight line was formed on each side of the screen. Then they placed two ropes of the same colour along the markers in order to test this assumption. The students also remarked that the two lines were actually 'diagonal' or 'slanted,' resulting in dismissing the earlier strip hypothesis. When the students were asked if they had expected the boundary markers to be arranged in this way, one pupil recalled the hypotheses previously formulated in class that the boundary lines would (or would not) pass through the observer (correct answer and oblique hypotheses, respectively). These hypotheses were then tested experimentally by moving the ropes over to the observer. The pupils concluded that the lines crossed each other at the observer.

Afterwards, the teacher asked the students the following question: 'Assuming that the observer position remains unchanged, could you have moved further apart from each other along the line behind the screen, and if so, how far could you have gone?' The aim of this question was to lead them to the idea of direction (i.e. the boundaries of the area are two infinite half-lines). There were some students saying that the line could stretch beyond the rope; however, not everyone in class was convinced. At this point, the experiment was repeated with two longer ropes of another colour and different lengths (changing the observer).[5] Note that from the outset, the pupils started positioning themselves farther away from the screen.

Two other observer locations (still intentionally chosen by the teacher on the right bisector of the screen) were used to repeat the same procedure. The students could see whether the observer changing position, coming closer to or moving away from the screen, would result in the hidden area increase or decrease in size. At the end of this session, the teacher instructed the children to straighten their arms along the lines delimiting the hidden area that varied as the observer moved toward or away from the screen. When the teacher asked the children if and how far the ropes could be stretched, they all replied in the affirmative, saying: 'Indefinitely' or 'As far as the eye can see.' Finally, when asked to make remarks on the experiment, the students arrived at the conclusion that two straight half-lines delimited the visible area. In addition, they noticed that the two half-lines extended indefinitely from the point corresponding to the observer location at the same time going through the edges of the screen. Once they returned to the classroom, the pupils were given an individual exercise to model the playground experiment. They had to colour in the hidden area on diagrams of various sizes (to force the pupils to produce a 'qualitative' diagram of the playground experiments without having to introduce the concept of scale) for different observer locations. The vast majority of children successfully completed the assigned task by colouring in the given area. (see Figure 3).

4.2.2. Session 2: emergence of the concept of angles as a delineator of the hidden area and identifying techniques for comparing and reproducing angles

In a collective discussion, the pupils were asked to compare the first two diagrams obtained at the end of the previous session (Figures 3a and b: same position of the observer with ropes of different lengths). The aim here was to make the pupils aware that the hidden area was the same for a given observer position, regardless of the length of the ropes. At this point, comparison of the different diagrams led to a socio-cognitive conflict (Mugny & Doise, 1978) due to the fact that the coloured areas could be different despite the hidden areas they modelled being the same. Table 2 presents excerpts from a debate showing this conflict and its regulation by the teacher (T: teacher, P: pupil) in one of the half-classes. Some of the pupils believed that the hidden area could be the same (see P2 and P4); however, not everyone shared this view. Whereas some pupils thought that the change in the size of the hidden area depended on whether the coloured area was smaller or larger (see P1 and P5), others believed that we could extend the boundaries as far as we want, and thus, the visible area was the same (see P2 and P6); these pupils managed to relate microspace

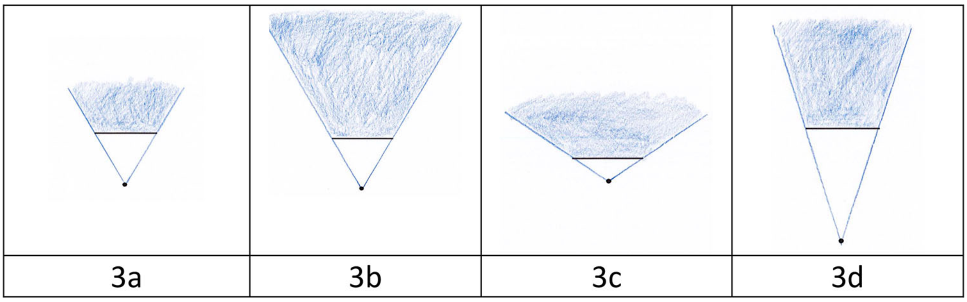


Figure 3. Examples of correct answer.

Table 2. Excerpt from debates in one of the half classes.

T: What is the biggest hidden area?
P1: This one, the right one.
T: Why?
P1: Because more is colored in.
P2: It's the same because all we did was make the frame bigger.
T: Why is it the same?
P2: Because if we continue the small one, we have the same as the big one.
T: What did we do with the ropes?
P3: Sometimes we couldn't go farther because we were blocked by the wall.
T: And if the wall hadn't been there?
P3: We would have been able to go farther.
T: What is the biggest hidden area?
P4: The two are the same.
P5: It's not the same because I measured both and it's not the same.
T: What two?
P5: The two lines.
T: What were they?
P5: Ropes.
P6: But if we continue them it will be the same.
P5: Oh, okay.

(the areas on the piece of paper) to mesospace (the areas modelled in the three-dimensional environment), as seen in the following table (see Table 2).

The teacher then handed out a sheet with diagrams each showing sides of unequal lengths (see Figures 3a and b). The students were asked to devise a technique for comparing the areas to see whether they were the same. They were thus encouraged to work out methods for comparing angles without even using the very term 'angle.' Angle comparison was thus induced by the problem itself and appeared necessary to the pupils. They were deliberately not given any algorithms and were encouraged to determine how to compare the angles on their own. For third-graders, three devices could be considered, namely templates, tracing paper, and an instrument with two articulated branches (e.g. scissors or a compass), each device highlighting one of the angles' conceptions. Templates emphasize the idea of a sector angle. When angles are compared using tracing paper, pupils manipulate pairs of half-lines. The use of an articulated instrument (scissors, a compass, or a bevel square made of two narrow strips of poster board attached at one end by a paper fastener) emphasizes the view of the angle as two half-lines (inclination of one direction with respect to another or the space between two directions). Moreover, opening and closing the branches of this instrument should promote a dynamic conception of angles.

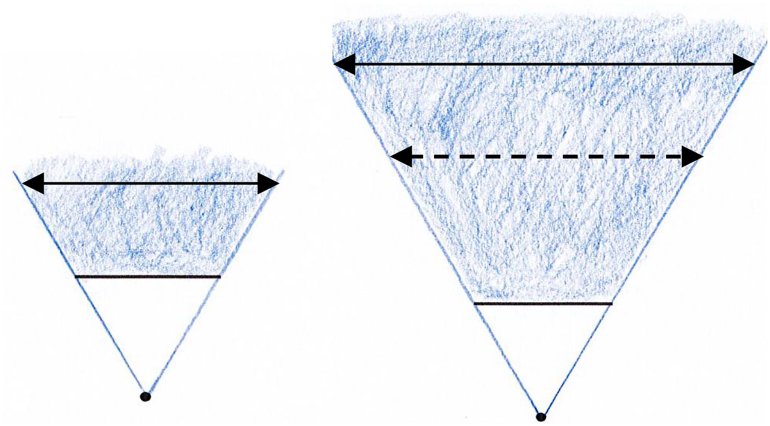


Figure 4. Examples of pupils' measurement procedures.

We hypothesized that the different techniques would appear spontaneously. Indeed, in accordance with Mitchelmore and White's proposals (2000), it is important that pupils use different angle comparison and reproduction techniques to tackle as many aspects of the angle as possible. The purpose of this practice is to help students to identify similarities between various physical situations expressed by the same geometric concept (the angle), thus enabling them to classify angles (inclination, sector, etc.). Not surprisingly, children thought of different tools of comparison. The graduated ruler was initially used by the majority of students to measure the length of two sides. Nevertheless, some students reacted against measuring the ropes, saying, 'That's useless because it goes on to infinity [gesturing in the direction of the rope].' Some pupils attempted to measure the 'space' between the two sides, but not everyone measured this 'space' at the same distance from the screen (see Figure 4). They focused on measures of length with which they were familiar. Others believed that it only made sense to measure this space at the same distance from the screen. This idea suggests that these pupils implicitly considered the concept of angles as a space between two directions, independently of the lengths of the sides.

Some children put one diagram on top of another, laid them against a windowpane and looked through the superimposed sheets of paper. Others cut out the coloured area and used it as a template. The teacher handed out tracing paper to all of the pupils, who managed to verify without difficulty that the sides of the hidden areas truly matched.

Despite this fact, comparison of the areas led to a debate:

P1: The two ropes are in the same place but that goes past; there is more white than color.

P2: Yes, but the white part could be colored in.

P1: But one of them goes farther toward infinity than the other.

P2: It can do the same as the other one because it goes to infinity.

This discussion helped them conclude that for a given observer position the length of the ropes does not affect the hidden area. Comparison with the other two diagrams (two different observer positions) led to the conclusion that, this time, the edges of the coloured areas were not on top of each other, so the hidden areas were different.

At that moment, the teacher introduced the students to the concept of angles by asking them which factors in fact determine the sides of the hidden area. It was no longer a question of considering each boundary separately, but rather of coordinating the two half-lines, since it is their spread that played a crucial role in determining the size of the hidden area. The children immediately referred to the hidden area as ‘a triangle’ (or a V-shape), but the teacher explained that it lacked a third side to be considered a triangle. Following a discussion, the students in each half-class were relatively quick to suggest the term ‘angle’ to denote the area in question. The students were unanimous in concluding that it is an angle that determines the hidden area and that, even though the first two diagrams depicted the angles of different side lengths, it was still the same angle. The teacher thus guided the students to employ the term ‘angle’ in verbally describing the situation where the observer moved toward or away from the screen: The closer one came to the screen, the larger the angle defining the area.

4.2.3. Session 3: mathematics session on angles and application exercises

The purpose of this session was to decontextualize the concept of angles. The teacher introduced the pupils to geometric terms such as angle, vertex and sides: ‘As you could see, what determines the hidden area is an angle whose vertex corresponds to the observer and whose sides go through the edges of the screen’. The pupils were then asked to define an angle, and both half-classes yielded similar definitions, which the teachers rephrased as: ‘An angle is the degree of openness between two half-lines (the sides) that extend from a common point (the vertex).’ To determine the extent of the pupils’ consensus on this concept, they were given an angle-copying exercise in which they were asked to come up with methods for comparing and replicating angles. Even though they had some manipulation difficulties, the students succeeded in crafting templates relying on different techniques. For example, they placed the edge of the paper so that it aligns with either of the angle’s sides and then folded it to coincide with the other side or cutting the paper so that the second side showed. Next, the teacher asked the pupils to find a more practical instrument than templates to compare and reproduce angles in order to introduce techniques that did not spontaneously appear during the hidden-area comparisons. Some pupils proposed to use scissors or a compass by placing the two blades on the sides of the angle. At this point, the teacher introduced to the students the bevel square (see Figure 5) and asked them to use this tool to copy the same angle, the task which they completed without difficulty. The teacher then used the instrument to trigger discussions, demonstrating various types of angles using a physical model for the dynamic angle.

The teacher started moving the arms of the bevel square thus changing the angle. When a right angle was formed (Position 2 in Figure 5), the students immediately recognized this familiar angle. Then, the notion of an acute angle (Position 1) and the notion of an ‘obtuse angle’ (Position 3) were introduced by the teacher. When the bevel square marked out a 180-degree angle, some students thought that the angle disappeared whereas others claimed that there was still an angle. Following a class debate, the teacher explained that this, too, was an angle and referred to it as a flat angle. When the bevel square’s arms formed the angle that was greater than 180°, there were some students saying that the angle has gone underneath, only considering the salient angle. This further generated a discussion, which led to the conclusion that two half-lines in fact form two angles – an open and a closed angle – referred to by the students as an outer angle and an inner angle, which in

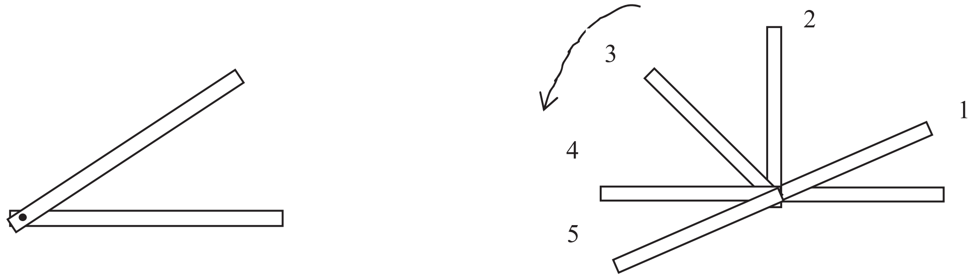


Figure 5. Dynamic bevel square.

turn corresponds to the scientific terms 'reflex angle' and 'salient angle,' respectively. The students deemed it necessary to mark angles to differentiate between the reflex angle and the salient angle. The teacher introduced a classical notation with an arc and then recalled the playground situation where 'The 'closed' angle corresponded to the part that could not be seen (except for the small triangle in front), and the 'open' angle was the part that could be seen.'

The session continued with individual comparison exercises (using templates, tracing paper, or a bevel square) designed to put these techniques into practice. These exercises presented equal angles with sides of different lengths, smaller angles with longer sides, etc. Production exercises were also proposed, such as drawing an angle, then a larger one and a smaller one. The last exercise was a reproduction task. Almost all the pupils successfully accomplished these different tasks. A collective correction phase allowed the pupils to revisit some of the difficulties they encountered concerning the concept of angles. Throughout this session, the pupils spontaneously made the connection with the playground situation. For example, while doing the angle-copying exercise, there were students who asked if they were to draw an angle with sides of the same length to which others replied that side length did not matter, evoking the ropes in the experiment on the playground.

4.3. Paper-and-pencil sequence

4.3.1. Session 1: elaborating the problem and experimentation on paper

This sequence is introduced as a geometry lesson, with the teacher administering an individual paper-and-pencil exercise. The pupils' task is to determine the points that can be connected to a particular point, O, while crossing Segment [AB] and then identify and colour the area containing all of the points with this property (Figure 6). Point O is analogous to the observer in the physical situation, and Segment [AB] is analogous to the screen. O is on the right bisector of Segment [AB], similar to the observer in the first sequence, who was located on the right bisector of the screen.

To explain this task, which is complex for pupils of this age, the teacher wrote examples on the board; that is, a point labelled O and a segment labelled [AB] were drawn on the blackboard. The teacher then added Point M, such that Segment [OM] crossed [AB], and asked a pupil to connect M and O using a ruler. The pupil noted that [OM] crossed [AB]. The same was done for another point called N, such that [ON] did *not* cross [AB] (Figure 6).

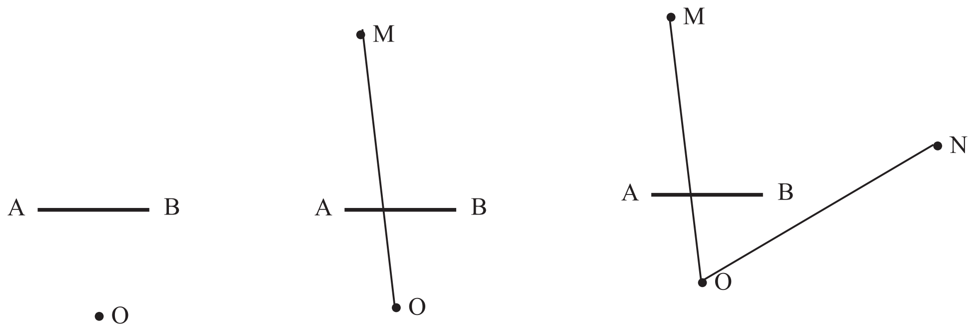


Figure 6. Examples drawn on the blackboard to explain the task.

The teacher then gave the pupils sheets of different sizes with O and [AB] and asked them to (individually) identify some points that could be connected to O while crossing Segment [AB] and then colour the area containing all such points. The pupils then carried out individually a paper-and-pencil experiment instead of experimenting on the playground. Having sheets of different sizes allowed the pupils to obtain coloured areas that differed in size but corresponded to the same portion of the plane and then to the same angle. To confront the pupils with variations in the area for which they were looking, the teacher asked them to delineate this area on the blackboard for three different locations of Point O on the right bisector of [AB] (their counterparts in the physical sequence were the observer's different positions).

We obtained a variety of responses (Figure 7), some of which were similar to those drawn in the visual-field situation. Somewhat surprisingly, the pupils were very motivated by this purely paper-and-pencil activity, which seems highly abstract. Less than one quarter of pupils found the correct area in this phase (Figure 7a). Some pupils coloured in areas that contained some but not all of the points that fit the instructions; i.e. they drew delineating half-lines that crossed Segment [AB] but did not go through its end points (Figure 7b). There were even some erroneous solutions, identical to those obtained in the physical sequence before experimentation, such as the strip solution (Figure 7c), a 'ball' (Figure 7e), and a half-plane (Figure 7f). We also obtained some intermediate proposals between the correct response and the strip (Figure 7d). As in the physical sequence, some pupils started with points near Segment [AB] before testing more distant points. Most pupils covered the area by placing points in many directions. A few pupils settled for drawing points only (Figure 7g), but most drew segments and connected their points to O in such a way that the majority of the drawings had lines 'fanning out' from O. However, this correct procedure seldom ended in success: Some only drew segments without colouring in the area (Figure 7h), and some did not go all the way out to the area's edges (Figure 7b). Other pupils only tested points situated on the strip 'perpendicular' to Segment [AB] (Figure 7c). Others still limited the area that they coloured to the point-by-point space actually explored (Figures 7e and i). Every pupil extensively explored the area, point by point, without looking outright for the area's boundaries. Notably, not a single pupil placed the ruler directly at Point O and at the end points of Segment [AB]. Finally, one pupil devised an interesting procedure: After having placed one point belonging to the desired area and drawing the segment linking this point to O, she extended this segment and drew another point along

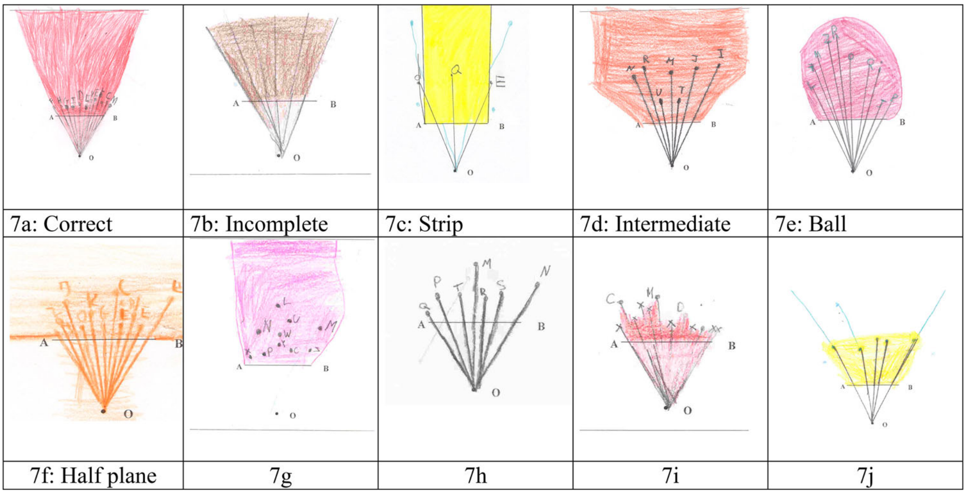


Figure 7. Examples of pupils' drawings.

the same line (Figure 7j). This pupil seems to have mobilized the idea of direction: If a given point is good, then all points in the same direction are also good.

The different productions were displayed on the board and the teacher organized a debate to engage the pupils in a critical analysis of these productions. For example, they searched for counterexamples (Point P, Figure 7c). This experimentation allowed them to exclude erroneous proposals and validate correct answers.

Looking at the correct drawings, the pupils could see that the coloured areas were not identical, either because they did not colour all the way out to the edges of their sheets or because the sheets were of different sizes. The discussion, regulated by the teacher, led to the idea that the length of the sides was not important (one can extend them out to 'infinity'). The pupils collectively concluded that the desired area is delineated by two half-lines originating at O and going through A and B, and that the lines could be lengthened at will. Next, the teacher had three pupils repeat the exercise on the blackboard for three different locations of O on the right bisector of [AB]. They discovered that the closer O was to [AB], the farther apart the half-lines were, and vice versa. For the time being, the pupils were only interested, here again, in the two boundaries of the area and not yet in the angle those boundary lines formed.

The teacher then proposed an exercise analogous to that of the physical sequence (see Figure 3) with the following instructions: 'On each drawing, colour the area containing all points that can be connected to O while crossing [AB].' In the same manner, the frames were different sizes for the same location of Point O, and the locations varied across frames. All of the pupils found the correct answer to this exercise.

4.3.2. Session 2: emergence of the concept of angles and identifying techniques for comparing and reproducing angles

This session is very similar to the second session of the physical sequence. The teacher displayed enlargements of the drawings made at the end of the previous session and then asked the pupils whether the area found was the same everywhere or whether it was larger in some

Table 3. Excerpts from discussions in the two half classes.

Explanation of a pupil of the 1st half class	Part of the whole class discussion in the 2nd half class
We cut out the two diagrams and put them back to back. We saw that one area was bigger than the other one [points to the colored part that went farther out]. But we said to ourselves, 'If we continue on to infinity it's the same size.' So we said, 'It's the same.'	T: [the fact that the cut-out areas fit right on top of each other] … What does this prove? P: That the answer is no [the areas are not the same] … Uh, it's yes. T: Are the areas the same? P: Yes. T: But this goes out farther than that [the teacher shows the hatched area]. Is it a problem that it goes farther out? P: No because if you continue this, it'll be the same. T: If you continue what? P: This area here, if I continue it [makes a gesture that extends the half lines], this will be the same size as that [she points to the biggest colored area]. T: Are you sure? P: Yes.

drawings. The teacher initially asked them to compare the first two drawings, in which O was in the same position with respect to [AB]. As in Session 2 of the physical sequence, some pupils focused on the coloured area, and this led to a socio-cognitive conflict.

T: Are the two red areas the same?

P1: No, because there, the frame is smaller.

T: Is there a smaller area or a bigger area?

P1: There is a smaller one [pointing to the smallest colored area].

P2: No, because if you continue the lines, it will be the same.

The teacher then asked them to identify techniques to check whether the areas were the same for a given location of O. The pupils relied on the same techniques as in the physical sequence, for example, cutting the areas out and superimposing the cutouts. Table 3 presents excerpts from the whole-class discussion concerning this last technique.

The collective debate also ended in the conclusion that, whenever the edges were on top of each other, the areas were the same, even if the coloured parts were different. For the other locations of Point O, the pupils used the superimposition technique directly and were immediately convinced that the areas were different.

4.3.3. Session 3: mathematics session on angles and application exercises

This third session is, as in the physical sequence, a decontextualized session, and the activities proposed to the pupils are identical. In both half-classes, the strategies and difficulties that occurred were very similar to those of the physical sequence. The definitions of 'angle' that were produced were also very similar, and the teachers rephrased them as previously mentioned (an angle is the degree of openness between two half lines (the sides) that extend from a common point: the vertex). As in the physical sequence, pupils frequently referred to the initial situation they experienced (here, the initial paper-and-pencil exercise). They also discussed the fact that two half-lines delineated two angles: the closed angle corresponding to the area containing all points that can be connected to O while crossing [AB] and the open angle corresponding to the area containing all points that can be connected to O without crossing [AB].

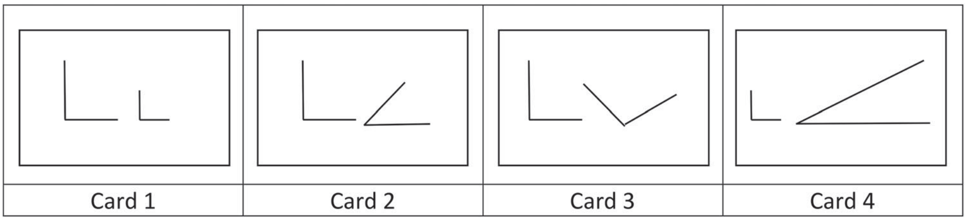


Figure 8. Pre- and posttest material.

Table 4. Number of correct answers in the initial test (IT) and the final test (FT).

	Card 1		Card 2		Card 3		Card 4	
	IT	FT	IT	FT	IT	FT	IT	FT
Physical	1	14	13	18	7	14	3	12
Paper-pencil	0	15	15	19	4	14	6	14

4.4. Evaluation results

Using various evaluations, we aimed at determining what the pupils had learned about the concept of angles and comparing the two sequences' respective impacts.

4.5. Pre- and posttest results

Recall that our sample consisted of 48 pupils from two classes and that each class was split into two equivalent groups (physical/paper-and-pencil) on the basis of the pretest. The corpus analyzed here pertains to 40 pupils who were present at all of the tests. To determine whether the pupils understood that the size of an angle does not depend on its sides, we had them compare four selected pairs of angles (Figure 8): two right angles with sides of different lengths (Card 1), two pairs of different-sized angles with sides of the same length (Cards 2 and 3), and two different-sized angles with sides of different lengths, the larger having shorter sides (Card 4).

The results are presented in Table 4.

Before the teaching sequences, the two groups were equivalent (by construction) on the comparison tasks. After the sequences, we observed that, for all cards, progress was significant in both groups, but there was no significant difference between the physical group and the paper-and-pencil group (For each card the Fisher exact test provides $p > .05$).

4.6. Results of the written test taken in class after the sequences

In addition to the pre- and posttests, the pupils took a written test in class that consisted of math exercises (comparisons of pairs of angles) and one exercise related to the situation experienced. Note that we have chosen to use paper-and-pencil exercises for these evaluations, but it would be interesting, in future research, to assess both groups in a physical environment and compare the groups' ability to solve new problems in physical and paper-and-pencil contexts. The corpus analyzed here pertains to the 48 pupils present for the final test (24 for each sequence).

Table 5. Percentage of correct answers.

Types of angles to compare	Physical	Paper-and-pencil
Equal angles with different-length sides (2 pairs)	73%	73%
Unequal angles with same-length sides (1 pair)	88%	83%
Unequal angles with different-length sides, the larger angle having the shorter side (2 pairs)	98%	94%
Reflex and salient angles with a sum of 360° (1 pair)	67%	58%

Table 6. Reproducing and drawing an angle (smaller, larger).

	Drawing	Reproducing	Smaller	Bigger	At least one successful size variation
Physical (24)	100% (24/24)	83% (20/24)	88% (21/24)	67% (16/24)	92% (22/24)
Paper-pencil (24)	96% (23/24)	96% (23/24)	83% (20/24)	88% (21/24)	92% (22/24)

4.6.1. Math exercises

For the comparison exercises, there were six pairs of angles, which are described in Table 5.

The results (Table 5) are comparable to those obtained from the pre- and posttests. Most pupils overcame the obstacle linked to side length. Regarding the reflex and salient item, the percentage of correct answers, although smaller, was very high for pupils of this age. Comparing the two sequences, we again found no significant differences.

The pupils then had to draw an angle, reproduce it using the technique of their choice (tracing paper, template, or bevel square), and draw one angle larger and one smaller than the first one. The results are presented in Table 6.

Overall, these exercises were completed very well. Once again, no significant difference was observed between the physical and paper-and-pencil groups (For each exercise the Fisher exact test provides $p > .05$).

4.6.2. Situation-specific exercise

In this exercise, the pupils had to colour in the area corresponding to the situation that they had experienced (physical/paper-and-pencil). This task was correctly performed by 19 out of 24 pupils in the physical sequence and by 21 out of 24 in the paper-and-pencil sequence. When asked, ‘What does this area become when the observer (or Point O) becomes closer to, or farther away from, the screen (or from segment [AB])?’ 17 out of 24 pupils for the physical sequence and 20 out of 24 for the paper-and-pencil sequence answered both questions correctly. Thus, after the teaching sequences, the vast majority of the pupils were able to master the functional relationship between the observer’s (or Point O’s) position on the right bisector of the screen (or Segment [AB]) and the angle delineating the area.

5. Discussion

The aim of this paper was to better understand the reasons for the physical sequence’s effectiveness. Was it due to the use of a physical situation in which pupils worked in mesospace and then modelled the situation in microspace and/or due to the task proposed (i.e. the comparison of regions of space delineated by an angle between two infinite directions)? To answer this question, we compared two teaching sequences: one physical, replicated from the 2013 study, and the other paper-and-pencil, confronting pupils with the 'same' task albeit limited to the microspace of a sheet of paper.

Let us first note that the results obtained in the present study confirm the effectiveness of the physical sequence and reinforce our previous results. In the two half-classes in which the physical sequence was implemented, pupils' hypotheses, reactions, formulations, the techniques they proposed to compare the areas, and the debates observed, involving different pupils and teachers, were very similar to those in our 2013 experiment. This can be explained by the fact that our research team precisely framed the preparation and teacher training in advance.

Present results showed that the paper-and-pencil sequence is also effective. Thus, both sequences enabled the pupils to overcome the common erroneous conception and grasp the concept of angles in three approximately one-hour sessions, which compares favourably to the time generally devoted to this concept. Moreover, comparison between the two sequences showed that they were *equally* effective. Indeed, comparison of posttest performances after the teaching sequences showed no significant differences between the physical and paper-and-pencil groups for the tasks proposed. This finding suggests that the effectiveness of the physical sequence was not linked solely to the work done in the mesospace provided by the school playground and its modelling; it also lies in the tasks proposed to the pupils, as both involve problem solving activities in which pupils had to compare areas of space delineated by an angle between two infinite directions. Indeed, fine-grained analysis of the unfolding of the two sequences revealed many similarities in pupils' activities.

In the first session, the areas the pupils drew initially were similar across the two sequences (Figures 2 and 7). This is quite surprising because in one case, they were hypotheses, while in the other, they were the results of paper-and-pencil experimentation. One can assume that the salience of right angles and the pervasiveness of the directions horizontal and vertical are the sources of the many strip-like productions in both classes. Similarities also concern the procedures used. In the physical situation, the pupils started by standing close to the screen, near its bisector, before moving farther away in response to the teacher's prompts. In the paper-and-pencil sequence, most pupils also began with points close to the segment before moving from the right bisector of [AB] and approaching the boundaries of the area. Hence, in the first session, we observed very similar behaviours across situations, although this session was supposed to be different in the two sequences.

Concerning the second and third sessions, which were designed so that the activities would be as alike as possible, they unfolded similarly in the two sequences, as expected. In both cases, the pupils had to delineate an area of space and consider infinite directions.[6] They were encouraged to compare different-sized coloured areas that modelled the same portions of the plane. These areas were delineated by infinite half-lines originating at the same point: the observer location in the physical sequence or Point O in the paper-and-pencil sequence. This process generated a similar socio-cognitive conflict in both sequences, which seems to be a key moment, as the pupils discussed the infinite nature of the area's boundaries extensively. Regardless of the length of the sides (and thus the size of the coloured areas), the modelled area goes on infinitely; hence, two drawn areas are identical if the boundaries can be superimposed. This idea of infinity was notably discussed during the technique-searching step, allowing the pupils to realize the irrelevance of the length of the sides. In both sequences, the comparison activities were meaningful because they are necessary to solve the socio-cognitive conflict. In both cases, the situations brought to bear salient angles (boundaries of the hidden area or of the area containing all points so that one could connect to O while crossing [AB]) and reflex angles (boundaries of the visible area or of the area containing all points so that one could connect to O while not crossing [AB]).

These similarities in the pupils' activities could partly explain the two sequences' effectiveness. First, both involve similar discussions about the possibility of extending the lines, which seems to be critical to invalidate the role of the sides' lengths. In both cases, pupils created an empirical referent that enabled them to discard the idea that the lengths of an angle's sides affect its size, as indicated by their frequent references to the situations they experienced over the two previous sessions (playground experiment/initial paper-and-pencil exercise). Second, in both sequences, working on delineating areas with certain physical or geometrical properties associated with the exchanges triggered by the bevel square probably helped the pupils become aware that a pair of half-lines delineates two angles. These angles, reflex and salient, correspond to the limits of the areas that can be seen (or not) or to the area containing all the points that can be connected to O while (or without) crossing [AB]. This could explain the progress observed in the posttest concerning reflex and salient angles in both sequences.

However, beyond the similarities discussed above, the two sequences also present differences that could influence their respective effectiveness. The only difference between the two sequences was supposed to be the work done in the mesospace and its modelling, which were present in the physical sequence but not in the paper-and-pencil one. This difference was inherent to the device and particularly concerned the space involved. In the physical sequence, the pupils experimented on the playground, to test the various paper-and-pencil hypotheses formulated, and in the classroom, when they had to compare the hidden areas. They therefore worked both in mesospace (playground) and microspace (sheet of paper), going back and forth between these two spaces, whereas in the paper-and-pencil sequence, the pupils worked only in microspace. One can assume that making connections between the two spaces in the physical sequence contributed more to leading the pupils to the idea that the sides could be extended 'to infinity.'[7] However, the results did not show any difference between the two sequences.

The fine-grained analysis of the unfolding of the two sequences showed two other differences in pupils' activities, which were not expected and could explain this equal efficiency. These differences concerned mainly the first session. The first one concerns the way in which the pupils explore the area. In the physical sequence, the pupils immediately focused on the hidden area's boundaries. This behaviour was induced partially by the initial road-safety problem. Indeed, the aim of the playground experiment was actually to choose between various hypotheses about the area's boundaries. In the paper-and-pencil sequence, the pupils explored the entire area, point by point, with multiple trials, without searching outright for the boundaries of the area, since they experimented without any predefined hypotheses. Although some pupils had trouble moving from a point-by-point analysis to a conceptualization of the whole area, as evidenced by the many erroneous drawings produced, we consider that this deeper exploration of the whole area can constitute an advantage of the paper-and-pencil sequence. The second difference concerned the number of trials each pupil undertook. In the paper-and-pencil sequence, exploration was individual, with each pupil making many attempts (mean number of attempts = 11). In contrast, in the physical situation, although the playground activity was organized so that every pupil assumed the observer's position at least once and was behind the screen or on the hidden area's boundaries several times, pupils' area exploration was not equitable. The amount of individual experimentation was thus greater in the paper-and-pencil sequence. This latter sequence thus required as much time as the physical one, despite the absence of playground experimentation.

One can assume that lengthy exploration of the whole area (point by point, with many attempts) enabled the pupils to grasp the problem better than in the physical sequence.

Finally, each sequence presents respective heuristic features. The physical sequence offered the advantage of allowing pupils to work in a vast space, conducive to the idea of infinite direction, while the paper-and-pencil sequence led to deeper exploration of the area because the pupils looked individually at one point after another. These features may explain the results obtained regarding the sequences' equal effectiveness.

In conclusion, our results showed that the effectiveness of the two sequences is mainly linked to the tasks proposed to the pupils (comparing areas of space delineated by an angle between two directions) and that their *equal* effectiveness can be explained by some specific features of each sequence (the work in the mesospace for the physical sequence, the lengthy exploration of the whole area for the paper-and-pencil one). More generally, these results suggest that problem-based activities that consider the main misconception (via the introduction of the idea of infinite direction) and include the use of investigative tools (here, the bevel square especially), whether physical or paper-and pencil based, are heuristic to grasp the concept of angles. Future research should conduct a deeper investigation of these different dimensions' respective role.

5.1. Implications for teaching

A question that arises here is whether these results question the utility of approaching angles using a physical situation in mesospace. As we have seen, such an approach, which goes from mesospace to microspace via modelling activities, is efficient, as many researchers have pointed out, including Berthelot and Salin (1998) in France, Fyhn (2006, 2008), and more recently, Bustang et al. (2013). Like these authors, we showed that such activities help students to render geometric knowledge meaningful and that they promote the acquisition of spatial knowledge. Indeed, the situation-specific exercise showed that a large number of pupils acquired spatial and geometric knowledge that allowed them to master the situation they had experienced. However, this approach is deemed difficult to implement and is seldom used in classrooms, as Berthelot and Salin (1998) have shown in France for example. Indeed, classroom constraints (number of pupils in the class, available time, classroom management) are such that some teachers limit, or even completely avoid, activities outside the classroom. Thus, some teachers, particularly beginners, do not dare to implement this type of situation, as it can seem more difficult to manage than traditional ones. Depending on teaching constraints, the paper-and-pencil situation might be more acceptable to some teachers, since it has pupils working in the smaller space of the classroom. This paper-and-pencil sequence, which has been elaborated as a transposition of the physical sequence, proved to be fully effective for learning the concept of angles, and if the aim is solely to meet the objectives of mathematics teaching, it offers an original and effective approach to this concept.

Nevertheless, the physical situation, in addition to permitting the discovery of the notion of angles, simultaneously helped the pupils to grasp the concepts of 'line of sight' and 'visual field,' which are fundamental in physics for understanding, for example, astronomical phenomena (Merle, 2002). Moreover, in this physical sequence, the pupils carried out real-world modelling activities, enhancing their modelling skills. Such skills are essential in physics as well as in mathematics, so it is critical to work on them as early as elementary school. Unfortunately, modelling is rarely undertaken in classrooms, although many studies have shown that pupils of this age are capable of building and manipulating models of the real world (English, 2017; Merle, 2002; Munier & Merle, 2009). Given that the two situations are equally effective at helping pupils to grasp the concept of angles, the choice between them should depend upon the teacher's objectives and the current teaching constraints.

A teaching sequence coupling the two sequences presented here could also be envisaged, starting with the paper-and-pencil sequence to introduce the angle concept, and then modelling a physical situation of the meso- or macrospace, involving the notion of angles (visual-field situation or another physical situation, as proposed, for example, by Fyhn [2008]; and Munier & Merle [2009]). In line with White and Mitchelmore's view, pupils would then be 'directed to new situations where they can use the concept' (2010). This approach would allow for an easy-to-implement introduction of angles, followed by specific modelling work that allows pupils to better construct this concept, build knowledge in physics, and develop modelling skills, an essential issue for the teaching of mathematics and physics.

Notes

1. In fact, the projection on the ground (two-dimensional) of this hidden three-dimensional area.
2. Note that in this paper, we use the word 'area' to refer to a region in space and not to the measure of its size.
3. Of course, the teacher did not use the mathematical expression 'right bisector.'
4. These photos were taken as part of our 2013 research. Already published in the *Journal of Mathematical Behaviour,* these photos are reused here because we believe that this overhead view is useful for understanding the experiment.
5. During the various experiments, the pupil acting as the observer changed so that every pupil had the opportunity to be the observer at least once.
6. Even if the areas that they had to compare did not correspond to the same reality.
7. The pupils spontaneously employed this term, as in 'We could lengthen the ropes as much as we wanted until infinity.'

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Disclosure statement

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