EVERYTHING YOU'VE ALWAYS WANTED TO KNOW ABOUT TIME-HARMONIC FIELDS, BUT WERE AFRAID TO ASK

(A Solution Manual to TIME-HARMONIC ELECTROMAGNETIC FIELDS, by R. F. Harrington, McGraw-Hill Book Co.)

by

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PREFACE

Most of the problems in "Time Harmonic Electromagnetic Fields" by R. F. Harrington are solved in this manual. The problems not done are 5-43, 5-44, 7-28, 8-11, 8-12, and 8-23. A few problems are referenced to other books or journal articles where the authors feel they are done more extensively than they would have been if presented here. During the course of this work various mistakes in the problems were found and corrections should appear in these solutions.

Finally the authors would like to express their grateful acknowledgement to Dr. Joseph Mautz for his invaluable help when consulted and to Dr. Roger Harrington for encouraging the project.

1-1 STOKES' THM:

$$\oint_C \bar{A}\cdot d\bar{l} = \iint_S \nabla\times\bar{A}\cdot d\bar{S}$$

S is any surface bounded by C where C is a closed contour.

DIVERGENCE THM:

$$\oiint_S \bar{A}\cdot d\bar{S} = \iiint_V \nabla\cdot\bar{A}\, dV$$

S is a <u>closed</u> surface containing V

a.) $\iint_S \left(\nabla\times\bar{\mathcal{E}} = -\frac{\partial\bar{B}}{\partial t}\right)\cdot dS$

$$\oint \bar{E}\cdot dl = -\iint_S \frac{\partial\bar{B}}{\partial t}\cdot d\bar{S}$$

b.) $\iint_S \left(\nabla\times\bar{H} = \frac{\partial\bar{D}}{\partial t} + \bar{J}\right)\cdot d\bar{S}$

$$\oint \bar{H}\cdot d\bar{l} = \iint_S \frac{\partial\bar{D}}{\partial t}\cdot d\bar{S} + \iint_S \bar{J}\cdot d\bar{S}$$

c.) $\iiint_V (\nabla\cdot\bar{B} = 0)\, dV$

$$\oiint_S \bar{B}\cdot d\bar{S} = 0$$

d.) $\iiint_V (\nabla\cdot\bar{D} = q_v)\, dV$

$$\oiint_S \bar{D}\cdot d\bar{S} = \iiint q_v\, dV$$

1-2 LORENTZ FORCE:

$$\bar{F} = q\bar{\mathcal{E}} + q\bar{v}\times\bar{B}$$

FROM DEFN. OF HALL COEFFICIENT:

$$\bar{v} = h\sigma\bar{\mathcal{E}} = h\bar{J}$$

$$\bar{F} = q\bar{\mathcal{E}} + qh\sigma\bar{\mathcal{E}}\times\bar{B}$$

$$\bar{\mathcal{E}}_{TOT} = \bar{F}/q$$

TOTAL CONDUCTION CURRENT $\mathcal{J}$:

$\sigma \sim \frac{coul^2}{nt \times sec\, m^2}$ $\quad \mathcal{J} = \sigma\bar{\mathcal{E}}_{TOT}$

$\therefore$

$h \sim \frac{m^3}{coul}$ $\quad \mathcal{J} = \sigma\bar{\mathcal{E}} + \sigma^2 h\bar{\mathcal{E}}\times\bar{B}$

$\mathcal{E} \sim \frac{nt}{coul}$

1-2 (cont.)

$J \sim \frac{coul}{sec\, m^2}$ For Cu, $h = -5.5\times10^{-11}$

$\sigma = 5.8\times10^{7}$

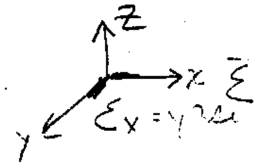
$\mathcal{J} - \frac{coul}{sec\, m^2}$ $\quad \frac{\sigma^2 h\bar{\mathcal{E}}\times\bar{B}}{\sigma\bar{\mathcal{E}}} = .01$

If $\mathcal{E}$ and B are at right angles,

$$B \approx \frac{.01}{\sigma h} = 3.13\ \frac{web}{m^2}$$

1-3 $\bar{\mathcal{E}} = \bar{u}_x\, y^2 \sin\omega t$

$\bar{H} = \bar{u}_y\, x\cos\omega t$



$\bar{J}^t = \nabla\times\bar{H} = \bar{u}_z\cos\omega t.$

$\bar{M}^t = -\nabla\times\bar{\mathcal{E}} = 2y\sin\omega t\ \bar{u}_z$

$$i^t = \int_0^1\int_0^{\sqrt{1-x^2}} \bar{u}_z\cos\omega t\, dy\, dx\cdot\bar{u}_z$$

$$= \cos\omega t\int_0^1 \sqrt{1-x^2}\, dx$$

$$= \frac{\pi}{4}\cos\omega t \qquad i = \iint \bar{J}\cdot dS$$

$$k^t = \int_0^1\int_0^{\sqrt{1-x^2}} 2y^2\sin\omega t\, dy\, dx$$

$$= \sin\omega t\int_0^1 (1-x^2)\, dx$$

$$= \frac{2}{3}\sin\omega t$$

1-4 $\bar{S}$ = Poynting vector = $\bar{E}\times\bar{H}$

$$\bar{S} = \begin{vmatrix} \bar{u}_x & \bar{u}_y & \bar{u}_z \\ y^2\sin\omega t & 0 & 0 \\ 0 & x\cos\omega t & 0 \end{vmatrix} = \bar{u}_z\, xy^2\sin\omega t\cos\omega t$$

Eq. 1-26,

$$\nabla\cdot(\bar{E}\times\bar{H}) + \bar{E}\cdot\bar{J}^t + \bar{H}\cdot\bar{M}^t = 0$$

$$0 + \cancelto{0}{(\bar{u}_x\cdot\bar{u}_z)}(\ ) + \cancelto{0}{(\bar{u}_y\cdot\bar{u}_z)}(\ ) = 0$$

Q.E.D.

1-5 $\nabla \cdot \bar{J} = -\dfrac{\partial \rho}{\partial t}$ (continuity eqn.)

$$\int_V \nabla \cdot \bar{J}\, dV = \int_S \bar{J} \cdot dA = i = \frac{dQ}{dt}$$

$$C = Q/V \text{ or } Q = CV$$

$$\therefore\ i = \frac{d(CV)}{dt} = C\frac{dV}{dt}$$

$$\oint \bar{E} \cdot d\bar{l} = -\frac{\partial \psi}{\partial t} = -v \text{ (Faraday)}$$

$$L = \psi / i$$

$$v = \frac{d}{dt}(Li) = L\frac{di}{dt}$$

1-6 a.) $I = 10 + j5$ (phasor)

$$i = \sqrt{2}\, \mathrm{Re}(I e^{j\omega t})$$

$$= \sqrt{2}\, \mathrm{Re}\left[(10 + j5)(\cos \omega t + j \sin \omega t)\right]$$

$$= \sqrt{2}\left[10 \cos \omega t - 5 \sin \omega t\right]$$

b.) $\mathcal{E} = \sqrt{2}\, \mathrm{Re}(E e^{j\omega t})$

$$= \sqrt{2}\, \mathrm{Re}\left[\bar{u}_x (5 + j3) e^{j\omega t} + \bar{u}_y (2 + j3) e^{j\omega t}\right]$$

$$= \sqrt{2}\left[\bar{u}_x (5 \cos \omega t - 3 \sin \omega t) + \bar{u}_y (2 \cos \omega t - 3 \sin \omega t)\right]$$

c.) $\mathcal{H} = \sqrt{2}\, \mathrm{Re}(H e^{j\omega t})$

$$= \sqrt{2}\left\{(\bar{u}_x + \bar{u}_y)\left[\cos(x+y) \cos \omega t - \sin(x+y) \sin \omega t\right]\right\}$$

1-7 $A = a + jb$

$B = c + jd$

$$\mathrm{Re}(A) + \mathrm{Re}(B) = a + c = \mathrm{Re}(A + B)$$

$$\mathrm{Re}(\alpha A) = \alpha a = \alpha\, \mathrm{Re}(A)$$

1-7 (cont.)

$$\frac{\partial}{\partial x} \mathrm{Re}(A) = \frac{\partial a}{\partial x} = \mathrm{Re}\left(\frac{\partial A}{\partial x}\right)$$

$$\int \mathrm{Re}(A)\, dx = \int a\, dx = \mathrm{Re}\left[\int A\, dx\right]$$

1-8 $\bar{H} = \bar{u}_x \sin y$

$$\bar{\mathcal{H}} = \sqrt{2}\, \bar{u}_x \sin y \cos \omega t$$

$$\bar{\mathcal{E}} = \frac{1}{\varepsilon_0 \varepsilon_r} \int_0^t \nabla \times \bar{\mathcal{H}}\, dt$$

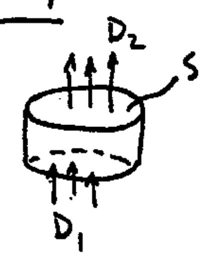
$$= \int_0^t \frac{-\sqrt{2} \cos y \cos \omega t}{\varepsilon_0 \varepsilon_r}\, dt$$

$$= \frac{\sqrt{2} \cos y}{\varepsilon_0 \varepsilon_r \omega} \sin \omega t$$

$$\bar{\mathcal{E}} = \sqrt{2}\, \mathrm{Re}\left[\bar{E} e^{j\omega t}\right]$$

$$\therefore\ E = \frac{-j \cos y}{\varepsilon_0 \varepsilon_r \omega}$$

1-9



$$\iiint_V (\nabla \cdot \bar{D} = \rho_v)\, dV$$

$$\oiint_S \bar{D} \cdot d\bar{S} = Q_v$$

$\oiint_S \bar{D} \cdot d\bar{s} = (\bar{D}_2 - \bar{D}_1) \cdot \bar{A}$ where A is the area of each face. Since the region is source free, $D_1 = D_2$ and thus $\oiint_S \bar{D} \cdot d\bar{s} = 0 = Q_v$.

1-10 $\bar{\mathcal{E}} = \sqrt{2}\, \mathrm{Re}\left[\bar{E} e^{j\omega t}\right]$

$$\bar{\mathcal{H}} = \sqrt{2}\, \mathrm{Re}\left[\bar{H} e^{j\omega t}\right]$$

$$\bar{\mathcal{J}} = \bar{\mathcal{E}} \times \bar{\mathcal{H}}$$

$$= 2\, \mathrm{Re}\left[\bar{E} e^{j\omega t}\right] \mathrm{Re}\left[\bar{H} e^{j\omega t}\right]$$

1-10 (cont.)

Let $\bar{E} = \bar{E}_1 + j\bar{E}_2$

$\bar{H} = \bar{H}_1 + j\bar{H}_2$

$$Re\left[\bar{E}e^{j\omega t}\right] Re\left[\bar{H}e^{j\omega t}\right] =$$

$$(\bar{E}_1 \cos\omega t - \bar{E}_2 \sin\omega t)(\bar{H}_1 \cos\omega t - \bar{H}_2 \sin\omega t)$$

$$= E_1H_1 \cos^2\omega t - H_1E_2 \sin\omega t \cos\omega t$$

$$- E_1H_2 \sin\omega t \cos\omega t - E_2H_2 \sin^2\omega t$$

$$\bar{S} = \bar{E}\times\bar{H}^* = (E_1H_1 + E_2H_2) + j(E_2H_1 - E_1H_2)$$

$$\bar{\mathcal{S}} = \left[E_1H_1(1+\cos 2\omega t) - \sin 2\omega t (E_1H_2 + E_2H_1) + E_2H_2(1-\cos 2\omega t)\right]$$

$$Re\left(S + \bar{E}\times\bar{H}\, e^{2j\omega t}\right) =$$

$$E_1H_1 + E_2H_2 + Re\left(\bar{E}\times\bar{H}\, e^{j2\omega t}\right)$$

$$\therefore\ = E_1H_1 + E_2H_2 + E_1H_1\cos 2\omega t$$

$$- E_2H_2 \cos 2\omega t - (E_1H_2 + E_2H_1)\sin 2\omega t$$

$$\therefore \bar{\mathcal{S}} = Re\left[\bar{S} + \bar{E}\times\bar{H}\, e^{j2\omega t}\right]$$

$\bar{\mathcal{S}}$ is not related to $\bar{S}$ by Eq 1-41 because $\mathcal{S}$ is not sinusoidal. ($\sin^2\omega t$ or $\cos^2\omega t$)

1-11

$$P_{in} = \int_0^1\int_0^1 (100\sin\pi y)(e^{-j\pi/6}\sin\pi y)\,dy\,dz$$

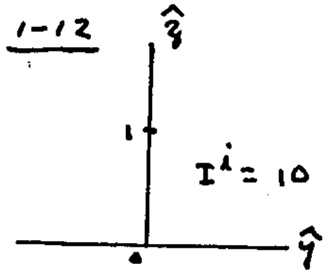
$$= 50e^{-j\pi/6}$$

$$= 50\left(\frac{\sqrt{3}}{2} - j\frac{1}{2}\right)$$

$$P_{diss} = 25\sqrt{3}$$

$$2\omega(W_m - W_e) = -25$$

1-12



$$\bar{E} = \bar{u}_z(1+j)$$

Complex power supplied by

$$\text{Source} = E\cdot I^i = 10(1+j)$$

Time average power $= 5(1+j)$

1-13

Conduction current $= \sigma\bar{E} = 5\times10^{-4}\,\bar{u}_x$

Free space displacement

Electric	Magnetic
$j\omega\varepsilon_0\bar{E}$	$j\omega\mu_0\bar{H}$
$= j\dfrac{5\times10^{-2}}{36\pi}\bar{u}_x$	$= j8\pi\,\bar{u}_y$

Polarization

$j\omega(\hat{\varepsilon}-\varepsilon_0)\bar{E}$	$j\omega(\hat{\mu}-\mu_0)\bar{H} =$
$= (5\times10^5 + j\,4.5\times10^8)\,\varepsilon_0\bar{u}_x$	$(2\times10^7 + j2.6\times10^8)\mu_0\,\bar{u}_y$

Displacement

$j\omega\hat{\varepsilon}\bar{E}$	$j\omega\hat{\mu}\bar{H}$

Dissipative —

$(\sigma + \omega\varepsilon'')\bar{E}$	$\omega\mu''\bar{H}$

Reactive —

$j\omega\varepsilon'\bar{E}$	$j\omega\mu'\bar{H}$

Induced —

$(\sigma + j\omega\hat{\varepsilon})\bar{E}$	$j\omega\hat{\mu}\bar{H}$

1-14 $\quad C = 300\,pf$

$$C = \frac{\varepsilon' A}{d}.\ \text{for air, } C = \frac{A}{d} = 300\times10^{-12}$$

$$Y = G + j\omega C = \frac{1}{(500-j)\times10^3}$$

$$G = \frac{1}{500\times10^3} = \frac{\omega\varepsilon'' A}{d} = 300\times10^{-6}\varepsilon''$$

1-14 (cont.)

$$\varepsilon'' = \frac{1}{5\times10^{11}\times300\times10^{-12}} = \frac{1}{150}$$

$$\varepsilon' = \frac{1}{(250)(300)}$$

$$\hat{\eta} = \omega\varepsilon'' + j\omega\varepsilon'$$

1-15

Power loss in wire $\sim \sigma E\cdot E^*$

Power loss in core $\sim \omega\mu'' H\cdot H^*$

If, for example, copper is used;

$\sigma = 5.8\times10^7$

From Fig. 1.12, it is clear that

$\sigma \gg \omega\mu''$ for all frequencies shown.

$\therefore$ Power loss in wire $\gg$

Power loss in core.

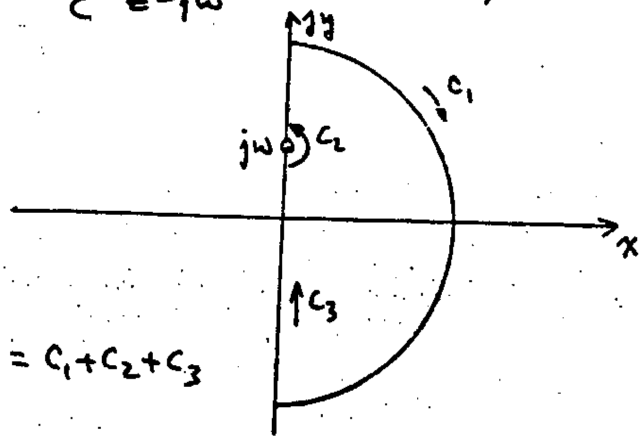
1-16

Assume $\hat{\varepsilon} = \varepsilon' - j\varepsilon''$ is an analytic function in ω and

$\therefore$ $\varepsilon', \varepsilon''$ satisfy Cauchy-Riemann equations.

Cauchy integral then is:

$$\oint_C \frac{\hat{\varepsilon}(\omega)}{z-j\omega}\,dz = 0 \quad , z = x+jy$$



$C = C_1 + C_2 + C_3$

1-16 (cont.)

The contribution from pole at $z = j\omega = 2\pi i\left[\frac{1}{2}\text{ residue at that p}\right.$

This is just $\hat{\varepsilon}(j\omega)$

To compute $\int_{C_1}$ let $z = R_0 e^{i\theta}$

$$\int_{C_1} \frac{\hat{\varepsilon}(\omega)}{z-j\omega}\,dz = \int_{C_1} \frac{\hat{\varepsilon}(\omega) R_0 e^{i\theta}}{R_0 e^{i\theta} - j\omega}\, jR_0 e^{i\theta}_0$$

$$\xrightarrow[R_0\to\infty]{} j\hat{\varepsilon}(\infty)\int_{\pi/2}^{-\pi/2} d\theta = j\pi\hat{\varepsilon}(\infty)$$

Since $\operatorname{Im}(\hat{\varepsilon}(\omega)) = 0$ at ∞,

$$\hat{\varepsilon}(\infty) = \operatorname{Re}\{\hat{\varepsilon}(\infty)\} = \varepsilon_0$$

For the rest of the contours,

$$\lim_{\substack{R_0\to\infty\\ r\to0}}\left[\int_{-R_0}^{\omega-r} \frac{\hat{\varepsilon}(jy)}{y-\omega}\,dy + \int_{\omega+r}^{R_0} \frac{\hat{\varepsilon}(jy)}{y-\omega}\,dy\right]$$

$$= \int_{-\infty}^{\infty} \frac{\hat{\varepsilon}(jy)}{y-\omega}\,dy = j\pi\left[\hat{\varepsilon}(\infty) - \hat{\varepsilon}(j\omega)\right]$$

Writing $\hat{\varepsilon}(j\omega)$ and $\hat{\varepsilon}(jy)$ in terms of real and imaginary parts,

$$\varepsilon'(\omega) = \varepsilon_0 + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\varepsilon''(y)}{y-\omega}\,dy$$

$$\varepsilon''(\omega) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\varepsilon'(y)}{y-\omega}\,dy$$

ε' = even function of ω

ε'' = odd function of ω

$$\varepsilon''(\omega) = -\frac{1}{\pi}\int_{-\infty}^{0} \frac{\varepsilon'(y)}{y-\omega}\,dy - \frac{1}{\pi}\int_{0}^{\infty} \frac{\varepsilon'(y)}{y-\omega}\,dy$$

$$\int_{-\infty}^{0} \frac{\varepsilon'(y)}{y-\omega}\,dy = \int_{\infty}^{0} \frac{\varepsilon'(-y)}{-(y+\omega)}\,(-dy)$$

$$= -\int_{0}^{\infty} \frac{\varepsilon'(y)}{y+\omega}\,dy$$

1-16 (cont.)

$$\varepsilon''(\omega) = \frac{-1}{\pi}\int_0^\infty \varepsilon'(y)\left[\frac{1}{y-\omega} - \frac{1}{y+\omega}\right] dy$$

$$= \frac{-2\omega}{\pi}\int_0^\infty \frac{\varepsilon'(y)}{y^2-\omega^2}\, dy$$

Similarly,

$$\varepsilon'(\omega) = \varepsilon_0 + \frac{2}{\pi}\int_0^\infty \frac{y\,\varepsilon''(y)}{y^2-\omega^2}\, dy$$

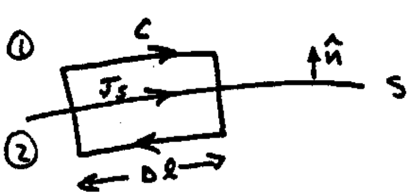
which is the first eqn. to be proven. $\varepsilon''(\omega)$ can be changed to the form given in the text by noting:

$$\omega\,\varepsilon_0 \int_0^\infty \frac{dy}{y^2-\omega^2} = 0$$

Thus,

$$\varepsilon''(\omega) = -\frac{2}{\pi}\int_0^\infty \frac{\omega\left[\varepsilon'(y) - \varepsilon_0\right]}{y^2-\omega^2}\, dy$$

1-17 Electric currents:

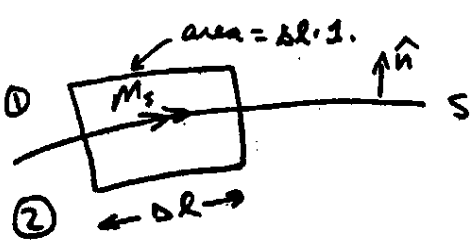


$$\oint_C \bar{H}\cdot\bar{dl} = (\bar{H}_1 - \bar{H}_2)\cdot\Delta l = \bar{J}_s\,\Delta l$$

$$\hat{n}\times\bar{A} = \bar{A}\cdot\bar{dl} \quad \text{so,}$$

$$n\times(\bar{H}_1 - \bar{H}_2) = \bar{J}_s$$

Magnetic currents:



1-17 (cont.)

$$\oint \bar{E}\cdot\bar{dl} = -\int \bar{M}\cdot\bar{dA}$$

$$(\bar{E}_1 - \bar{E}_2)\cdot\Delta l = -\bar{M}_s\cdot\Delta l$$

$$\therefore\ \hat{n}\times(\bar{E}_2 - \bar{E}_1) = \bar{M}_s$$

$$\text{or } (\bar{E}_1 - \bar{E}_2)\times\hat{n} = \bar{M}_s$$

2-1

$E_z = E_0 e^{-jkz}$ satisfies $\nabla^2 \bar{E} + k^2 \bar{E} = 0$

but does not satisfy $\nabla \cdot \bar{E} = 0$

$$\nabla_z^2 E_z + k_z^2 E_z = (-jk)^2 E_0 e^{-jkz} + k^2 E_0 e^{-jkz} = 0$$

$$\nabla \cdot \bar{E} = \frac{\partial E_z}{\partial z} = -jk E_0 e^{-jkz} \neq 0$$

It does not satisfy

$$\nabla \times \nabla \times \bar{E} - k^2 \bar{E} = 0$$

$$= \nabla(\nabla \cdot \bar{E}) - k^2 \bar{E} - \nabla^2 \bar{E} \neq 0$$

(Since $\nabla \cdot \bar{E} \neq 0$)

$\therefore$ This is not a possible electromagnetic field.

2-2

$$\nabla \times \bar{E} = -\hat{z}\bar{H} \quad \text{or} \quad \hat{z}^{-1}\nabla \times \bar{E} = -\bar{H} \qquad ①$$

$$\nabla \times \bar{H} = \hat{y}\bar{E} \quad \text{or} \quad \hat{y}^{-1}\nabla \times \bar{H} = \bar{E} \qquad ②$$

Taking curl of ①,

$$\nabla \times (\hat{z}^{-1}\nabla \times \bar{E}) = -\nabla \times \bar{H} = -\hat{y}\bar{E}$$

or $\nabla \times (\hat{z}^{-1}\nabla \times \bar{E}) + \hat{y}\bar{E} = 0$

Taking curl of ②,

$$\nabla \times (\hat{y}^{-1}\nabla \times \bar{H}) = \nabla \times \bar{E} = -\hat{z}\bar{H}$$

or $\nabla \times (\hat{y}^{-1}\nabla \times \bar{H}) + \hat{z}\bar{H} = 0$

These eqns. are valid for non-isotropic media.

Eqns. 2-5 do not hold for inhomogeneous media.

2-3

a.) $k = \sqrt{-\hat{z}\hat{y}} = \sqrt{-j\omega\mu(\sigma + j\omega\varepsilon)}$

for $\sigma = 0$, $\varepsilon = \varepsilon_0 \varepsilon_r$;

$$k = \sqrt{\omega^2 \mu_0 \varepsilon_0 \varepsilon_r} = k_0 \sqrt{\varepsilon_r}$$

2-3 (cont.)

b.) $$\eta = \sqrt{\frac{\hat{z}}{\hat{y}}} = \sqrt{\frac{j\omega\mu_0}{j\omega\varepsilon_0\varepsilon_r}} = \sqrt{\frac{\mu_0}{\varepsilon_0}}\frac{1}{\sqrt{\varepsilon_r}} = \frac{\eta_0}{\sqrt{\varepsilon_r}}$$

c.) $$\lambda = \frac{2\pi}{k} = \frac{2\pi}{k_0\sqrt{\varepsilon_r}} = \frac{\lambda_0}{\sqrt{\varepsilon_r}}$$

d.) $$v_p = \frac{\omega}{k} = \frac{\omega_0}{k_0\sqrt{\varepsilon_r}} = \frac{c_0}{\sqrt{\varepsilon_r}}$$

2-4

Eqn 1-35:

$$P_s = P_f + P_d + \frac{\partial}{\partial t}(w_e + w_m)$$

or $$-(\bar{\mathcal{E}} \cdot \bar{\mathcal{J}}^i + \bar{\mathcal{H}} \cdot \bar{\mathcal{M}}^i) = \nabla \cdot \bar{S} + \sigma \mathcal{E}^2 + \frac{\partial}{\partial t}(w_e + w_m)$$

$$\bar{\mathcal{E}} \cdot \sigma \bar{\mathcal{E}} = \nabla \cdot \bar{S} + \sigma \mathcal{E}^2 + \frac{\partial}{\partial t}(w_e + w_m)$$

$$\nabla \cdot \bar{S} + \frac{\partial}{\partial t}(w_e + w_m) = 0 \qquad ①$$

$$\bar{S} = \bar{\mathcal{E}} \times \bar{\mathcal{H}}$$

$$\nabla \cdot \bar{S} = \frac{4k}{\eta} E_0^2 \cos(\omega t - kz)\sin(\omega t - kz)$$

$$\frac{\partial}{\partial t}(w_e + w_m) = -4\omega\varepsilon E_0^2 \cos(\omega t - kz)\sin(\omega t - kz)$$

$$\frac{4k}{\eta} = \frac{4\omega\sqrt{\mu\varepsilon}\sqrt{\varepsilon}}{\sqrt{\mu}} = 4\omega\varepsilon$$

Thus ① is satisfied for Eqn 2-18.

Eqns 2-21:

$$\nabla \cdot \bar{S} = \frac{-2kE_0^2}{2\eta}\cos 2kz \sin 2\omega t$$

$$\frac{\partial}{\partial t}(w_e + w_m) = -\omega 2\varepsilon E_0^2 \sin^2 kz \cdot (\cos\omega t \sin\omega t)$$

$$+ 2\omega\varepsilon E_0^2 \cos^2 kz \sin\omega t \cos\omega t$$

2-4 (cont.)

① becomes:

$$-\frac{kE_0^2}{\eta}\cos 2kz\sin 2\omega t - \omega\varepsilon E_0^2\cos\omega t\sin\omega t$$
$$+\omega\varepsilon E_0^2\cos 2kz(\cos\omega t\sin\omega t)$$
$$+\omega\varepsilon E_0^2\cos\omega t\sin\omega t$$
$$+\omega\varepsilon E_0^2\cos 2kz\sin\omega t\cos\omega t$$
$$= -\frac{kE_0^2}{\eta}\cos 2kz\sin 2\omega t + \omega\varepsilon E_0^2\cos 2kz\cos 2\omega t$$

$= 0$ since $\frac{k}{\eta} = \omega\varepsilon$

Eqns. 2-27:

$$\nabla\cdot\bar{S} = 0$$
$$\frac{\partial}{\partial t}(w_e + w_m) = 0$$

$\therefore$ eqn (1) is satisfied.

Eqns. 2-29:

$$\nabla\cdot\bar{S} = 0$$
$$\frac{\partial}{\partial t}(w_m + w_e) = 0$$

$\therefore$ eqn (1) is satisfied.

2-5 velocity of energy propagation $= v_e$

$$\bar{v}_e = \frac{\mathcal{S}}{w_e + w_m} = \frac{\bar{\mathcal{E}}\times\bar{\mathcal{H}}^*}{w_e + w_m}$$

$$\mathcal{E}_x = \sqrt{2}E_0\sin kz\cos\omega t$$

$$\mathcal{H}_y = -\sqrt{2}\frac{E_0}{\eta}\cos kz\sin\omega t$$

$$w_e = \frac{\varepsilon\mathcal{E}^2}{2} = \varepsilon E_0^2\sin^2 kz\cos^2\omega t$$

$$w_m = \frac{\mu\mathcal{H}^2}{2} = \varepsilon E_0^2\cos^2 kz\sin^2\omega t$$

$$\mathcal{S} = -\frac{E_0^2}{2\eta}\sin 2kz\sin 2\omega t$$

2-5 (cont.)

$$v_e = -\frac{\sin 2kz\sin 2\omega t/2\eta}{A}$$

$$A = \varepsilon\,\frac{1}{4}[1-\cos 2kz][1+\cos 2\omega t] + \frac{1}{4}[1+\cos 2kz][1-\cos 2\omega t]$$

$$v_e = -\frac{\sin 2kz\sin 2\omega t}{\frac{1}{\sqrt{\mu\varepsilon}}(1-\cos 2kz\cos 2\omega t)}$$

$$\frac{\sin 2kz\sin 2\omega t}{1-\cos 2kz\cos 2\omega t}$$
$$= 1 - \frac{1-2\cos(kz-\omega t)}{1-\cos(kz+\omega t)-\cos(kz-\omega t)}$$
$$= 1 - \Sigma$$

$$\therefore v_e \le \frac{1}{\sqrt{\mu\varepsilon}}$$

2-6

$$E_x = Ae^{-jkz} + Ce^{jkz}$$
$$= A\cos kz - jA\sin kz + C\cos kz + jC\sin kz$$
$$= (A+C)\cos kz + j(C-A)\sin kz$$
$$= \left((A+C)^2\cos^2 kz + (C-A)^2\sin^2 kz\right)^{1/2} e^{j\tan^{-1}\theta}$$
$$= \left[(A^2+C^2+2AC)\cos^2 kz + (C^2+A^2-2AC)\sin^2 kz\right]^{1/2} \cdot e^{j\tan^{-1}\theta}$$
$$= \left(A^2+C^2+2AC\cos 2kz\right)^{1/2} e^{j\tan^{-1}\theta}$$

$$\theta = \frac{(C-A)\sin kz}{(A+C)\cos kz} = -\frac{(A-C)}{(A+C)}\tan kz$$

2-6 (cont.)

$$e^{j\tan\phi} = e^{-j\tan^{-1}\frac{(A-C)}{(A+C)}\tan kz}$$

For phase velocity v_p:

$$\omega - \frac{d}{dt}\left[\tan^{-1}\frac{(A-C)}{(A+C)}\tan kz\right] = 0$$

$$\omega - \frac{\frac{(A-C)}{(A+C)}k\sec^2 kz\, v_p}{1+\left(\frac{A-C}{A+C}\right)^2\tan^2 kz} = 0$$

$$\therefore v_p = \frac{1}{\sqrt{\mu\varepsilon}}\left[\frac{A+C}{A-C}\cos^2 kz + \frac{A-C}{A+C}\sin^2 kz\right]$$

2-7 $\bar{E} = (\bar{u}_x + j\bar{u}_y)E_0 \sin kz$

$$\bar{H} = (\bar{u}_x + j\bar{u}_y)\frac{E_0}{\eta}\cos kz$$

$$Z^+_{xy} = \frac{E_x}{H_y} = \frac{E_0 \sin kz}{\frac{jE_0}{\eta}\cos kz} = -j\eta\tan kz$$

$$Z^+_{yx} = -\frac{E_y}{H_x} = \frac{-jE_0\sin kz}{\frac{E_0}{\eta}\cos kz} = -j\eta\tan kz$$

$Z^+_{xy} = Z^+_{yx}$ only in isotropic, homogeneous media.

2-8 Let $\bar{E} = (A\bar{u}_x + B\bar{u}_y)e^{-jkz}$

$A = |A|e^{ja}$, $B = |B|e^{jb}$

$$\mathcal{E}_x = \sqrt{2}\,|A|\cos(\omega t - kz + a)$$

$$\mathcal{E}_y = \sqrt{2}\,|B|\cos(\omega t - kz + b)$$

The wave is circularly polarized for $|A| = |B|$ and:

$a = 0$, $b = \pi/2$ or $a = \pi/2$, $b = 0$

In either case,

$$A = Be^{j\pi/2} = jB$$

and $jA = B$ or $A = -jB$

2-9 $\bar{E} = (A\bar{u}_x + B\bar{u}_y)e^{-jkz}$

Let $A = B = |A|e^{j\pi/2}$

$$\bar{E} = [\bar{u}_x A(1+j) + \bar{u}_y A(1+j)]e^{-jkz}$$

$$= [\bar{u}_x A + j\bar{u}_y A]e^{-jkz} + [j\bar{u}_x A + \bar{u}_y A]e^{-jkz}$$

These represent two circularly polarized waves and thus $\bar{E}$ is a uniform plane wave.

2-10 From eqn. 2-25,

$$\bar{E} = (\bar{u}_x A + \bar{u}_y B)e^{-jkz}$$

Let $A = a + jb$

$B = c + jd$

$$\bar{E} = [(a+jb)\bar{u}_x + (c+jd)\bar{u}_y]e^{-jkz}$$

$$= [(a\bar{u}_x + c\bar{u}_y) + j(b\bar{u}_x + d\bar{u}_y)]e^{-jkz}$$

$$= (\bar{E}_1 + j\bar{E}_2)e^{-jkz}$$

where $\bar{E}_1 = Re(A)\bar{u}_x + Re(B)\bar{u}_y$

$\bar{E}_2 = Im(A)\bar{u}_x + Im(B)\bar{u}_y$

2-11 $\bar{E} = Re(\bar{E}) + j\,Im(\bar{E})$

$$\mathcal{E}_x = A\cos(kz)e^{j\omega t}$$

$$\mathcal{E}_y = B\sin(kz)e^{j\omega t}$$

$$\mathcal{E}_x^2 = \mathcal{E}_x\mathcal{E}_x^* = (a^2+b^2)\cos^2(kz)$$

$$\mathcal{E}_y^2 = \mathcal{E}_y\mathcal{E}_y^* = (c^2+d^2)\sin^2(kz)$$

$$\therefore \frac{\mathcal{E}_x^2}{(a^2+b^2)} + \frac{\mathcal{E}_y^2}{(c^2+d^2)} = 1$$

which shows the elliptic property.

If $a = c$, $b = d$; the wave would be circularly polarized.

$K = \sqrt{-\hat{Z}\hat{Y}} = \sqrt{-j\omega\hat{\varepsilon}(j\omega\mu_0)}$

$\hat{\varepsilon} = \varepsilon' - j\varepsilon''$

2-12

a.) Polystyrene: $\varepsilon' = \varepsilon_0\varepsilon_r'$; $\varepsilon'' = \varepsilon_0\varepsilon_r''$

$k = \omega\sqrt{\mu_0\varepsilon_0(\varepsilon_r' - j\varepsilon_r'')}$

let $x = \varepsilon'/\varepsilon_0 \Rightarrow \varepsilon' = x\varepsilon_0$

$y = \varepsilon''/\varepsilon_0 \Rightarrow \varepsilon'' = y\varepsilon_0$

$k = \omega\varepsilon_0\sqrt{\mu_0(x-jy)}$

$= 6.24\times10^{-14} f\sqrt{(x-jy)}$

$\eta = \sqrt{\dfrac{\mu_0}{\varepsilon_0(\varepsilon'-j\varepsilon'')}} = \dfrac{1.31\times10^8}{\sqrt{x-jy}}$

$f\times10^6$	10	100	1000
$\varepsilon''/\varepsilon_0$	.0003	.0005	.0008
$\varepsilon'/\varepsilon_0$	2.6	2.55	2.55
$\sqrt{x-jy}$	$2.6\angle .003$	$2.5\angle .0056$	$2.55\angle .009$
$k\times10^{-7}$	$16.2 - j.0008$	$159 - j.016$	$1591 - j.25$
$\eta\times10^8$	$.5 + j.000026$	$.514 + j5.\times10^{-5}$	$.514 + j8.\times10^{-5}$

b.) Plexiglass:

$f\times10^6$	10	100	1000
$\varepsilon'/\varepsilon_0$	2.7	2.67	2.65
$\varepsilon''/\varepsilon_0$	.027	.020	.015
$\sqrt{x-jy}$	$1.64\angle -.286$	$1.63\angle -.215$	$1.627\angle -.162$
$k\times10^{-7}$	$10.23 - j.05$	$101.7 - j.381$	$1015 - j2.87$
$\eta\times10^8$	$.799 + j.004$	$.804 + j.003$	$.805 + j.002$

c.) Ferramic A:

$k = \omega\sqrt{\varepsilon_r\mu_0(\mu' - j\mu'')}$

2-12 (cont.)

let $x = \mu'/\mu_0 \Rightarrow \mu' = x\mu_0$

$y = \mu''/\mu_0 \Rightarrow \mu'' = y\mu_0$

$k = \mu_0\sqrt{\varepsilon_r(x-jy)}$

$= 2.35\times10^4 f\sqrt{x-jy}$

$\eta = \sqrt{\dfrac{\mu_0(\mu'-j\mu'')}{\varepsilon_r}} = .133\sqrt{x-jy}$

$f\times10^6$	10	100	1000
μ'/μ_0	25	7	1.5
μ''/μ_0	2	10	2
$\sqrt{x-jy}$	$25.08\angle -2.28$	$12.2\angle -27.5$	$2.5\angle -26.6$
$k\times10^{12}$	$5.89 - j.234$	$25.4 - j13.24$	$52.5 - j26.3$
η	$3.33 - j.133$	$1.44 - j.749$	$.297 - j.149$

d.) Copper, $\sigma = 5.8\times10^7$

$k' = \sqrt{\dfrac{\omega\mu\sigma}{2}} = k''$

$\eta = \sqrt{\dfrac{\omega\mu}{\sigma}}\angle \pi/4$

$f\times10^6$	10	100	1000
$k\times10^4$	$4.78 - j4.78$	$15.1 - j15.1$	$47.8 - j47.8$
$\eta\times10^{-3}$	$1.17 + j1.17$	$3.28 + j3.28$	$11.7 + j11.7$

2-13 $\eta = \sqrt{\frac{\hat{z}}{\hat{y}}}$

From Eqn 2-31, $\hat{z} = jk\eta$

and $\hat{y} = \frac{jk}{\eta}$

$$\eta = \frac{jk}{y} = \frac{jk}{j\omega\varepsilon} \qquad \sigma = 0, \quad \varepsilon'' = 0$$

$$= \frac{k}{\omega\varepsilon} = \frac{k' - jk''}{\omega\varepsilon}$$

2-14 Eqn 1-79, $Q = \frac{\omega\varepsilon'}{\sigma + \omega\varepsilon''}$

$$k = \sqrt{-j\omega\mu(\sigma + j\omega\varepsilon)}$$

$$= \sqrt{-j\omega\mu\sigma + \omega^2\mu(\varepsilon' - j\varepsilon'')}$$

$$= \sqrt{-j\omega\mu(\sigma + \omega\varepsilon'') + \omega^2\mu\varepsilon'}$$

$$= \omega\sqrt{\mu\varepsilon'}\left[1 - j\left(\frac{\sigma + \omega\varepsilon''}{\omega\varepsilon'}\right)\right]^{1/2}$$

$$= \omega\sqrt{\mu\varepsilon'}\left[1 - \frac{1}{2}\frac{j}{Q} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}\left(-\frac{1}{Q^2}\right) - \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}\frac{j}{Q^3}\right]$$

$$k' \approx \omega\sqrt{\mu\varepsilon'}\left(1 + \frac{1}{8Q^2}\right)$$

$$k'' \approx \frac{\omega\sqrt{\mu\varepsilon'}}{2Q}\left(1 - \frac{1}{8Q^2}\right)$$

$$\approx \frac{\omega\varepsilon''}{2}\sqrt{\frac{\mu}{\varepsilon'}}\left(1 - \frac{1}{8Q^2}\right)$$

$$\eta = \sqrt{\frac{\hat{z}}{\hat{y}}} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega(\varepsilon' - j\varepsilon'')}} = \sqrt{\frac{j\omega\mu}{\sigma + \omega\varepsilon'' + j\omega\varepsilon'}}$$

2-14 (cont.)

$$\eta = \sqrt{\frac{\mu}{\varepsilon'}}\left[\frac{1}{1 + \frac{1}{jQ}}\right]^{1/2} \qquad Q = \frac{\omega\varepsilon'}{\sigma + \omega\varepsilon''}$$

$$= \sqrt{\frac{\mu}{\varepsilon'}}\left[1 + \frac{1}{jQ}\right]^{-1/2}$$

$$= \sqrt{\frac{\mu}{\varepsilon'}}\left[1 - \frac{1}{j2Q} + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2} - 1\right)}{2!}\left(-\frac{1}{Q^2}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2} - 1\right)\left(-\frac{1}{2} - 2\right)}{3!}\left(\frac{-1}{jQ^3}\right)\right]$$

$$R = \sqrt{\frac{\mu}{\varepsilon'}}\left[1 - \frac{3}{8Q^2}\right]$$

$$X = \sqrt{\frac{\mu}{\varepsilon'}}\left(\frac{1}{2Q}\right)\left(1 - \frac{5}{8Q^2}\right)$$

$$= \sqrt{\frac{\mu}{\varepsilon'}}\,\frac{1}{2}\left(\frac{\sigma + \omega\varepsilon''}{\omega\varepsilon'}\right)\left[1 - \frac{5}{8Q^2}\right]$$

$$\approx \frac{\varepsilon''}{2\varepsilon'}\sqrt{\frac{\mu}{\varepsilon'}}\left(1 - \frac{5}{8Q^2}\right)$$

2-15

$$\varepsilon = \varepsilon' \qquad Q = \frac{\omega\varepsilon'}{\sigma} \qquad k = k' - jk''$$

$$k = \sqrt{-j\omega\mu(\sigma + j\omega\varepsilon)}$$

$$= \sqrt{-j\omega\mu(\sigma + j\omega\varepsilon')}$$

$$= \sqrt{-j\omega\mu\sigma + \omega^2\mu\varepsilon'}$$

$$= \sqrt{\omega\mu\sigma}\left[-j + Q\right]^{1/2}$$

$$= \sqrt{-j\omega\mu\sigma}\left[1 + jQ\right]^{1/2}$$

$$= \sqrt{-j\omega\mu\sigma}\left[1 + \frac{jQ}{2} + \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)}{2!}(jQ)^2 + \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)\left(\frac{1}{2} - 2\right)}{3!}(jQ)^3\right]$$

$$= \sqrt{\omega\mu\sigma}\left[\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right]\left[1 + \frac{jQ}{2}\right.$$

2-15 (cont.)

$$k = \sqrt{\omega\mu\sigma}\left[\left(\frac{1}{\sqrt{2}} + \frac{Q}{2\sqrt{2}}\right) + j\left(-\frac{1}{\sqrt{2}} + \frac{Q}{2\sqrt{2}}\right)\right]$$

$$k' = \sqrt{\frac{\omega\mu\sigma}{2}}\left(1 + \frac{Q}{2}\right)$$

$$k'' = \sqrt{\frac{\omega\mu\sigma}{2}}\left(1 - \frac{Q}{2}\right)$$

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = \sqrt{\frac{\omega\mu}{\sigma}}\left(\frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}}\right)\left[1 + jQ\right]^{-1/2}$$

$$= \sqrt{\frac{\omega\mu}{2\sigma}}\,(1+j)\left(1 - \frac{jQ}{2}\right)$$

$$= \sqrt{\frac{\omega\mu}{2\sigma}}\left(1 - \frac{jQ}{2} + j + \frac{Q}{2}\right)$$

$$R = \sqrt{\frac{\omega\mu}{2\sigma}}\left(1 + \frac{Q}{2}\right)$$

$$X = \sqrt{\frac{\omega\mu}{2\sigma}}\left(1 - \frac{Q}{2}\right)$$

2-16

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} \qquad (\varepsilon = 0)$$

$$\delta = \frac{1}{\sqrt{\pi f \mu\sigma}} = \frac{\sqrt{2}}{\sqrt{\omega\mu\sigma}}$$

$$\eta = \sqrt{\frac{\omega\mu}{\sigma}}\left(\frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}}\right)$$

$$= \sqrt{\frac{\omega\mu}{2\sigma}}\,(1+j) = R\,(1+j)$$

$$k = \sqrt{-j\omega\mu\sigma} = \sqrt{\frac{\omega\mu\sigma}{2}}\,(1-j)$$

$$= \frac{1}{\delta}(1-j)$$

$$R = Re(\eta) = Re\left\{\sqrt{\frac{\omega\mu}{\sigma}}\left(\frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}}\right)\right\}$$

$$= \sqrt{\frac{\omega\mu}{2\sigma}} = \frac{1}{\sigma\delta}$$

2-17

$$R = \frac{1}{\sigma\delta} = \frac{\sqrt{\omega\mu\sigma}}{\sqrt{2}\,\sigma}$$

$$R = \frac{\sqrt{\omega\mu}}{\sqrt{2\sigma}} = 1.99\times10^{-3}\frac{\sqrt{f}}{\sqrt{\sigma}}$$

Values of σ used were:

σ (copper) $= 5.8\times10^{7}$

σ (silver) $= 6.23\times10^{7}$

σ (gold) $= 4.07\times10^{7}$

σ (aluminum) $= 3.73\times10^{7}$

σ (brass) $= 1.57\times10^{7}$

Thus the formulas for the intrinsic wave resistance are easily obtained.

2-18

a.) $f = 60\,\text{Hz}$.

$$P_d = |H|^2 R = R = 2.61\times10^{-7}\sqrt{60}$$

$$= 2.02\times10^{-6}$$

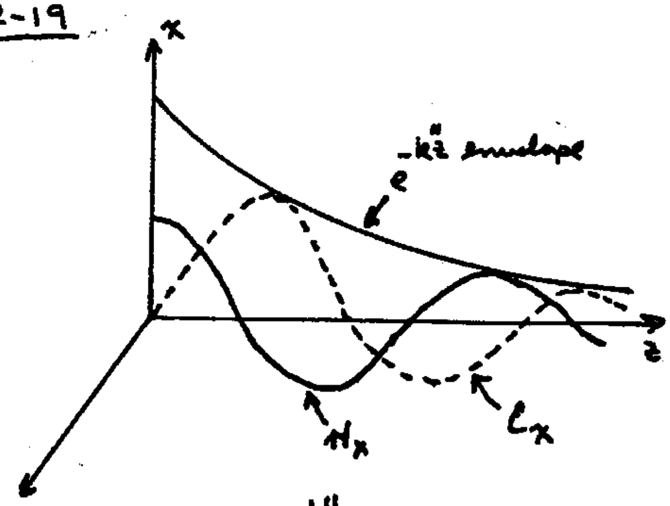
b.) $f = 10^{6}\,\text{Hz}$

$$P_d = 2.52\times10^{-4}$$

c.) $f = 10^{9}\,\text{Hz}$

$$P_d = 8.25\times10^{-3}$$

2-19



$$\mathcal{E}_x = \sqrt{2}E_0\, e^{-k''z}\cos(\omega t - k'z)$$

$$H_x = \frac{\sqrt{2}E_0}{|\eta|}\, e^{-k''z}\sin(\omega t - k'z - \zeta)$$

2-19 (cont.)

$$k = k' - jk''$$
$$\eta = |\eta| e^{j\xi}$$

For a circularly polarized wave in dissipative media the amplitude decays exponentially with z and H_x lags E_x by $\pi/2 - \xi$.

2-20 From Eqn 2-47:

$$SWR = \frac{1+|\Gamma|}{1-|\Gamma|}$$

From Eqn 2-45:

① = air ② = dielectric $\quad \Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$

$$\Gamma = \frac{\sqrt{\frac{\mu_0}{\varepsilon_0 \varepsilon_r}} - \sqrt{\frac{\mu_0}{\varepsilon_0}}}{\sqrt{\frac{\mu_0}{\varepsilon_0 \varepsilon_r}} + \sqrt{\frac{\mu_0}{\varepsilon_0}}} = \frac{1-\sqrt{\varepsilon_r}}{1+\sqrt{\varepsilon_r}}$$

$$SWR = \frac{1 + \left|\frac{1-\sqrt{\varepsilon_r}}{1+\sqrt{\varepsilon_r}}\right|}{1 - \left|\frac{1-\sqrt{\varepsilon_r}}{1+\sqrt{\varepsilon_r}}\right|} \quad \text{since } \varepsilon_r > 1$$

$$= \frac{1 + \frac{\sqrt{\varepsilon_r}-1}{\sqrt{\varepsilon_r}+1}}{1 - \left|\frac{\sqrt{\varepsilon_r}-1}{\sqrt{\varepsilon_r}+1}\right|} = \sqrt{\varepsilon_r}$$

2-21 ① — air Antenna Wave Imp. $\eta = \sqrt{\frac{\mu}{\varepsilon}}$

② — water $\eta_1 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$

$\eta_1 = 120\pi \qquad \Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$

$\eta_2 = 360\pi$

2-21 (cont.)

$$\Gamma = \frac{240\pi}{480\pi} = \frac{1}{2}$$

$$S_{trans.} = S_{inc}(1 - |\Gamma|^2) = \frac{3}{4} S_{inc}$$

$$S_{refl.} = |\Gamma|^2 S_{inc} = \frac{1}{4} S_{inc}$$

75% of the power is transmitted and 25% is reflected.

2-22

Brewster angle $= \theta_i = \tan^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$

$$\varepsilon_2 = \varepsilon_r \varepsilon_1$$

$$\theta_t = \cos^{-1}\left[\sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\sqrt{\frac{\varepsilon_1}{\varepsilon_1+\varepsilon_2}}\right]$$

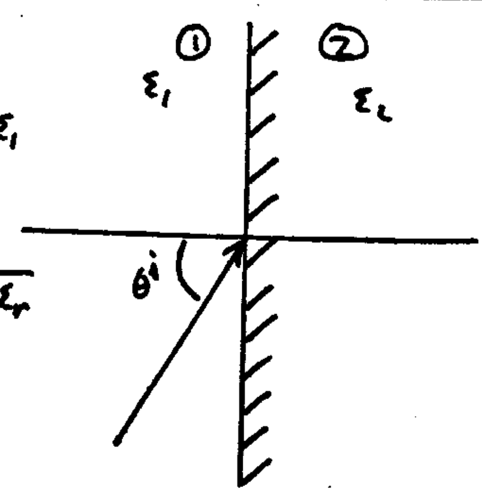
$$\theta_c = \sin^{-1}\sqrt{\frac{\varepsilon_1\mu_1}{\varepsilon_2\mu_2}} = \sin^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = \sin^{-1}\sqrt{\varepsilon_r}$$

	WATER	GLASS	POLYSTYRENE
ε_r	81	9	2.56
θ_i	83.7°	71.6°	58.0°
θ_t	89.3°	83.9°	70.7°
θ_c	6.38°	19.5°	38.7°

2-23

$$\varepsilon_2 = \varepsilon_r \varepsilon_1$$

$$\theta^i = \sin^{-1}\sqrt{\varepsilon_r}$$



2-23 (cont.)

Any wave incident at an angle greater than θ^i will be totally reflected. Values of α at critical angle are:

Water : $\pm j\alpha = k\cos 6.38° = .994k$

Glass : $\pm j\alpha = k\cos 19.5° = .943k$

Polystyrene: $\pm j\alpha = k\cos 38.7° = .78k$

$e^{-\alpha z} \Rightarrow \frac{1}{\alpha}$ is the value of z which gives an attenuation of $1/e$

Water: $z = \dfrac{1.006}{k} = .16\lambda_w$

Glass: $z = \dfrac{1.06}{k} = .17\lambda_g$

Polystyrene: $z = \dfrac{1.28}{k} = .2\lambda_p$

Where the λ's are the different wavelengths in the various materials.

2-24 $\gamma = \alpha + j\beta = \sqrt{(R+j\omega L)(G+j\omega C)}$

$$= (j\omega L\, j\omega C)^{1/2}\left[\left(1+\frac{R}{j\omega L}\right)\left(1+\frac{G}{j\omega C}\right)\right]^{1/2}$$

$$\approx j\omega\sqrt{LC}\left[1+\frac{R}{2j\omega L}+\frac{G}{2j\omega C}\right]$$

$$\alpha \approx \frac{R}{2\sqrt{L/C}} + \frac{G\sqrt{L/C}}{2}$$

$$\beta \approx \omega\sqrt{LC}$$

2-25 $Z_0 = \sqrt{\dfrac{R+j\omega L}{G+j\omega C}}$

$$\gamma = \sqrt{(R+j\omega L)(G+j\omega C)}$$

2-25 (cont.)

$$G + j\omega C = \frac{\gamma}{Z_0}$$

Similarly for a plane wave:

$$\gamma = \sqrt{\hat{z}\hat{y}} \qquad \eta = \sqrt{\frac{\hat{z}}{\hat{y}}}$$

$$\gamma = \hat{y}\eta$$

$$G + j\omega C = \frac{\hat{y}\eta}{Z_0} = \frac{\eta}{Z_0} j\omega(\varepsilon' - j\varepsilon'')$$

$$G = \frac{\omega\eta\varepsilon''}{Z_0} \qquad C = \frac{\omega\eta\varepsilon'}{Z_0}$$

$$G = \frac{\omega\varepsilon''}{\varepsilon'}C$$

If the frequency is high enough the curvature of the wire is unimportant so for a conductor of thickness δ;

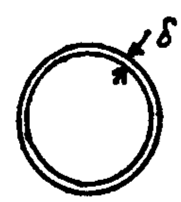
$$R_{h.f.} = \frac{R_s}{\pi\delta} = \frac{1}{\pi\sigma\delta^2} \text{ ohms/meter}$$

$$R_{D.C.} = \frac{1}{\pi\delta^2\sigma} \text{ ohms/meter}$$

$R_{h.f.} = R_{D.C.}$ (for wires of thickness $= \delta$).

2-26

a.) $R \approx \dfrac{1}{\pi d\sigma\delta}$



which for two lines becomes:

$$R \approx \frac{2}{\pi d\sigma\delta} \quad \text{but } \mathcal{R} = \frac{1}{\sigma\delta}$$

$$\text{so } R \approx \frac{2\mathcal{R}}{\pi d}$$

2-26 (cont.)

b.)

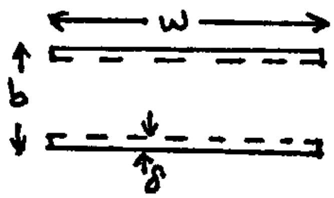
$$R_1 = \frac{1}{2\pi a \sigma \delta}$$

$$R_2 = \frac{1}{2\pi b \sigma \delta}$$

$$R = R_1 + R_2 = \frac{\pi \sigma \delta (a+b)}{2\pi^2 \delta^2 \sigma^2 a b}$$

$$= \frac{\mathcal{R}(a+b)}{2\pi a b}$$

c.)



Resistance of one plate is $R_1 = \dfrac{1}{w\delta\sigma}$

$$= \frac{\mathcal{R}}{w}$$

For both plates: $R = \dfrac{2\mathcal{R}}{w}$.

2-27

$$\bar{E} = \eta \bar{H} \times \bar{u}_z$$

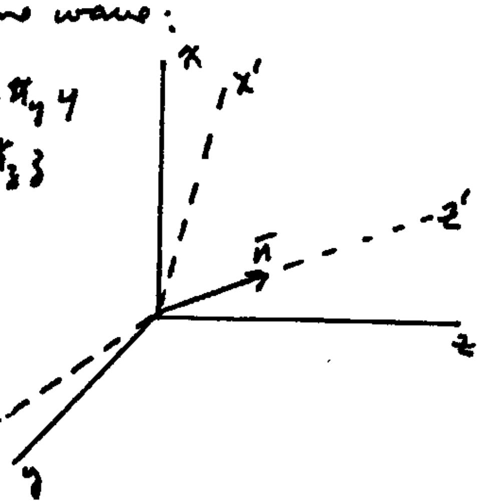
$$\bar{H} = \frac{1}{\eta} \bar{u}_z \times \bar{E}$$

$$\nabla \times \bar{E} = -j\omega\mu \bar{H}, \quad \nabla \times \bar{H} = (\sigma + j\omega\varepsilon)\bar{E}$$

For a plane wave:

$$z' = n_x x + n_y y + n_z z$$

$$= \bar{n} \cdot \bar{r}$$



2-27 (cont)

$$\frac{\partial}{\partial x} = \frac{\partial z'}{\partial x} \cdot \frac{\partial}{\partial z'} = n_x \cdot \frac{\partial}{\partial z'}, \text{ etc.}$$

$$\bar{\nabla} = \bar{n} \frac{\partial}{\partial z'}$$

$$\bar{n} \times \frac{\partial}{\partial z'} \bar{E} = -j\omega\mu \bar{H}$$

$$\bar{n} \times \frac{\partial}{\partial z'} \bar{H} = (\sigma + j\omega\varepsilon)\bar{E}$$

Assuming the z' variation is e^{-jkz}

$$k = \sqrt{-j\omega\mu(\sigma + j\omega\varepsilon)}$$

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}$$

$$\bar{H} = \frac{-jk}{-j\omega\mu} \bar{n} \times \bar{E} = \frac{1}{\eta} \bar{n} \times \bar{E}$$

$$\bar{E} = \frac{-jk}{\sigma + j\omega\varepsilon} \bar{n} \times \bar{H} = \eta \bar{H} \times \bar{n}$$

2-28

$$\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial z^2} + k^2 E = 0$$

Assuming variation of E with z is $e^{-\gamma z}$:

$$\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + (k^2 + \gamma^2)E = 0$$

Field is also uniform in x-direction.

$$\frac{\partial^2 E}{\partial y^2} + (k^2 + \gamma^2)E = 0$$

or $\dfrac{\partial^2 E}{\partial y^2} + k_y^2 E = 0 \qquad k_y^2 = k^2 + \gamma^2$

Since $E_z = 0$,

$$E_x = A \sin k_y y + B \cos k_y y$$

$E_x = 0$ at $y = 0$ and $y = b$

2-28 (cont.)

$k_y = \frac{n\pi}{b}$, $B = 0$.

$E_x = A \sin \frac{n\pi y}{b} e^{-\gamma z}$ for the TE modes.

For the TM case, $H_z = 0$ and the equation satisfied by H_x is:

$$\frac{\partial^2 H_x}{\partial y^2} + (k^2 + \gamma^2) H_x = 0$$

$$k_y^2 = k^2 + \gamma^2$$

$$H_x = [A \sin k_y y + B \cos k_y y] e^{-\gamma z}$$

$$-\frac{\partial H_x}{\partial y} = j\omega\varepsilon E_z$$

$E_z = 0$ at $y = 0$, $y = b$

$\therefore A = 0$ and $k_y = n\pi/b$

$$H_x = B \cos \frac{n\pi}{b} e^{-\gamma z}$$

cutoff occurs when: $\gamma^2 = K_y^2 - K^2$

$$k_c^2 = \left(\frac{n\pi}{b}\right)^2 = k^2 \quad \text{or} \quad K_y = K$$

$K = \omega\sqrt{\mu\varepsilon}$

$$f_c = \frac{n}{2b\sqrt{\mu\varepsilon}} \qquad \omega_c = \frac{K}{\sqrt{\mu\varepsilon}} = \frac{n\pi}{b\sqrt{\mu\varepsilon}}$$

2-29 For TE_n modes,

$$E_x = E_0 \sin(k_c y) e^{-\gamma z} \qquad \gamma = j\beta$$

$$H_y = \frac{\gamma}{j\omega\mu} E_0 \sin(k_c y) e^{-\gamma z}$$

$$P = \int_0^b E_x H_y^* dy = \int_0^b \frac{\gamma}{-j\omega\mu} E_0^2 \sin^2 k_c y \, dy$$

$$= \frac{\gamma b}{-j\omega\mu} \frac{E_0^2}{2} \qquad \gamma^2 = k_c^2 - k^2$$

$$-\beta^2 = -k^2 + k_c^2$$

2-29 (cont.)

$$\beta^2 = \left(\frac{2\pi}{c_0}\right)^2 \left[f^2 - f_c^2\right]$$

$$\beta = \omega\sqrt{\mu_0\varepsilon_0} \left[1 - \left(\frac{f_c}{f}\right)^2\right]^{1/2}$$

$$P = \frac{E_0^2 \beta b}{2\omega\mu} = \frac{E_0^2 b}{2\eta} \left[1 - (f_c/f)^2\right]^{1/2}$$

For TM modes,

$$H_x = H_0 \cos k_c y \, e^{-\gamma z}$$

$$E_y = \frac{1}{j\omega\varepsilon} \frac{\partial H_x}{\partial z} = -\frac{\gamma H_0}{j\omega\varepsilon} H_0 \cos k_c y e^{-\gamma z}$$

$$P = -\int_0^b E_y H_x^* dy = \frac{H_0^2}{2} \frac{b\beta}{\omega\varepsilon}$$

$$= \frac{H_0^2}{2} \eta b \left[1 - \left(\frac{f_c}{f}\right)^2\right]^{1/2}$$

2-30 For TM modes: $(H_x, E_y, E_z \neq 0)$

$$P_d = \int_0^1 |H_x|^2 R \, dz \text{ (over unit area)} \times 2$$

$$= 2|H_0|^2 R$$

$$\alpha_c = \frac{P_d}{2P_f} = \frac{2R}{b\eta\sqrt{1 - (f_c/f)^2}}$$

For TE modes: $(E_x, H_y, H_z \neq 0)$

$$P_d = 2|H_z|^2 R = 2\left|\frac{k_c}{\omega\mu}\right|^2 E_0^2 R$$

$$= \left(\frac{f_c}{f}\right)^2 \left|\frac{k}{\omega\mu}\right|^2 2R$$

$$\alpha = \frac{P_d}{2P_f} = \left(\frac{f_c}{f}\right)^2 \left|\frac{k}{\omega\mu}\right|^2 \frac{R}{\frac{b}{2\eta}\left[1 - (f_c/f)^2\right]^{1/2}}$$

$$= \left(\frac{f_c}{f}\right)^2 \frac{2R}{b\eta\sqrt{1 - (f_c/f)^2}}$$

2-31 For TM_0 mode: $\begin{vmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$

$$H_x = H_0 e^{-\gamma z}$$

$$? \quad E_y = \frac{1}{j\omega\varepsilon}\frac{\partial H_x}{\partial z} = \frac{-\gamma}{j\omega\varepsilon} H_0 e^{-\gamma z}$$

$$E_z = \frac{1}{j\omega\varepsilon}(-)\frac{\partial H_x}{\partial y} = 0$$

$\therefore$ we have a TEM mode and $f_c = 0$

$$\therefore \alpha_c = \frac{P_d}{2P_f} = \frac{R|H_0|^2}{\left(\frac{\beta}{\omega\varepsilon}\right)|H_0|^2 b} = \frac{R}{b\eta}$$

From Eqn. 2-24:

$$\alpha = \frac{R}{2\sqrt{L/C}} = \frac{R\omega\varepsilon}{2\beta} = \frac{2R\omega\varepsilon}{2\beta W}$$

$$\text{(using Eqn 2-26)} \quad = \frac{R}{\eta W}$$

2-32 For TE_{01} mode:

$$E_x = E_0 \sin\frac{\pi y}{b} e^{-j\beta z}$$

$$H_y = \frac{\gamma}{j\omega\mu} E_0 \sin\frac{\pi y}{b} e^{-j\beta z}$$

$$H_z = \frac{\pi}{bj\omega\mu} E_0 \cos\frac{\pi y}{b} e^{-j\beta z}$$

$$W_e = \frac{1}{2}\int_0^a\int_0^b \varepsilon |E|^2 dy\,dx$$

$$= \frac{\varepsilon}{2}|E_0|^2\frac{ab}{2} = \frac{\varepsilon ab}{4}|E_0|^2$$

$$W_m = \frac{\mu E_0^2}{2}\frac{ab}{2}\left[\frac{\gamma^2 + \left(\frac{\pi}{b}\right)^2}{\omega^2\mu^2}\right]$$

$$= \frac{\mu ab}{4}|E_0|^2\frac{1}{\eta^2} = \frac{\varepsilon}{4}ab|E_0|^2$$

$$\oint\!\!\oint \bar{S}\cdot d\bar{s} + \bar{P}_d + j2\omega(\bar{W}_m - \bar{W}_e) = 0$$

If there is no dissipation in the

2-32 (cont.)

guide walls we can choose a surface S at some pt. in the guide where $E = 0$,

$$\oint\!\!\oint \bar{S}\cdot d\bar{s} = 0$$

and $\bar{W}_m = \bar{W}_e$.

This is most easily seen for a pure standing wave.

2-33 $v_e = \dfrac{\bar{S}_z}{\bar{W}}$

$$S_z = E \times H^* = \sqrt{2}\,\mathrm{Re}\left[E_0 \sin\frac{\pi y}{b} e^{j(\omega t - \beta z)}\right] \cdot \sqrt{2}\,\mathrm{Re}\left(E_0 \sin\frac{\pi y}{b} e^{j(\omega t-\beta z)}\frac{\beta}{\omega\mu}\right]$$

$$= 2E_0^2 \sin^2\frac{\pi y}{b}\cos(\omega t - \beta z)\frac{\beta}{\omega\mu}$$

$$\bar{S}_z = \frac{1}{2}\int_0^a\int_0^b S_z\,dy\,dx$$

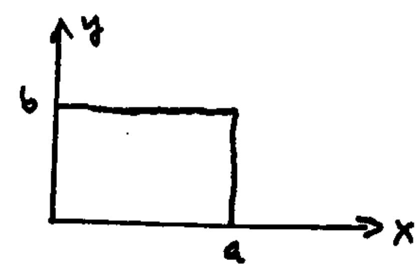
$$= E_0^2\frac{ba}{2}\frac{\beta}{\omega\mu}$$

$$v_e = \frac{E_0^2 ba\beta 2}{2\omega\mu\varepsilon|E_0|^2 ab} = \frac{\beta}{\omega\mu\varepsilon}$$

$$= \frac{(\eta)^{-1}}{\varepsilon}\left[1 - \left(\frac{f_c}{f}\right)^2\right]^{1/2}$$

$$= \frac{1}{\sqrt{\mu\varepsilon}}\left[1 - (f_c/f)^2\right]^{1/2}$$

2-34



$$V = \int_0^x E_x dx\Big|_{y=b/2} = \int_0^x E_0 e^{-\gamma z} dx = aE_0 e^{-\gamma z}$$

2-34 (cont.)

$$I = \int_0^b H_y\,dy\Big|_{x=0} = \int_0^b \frac{E_0}{Z_0}\sin\frac{\pi y}{b}\, e^{-\gamma z}\,dy$$

$$= -\frac{b}{\pi}\frac{E_0}{Z_0}\cos\frac{\pi y}{b}\, e^{-\gamma z}\Big|_0^b$$

$$= \frac{2bE_0}{\pi Z_0}\, e^{-\gamma z}, \quad \gamma = j\beta$$

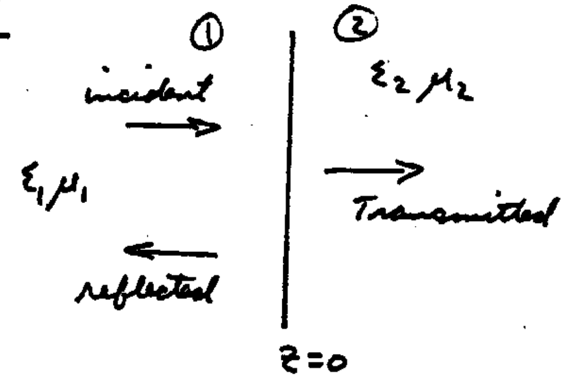
$$P = \frac{|E_0|^2 ab}{2Z_0}, \quad VI^* = \frac{2ab|E_0|^2}{\pi Z_0}$$

$P \neq VI^*$ because of the arbitrariness in the definitions of V and I.

$$Z_{VI} = V/I = \frac{aZ_0\pi}{2b} \sim \frac{j\omega\mu}{\gamma}$$

$$= \frac{\eta}{\sqrt{1-(f_c/f)^2}}$$

2-35



$$E_x^1 = E_0\left(e^{-jk_1z} + \Gamma e^{jk_1z}\right)\sin\frac{\pi y}{b}$$

$$H_y^1 = \frac{E_0}{Z_{01}}\left(e^{-jk_1z} - \Gamma e^{jk_1z}\right)\sin\frac{\pi y}{b}$$

$$E_x^2 = E_0 T e^{-jk_2z}\sin\frac{\pi y}{b}$$

$$H_y^2 = \frac{E_0}{Z_{02}} T e^{-jk_2z}\sin\frac{\pi y}{b}$$

Now apply continuity of wave

2-35 (cont.)

impedance at interface:

$$Z_{02}\Big|_{z=0} = \frac{E_x^1}{H_y^1}\Big|_{z=0} = Z_{01}\frac{1+\Gamma}{1-\Gamma}$$

$$\therefore \Gamma = \frac{Z_{02}-Z_{01}}{Z_{02}+Z_{01}}$$

and $$T = \frac{2Z_{02}}{Z_{01}+Z_{02}}$$

2-36 There is no reflected wave when $Z_{02} = Z_{01}$.

$$Z_{01} = \frac{\eta_1}{\sqrt{1-(f_{c1}/f)^2}} = Z_{02} = \frac{\eta_2}{\sqrt{1-(f_{c2}/f)^2}}$$

$$f_c = \frac{1}{2b\sqrt{\mu\varepsilon}}$$

$$f_{c1}^2 = \frac{1}{4b^2\mu_1\varepsilon_1}, \quad f_{c2}^2 = \frac{1}{4b^2\mu_2\varepsilon_2}$$

$$f_{c2}^2 = \frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2} f_{c1}^2$$

$$\eta_1^2\left[1-\left(\frac{f_{c2}}{f}\right)^2\right] = \eta_2^2\left[1-\left(f_{c1}/f\right)^2\right]$$

$$\frac{\mu_1}{\varepsilon_1}\left[1-\frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}\left(\frac{f_{c1}}{f}\right)^2\right] = \frac{\mu_2}{\varepsilon_2}\left[1-\left(\frac{f_{c1}}{f}\right)^2\right]$$

$$\frac{\mu_1}{\varepsilon_1} - \frac{\mu_2}{\varepsilon_2} = \left(\frac{f_{c1}}{f}\right)^2\left[\frac{\mu_1^2}{\mu_2\varepsilon_2} - \frac{\mu_2}{\varepsilon_2}\right]$$

$$\frac{\mu_1\varepsilon_2 - \mu_2\varepsilon_1}{\varepsilon_1\varepsilon_2} = \left(\frac{f_{c1}}{f}\right)^2\left[\frac{\mu_1^2-\mu_2^2}{\mu_2\varepsilon_2}\right]$$

$$\therefore \frac{f}{f_{c1}} = \sqrt{\frac{(\mu_1^2-\mu_2^2)\,\varepsilon_1}{\mu_2(\mu_1\varepsilon_2-\mu_2\varepsilon_1)}}$$

2-37 For TM mode:

$$Z_{01} = \frac{\beta_1}{\omega\varepsilon} = \frac{\omega\sqrt{\mu\varepsilon}}{\varepsilon}\left[1-\left(\frac{f_{c1}}{f}\right)^2\right]^{1/2}$$

$$= \eta_1\left[1-\left(f_{c1}/f\right)^2\right]^{1/2}$$

$$= Z_{02} = \eta_2\left[1-\left(f_{c2}/f\right)^2\right]^{1/2}$$

$$f_{c1}^2 = \frac{1}{4b^2\mu_1\varepsilon_1} \qquad f_{c2}^2 = \frac{1}{4b^2\mu_2\varepsilon_2} = \frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2} f_{c1}^2$$

$$\frac{\mu_1}{\varepsilon_1}\left[1-\left(\frac{f_{c1}}{f}\right)^2\right] = \frac{\mu_2}{\varepsilon_2}\left[1-\left(\frac{f_{c1}}{f}\right)^2\frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}\right]$$

$$\frac{\mu_1}{\varepsilon_1} - \frac{\mu_2}{\varepsilon_2} = \left(\frac{f_{c1}}{f}\right)^2\left[\frac{\mu_1}{\varepsilon_1} - \frac{\mu_1\varepsilon_1}{\varepsilon_2^2}\right]$$

$$\frac{\mu_1\varepsilon_2 - \mu_2\varepsilon_1}{\varepsilon_1\varepsilon_2} = \left(\frac{f_{c1}}{f}\right)^2\left[\frac{\varepsilon_2^2-\varepsilon_1^2}{\varepsilon_1\varepsilon_2^2}\right]\mu_1$$

$$\therefore \frac{f}{f_{c1}} = \sqrt{\frac{\mu_1(\varepsilon_1^2-\varepsilon_2^2)}{\varepsilon_2(\mu_1\varepsilon_2-\mu_2\varepsilon_1)}}$$

2-38 $b=c$, $a=b/2$ in Fig 2-19

$$f_r = 10^9$$

$$b = \frac{1}{f_r\sqrt{2\varepsilon_r\varepsilon_0\mu_0}}, \quad Q_c = \frac{1.11\sqrt{\mu_0}}{R(1+b/2a)\sqrt{\varepsilon_0\varepsilon_r}}$$

Assume walls are made of copper,

$$R = 2.61\times10^{-7}\sqrt{f} = 8.25\times10^{-3}$$

a.) air, $\varepsilon_r = 1$

$$b = \frac{3\times10^8}{(10^9)\sqrt{2}} = .212 \text{ meter}$$

so $a = .106$ meter

2-38 (cont.)

$$Q_c = \frac{(1.11)(377)}{(8.25\times10^{-3})(2)} = 2.54\times10^4$$

b.) Polystyrene, $\varepsilon_r = 2.56$

$$b = \frac{3\times10^8}{(10^9)\sqrt{2(2.56)}} = .133\,\text{m}$$

$$a = .066\text{ m.}$$

$$Q = \frac{(1.11)(377)}{(8.25\times10^{-3})(2)\sqrt{2.56}} = 1.58\times10^4$$

2-39 $E_x = E_0 \sin\frac{\pi y}{b}\sin\frac{\pi z}{c}$

$$H_y = \frac{jbE_0}{\eta\sqrt{b^2+c^2}}\sin\frac{\pi y}{b}\cos\frac{\pi z}{c}$$

$$H_z = \frac{-jcE_0}{\eta\sqrt{b^2+c^2}}\cos\frac{\pi y}{b}\sin\frac{\pi z}{c}$$

$$V = \int_0^a E_x\Big|_{y=b/2,\ z=c/2} dx = aE_0$$

$$I = \oint H\cdot dl = \int_0^b H_y\Big|_{z=0} dy + \int_0^c -H_z\,dz$$

$$+ \int_b^0 -H_y\,dy + \int_c^0 H_z\,dz$$

$$= 4\int_0^b \frac{jbE_0}{\eta\sqrt{b^2+c^2}}\sin\frac{\pi y}{b}\,dy$$

$$= \frac{j4bE_0}{\eta\sqrt{b^2+c^2}}\left(\frac{-b}{\pi}\right)\cos\frac{\pi y}{b}\Big|_0^b$$

$$= \frac{j8E_0}{\pi\eta\sqrt{b^2+c^2}}$$

2-39 (cont.)

$$P_d = \frac{R|E_0|^2}{2\eta^2(b^2+c^2)}\left[bc(b^2+c^2) + 2a(b^3+c^3)\right]$$

Formulas for G and R are obtained directly.

2-40

$$A_r = A_z \cos\theta$$
$$A_\theta = -A_z \sin\theta$$

$$\overline{H} = \nabla \times \overline{A}$$

$$\overline{H} = \frac{1}{r^2 \sin\theta}\begin{vmatrix} \overline{u}_r & r\overline{u}_\theta & r\sin\theta\,\overline{u}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_z\cos\theta & -rA_z\sin\theta & 0 \end{vmatrix}$$

$$H_\phi = \frac{1}{r}\left[-\frac{\partial}{\partial r}(rA_z\sin\theta) - \frac{\partial}{\partial\theta}(A_z\cos\theta)\right]$$

$$= \frac{1}{r}\left[-A_z\sin\theta - r\frac{\partial A_z}{\partial r}\sin\theta + A_z\sin\theta\right]$$

$$H_\phi = -\frac{\partial A_z}{\partial r}\sin\theta = -\sin\theta\, C\frac{\partial}{\partial r}\frac{e^{-jkr}}{r}$$

$$= -C\sin\theta\left[\frac{-jke^{-jkr}r + e^{-jkr}}{r^2}\right]$$

$$= jkC\sin\theta\,\frac{e^{-jkr}}{r} \quad \text{for large } r$$

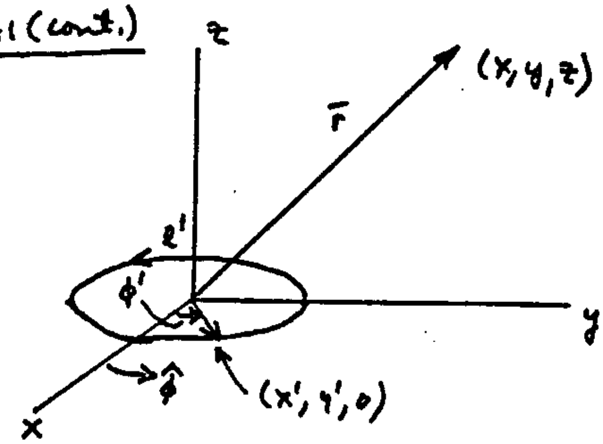
$$H_\phi = jk\sin\theta\, A_z \quad \text{(far field)}$$

$$E_\theta = \eta H_\phi = j\omega\mu\sin\theta\, A_z$$

2-41

$$\overline{A} = A_\phi = \frac{1}{4\pi}\oint_l \frac{J(\phi')}{|\overline{r}-\overline{r}'|}e^{-jk|\overline{r}-\overline{r}'|}\,dl'$$

2-41 (cont.)



$$|\overline{r}-\overline{r}'| = (x-x')^2 + (y-y')^2 + (z)^2$$

$$= \left[(r\sin\theta\cos\phi - a\cos\phi')^2 + (r\sin\theta\sin\phi - a\sin\phi')^2 + (r\cos\theta)^2\right]^{1/2}$$

which when $\phi = 0$ becomes,

$$|\overline{r}-\overline{r}'| = \left[r^2\sin^2\theta - 2ra\sin\theta\cos\phi' + a^2\cos^2\phi' + a^2\sin^2\phi' + r^2\cos^2\theta\right]^{1/2}$$

$$= \left[r^2 + a^2 - 2ra\sin\theta\cos\phi'\right]^{1/2}$$

$$J(\phi') = I, \quad dl' = a\cos\phi'\,d\phi'$$

$$A_\phi = \frac{Ia}{4\pi}\int_0^{2\pi}\frac{e^{-jk\sqrt{r^2+a^2-2ra\sin\theta\cos\phi'}}}{\sqrt{r^2+a^2-ra2\sin\theta\cos\phi'}}\cos\phi'\,d\phi'$$

$$f(a) = \frac{e^{-jk\alpha}}{\alpha}, \quad \alpha = \sqrt{r^2+a^2-2ra\sin\theta\cos\phi'}$$

$$af'(a)\Big|_{a=0} = \frac{\partial f}{\partial\alpha}\frac{d\alpha}{da}$$

$$= a\left\{\frac{\left[-jk\alpha e^{-jk\alpha} - e^{-jk\alpha}\right](2a - 2r\sin\theta\cos\phi')}{\alpha^2 \cdot 2\alpha}\right\}\Bigg|_{a=0}$$

$$\alpha(0) = r$$

Computing second term of Maclaurin series we get:

2-41 (cont.)

$$\frac{a e^{-jkr}(jkr+1)(2r\sin\theta\cos\phi')}{2r^3}$$

$$= a e^{-jkr}\left(\frac{jk}{r}+\frac{1}{r^2}\right)\sin\theta\cos\phi'$$

$$A_\phi = \frac{Ia^2}{4\pi}\int_0^{2\pi} e^{-jkr}\left(\frac{jk}{r}+\frac{1}{r^2}\right)\sin\theta\cos^2\phi' \,(\cdot\, d\phi')$$

$$= \frac{I\pi a^2}{4\pi} e^{-jkr}\left(\frac{jk}{r}+\frac{1}{r^2}\right)\sin\theta$$

2-42

$$A_\phi = \frac{IS}{4\pi} e^{-jkr}\left(\frac{jk}{r}+\frac{1}{r^2}\right)\sin\theta$$

$$H_r = \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(r\sin\theta A_\phi)$$

$$= \frac{1}{r\sin\theta}\left[\cos\theta A_\phi + \sin\theta\frac{\partial A_\phi}{\partial\theta}\right]$$

$$= \frac{IS}{4\pi} e^{-jkr}\left(\frac{jk}{r^2}+\frac{1}{r^3}\right)\cos\theta$$

$$+ \frac{IS}{4\pi} e^{-jkr}\left(\frac{jk}{r^2}+\frac{1}{r^3}\right)\cos\theta$$

$$= \frac{IS}{2\pi} e^{-jkr}\left(\frac{jk}{r^2}+\frac{1}{r^3}\right)\cos\theta$$

$$H_\theta = -\frac{1}{r}\frac{\partial}{\partial r}(rA_\phi) = -\frac{A_\phi}{r} - \frac{\partial A_\phi}{\partial r}$$

$$= \frac{IS}{4\pi} e^{-jkr}\sin\theta\left[-\frac{jk}{r^2}-\frac{1}{r^3} + \frac{jk}{r^2}+\frac{2}{r^3}\right\} + jk\left(\frac{jk}{r}+\frac{1}{r^2}\right)\Bigg]$$

$$= \frac{IS}{4\pi} e^{-jkr}\left[\frac{-k^2}{r}+\frac{jk}{r^2}+\frac{1}{r^3}\right]\sin\theta$$

2-42 (cont.)

$$R_r = E_\phi \times H_\theta^* / I^2$$

$$= \frac{\eta |H_\theta|^2}{I^2}$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{\eta S^2}{(4\pi)^2}\left(\frac{k^2}{r}\right)^2 \sin^2\theta\, r^2 \sin\theta\, d\theta\, d\phi$$

$$= \eta \frac{2\pi}{3}\left(\frac{kS}{\lambda}\right)^2 \quad \boxed{\int_0^{\pi}\sin^3\theta = \frac{4}{3}}$$

2-43

From Eqn 2-113:

$$E_\theta = \frac{Il}{4\pi} e^{-jkr}\left(\frac{j\omega\mu}{r}+\frac{\eta}{r^2}+\frac{1}{j\omega\varepsilon r^3}\right)\cdot(\sin\theta)$$

$$H_\phi = \frac{Il}{4\pi} e^{-jkr}\left(\frac{jk}{r}+\frac{1}{r^2}\right)\sin\theta$$

From problem 2-42,

$$E_\phi = \frac{kIS}{4\pi} e^{-jkr}\left(\frac{k\eta}{r}-\frac{j\eta}{r^2}\right)\sin\theta$$

$$H_\theta = \frac{kIS}{4\pi} e^{-jkr}\left(-\frac{k}{r}+\frac{j}{r^2}+\frac{1}{kr^3}\right)\sin\theta$$

The first two terms of E_θ, E_ϕ and H_θ, H_ϕ indicate that they are circularly polarized.

2-44

$$\frac{\pi R_r}{\eta} = \frac{1}{2}\int_0^{\pi} \frac{\left[\cos\left(\frac{kL}{2}\cos\theta\right) - \cos\frac{kL}{2}\right]^2}{\sin\theta}\, d\theta$$

$$= \int_0^{\pi/2}\left\{\frac{\cos^2\left(\frac{kL}{2}\cos\theta\right)+\cos^2\frac{kL}{2}}{\sin\theta} - \frac{2\cos\left(\frac{kL}{2}\cos\theta\right)\cos\frac{kL}{2}}{\sin\theta}\right\} d\theta$$

2-44 (cont.)

Let $\cos\theta = u$

$$\frac{\pi R_r}{\eta} = \int_0^1 \frac{\cos^2\frac{kL}{2}u + \cos^2\frac{kL}{2} - 2\cos\frac{kLu}{2}\cos\frac{kL}{2}}{1-u^2}\,du$$

$$= \frac{1}{2}\int_{-1}^{1} \frac{\cos^2\frac{kLu}{2} + \cos^2\frac{kL}{2} - 2\cos\frac{kLu}{2}\cos\frac{kL}{2}}{1+u}\,du$$

$$= \frac{1}{4}\int_{-1}^{1} \frac{1+\cos kLu + 1 + \cos kL}{1+u}\,du$$

$$-\frac{1}{2}\int_{-1}^{1} \frac{\cos\left[\frac{kL}{2}(1+u)\right] + \cos\left[\frac{kL}{2}(1-u)\right]}{1+u}\,du$$

(Let $(1+u)kL = v$)

$$= \frac{1}{4}\int_0^{2kL} \frac{2+\cos kL + \cos(v-kL)}{v}\,dv$$

$$-\frac{1}{2}\int_0^{kL} \frac{\cos v}{v}\,dv - \frac{1}{2}\int_{-1}^{1} \frac{\cos\left[\frac{kL}{2}(1-u)\right]}{1+u}\,du$$

$$= \frac{1}{2}\int_0^{kL} \frac{1+\cos kL - \cos v}{v}\,dv$$

$$+\frac{1}{4}\int_0^{2kL} \frac{-\cos kL + \cos(v-kL)}{v}\,dv$$

$$-\frac{1}{2}\int_0^{1} \frac{\cos kL(1-v)}{v}\,dv$$

provided $v = \frac{1+u}{2}$

$$= \frac{1}{2}\int_0^{kL} \frac{1+\cos kL - \cos v}{v}\,dv$$

$$+\frac{1}{4}\int_0^{2kL} \frac{-\cos kL + \cos v\cos kL + \sin v \sin kL}{v}\,dv$$

$$-\frac{1}{2}\int_0^{kL} \frac{\cos kL\cos x + \sin kL \sin x}{x}\,dx$$

where $x = kLv$

2-44 (cont.)

$$= \frac{1}{2}\int_0^{kL} \frac{1-\cos v}{v}\,dv\,(1+\cos kL)$$

$$-\int_0^{kL} \frac{\sin v \sin kL}{v}\,dv$$

$$+\frac{1}{2}\left[\sin kL \int_0^{2kL}\frac{\sin v}{v}\,dv - \cos kL\int_0^{2kL}\frac{1-\cos v}{v}\,dv\right]$$

$\therefore$

$$\frac{\pi R_r}{\eta} = \frac{1}{2}\Big[C + \log kL - Ci\,kL$$

$$+\sin kL\left(\frac{1}{2}Si\,kL - Si\,kL\right)$$

$$+\frac{1}{2}\cos kL\left(C + \log\frac{kL}{2} + Ci\,2kL - 2Ci\,kL\right)\Big]$$

2-45

$$I(z) = I_m \sin k(z + l/2)$$

$$A_z = \frac{1}{4\pi}\int_{-l/2}^{l/2} I_m \sin k(z+l/2)\frac{e^{-jkR}}{R}\,dz$$

$E_\theta = j\omega\mu \sin\theta A_z$ for large r

$$\approx \frac{j\omega\mu\sin\theta\, e^{-jkr}}{4\pi r}$$

$$\cdot\int_{-l/2}^{l/2} I_m \sin k(z'+l/2)\, e^{jkz'\cos\theta}\,dz'$$

$$\mathcal{I} = \int_{-l/2}^{l/2} \sin k(z'+l/2)\, e^{jkz'\cos\theta}\,dz'$$

$$= \int_{-l/2}^{l/2} \frac{e^{jk(z'+l/2)} - e^{-jk(z'+l/2)}}{2j}\, e^{jkz'\cos\theta}\,dz'$$

$$= \int_{-l/2}^{l/2}\left\{\frac{e^{jkz'(1+\cos\theta)}e^{jkl/2}}{2j} - \frac{e^{-jkz'(1-\cos\theta)}e^{jkl/2}}{2j}\right\}dz'$$

2-45 (cont.)

$$\mathcal{J} = \frac{1}{2j}\left[\frac{e^{jkl/2}e^{jkz'(1+\cos\theta)}}{jk(1+\cos\theta)} + \frac{e^{-jkz'(1-\cos\theta)}e^{-jkl/2}}{jk(1-\cos\theta)}\right]\Bigg|_{-l/2}^{l/2}$$

$$= \frac{-1}{2k}\left[\frac{e^{jkl/2}e^{jkl/2(1+\cos\theta)}}{1+\cos\theta} - \frac{e^{j\frac{kl}{2}}e^{-j\frac{kl}{2}(1+\cos\theta)}}{1+\cos\theta} + \frac{e^{-j\frac{kl}{2}(1-\cos\theta)}e^{-jkl/2}}{1-\cos\theta} - \frac{e^{j\frac{kl}{2}(1-\cos\theta)}e^{-jkl/2}}{1-\cos\theta}\right]$$

$$= \frac{-1}{2k}\left[\frac{2je^{jkl/2}}{1+\cos\theta}\sin\left[\frac{kl}{2}(1+\cos\theta)\right] - \frac{2je^{-j\frac{kl}{2}}\sin\left[\frac{kl}{2}(1-\cos\theta)\right]}{1-\cos\theta}\right]$$

$$= \frac{-2j}{2k}\left[\frac{e^{jkl/2}}{1+\cos\theta}\left\{\sin\frac{kl}{2}\cos\left(\frac{kl}{2}\cos\theta\right) + \cos\frac{kl}{2}\sin\left(\frac{kl}{2}\cos\theta\right)\right\} - e^{-j\frac{kl}{2}}\left\{\sin\frac{kl}{2}\cos\left(\frac{kl}{2}\cos\theta\right) - \cos\frac{kl}{2}\sin\left(\frac{kl}{2}\cos\theta\right)\right\}\right]$$

$$\frac{-j}{k}\left[\sin\frac{kl}{2}\cos\left(\frac{kl}{2}\cos\theta\right)\left\{\frac{e^{jkl/2}}{\sin^2\theta} - \frac{e^{jkl/2}\cos\theta - e^{-jkl/2} - e^{-jkl/2}\cos\theta}{\sin^2\theta}\right\}\right.$$

2-45 (cont.)

$$\left.\cdot\cos\frac{kl}{2}\sin\left(\frac{kl}{2}\cos\theta\right)\left\{\frac{e^{jkl/2}}{\sin^2\theta} - \frac{e^{jkl/2}\cos\theta + e^{-jkl/2} + e^{-jkl/2}\cos\theta}{\sin^2\theta}\right\}\right]$$

$$= \frac{-j}{k}\left[\frac{\sin\frac{kl}{2}\cos\left(\frac{kl}{2}\cos\theta\right)}{\sin^2\theta}\left\{2j\sin\frac{kl}{2} - 2\cos\theta\cos\frac{kl}{2}\right\} + \frac{\cos\frac{kl}{2}\sin\left(\frac{kl}{2}\cos\theta\right)}{\sin^2\theta}\left\{2\cos\frac{kl}{2} - 2j\cos\theta\sin\frac{kl}{2}\right\}\right]$$

For $l = \frac{n\lambda}{2}$, $\frac{kl}{2} = \frac{n\pi}{2}$

$$\left.\begin{aligned}\sin\frac{n\pi}{2} &= 1\\ \cos\frac{n\pi}{2} &= 0\end{aligned}\right\}\text{ for } n \text{ odd}$$

$$\left.\begin{aligned}\sin\frac{n\pi}{2} &= 0\\ \cos\frac{n\pi}{2} &= 1\end{aligned}\right\}\text{ for } n \text{ even}$$

$$\mathcal{J} = \frac{2}{k}\frac{\cos\left(\frac{n\pi}{2}\cos\theta\right)}{\sin^2\theta}, \quad n \text{ odd}$$

$$= \frac{-2j\sin\left(\frac{n\pi}{2}\cos\theta\right)}{\sin^2\theta}, \quad n \text{ even}$$

$$E_\theta = \frac{j\omega\mu\sin\theta}{4\pi r}e^{-jkr}I_m\mathcal{J}, \quad \eta = \frac{\omega\mu}{k}$$

$$= \frac{j\eta}{2\pi r}I_m e^{-jkr}\frac{\cos\left(\frac{n\pi}{2}\cos\theta\right)}{\sin\theta}, \quad n \text{ odd}$$

$$= \frac{\eta I_m}{2\pi r}e^{-jkr}\frac{\sin\left(\frac{n\pi}{2}\cos\theta\right)}{\sin\theta}, \quad n \text{ even.}$$

2-46 $E_\theta = \eta H_\phi$

$$P = \int_0^{2\pi}\int_0^{\pi} E_\theta H_\phi^* r^2 \sin\theta \, d\theta \, d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{|E_\theta|^2}{\eta} r^2 \sin\theta \, d\theta \, d\phi$$

$$P = \int_0^{\pi} \frac{2\pi I_m^2}{(2\pi r)^2} \eta \frac{\cos^2\left(\frac{n\pi}{2}\cos\theta\right)}{\sin^2\theta} r^2 \sin\theta \, d\theta$$

$$= \int_0^{\pi} \frac{\eta I_m^2}{2\pi} \frac{\cos^2\left(\frac{n\pi}{2}\cos\theta\right)}{\sin\theta} d\theta, \quad n \text{ even}$$

$$= \int_0^{\pi} \frac{\eta I_m^2}{2\pi} \frac{\sin^2\left(\frac{n\pi}{2}\cos\theta\right)}{\sin\theta} d\theta, \quad n \text{ odd}$$

$$\int_0^{\pi} \frac{\cos^2\left(\frac{n\pi}{2}\cos\theta\right)}{\sin\theta} d\theta \qquad u = \cos\theta, \quad du = -\sin\theta \, d\theta$$

$$= \frac{1}{2}\int_{-1}^{1} \frac{(1+\cos n\pi u)}{1-u^2} du$$

$$= \frac{1}{4}\int_{-1}^{1} (1+\cos n\pi u)\left(\frac{1}{1+u} + \frac{1}{1-u}\right) du$$

$$= \frac{1}{2}\int_{-1}^{1} \frac{1+\cos n\pi u}{1+u} du \qquad \text{let } v = \pi(1+u)$$

$$= \frac{1}{2}\int_0^{2\pi} \frac{1-\cos v}{v} dv =$$

$$= \frac{1}{2}\left[\log 2n\pi\gamma - \mathrm{Ci}\, 2n\pi\right]$$

$$R_r = \frac{P}{I_m^2} = \frac{\eta}{4\pi}\left[\log\gamma + \log 2n\pi - \mathrm{Ci}\, 2n\pi\right]$$

$$I_i = I_m \sin k(z + L/2)$$

2-46 (cont.)

$$R_i = \frac{R_r}{\sin^2 k(z+L/2)}$$

$$= \frac{R_r}{\sin^2 \frac{2\pi}{\lambda}\left(a\lambda + \frac{n\lambda}{4}\right)} \qquad L = \frac{\lambda n}{2}, \quad z = a\lambda$$

$$= \frac{R_r}{\sin^2 2\pi(a + n/4)}$$

For $a = 0, n = 1$:

$$R_i = \frac{R_r}{\sin^2 \frac{2\pi}{4}} = R_r$$

3-1

Let $E_x = \begin{cases} -\dfrac{\eta J_0}{2} e^{-jkz}, & z>0 \\ -\dfrac{\eta J_0}{2} e^{jkz}, & z<0 \end{cases}$

to show that there exists a current sheet $\bar{J} = \bar{u}_x J_0$.

$$H_y = -\bar{E}_x \times \bar{u}_z$$

$$= -\frac{J_0}{2} e^{-jkz}, \quad z>0 \quad (1)$$

$$= +\frac{J_0}{2} e^{jkz}, \quad z<0 \quad (2)$$

$$\bar{J}\Big|_{z=0} = \bar{u}_z \times \left[\bar{H}^{(1)} - \bar{H}^{(2)}\right]\Big|_{z=0}$$

$$= \bar{u}_x\left[\frac{J_0}{2} + \frac{J_0}{2}\right] = J_0 \bar{u}_x$$

3-2 Let $E_x = -\dfrac{M_0}{2} \sin\dfrac{\pi y}{b} e^{-j\beta z}, \quad z>0$

$$= \frac{M_0}{2} \sin\frac{\pi y}{b} e^{j\beta z}, \quad z<0$$

$$\bar{M} = \left[\bar{E}^{(1)} - \bar{E}^{(2)}\right] \times \bar{n}$$

$$= \left[E^{(1)} - E^{(2)}\right] \bar{u}_x \times \bar{u}_z$$

$$M_y\Big|_{z=0} = \frac{M_0}{2} \sin\frac{\pi y}{b} + \frac{M_0}{2} \sin\frac{\pi y}{b}$$

$$= M_0 \sin\frac{\pi y}{b}$$

3-3 Since $\bar{M}_s = \bar{u}_y A \sin\dfrac{\pi y}{b}$ from 3-2;

$$E_x = -\frac{A}{2} \sin\frac{\pi y}{b} e^{-j\beta z}, \quad z>0$$

$$= \frac{A}{2} \sin\frac{\pi y}{b} e^{j\beta z}, \quad z<0$$

and the electric field due to current sheet $\bar{J} = \bar{u}_x \dfrac{A}{Z_0} \sin\dfrac{\pi y}{b}$

3-3 (cont.)

can be obtained from Eqn. 3-3:

$$E_x = -\frac{A}{2} \sin\frac{\pi y}{b} e^{-j\beta z}, \quad z>0$$

$$= -\frac{A}{2} \sin\frac{\pi y}{b} e^{+j\beta z}, \quad z<0$$

Adding the two fields:

$$E_x = -A \sin\frac{\pi y}{b} e^{-j\beta z}, \quad z>0$$

$$= 0, \quad z<0$$

3-4

$$E_x^{(1)} = -\frac{J_0 Z_0}{2} \sin\frac{\pi y}{b} e^{-j\beta z}, \quad z>0$$

$$= -\frac{J_0 Z_0}{2} \sin\frac{\pi y}{b} e^{j\beta z}, \quad z<0$$

$$E_x^{(2)} = \frac{J_0 Z_0}{2} \sin\frac{\pi y}{b} e^{-j\beta(z+2d)}$$

$E_x = E_x^{(1)} + E_x^{(2)}$ (no discontinuity at $z=0$)

$$E_x = 0 \text{ at } z=-d$$

$$E_x = -j J_0 Z_0 \sin\frac{\pi y}{b} \sin\beta(z+d) e^{-j\beta d}, \quad z<0$$

$$E_x = -\frac{J_0 Z_0}{2} \sin\frac{\pi y}{b} e^{-j\beta z} + \frac{J_0 Z_0}{2} \sin\frac{\pi y}{b} e^{-j\beta(z+2d)}, \quad z>0$$

$$= \frac{J_0 Z_0}{2} \sin\frac{\pi y}{b}\left(e^{-j2\beta d} - 1\right) e^{-j\beta z}, \quad z>0$$

3-5 For the configuration of problem 2-28, the fields are written as:

$$(TE_n)\ E_x = E_0 \sin\frac{n\pi y}{b}\, e^{-\gamma z},\quad n=1,2,\ldots$$

$$(TM_n)\ H_x = H_0 \cos\frac{n\pi y}{b}\, e^{-\gamma z},\quad n=0,1,\ldots$$

The dual fields for TE_n configuration are

$$H_x = H_0 \sin\frac{n\pi y}{b}\, e^{-\gamma z},\quad n=1,2,\ldots$$

This would be a field for the TM_n case when the field strength is maximum at $\pm b/2$ as conductors are placed at $y=\pm b/2$

Similarly the dual fields for the TM_n configuration are:

$$E_x = -E_0 \cos\frac{n\pi y}{b}\, e^{-\gamma z},\quad n=0,1,\ldots$$

which would be the field for a TE_n case only when the field is $E_x=0$ achieved by placing conductors at $y=\pm b/2$

3-6 The dual fields from a magnetic current loop come from an electric wire loop (Fig. 2-26). Hence the fields for the magnetic current loop having a z-directed moment KS are:

$$E_r = -\frac{KS}{2\pi}\, e^{-jkr}\left(\frac{jk}{r^2}+\frac{1}{r^3}\right)\cos\theta$$

$$E_\theta = -\frac{KS}{4\pi}\, e^{-jkr}\left(-\frac{k^2}{r}+\frac{jk}{r^2}+\frac{1}{r^3}\right)\sin\theta$$

3-6 (cont.)

$$H_\phi = \frac{KS}{4\pi\eta}\, e^{-jkr}\left(\frac{k^2}{r}-\frac{jk}{r^2}\right)\sin\theta$$

The fields due to a current dipole of moment Il are given by Eqn. 2-113:

$$H_\phi = \frac{Il}{4\pi}\, e^{-jkr}\left(\frac{jk}{r}+\frac{1}{r^2}\right)\sin\theta$$

The two H_ϕ's can only be equal if $\frac{KS}{4\pi\eta}(-jk) = \frac{Il}{4\pi}$

or $Il = -j\omega\varepsilon KS$

3-7 Z_{TS} = impedance of twin slot line

Z_{cp} = impedance of collinear plate

For twin slot line:

$$I = 2\int_{c_1}\bar{H}_{TS}\cdot d\bar{l},\quad V=\int_{c_2}\bar{E}_{TS}\cdot d\bar{l}$$

$$Z_{TS} = \frac{\int_{c_2}\bar{E}_{TS}\cdot d\bar{l}}{2\int_{c_1}\bar{H}_{TS}\cdot d\bar{l}}$$

For collinear plate:

$$I = 2\int_{c_2}\bar{H}_{cp}\cdot d\bar{l},\quad V=-\int_{c_1}\bar{E}_{cp}\cdot d\bar{l}$$

$$Z_{cp} = \frac{-\int_{c_1}\bar{E}_{cp}\cdot d\bar{l}}{2\int_{c_2}\bar{H}_{cp}\cdot d\bar{l}}$$

As $r\to\infty$, $E_{TS} = \eta H_{TS}$

and $E_{cp} = -\eta H_{cp}$

3-7 (cont.)

$$\therefore Z_{TS} \cdot Z_{cp} = \frac{\int_{c_2} \bar{E}_{TS} \cdot d\bar{l} \int_{c_1} -\bar{E}_{cp} \cdot d\bar{l}}{4 \int_{c_1} \bar{H}_{TS} \cdot d\bar{l} \int_{c_2} \bar{H}_{cp} \cdot d\bar{l}}$$

$$= \frac{\eta^2}{4}$$

$$Z_{TS} = \frac{\eta^2}{4 Z_{cp}}$$

For both the slotted line and collinear plate problems we need to solve the field equations:

① $\nabla^2 \bar{E} = k_x^2 \bar{E}$

② $\nabla^2 \bar{H} = k_x^2 \bar{H}$

on the collinear plate, ② is solved subject to:

$H_y = H_z = 0$ at $x = 0$ from symmetry.

$H_x = 0$ at $x = 0$ because $\hat{n} \cdot \bar{H} = 0$ at perfect conductor.

For twin slotted line ① is solved subject to:

$E_y = E_z = 0$ at $x = 0$ because $\bar{E}_{tan} = 0$ at a perfect conductor and $E_x = 0$ at $x = 0$ from symmetry.

3-8

$$H_y^- = -\frac{J_0 Z_0}{2} \sin\frac{\pi y}{b} e^{j\beta z}, \quad z > 0$$

$$H_y^+ = \frac{J_0 Z_0}{2} \sin\frac{\pi y}{b} e^{-j\beta z}, \quad z < 0$$

$$\bar{J}_s = -\bar{u}_x \left[H_y^- - H_y^+ \right] \Big|_{z=0}$$

3-8 (cont.)

$$\bar{J}_s = \bar{u}_x J_0 \sin\frac{\pi y}{b}$$

These are the incoming waves instead of the outgoing waves developed in section 3-2. As in section 2-9 it was reasoned that waves must travel outward from the source, not inward. The above solutions do not vanish as $z \to \infty$ and therefore cannot be solutions in loss-free media. Any $\bar{H}$ field of the form:

$$H_y^- = -\frac{J_0 Z_0}{2} \sin\frac{\pi y}{b} f(\beta z), \quad z > 0$$

$$H_y^+ = \frac{J_0 Z_0}{2} \sin\frac{\pi y}{b} f(-\beta z), \quad z < 0$$

is also a solution which gives the required current distribution if $f(\beta z)\big|_{z=0} = f(-\beta z)\big|_{z=0} = 1$

but are not physically realizable unless $\lim_{z \to \infty} f(\beta z) = \lim_{z \to \infty} f(-\beta z) = 0$

3-9 Start with fields given by Eqn 2-113 and find the current sheets.

$$E_r = \frac{I\ell}{2\pi} e^{-jkr} \left(\frac{\eta}{r^2} + \frac{1}{j\omega\varepsilon r^3} \right) \cos\theta$$

$$E_\theta = \frac{I\ell}{4\pi} e^{-jkr} \left(\frac{j\omega\mu}{r} + \frac{\eta}{r^2} + \frac{1}{j\omega\varepsilon r^3} \right) \sin\theta$$

$$H_\phi = \frac{I\ell}{4\pi} e^{-jkr} \left(\frac{jk}{r} + \frac{1}{r^2} \right) \sin\theta$$

$$\bar{J}_s = \hat{n} \times (\bar{H}_1 - \bar{H}_2) = \hat{n} \times \bar{H}_\phi = -\bar{u}_\theta H_\phi$$

3-9 (cont.)

$H_2 = 0$ inside current sphere

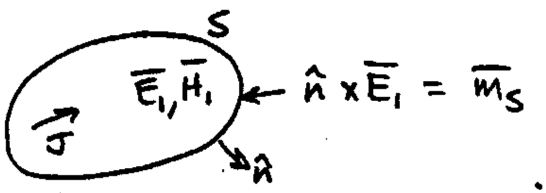
$$\bar{J} = -\bar{u}_\theta \frac{I\ell}{4\pi} e^{-jka}\left(\frac{jk}{a} + \frac{1}{a^2}\right)\sin\theta$$

$$\bar{m} = (E_1 - E_2) \times \hat{n} \qquad (E_2 = 0)$$

$$\bar{m} = \bar{u}_\theta \times E_r = -\bar{u}_\phi E_\theta$$

$$\bar{m} = -\bar{u}_\phi \frac{I\ell}{4\pi} e^{-jka}\left(\frac{j\omega\mu}{a} + \frac{\eta}{a^2} + \frac{1}{j\omega\varepsilon a^2}\right) \cdot (\sin\theta)$$

3-10 Given a surface S enclosing a source $\bar{J}$:



Fields $\bar{E}_1, \bar{H}_1$ are uniquely specified since $\hat{n} \times \bar{E}$ is known over S from problem statement and $\bar{J}$ is given. Also since $\hat{z}\bar{H}_1 = -\nabla \times \bar{E}_1$,

$$\nabla \times \bar{H}_1 = \hat{y}\bar{E}_1 + \bar{J}$$

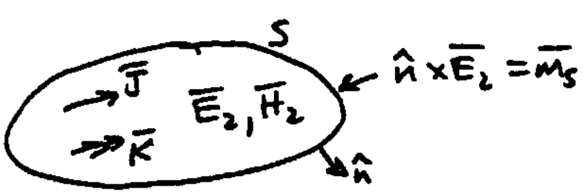
$$= \hat{y}\bar{E}_1 + \left(-\hat{y}\bar{E}_1 - \frac{1}{\hat{z}}\nabla\times\nabla\times\bar{E}_1\right)$$

$$= \hat{y}E_1 - \hat{y}E_1 - \frac{1}{\hat{z}}\left(\hat{z}\,\nabla\times\bar{H}_1\right)$$

$$= \nabla\times\bar{H}_1$$

Thus (E_1, H_1) is supported by $\bar{J}$

If we now have:



3-10 (cont.) Let $\bar{E}_1 = \bar{E}_2$

$$\bar{H}_2 = \frac{1}{\hat{z}}(-\nabla\times\bar{E}_1) - \bar{K}$$

$$\nabla\times\bar{H}_2 = \hat{y}\bar{E}_1 + \bar{J}$$

$$= \hat{y}\bar{E}_1 + \left(-\hat{y}E_1 - \frac{1}{\hat{z}}\nabla\times\nabla\times\bar{E}_1\right) - \bar{K}$$

$$= \nabla\times\bar{H}_1 - \bar{K}$$

so $\bar{H}_1 \neq \bar{H}_2$

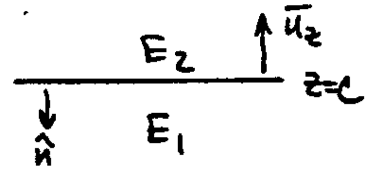
Sources $\bar{J}$ and $\bar{K}$ support $(\bar{E}_1, \bar{H}_2)$ which is in the same class as (E_1, H_1) because of uniqueness imposed by $\hat{n}\times\bar{E}_1 = \bar{m}_s$ in both cases.

3-11 $E_x = E_0 \sin\frac{\pi y}{b} \sinh \gamma z$

at wall surface, $z = c$, $\hat{n} = -\bar{u}_z$

Eqns 1-86 are:

$$\left[E^1 - E^2\right]\times\hat{n}\Big|_{z=c} = \bar{m}_s$$



$$E^2_t = 0$$

$$E^1 = E_x \bar{u}_x\Big|_{z=c}$$

$$\bar{m}_s = (E_x\bar{u}_x)\times(-\bar{u}_z)\Big|_{z=c}$$

$$= -\bar{u}_y E_x\Big|_{z=c}$$

$$= -\bar{u}_y E_0 \sin\frac{\pi y}{b}\sinh\gamma c$$

$\bar{m}_s$ uniquely specifies the field E_x as given. For a good dielectric (low loss):

$$k = k' - jk''$$

$$= \omega\sqrt{\mu\varepsilon'} - j\frac{\omega\varepsilon''}{2}\sqrt{\frac{\mu}{\varepsilon'}} \quad \text{from Eqn 2-39.}$$

$$= \omega\sqrt{\frac{\mu}{\varepsilon'}}\left[\varepsilon' - j\frac{\omega\varepsilon''}{2}\right], \quad \varepsilon'' << \varepsilon'$$

3-11 (cont.) So $k^2 = \omega^2 \mu \varepsilon'$

$$\gamma = \sqrt{\left(\frac{\pi}{b}\right)^2 - k^2} = \sqrt{\left(\frac{\pi}{b}\right)^2 - \omega^2 \mu \varepsilon'}$$

at resonance,

$$\omega_r = \frac{2\pi}{2bc}\sqrt{\frac{b^2+c^2}{\mu\varepsilon'}}$$

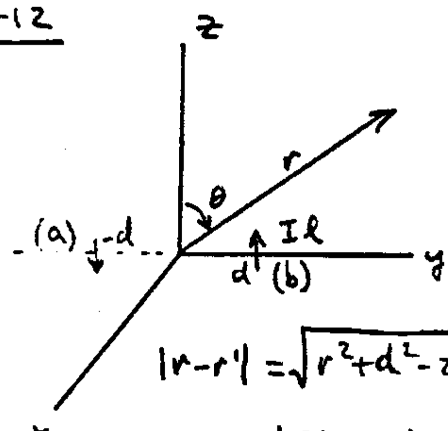
$$\gamma = \sqrt{\left(\frac{\pi}{b}\right)^2 - \left(\frac{\pi}{bc}\right)^2 (b^2+c^2)}$$

$$\sinh \gamma c = \sinh \sqrt{\left(\frac{c\pi}{b}\right)^2 - \left(\frac{c\pi}{b}\right)^2 - \pi^2}$$

$$= \sinh i\pi = 0$$

so as ε'' becomes much smaller than ε', $m_s \to 0$.

3-12



$$|r - r'| = \sqrt{r^2 + d^2 - 2rd\sin\theta\sin\phi}$$

$$= r - d\sin\theta\sin\phi, \quad r \gg d$$

$$A_z \approx \frac{e^{-jkr}}{4\pi r} I\ell\, e^{jkd\sin\theta\sin\phi}$$

$E_\theta = j\omega\mu \sin\theta A_z$ in the far-field

Adding the contribution from the image we obtain:

$$E_\theta = \frac{j\omega\mu I\ell e^{-jkr}}{4\pi r}\sin\theta \left[e^{jkd\sin\theta\sin\phi} - e^{-jkd\sin\theta\sin\phi}\right]$$

$$= j\frac{\eta I\ell e^{-jkr}}{2\lambda r} 2j \sin(kd\sin\theta\sin\phi)\sin\theta$$

3-12 (cont.)

$$E_\theta = -\frac{\eta I\ell e^{-jkr}}{\lambda r}\sin(kd\sin\theta\sin\phi)\sin\theta$$

$R_r = R_r$ of single element plus R_r of its image.

$\tilde{P}_f$ of single current element is from Eqn 2-116:

$$\tilde{P}_f = \frac{\eta 2\pi}{3}\left|\frac{I\ell}{\lambda}\right|^2, \quad R_r = \frac{\tilde{P}_f}{|I|^2}$$

$$R_r = \frac{2}{3}\frac{\eta\pi\ell^2}{\lambda^2}$$

Now from reciprocity,

$$\tilde{P}_f \text{ of image} = \mathrm{Re}\left\{\int E_a|_b \cdot J_b \, d\tau\right\}$$

$$J_b = I\ell\,\delta(y-d)$$

$$P = \int E_a \cdot J_b \, d\tau$$

$$= \frac{-(I\ell)^2 e^{-jkr}}{4\pi}\left(\frac{j\omega\mu}{r} + \frac{\eta}{r^2} + \frac{1}{j\omega\varepsilon r^3}\right)$$

from Eqn. 2-113 at $\theta = \pi/2$, $r = 2d$.

Note that $k\eta = \omega\mu$, $\omega\varepsilon = \frac{k}{\eta}$

$$P = \frac{-I^2\ell^2}{4\pi}\left[(\cos 2kd - j\sin 2kd)\frac{jk\eta}{2d} + (\cos 2kd - j\sin 2kd)\frac{\eta}{(2d)^2} + (\cos 2kd - j\sin 2kd)\frac{\eta}{jk(2d)^3}\right]$$

$$\tilde{P}_f = \mathrm{Re}(P) = \frac{I^2\ell^2}{4\pi}\left[-\sin 2kd\left(\frac{k\eta}{2d}\right) - \cos 2kd\frac{\eta}{(2d)^2} + \sin 2kd\frac{\eta}{k(2d)^3}\right]$$

3-12 (cont.) Now factoring out η and k^2 we get:

$$\tilde{P}_f = \frac{\eta \pi I^2 \ell^2}{\lambda^2}\left[-\frac{\sin 2kd}{2kd} - \frac{\cos 2kd}{(2kd)^2} + \frac{\sin 2kd}{(2kd)^3}\right]$$

And adding this $\tilde{P}_f$ to that of a single element and dividing by $|I|^2$:

$$R_r = \frac{\eta \pi \ell^2}{\lambda^2}\left[\frac{2}{3} - \frac{\sin 2kd}{2kd} - \frac{\cos 2kd}{(2kd)^2} + \frac{\sin 2kd}{(2kd)^3}\right]$$

Let $A = -\frac{\sin 2kd}{2kd} - \frac{\cos 2kd}{(2kd)^2} + \frac{\sin 2kd}{(2kd)^3}$

$$A = \left[-1 + \frac{(2kd)^2}{3!} \pm \ldots - \frac{1}{(2kd)^2} + \frac{1}{2!} - \frac{(2kd)^2}{4!} + \ldots + \frac{1}{(2kd)^2} - \frac{1}{3!} + \frac{(2kd)^2}{5!} - \ldots\right]$$

$$= -\frac{2}{3} + \frac{8}{15}(kd)^2 \text{ as } kd \to 0$$

$$R_r = \frac{\eta \pi \ell^2}{\lambda^2}\left[\frac{2}{3} - \frac{2}{3} + \frac{8}{15}\frac{(2\pi d)^2}{\lambda^2}\right]$$

$$= \frac{\eta\, 32 \pi^3 \ell^2 d^2}{15 \lambda^4} \text{ as } kd \to 0$$

For small d,

$$g = \frac{4\pi r^2 \eta\, |H_\phi|^2}{\tilde{P}_f} \quad \text{at } \theta = \phi = \frac{\pi}{2}$$

$$= \frac{4\pi r^2 \eta \left|\frac{I\ell}{\lambda r}\right|^2 \sin^2 kd}{\eta \pi \left|\frac{I\ell}{\lambda}\right|^2 \left[\frac{2}{3} + A\right]}$$

3-12 (cont.)

$$g \approx \frac{4(kd)^2}{\frac{8}{15}(kd)^2} = 7.5$$

For $d = \lambda/4$, $kd = \pi/2$

$$g = \frac{4\,(1)}{\left[\frac{2}{3} - 0 + \frac{1}{\pi^2} + 0\right]} = 5.21$$

For d large,

$$g = \frac{4 \sin^2 kd}{\left[\frac{2}{3} - \frac{\sin 2kd}{2kd} - \frac{\cos 2kd}{(2kd)^2} + \frac{\sin 2kd}{(2kd)^3}\right]}$$

$$= \frac{4}{2/3} = 6.$$

3-13 The dual of a small loop of electric current is a magnetic current of moment $K\ell$ such that $K\ell = j\omega\mu I S$.

$$E_\phi = \frac{K\ell}{\lambda r} e^{-jkr} \sin(kd \cos\theta) \sin\theta$$

The results of the electric and magnetic currents are related by Eqn. 3-19 (its dual).

$$E_\phi = \frac{j\omega\mu k}{\lambda} \frac{e^{-jkr}}{r} I S \sin(kd\cos\theta)\sin\theta$$

$$= \frac{j\eta 2\pi I S}{\lambda^2 r} e^{-jkr} \sin(kd\cos\theta)\sin\theta$$

as $kd \to 0$,

$$E_\phi = \frac{j\eta e^{-jkr} \pi I S kd}{\lambda^2 r} \sin 2\theta$$

Finding time average power $\tilde{P}_f$ as:

3-13 (cont.)

$$\tilde{P}_f = \int_0^{\pi}\int_0^{\pi/2} -E_\phi H_\theta^* r^2 \sin\theta \, d\theta \, d\phi$$

$$= \underbrace{\eta 2\pi k^2 \left(\frac{IS}{\lambda}\right)^2}_{A} \int_0^{\pi/2} \sin^2(kd\cos\theta)\sin^3\theta \, d\theta$$

Let $u = \cos\theta$

$du = -\sin\theta \, d\theta$

$$\tilde{P}_f = -A\int_1^0 \sin^2(kdu)(1-u^2)\,du$$

$$= A\int_0^1 \sin^2(kdu)(1-u^2)\,du$$

$$= A\left[\int_0^1 \tfrac{1}{2}(1-\cos 2kdu)\,du - \int_0^1 u^2\sin^2(kdu)\,du\right]$$

$$= A\left\{ \frac{u}{2}\Big|_0^1 - \frac{1}{4kd}\sin 2kdu\Big|_0^1 - \frac{u^3}{6}\Big|_0^1 + \frac{2u\cos 2kdu}{(2kd)^2} + \frac{((2kd)^2u^2-2)\sin 2kdu}{(2kd)^3}\Bigg|_0^1 \right.$$

$$= A\left\{ \frac{1}{2} - \frac{1}{2}\frac{\sin 2kd}{2kd} - \frac{1}{6} + \frac{2\cos 2kd}{2(2kd)^2} + \frac{[(2kd)^2-2]\sin 2kd}{2(2kd)^3} \right\}$$

$$\tilde{P}_f = A\left\{ \frac{1}{3} - \frac{1}{2}\frac{\sin 2kd}{2kd} + \frac{\cos 2kd}{(2kd)^2} + \frac{1}{2}\frac{\sin 2kd}{2kd} - \frac{2\sin 2kd}{2(2kd)^3} \right\}$$

$$= 2\pi\eta k^2\left(\frac{IS}{\lambda}\right)^2\left[\frac{1}{3} + \frac{\cos 2kd}{(2kd)^2} - \frac{\sin 2kd}{(2kd)^3}\right]$$

$$R_r \xrightarrow[kd\to 0]{} 2\pi\eta\left(\frac{kS}{\lambda}\right)^2\left[\frac{1}{3} + \frac{1}{(2kd)^2} - \frac{1}{2} + \frac{(2kd)^2}{4!} - \cdots - \frac{1}{(2kd)^2} + \frac{1}{6} - \frac{(2kd)^2}{5!} + \cdots\right]$$

3-13 (cont.)

$$= 2\pi\eta\left(\frac{kS}{\lambda}\right)^2\left[\frac{(2kd)^2}{24} - \frac{(2kd)^2}{120} \pm \cdots\right]$$

$$= 2\pi\eta\left(\frac{kS}{\lambda}\right)^2(kd)^2\left(\frac{1}{30}\right)$$

$$= \frac{\pi\eta}{15}\left(\frac{kSkd}{\lambda}\right)^2$$

3-14

$$F_z = \frac{K\ell}{4\pi}\frac{e^{-jkr}}{r}\left[e^{jkd\sin\phi\sin\theta} + e^{-jkd\sin\phi\sin\theta}\right]$$

$$= \frac{K\ell}{2\pi}\frac{e^{-jkr}}{r}\cos(kd\sin\phi\sin\theta)$$

$K\ell = j\omega\mu IS$ from dual of Eqn 3-19.

$E_\phi = -j\omega\varepsilon\eta\sin\theta F_z$ from dual of Eqn. 2-123.

$$E_\phi = (-j\omega\varepsilon\eta)j\omega\mu IS\frac{e^{-jkr}}{2\pi r}\sin\theta \cdot \cos(kd\sin\phi\sin\theta)$$

$$= \frac{\omega^2\mu\varepsilon\eta IS e^{-jkr}}{2\pi r}\sin\theta\cos(kd\sin\phi\sin\theta)$$

R_r = R_r of image plus R_r of single current loop. Replace electric current loops with dual magnetic current elements. R_r of single current loop is from problem 2-42:

$$R_r = \frac{\eta 2\pi}{3}\left(\frac{kS}{\lambda}\right)^2$$

3-14 (cont.)

As in prob. 3-12,

$$\tilde{P}_f = Re\left\{\int H_a \cdot M_b \, d\tau\right\}$$

H_a from dual of Eqn 2-113 :

$$H_\theta = -\frac{Kl}{4\pi} e^{-jkr}\left(\frac{j\omega\varepsilon}{r} + \frac{1}{\eta r^2} + \frac{1}{j\omega\mu r^3}\right)$$

at $\theta = \pi/2$, $r = 2d$.

$$P = \int H_a \cdot M_b \, d\tau$$

$$= \frac{\omega^2\mu^2 (IS)^2}{4\pi} e^{-jk2d}\left(\frac{jk}{\eta r} + \frac{1}{\eta r^2} + \frac{1}{jk\eta r^3}\right)$$

$$\omega^2\mu^2 = \frac{k^2\mu}{\varepsilon} = (k\eta)^2$$

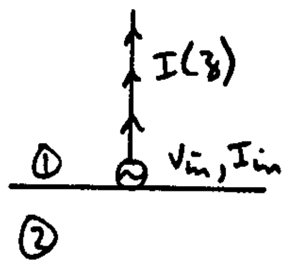
$$P = \frac{k^2 (kIS)^2 \eta^2}{\eta 4\pi}\left(\cos 2kd - j\sin 2kd\right)\left(\frac{j}{2kd} + \frac{1}{(2kd)^2} + \frac{1}{j(2kd)^3}\right)$$

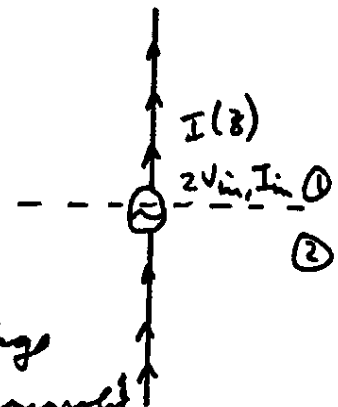
$$\tilde{P}_f = Re(P) = \pi\eta\left(\frac{kS}{\lambda}\right)^2 I^2\left(\frac{\sin 2kd}{2kd} + \frac{\cos 2kd}{(2kd)^2} - \frac{\sin 2kd}{(2kd)^3}\right)$$

Adding this to $\tilde{P}_f$ for single element and dividing by $|I^2|$ we get:

$$R_r = \eta\pi\left(\frac{kS}{\lambda}\right)^2\left[\frac{2}{3} + \frac{\sin 2kd}{2kd} + \frac{\cos 2kd}{(2kd)^2} - \frac{\sin 2kd}{(2kd)^3}\right]$$

3-15





The applied voltage for the dipole (monopole plus its image) is twice that of the monopole alone. Fields of monopole are same as those of dipole in region ① only.

$$R_D = \frac{\tilde{P}_{fD}}{|I_{in}|^2} = 2\left|\frac{V_{in}}{I_{in}}\right|$$

D - denotes dipole

m - denotes monopole

$$R_m = \frac{\tilde{P}_{fm}}{|I_{in}|^2} = \left|\frac{V_{in}}{I_{in}}\right|$$

$$\therefore\ R_m = \frac{1}{2} R_D$$

$$\frac{g_D}{g_m} = \frac{P_{fm}}{P_{fD}}$$ Since fields are same for both in region ①.

$$g_m = \frac{g_D P_{fD}}{P_{fm}} = 2 g_D$$

3-16

$$\bar{E}_1 = E_\rho \bar{u}_\rho = \frac{-V}{\rho \ln\left(\frac{b}{a}\right)} \bar{u}_\rho$$

$$M_S = \bar{E}_1 \times \bar{u}_z = \frac{V}{\rho \ln \frac{b}{a}} \bar{u}_\phi$$

This is a loop of magnetic current

3-16 (cont.)

which, if $b \ll \lambda$, acts as an electric dipole. The current can be represented as a continuous distribution of magnetic current filaments of strength $dK = M_\phi \, d\rho$. The total moment of the source is then:

$$KS = \int_a^b \pi \rho^2 \, dK = \frac{\pi V (b^2 - a^2)}{2 \ln b/a}$$

The dual equivalent electric current element must satisfy the equality: $Il = -j\omega\varepsilon KS$

$$H_\phi = \frac{jIl}{2\pi} \frac{e^{-jkr}}{r} \sin\theta$$

$$= \frac{\omega\varepsilon\pi V (b^2 - a^2) e^{-jkr} \sin\theta}{4\lambda r \ln b/a}$$

$$= \frac{1}{2} \text{ of Eqn. 3-20.}$$

Radiated power $= P = \int_0^{2\pi} d\phi \int_0^{\pi} \eta |H_\phi|^2 \sin^3\theta \, d\theta$

$$P = \left| \frac{\omega\varepsilon\pi V (b^2 - a^2)}{4\lambda \ln b/a} \right|^2 2\pi\eta \left(\frac{4}{3}\right)$$

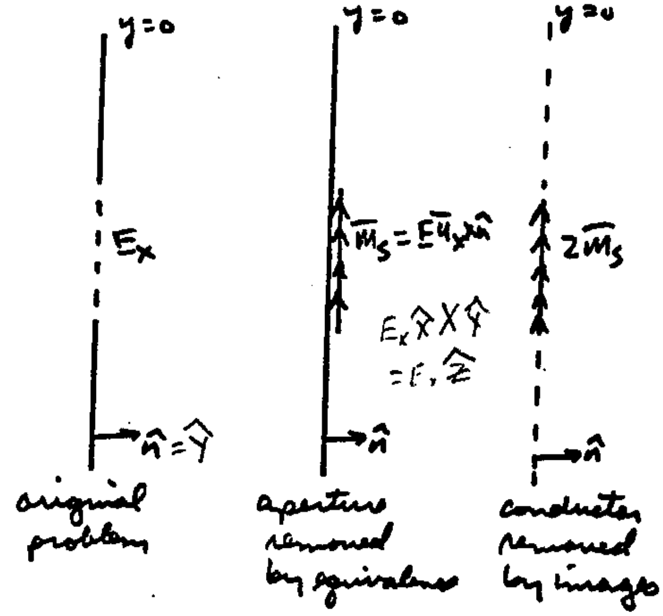
$$= \frac{1}{2} \tilde{P}_f \quad (\text{of Eqn 3-21})$$

$$G_r = \frac{\tilde{P}_f}{V^2} = \frac{1}{2} \text{ of Eqn 3-23.}$$

3-17 Using the equivalence principle we can replace the aperture by a patch of magnetic

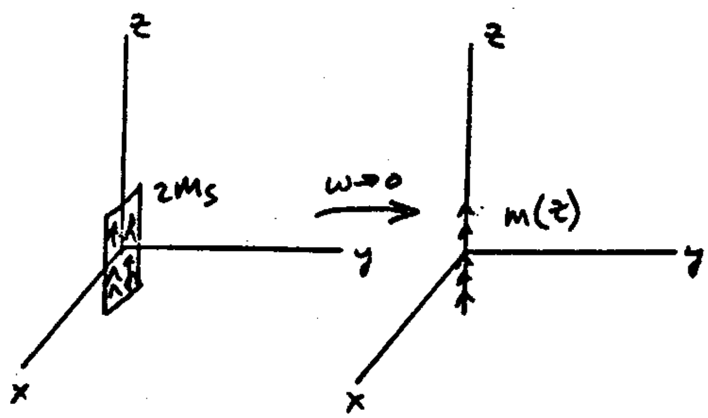
3-17 (cont.)

current.



$$M_s = \frac{2V_m}{w} \sin\left[k\left(\frac{L}{2} - |z|\right)\right]$$

As $w \to 0$ the equivalent magnetic current behaves as a line filament:



$m(z)$ replaces $I(z)$

Using the dual of Eqn. 2-125 and making the following substitutions:

$I_m \to 2V_m$, $E_\theta \to H_\theta$ $\quad y > 0$

$\eta \to 1/\eta$ $\quad E_\theta \to -H_\theta$ $\quad y < 0$

$$\frac{jV_m e^{-jkr}}{\eta\pi r} \frac{\cos\left(\frac{kL}{2}\cos\theta\right) - \cos\left(\frac{kL}{2}\right)}{\sin\theta} = \begin{cases} H_\theta, & y > 0 \\ -H_\theta, & y < 0 \end{cases}$$

3-17 (cont.)

$$G_r = \tilde{P}_f / |V_m|^2$$

using Eqn 2-127,

$$\tilde{P}_{f(slot)} = \frac{4|V_m|^2}{\eta 2\pi} \int_0^{\pi} \frac{\left[\cos\left(\frac{kL}{2}\cos\theta\right) - \cos\frac{kL}{2}\right]^2}{\sin\theta} d\theta$$

and Eqn. 1-129,

$$R_{r(wire)} = \frac{\eta}{2\pi} \int_0^{\pi} \frac{\left[\cos\left(\frac{kL}{2}\cos\theta\right) - \cos\frac{kL}{2}\right]^2}{\sin\theta} d\theta$$

$$\eta^2 \tilde{P}_{f(slot)} = 4|V_m|^2 R_{r(wire)}$$

$$\therefore G_{r(slot)} = \frac{4 R_{r(wire)}}{\eta^2}$$

3-18

$$\bar{F} = \frac{1}{4\pi} \int_{-L/2}^{L/2} \int_{-w/2}^{w/2} 2\bar{m} \frac{e^{-jkr}}{r} e^{jkr'\cos\xi} dx'dy'$$

for the far field.

$$\bar{m}(r') = E_x \bar{u}_x \times \bar{u}_y$$

$$= \frac{V_m}{w} \sin\left[k\left(\frac{L}{2} - |z'|\right)\right] \bar{u}_z$$

$$r'\cos\xi = (x'\cos\phi + y'\sin\phi\sin\theta + z'\cos\theta)$$

$$y' = 0$$

$$F_z = \frac{1}{4\pi} \int_{-L/2}^{L/2} dz' \int_{-w/2}^{w/2} dx' \frac{2V_m}{w} \sin\left[k\left(\frac{L}{2} - |z'|\right)\right] \frac{e^{-jkr}}{r} e^{jk(x'\cos\phi\sin\theta + z'\cos\theta)}$$

$$= \frac{V_m}{2\pi w} \frac{e^{-jkr}}{r} \int_{-w/2}^{w/2} e^{jkx'\cos\phi\sin\theta} dx' \Bigg[$$

3-18 (cont.)

$$\int_0^{L/2} \sin k\left(\frac{L}{2} - z'\right) + \int_{-L/2}^{0} \sin k\left(\frac{L}{2} - z'\right)\Bigg] e^{jkz'\cos\theta} dz'$$

$$F_z = \frac{V_m}{2\pi w} \frac{e^{-jkr}}{r} \left.\frac{e^{jkx'\cos\phi\sin\theta}}{jk\cos\phi\sin\theta}\right|_{-w/2}^{w/2}$$

$$\cdot \left[\int_0^{L/2} \sin k\left(\frac{L}{2} - z'\right) + \int_{-L/2}^{0} \sin k\left(\frac{L}{2} + z'\right)\right] e^{jkz'\cos\theta} dz'$$

Carrying out the integration of the last terms above we have,

$$\int_0^{L/2} \sin k\left(\frac{L}{2} - z'\right) e^{jkz'\cos\theta} dz'$$

$$= \int_0^{L/2} \frac{e^{j\frac{kL}{2}} e^{-jkz'(1-\cos\theta)} - e^{-j\frac{kL}{2}} e^{jkz'(1+\cos\theta)}}{2j} dz'$$

$$= \frac{1}{2j} \left[\frac{e^{j\frac{kL}{2}} e^{-jkz'(1-\cos\theta)}}{-jk(1-\cos\theta)} - \frac{e^{-j\frac{kL}{2}} e^{jkz'(1+\cos\theta)}}{jk(1+\cos\theta)}\right]_0^{L/2}$$

$$= \frac{1}{2j} \left[\frac{e^{j\frac{kL}{2}\cos\theta}}{-jk(1-\cos\theta)} - \frac{e^{j\frac{kL}{2}\cos\theta}}{jk(1+\cos\theta)} + \frac{e^{j\frac{kL}{2}}}{jk(1-\cos\theta)} + \frac{e^{-j\frac{kL}{2}}}{jk(1+\cos\theta)}\right],$$

$$\int_{-L/2}^{0} \sin k\left(\frac{L}{2} + z'\right) e^{jkz'\cos\theta} dz'$$

$$= \frac{1}{2j} \int_{-L/2}^{0} e^{j\frac{kL}{2}} e^{jkz'(1+\cos\theta)} - e^{-j\frac{kL}{2}} e^{-jkz'(1-\cos\theta)} dz'$$

$$= \frac{1}{2j} \left[\frac{e^{j\frac{kL}{2}}}{jk(1+\cos\theta)} + \frac{e^{-j\frac{kL}{2}}}{jk(1-\cos\theta)} - \frac{e^{-j\frac{kL}{2}\cos\theta}}{jk(1+\cos\theta)} - \frac{e^{-j\frac{kL}{2}\cos\theta}}{jk(1-\cos\theta)}\right]$$

3-18 (cont.)

$$\therefore F_z = \frac{V_m}{2\pi w}\frac{e^{-jkr}}{r}\frac{2\sin\left(\frac{kw}{2}\cos\phi\sin\theta\right)}{k\cos\phi\sin\theta}$$

$$\cdot\left[\frac{e^{j\frac{kL}{2}\cos\theta}+e^{-j\frac{kL}{2}\cos\theta}-e^{j\frac{kL}{2}}+e^{-j\frac{kL}{2}}}{k\sin^2\theta}\right]$$

$$F_z = \frac{V_m}{\pi}\frac{e^{-jkr}}{r}\frac{\sin\left(\frac{kw}{2}\cos\phi\sin\theta\right)}{\left(\frac{kw}{2}\sin\theta\cos\phi\right)}$$

$$\cdot\frac{\cos\left(\frac{kL}{2}\cos\theta\right)-\cos\left(\frac{kL}{2}\right)}{k\sin^2\theta}$$

$$H_\theta = j\omega\varepsilon\sin\theta F_z \qquad , y>0$$

$$=\frac{jV_m}{\eta\pi r}e^{-jkr}\frac{\sin\left(\frac{kw}{2}\cos\phi\sin\theta\right)}{\left(\frac{kw}{2}\cos\phi\sin\theta\right)}$$

$$\cdot\frac{\cos\left(\frac{kL}{2}\cos\theta\right)-\cos\left(\frac{kL}{2}\right)}{\sin\theta}$$

$$\frac{jV_m}{\eta\pi r}e^{-jkr}f(\theta,\phi) = H_\theta,\ y>0$$
$$= -H_\theta,\ y<0$$

$$f(\theta,\phi) = \frac{\sin\left(\frac{kw}{2}\cos\phi\sin\theta\right)}{\frac{kw}{2}\cos\phi\sin\theta}\frac{\left(\cos\left(\frac{kL}{2}\cos\theta\right)-\cos\frac{kL}{2}\right)}{\sin\theta}$$

3-19

$$E_x = \cos\frac{\pi z}{b}$$

$$\overline{m}_s = 2\overline{E}_x\times\overline{u}_y = 2\cos\frac{\pi z}{b}\,\overline{u}_z$$

$$F_z = \frac{1}{4\pi}\int_{-a/2}^{a/2}dx'\int_{-b/2}^{b/2}dz'\frac{e^{-jkr}}{r}e^{jk[x'\cos\phi\sin\theta+z'\cos\theta]}$$

$$\cdot\left\{e^{j\frac{\pi z'}{b}}+e^{-j\frac{\pi z'}{b}}\right\}$$

$$=\frac{e^{-jkr}}{2\pi r}\frac{\sin\left[\frac{ka}{2}\cos\phi\sin\theta\right]}{\ldots}\ (x)$$

3-19 (cont.)

$$\cdot\left[\frac{e^{jz'[\pi/b+k\cos\theta]}}{j[\pi/b+k\cos\theta]}+\frac{e^{jz'[k\cos\theta-\pi/b]}}{j[k\cos\theta-\pi/b]}\right]$$

This expression evaluated from $-b/2$ to $+b/2$.

$$F_z = \frac{e^{-jkr}}{2\pi r}\frac{\sin\left(\frac{ka}{2}\cos\phi\sin\theta\right)}{k\cos\phi\sin\theta}$$

$$\cdot 2\cos\left(\frac{kb}{2}\cos\theta\right)\left[\frac{1}{k\cos\theta+\frac{\pi}{b}}-\frac{1}{k\cos\theta-\frac{\pi}{b}}\right]$$

$$=\frac{e^{-jkr}}{r}\frac{\sin\left(\frac{ka}{2}\cos\phi\sin\theta\right)}{\frac{k}{2}\cos\phi\sin\theta}\frac{b\cos\left(\frac{kb}{2}\cos\theta\right)}{(\pi^2-k^2b^2\cos^2\theta)}$$

$$H_\theta = j\omega\varepsilon\sin\theta F_z$$

$$=\frac{2j}{\eta}\frac{e^{-jkr}}{r}\frac{b\sin\left(\frac{ka}{2}\cos\phi\sin\theta\right)}{\cos\phi}\frac{\cos\left(\frac{kb}{2}\cos\theta\right)}{(\pi^2-k^2b^2\cos^2\theta)}$$

3-20

$$E_z^i = E_0 e^{jk(x\cos\phi_0+y\sin\phi_0)}$$

$$\overline{m} = 2\hat{n}\times\overline{E}^i = 2\overline{u}_x\times E_z^i\,\overline{u}_z$$

$$= -2E_0 e^{jky'\sin\phi_0}\,\overline{u}_y\Big|_{x=0}$$

$$d\overline{F} = -\frac{1}{4\pi r}\overline{m}\,e^{-jk|r-r'|}$$

$$|r-r'| = r-y'\sin\phi$$

$$dF_y = -\frac{1}{2\pi r}E_0 e^{-jkr}e^{jky'\sin\phi_0}e^{jky'\sin\phi}\,dS$$

$$dE_z^{(s)} = -\nabla_{(r)}\times d\overline{F}_y$$

$$= -\frac{jke^{-jkr}}{2\pi r}E_0 e^{jky'(\sin\phi+\sin\phi_0)}$$

$$\cdot\sin(\pi/2+\phi)\,dS$$

$$= -\frac{jkE_0}{2\pi r}e^{jky'(\sin\phi+\sin\phi_0)}\cos\phi\,dS$$

3-20 (cont.)

$$E_z^s = \frac{-jkE_0}{2\pi r} e^{-jkr} \cos\phi \cdot \int_{-a/2}^{a/2} dy' \int_{-b/2}^{b/2} dz' \, e^{jky'(\sin\phi + \sin\phi_0)}$$

$$= \frac{-jkE_0 b}{2\pi r} e^{-jkr} \cos\phi \left[\frac{e^{j\frac{ka}{2}(\sin\phi + \sin\phi_0)}}{jk(\sin\phi + \sin\phi_0)} - \frac{e^{-j\frac{ka}{2}(\sin\phi + \sin\phi_0)}}{jk(\sin\phi + \sin\phi_0)} \right]$$

$$= \frac{kE_0 ab e^{-jkr} \cos\phi}{j 2\pi r} \frac{\sin\left[\frac{ka}{2}(\sin\phi + \sin\phi_0)\right]}{\frac{ka}{2}(\sin\phi + \sin\phi_0)}$$

At $\phi = \phi_0$, maximum backscatter occurs.

$$E_z^s = \frac{kE_0 ab e^{-jkr}}{j 2\pi r} \frac{\sin[ka \sin\phi_0] \cos\phi_0}{ka \sin\phi_0}$$

$$\tilde{S}^i = |E_0|^2/\eta$$

$$\tilde{S}^s = \frac{1}{\eta} \left| \frac{kE_0 ab}{2\pi r} \frac{\sin[ka \sin\phi_0] \cos\phi_0}{ka \sin\phi_0} \right|$$

$$A_e = \lim_{r\to\infty} \left(4\pi r^2 \frac{\tilde{S}^s}{\tilde{S}^i} \right)$$

$$= 4\pi \left[\frac{ab \cos\phi_0 \sin\left(\frac{ka}{2} \sin\phi_0\right)}{\lambda ka \sin\phi_0} \right]^2$$

3-21

$$H_z^i = H_0 e^{jk(x\cos\phi_0 + y \sin\phi_0)}$$

$$\bar{M}_s = 2\hat{n} \times \bar{E}^i$$

$$\bar{E}^i = \eta H_z^i$$

3-21 (cont.)

$$E_z^i = \eta H_0 e^{jk(x\cos\phi_0 + y\sin\phi_0)}$$

$$M_z = 2\eta H_0 e^{jk(y \sin\phi_0)} \cos\phi_0$$

$$F_y = \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} \frac{e^{-jkr}}{4\pi r} 2\eta H_0 e^{jk(y\sin\phi_0)} \cdot e^{jk(y\sin\phi)} \cos\phi_0 \, dz\, dy$$

$$E_z^s = \eta H_z^s = -\nabla \times F_y$$

At $\theta = \frac{\pi}{2}$ in far field, $E_\theta \Rightarrow -E_z$

$$-E_z = jkF_\phi = jkF_y$$

from Eqn 3-97.

$\therefore$ after carrying out integral for F_y we have:

$$H_z^s = \frac{E_z^s}{\eta} = \frac{jkH_0 ab e^{-jkr}}{2\pi r} (\cdot) \frac{\sin\left[\frac{ka}{2}(\sin\phi + \sin\phi_0)\right] \cos\phi_0}{\frac{ka}{2}(\sin\phi + \sin\phi_0)}$$

At $\phi = \phi_0$ for max. backscatter:

$$H_z^s = \frac{jkH_0 ab e^{-jkr}}{2\pi r} \frac{\sin[ka \sin\phi_0] \cos\phi_0}{ka \sin\phi_0}$$

$$\tilde{S}^i = |H_0|^2 \eta$$

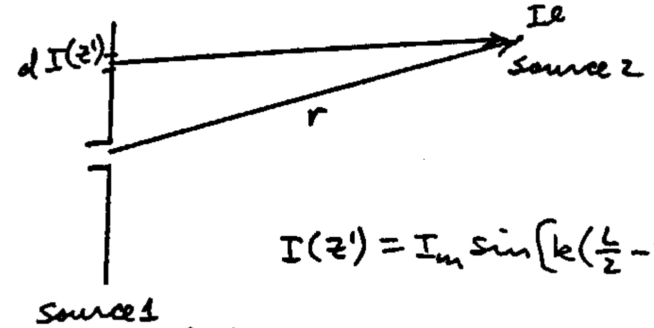
$$\tilde{S}^i = \eta \left| \frac{kH_0 ab}{2\pi r} \frac{\sin[ka\sin\phi_0]\cos\phi_0}{ka \sin\phi_0} \right|^2$$

$$A_e = \lim_{r\to\infty} \left(\frac{4\pi r^2 \tilde{S}^s}{\tilde{S}^i} \right)$$

$$= 4\pi \left[\frac{ab \cos\phi_0 \sin[ka \sin\phi_0]}{\lambda ka \sin\phi_0} \right]^2$$

which is the same as problem 3-20.

3-22



$$I(z') = I_m \sin\left[k\left(\frac{L}{2} - |z|\right)\right]$$

$$\iiint \bar{E}_1 \cdot \bar{J}_2 \, d\tau = \iiint \bar{E}_2 \cdot \bar{J}_1 \, d\tau$$

(1) $E_{1\theta} = \int E_{2z} \cdot I_z \, dz'$ (since source 2 is a delta function)

Far field of a Hertzian dipole is:

$$E_\theta = \frac{\eta j I\ell e^{-jkr}}{2\lambda r} \sin\theta$$

$$E_{2z} = \frac{\eta j I\ell e^{-jkr} e^{jkz'\cos\theta}}{2\lambda r} \sin\theta$$

Substituting into (1),

$$E_{1\theta} = \frac{e^{-jkr}}{r} \int_{-L/2}^{L/2} \frac{\eta j \overset{1}{I\ell}}{2\lambda r} e^{jkz'\cos\theta}$$

$$\cdot I_m \sin\left[k\left(\frac{L}{2} - |z'|\right)\right] \sin\theta \, dz'$$

or,

$$E_{1\theta} = \frac{j\eta I_m e^{-jkr}}{2\pi r}\left[\frac{\cos\left(\frac{kL}{2}\cos\theta\right) - \cos\left(\frac{kL}{2}\right)}{\sin\theta}\right]$$

which is Eqn. 2-125.

3-23

From Eqn 3-39,

$$\langle a, b\rangle = V^b I^a \quad \text{(b is a voltage source)}$$

I^a is the current at b when source a is applied. When Eqn 3-38 is satisfied,

$$\langle a, b\rangle = \langle b, a\rangle$$

3-23 (cont.)

or $V^b I^a_{V_a} = V^a I^b_{V_b}$

When port 1 (Fig 3-18) is excited and the short circuit current at port 2 is observed,

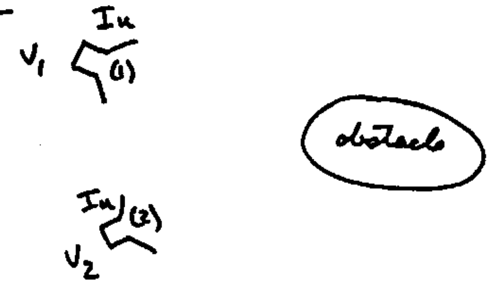
$$V^a y_{21} = I^a_{V_a}$$

Similarly,

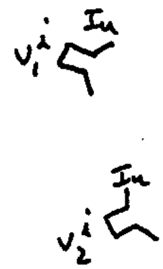
$$V^b y_{12} = I^b_{V_b}$$

$$\therefore y_{21} = y_{12}$$

3-24



case I



case II

$|I_u| = 1$

For case II,

$V_1^i = V$ at (1) when I_u is applied at (2)

$V_2^i = V$ at (2) when I_u is applied at (1)

$V_1^i I_u = V_2^i I_u$ from reciprocity

For case I, $I_u V_1 = V_2 I_u$

$$I_u(V_1^i + V_1^s) = (V_2^i + V_2^s) I_u$$

$$V_2^s I_u = V_1^s I_u$$

$$V_2^s = V_1^s$$

3-25

$$Z = -\frac{\langle a,a\rangle}{I^2}$$

$$E_x^a = -\frac{J_0 Z_0}{2}\sin\frac{\pi y}{b} \quad \text{at } z=0$$

$$J_x^a = J_0 \sin\frac{\pi y}{b}$$

$$\langle a,a\rangle = \iint E_x^a \cdot J_x^a \, dS$$

$$= \int_0^b\int_0^a -\frac{J_0^2 Z_0}{2}\sin^2\frac{\pi y}{b}\,dx\,dy$$

$$= -\frac{J_0^2 ab Z_0}{4}$$

I = total current of sheet

$$= \int_0^b\int_0^a J_0 \sin\frac{\pi y}{b}\,dx\,dy = \frac{2abJ_0}{\pi}$$

$$I^2 = \frac{4(abJ_0)^2}{\pi^2}$$

$$\therefore Z = \frac{\frac{J_0^2 ab Z_0}{4}}{\frac{4(abJ_0)^2}{\pi^2}} = \frac{Z_0\pi^2}{16ab}$$

3-26

$$Z = -\frac{\langle a,a\rangle}{I^2}$$

$$J_x^a = J_0 \sin\frac{\pi y}{b}$$

$$E_x^a = -\frac{J_0 Z_0}{2}\left(1-e^{-j2\beta d}\right)\sin\frac{\pi y}{b}$$

(at $z=0$)

$$\langle a,a\rangle = \iint E_x^a \cdot J_x^a \, dS$$

$$= \int_0^b\int_0^a -\frac{J_0^2 Z_0}{2}\sin^2\frac{\pi y}{b}\left(1-e^{-j2\beta d}\right)dx\,dy$$

$$= -\frac{J_0^2 ab Z_0}{4}\left(1-e^{-j2\beta d}\right)$$

3-26 (cont.)

I = total current as in 3-25

$$I = \frac{2abJ_0}{\pi}$$

$$Z = \frac{Z_0\pi^2}{16ab}\left(1-e^{-j2\beta d}\right)$$

3-27

$$\iint (E^a\times\nabla\times E^b - E^b\times\nabla\times E^a)\,dS$$

$$= \iiint (E^b\cdot\nabla\times\nabla\times E^a - E^a\cdot\nabla\times\nabla\times E^b)\,d\tau$$

$$\nabla\times E = -j\omega\mu H - M$$

$$\nabla\times H = j\omega\varepsilon E + J$$

$$\nabla\times\nabla\times E = k^2E - j\omega\mu J - \nabla\times M$$

$$\iint E^a\times(-j\omega\mu H^b - M^b) - E^b\times(-j\omega\mu H^a - M^a)\,dS$$

$$= \iiint E^b\cdot(k^2E^a - j\omega\mu J^a - \nabla\times M^a)$$

$$- E^a\cdot(k^2E^b - j\omega\mu J^b - \nabla\times M^b)\,d\tau$$

Use identity:

$$\nabla\cdot(A\times B) = B\cdot\nabla\times A - A\cdot\nabla\times B,$$

we obtain:

$$-j\omega\mu\iint(E^a\times H^b - E^b\times H^a)\,dS$$

$$+\iint(E^b\times M^a - E^a\times M^b)\,dS =$$

~~we obtain~~:

$$j\omega\mu\iiint -E^b\cdot J^a + E^a\cdot J^b\,d\tau$$

$$+\iiint \nabla\cdot(E^b\times M^a) - \nabla\cdot(E^a\times M^b)\,d\tau$$

$$-\iiint M^a\cdot\underbrace{\nabla\times E^b}_{-j\omega\mu H^b - M^b} - M^b\cdot\underbrace{\nabla\times E^a}_{-j\omega\mu H^a - M^a}\,d\tau$$

Note $M^a\cdot M^b = 0$

3-27 (cont.)

Using the divergence theorem and cancelling $j\omega\mu$ we get:

$$\iint (E^a \times H^b - E^b \times H^a)\, dS$$

$$= \iiint E^a \cdot J^b - E^b \cdot J^a + H^b \cdot M^a - H^a \cdot M^b$$

3-28

$$\nabla \cdot (\bar{A} \times \phi \nabla \times \bar{B})$$

$$= \nabla \times \bar{A} \cdot \phi \nabla \times \bar{B} - \bar{A} \cdot \nabla \times \phi \nabla \times \bar{B}$$

$$(1)\ \iint (\bar{A} \times \phi \nabla \times \bar{B}) \cdot dS$$

$$= \iiint (\nabla \times \bar{A} \cdot \phi \nabla \times \bar{B} - \bar{A} \cdot \nabla \times \phi \nabla \times \bar{B})\, d\tau$$

Interchanging A and B,

$$(2)\ \iint (\bar{B} \times \phi \nabla \times \bar{A})\, dS$$

$$= \iiint (\nabla \times \bar{B} \cdot \phi \nabla \times \bar{A} - \bar{B} \cdot \nabla \times \phi \nabla \times \bar{A}$$

Subtracting (2) from (1),

$$\iint \phi (\bar{A} \times \nabla \times \bar{B} - \bar{B} \times \nabla \times \bar{A})\, dS$$

$$= \iiint (\bar{B} \cdot \nabla \times \phi \nabla \times \bar{A} - \bar{A} \cdot \nabla \times \phi \nabla \times \bar{B})\, d\tau$$

Let $A = E^a$, $B = E^b$, $\phi = \frac{1}{\hat{z}}$

$$\iint \frac{1}{\hat{z}} (E^a \times \nabla \times E^b - E^b \times \nabla \times E^a)\, dS$$

$$= \iiint (E_b \cdot \nabla \times \frac{1}{\hat{z}} \nabla \times E^a - E^a \cdot \nabla \times \frac{1}{\hat{z}} \nabla \times E^b)\, d\tau$$

$$\iint (E^a \times H^b - E^b \times H^a)\, dS$$

$$= \iiint (E^b \cdot \nabla \times \nabla \times E^a - E^a \cdot \nabla \times \nabla \times E^b)\, d\tau$$

which from results of problem 3-27 reduces to Eqn. 3-25.

3-29

Green's second identity,

$$\iint (\bar{A} \times \nabla \times \bar{B} - \bar{B} \times \nabla \times \bar{A})\, dS$$

$$= \iiint (\bar{B} \cdot \nabla \times \nabla \times \bar{A} - \bar{A} \cdot \nabla \times \nabla \times \bar{B})\, d\tau$$

Let $\bar{A} = E$, $B = G_1$

$$\iint (E \times \nabla \times G_1) - (G_1 \times \nabla \times E)\, dS$$

$$= \iiint (G_1 \cdot \nabla \times \nabla \times E - E \cdot \nabla \times \nabla \times G_1)\, d\tau$$

From Eqn 3-47, 3-48,

$$G_1 = \bar{C} \phi$$

$$\phi = \frac{e^{-jk|r-r'|}}{|r-r'|}$$

Volume integral becomes;

$$\iiint G_1 \cdot (\nabla(\nabla \cdot E) + k^2 E)\, d\tau$$

$$- \iiint E \cdot (\nabla(\nabla \cdot G_1) + k^2 G_1\, d\tau$$

In the region enclosed by the volume $\nabla \cdot E = 0$.

Also $G_1 (k^2 E) = E \cdot (k^2 G_1)$

So the volume integral further reduces to:

$$- \iiint E \cdot (\nabla(\nabla \cdot G_1))\, d\tau$$

$$= - \iiint \nabla \cdot (E \nabla \cdot G_1)\, d\tau - \iiint E \cdot G_1\, d\tau$$

$$= \underbrace{- \iint E \nabla \cdot G_1\, dS}_{\text{by divergence thm}} \ \underbrace{- 4\pi \bar{C} \cdot E}_{\text{by approx. as } |r-r'| \to 0}$$

Remembering this equals original surface integral above we obtain

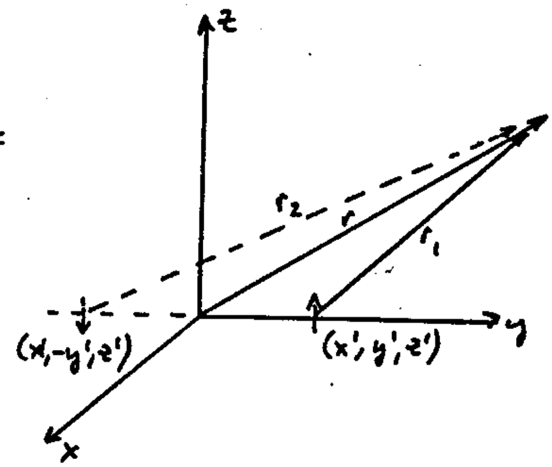
3-29 (cont.)

a final result as:

$$-4\pi \bar{C} \cdot E =$$

$$\iint (E \times \nabla \times G_1 - G_1 \times \nabla \times E + E\,\nabla \cdot G_1)\, dS$$

3-30



$$\bar{C} = \bar{u}_z$$

$$r_1 = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$

$$r_2 = \sqrt{(x-x')^2 + (y+y')^2 + (z-z')^2}$$

Finding the vector potential A,

$$A_z = \frac{1}{4\pi} \iiint \frac{\delta(x',y',z')\, e^{-jk|r_1 - r'|}}{r_1} - \frac{\delta(x',-y',z')\, e^{-jk|r_2 - r'|}}{r_2}\, d\tau'$$

$$= \frac{e^{-jkr_1}}{4\pi r_1} - \frac{e^{-jkr_2}}{4\pi r_2}$$

$$\bar{H} = \nabla \times \bar{A} = \bar{G}_4$$

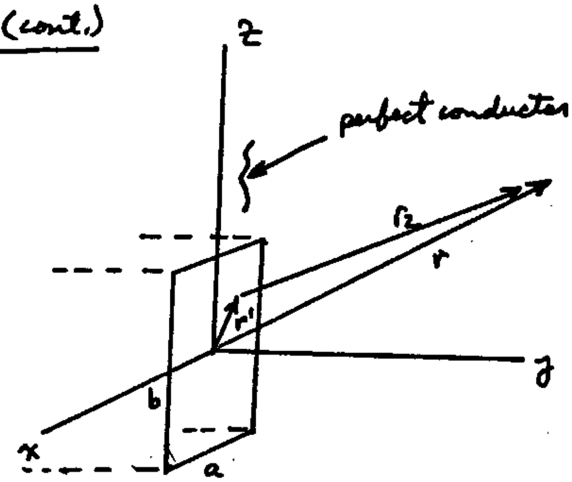
$$\bar{G}_4 = \nabla \times \bar{u}_z \left(\frac{e^{-jkr_1}}{r_1} - \frac{e^{-jkr_2}}{r_2} \right)$$

3-31

$$\bar{E} = \bar{u}_x \cos \frac{\pi z}{b}$$

$$\bar{G}_4 = \nabla \times \bar{u}_z \left(\frac{-e^{-jkr_2}}{r_2} \right)$$

3-31 (cont.)



Eqn 3-75:

$$4\pi \bar{C} \cdot \nabla' \times \bar{E} = \iint (\bar{G}_4 \times \nabla \times \bar{E}) \cdot \bar{dS}$$

$$\bar{C} = \bar{u}_z$$

$$\nabla \times \bar{E} = \frac{1}{\eta} \sin \frac{\pi y}{b}\, \bar{u}_y$$

$$\iint (\bar{G}_4 \times \nabla \times \bar{E}) \cdot \bar{dS}$$

$$= \iint \nabla \times \bar{u}_z \left(\frac{-e^{-jkr_2}}{r_2} \right) \times \frac{1}{\eta} \sin \frac{\pi y}{b}\, \bar{u}_y \cdot \bar{dS}$$

$$= \iint \nabla \times \bar{u}_x \left(\frac{e^{-jkr_2}}{r_2 \eta} \sin \frac{\pi y}{b} \right) \cdot \bar{dS}$$

$$= \nabla \times \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} \frac{e^{-jkr}}{r}\, e^{jk(x'\cos\phi \sin\phi + z'\cos\phi)}\, dx\, dz$$

which is the same integral as for $4\pi \bar{F}$ in problem 3-19 and if the far field expression for curl is used ($j\omega \varepsilon \sin\theta$) we get the same answer as in problem 3-19.

3-32

$$\bar{F} = \frac{\bar{K} \ell\, e^{-jk|r-r'|}}{4\pi |r-r'|}$$

$$\psi = \frac{e^{-jk|r-r'|}}{4\pi |r-r'|}$$

3-32 (cont.)

By duality:

$j\omega\mu \rightarrow j\omega\varepsilon$

$j\omega\varepsilon \rightarrow j\omega\mu$

$E \rightarrow H$

$A \rightarrow F$

Eqns. 3-65 become:

$$\Gamma_{ii} = \left(-j\omega\varepsilon + \frac{1}{j\omega\mu}\frac{\partial^2}{\partial i^2}\right)\psi$$

$$\Gamma_{ij} = \frac{1}{j\omega\mu}\frac{\partial^2\psi}{\partial i\partial j} \qquad i \neq j$$

This can be very easily seen by taking the duals of Eqns. 2-111 which are:

$$\bar{H} = -j\omega\varepsilon\bar{F} + \frac{1}{j\omega\mu}\nabla(\nabla\cdot\bar{F})$$

$$\bar{E} = -\nabla\times\bar{F}$$

and finding $\bar{H}$ for $\bar{K}\ell; \bar{u}_x, \bar{u}_y$, and $\bar{u}_z$ directed separately then,

$$\bar{H} = [\Gamma]\,\bar{K\ell} \quad \text{where } [\Gamma]$$

is given above.

3-33 $\qquad \bar{A} = [\Gamma]\,\bar{I\ell}$

If $\bar{I\ell}$ is $\bar{u}_x$ directed,

$$A_x = \frac{I\ell\, e^{-jk|r-r'|}}{4\pi|r-r'|} \qquad \psi = \frac{e^{-jk|r-r'|}}{4\pi|r-r'|}$$

$$\bar{H} = \nabla\times\bar{A} = \frac{\partial A_x}{\partial z}\bar{u}_y - \frac{\partial A_x}{\partial y}\bar{u}_z$$

$$H_z = -\frac{\partial}{\partial y}\psi\, I\ell_x$$

$$H_y = \frac{\partial\psi}{\partial z} I\ell_x$$

3-33 (cont.)

For $\bar{I\ell}$ $\bar{u}_y$ directed,

$$H_x = -\frac{\partial\psi}{\partial z} I\ell_y$$

$$H_z = \frac{\partial\psi}{\partial x} I\ell_y$$

For $\bar{I\ell}$ $\bar{u}_z$ directed,

$$H_x = \frac{\partial\psi}{\partial y} I\ell_z$$

$$H_y = -\frac{\partial\psi}{\partial x} I\ell_z$$

$\therefore$ we can write:

$$\Gamma_{xx} = \Gamma_{yy} = \Gamma_{zz} = 0$$

$$[\Gamma] = \begin{bmatrix} 0 & -\frac{\partial\psi}{\partial z} & \frac{\partial\psi}{\partial y} \\ \frac{\partial\psi}{\partial z} & 0 & -\frac{\partial\psi}{\partial x} \\ -\frac{\partial\psi}{\partial y} & \frac{\partial\psi}{\partial x} & 0 \end{bmatrix}$$

3-34 $\qquad E_z^i = E_0 e^{jk(x\cos\phi_0 + y\sin\phi_0)}$

$$\bar{J} = 2\hat{n}\times\bar{H}^i$$

$$J_y = -\frac{E_0}{\eta} e^{jky'\sin\phi_0}\sin(\pi/2+\phi_0)$$

$$= -\frac{E_0}{\eta} e^{jky'\sin\phi_0}\cos\phi_0$$

$$A_y = \frac{-e^{-jkr}}{4\pi r}\frac{E_0\cos\phi_0}{\eta}\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{-\frac{b}{2}}^{\frac{b}{2}} e^{jky'\sin\phi_0}\, e^{jky'\sin\phi}\, dy'dz'$$

$$E_z^s = \nabla\times\bar{A} =$$

$$= \frac{-e^{-jkr}}{4\pi r}\frac{E_0 b a\omega\mu(j)\cos\phi_0}{\eta} \cdot 2\left[\frac{\sin\left(\frac{ka}{2}(\sin\phi_0+\sin\phi)\right)}{\frac{ka}{2}(\sin\phi+\sin\phi_0)}\right]$$

3-34 (cont.)

$$E_z^s = \frac{e^{-jkr}}{2\pi r}\frac{E_0 abk\cos\phi_0}{j}\left[\frac{\sin\left[\frac{ka}{2}(\sin\phi+\sin\phi_0)\right]}{\frac{ka}{2}(\sin\phi+\sin\phi_0)}\right]$$

The echo area is found at $\phi=\phi_0=0$ and the same procedure followed as in problem 3-20 which gives the same result.

3-35

$$H_z^i = H_0 e^{jk(x\cos\phi_0+y\sin\phi_0)}$$

$$\bar{J} = \hat{n}\times\bar{H}^i = H_0 e^{jky'\sin\phi_0} = -J_y$$

$$A_y = \frac{-e^{-jkr}}{4\pi r}H_0\int_{-b/2}^{b/2}\int_{-a/2}^{a/2} e^{jky'\sin\phi_0}e^{jky'\sin\phi}\,dy'dz'$$

$$H_z^s = -jkA_y\cos\phi$$

$$= \frac{jkH_0abe^{-jkr}}{2\pi r}\frac{\sin\left[\frac{ka}{2}(\sin\phi+\sin\phi_0)\right]}{\frac{ka}{2}(\sin\phi+\sin\phi_0)}\cos\phi$$

The echo area is found in the same way as problem 3-21 at $\phi=\phi_0=0$ after which the same A_e is obtained.

3-36

$$\psi = e^{-jky}, \quad \bar{A} = \psi\,\bar{u}_z$$

$E_x = 0$ $\qquad H_x = -jke^{-jky}$

$E_y = 0$ $\qquad H_y = 0$

$E_z = -\hat{z}e^{-jky}$ $\qquad H_z = 0$

This field is TM to z. It is a uniform plane wave travelling in the $+\hat{y}$ direction and is linearly polarized. It is also TEM to its direction of propagation.

3-37

$$\psi = e^{-jkx} \qquad \bar{F} = \bar{u}_z\psi$$

$E_x = 0$ $\qquad H_x = 0$

$E_y = -jke^{-jkx}$ $\qquad H_y = 0$

$E_z = 0$ $\qquad H_z = -\hat{y}e^{-jkx}$

This field is TEM to the x-direction. It is a uniform plane wave travelling in the $+x$ direction and linearly polarized.

3-38

$$\bar{c} = \bar{u}_x, \quad \psi^a = e^{-jkz}$$

$$\psi^f = je^{-jkz}$$

$$\nabla\times(\bar{c}\psi^f) = ke^{-jkz}\bar{u}_y$$

$$\nabla\times\nabla\times(\bar{c}\psi^a) = k^2e^{-jkz}\bar{u}_x$$

$$\bar{E} = -ke^{-jkz}\bar{u}_y - \hat{z}e^{-jkz}\bar{u}_x$$

$$\nabla\times(\bar{c}\psi^a) = -jke^{-jkz}\bar{u}_y$$

$$\nabla\times\nabla\times(\bar{c}\psi^f) = jk^2e^{-jkz}\bar{u}_x$$

$$\bar{H} = -jke^{-jkz}\bar{u}_y - \hat{y}je^{-jkz}\bar{u}_x$$

$E_x = -\hat{z}e^{-jkz}$ $\qquad H_x = -\hat{y}je^{-jkz}$

$E_y = -ke^{-jkz}$ $\qquad H_y = -jke^{-jkz}$

$$\frac{E_x}{H_y} = \frac{-\hat{z}}{-jk} = \sqrt{\frac{\hat{z}}{\hat{y}}} = \eta$$

$$\frac{E_y}{H_x} = \frac{-k}{-\hat{y}j} = \frac{\sqrt{-\hat{z}\hat{y}}}{\hat{y}j} = -\eta$$

Thus we have a uniform plane wave TEM to z

3-38 (cont.)

travelling in the $+z$ direction which is circularly polarized.

3-39

In the radiation zone, Eqns 3-4 become:

$$\bar{E} = -\nabla \times \bar{F} + \frac{1}{\hat{y}}(\nabla \times \nabla \times \bar{A})$$

$$\bar{H} = \nabla \times \bar{A} + \frac{1}{\hat{z}}(\nabla \times \nabla \times \bar{F})$$

$$\bar{A} = \frac{e^{-jkr}}{4\pi r}\iiint J(r')\, e^{jkr'\cos\xi}\, d\tau'$$

$$\bar{F} = \frac{e^{-jkr}}{4\pi r}\iiint m(r')\, e^{jkr'\cos\xi}\, d\tau'$$

$$\nabla \times \bar{F} = -\bar{u}_\theta \frac{1}{r}\left[\frac{\partial}{\partial r}(rF_\phi)\right] + \bar{u}_\phi \frac{1}{r}\left[\frac{\partial}{\partial r}(rF_\theta)\right]$$

$$= \frac{-\bar{u}_\theta}{4\pi r}(-jk)e^{-jkr}\iiint m_\phi(r')e^{jkr'\cos\xi}\, d\tau'$$

$$+ e^{-jkr}\frac{\bar{u}_\phi(-jk)}{4\pi r}\iiint m_\theta(r')e^{jkr'\cos\xi}\, d\tau'$$

$$= \bar{u}_\theta(jk)F_\phi - \bar{u}_\phi(jk)F_\theta$$

Similarly,

$$\nabla \times \nabla \times \bar{A} = \bar{u}_\theta k^2 A_\theta + \bar{u}_\phi k^2 A_\phi$$

So,

$$\bar{E} = -\bar{u}_\theta(jk)F_\phi + \bar{u}_\phi(jk)F_\theta + \bar{u}_\theta\frac{k^2}{\hat{y}}A_\theta + \bar{u}_\phi\frac{k^2}{\hat{y}}A_\phi$$

3-39 (cont.)

Separating components we get:

$$E_\theta = -j\omega\mu A_\theta - jkF_\phi$$

$$E_\phi = -j\omega\mu A_\phi + jkF_\theta$$

4-1

$$\psi = \int_{k_x}\int_{k_y} f(k_x,k_y)\, h(k_x x)\, h(k_y y)\, h(k_z z)\, dk_x dk_y$$

$$\frac{\partial^2 \psi}{\partial x^2} = k_x^2 \iint_{k_x k_y} f(k_x,k_y)\, h''(k_x x)\, h(k_y y)\, h(k_z z)\, dk_x dk_y$$

$$\therefore \frac{\partial^2 \psi}{\partial x^2} + k_x^2 \psi =$$

$$= \int_{k_x}\int_{k_y} f(k_x,k_y)\left[h''(k_x x) + k_x^2 h(k_x x)\right] h(k_y y) \cdot h(k_z z)\, dk_x dk_y$$

Now $h''(k_x x) + k_x^2 h(k_x x) = 0$ from separation of variables. This is also true for the other two components thus the Helmholtz equation is satisfied.

4-2

$$\sin kx = \sin(\beta - j\alpha)x$$

$$= \sin \beta x \cos(j\alpha)x - \cos\beta x \sin(j\alpha)x$$

$$= \sin\beta x \cosh \alpha x - j \cos\beta x \sinh \alpha x$$

$$\sin(jx) = j\sinh x$$

$$\cos(jx) = \cosh x$$

Similarly,

$$\cos kx = \cos(\beta x - j\alpha x)$$

$$= \cos\beta x \cosh \alpha x + j \sin\beta x \sinh \alpha x$$

4-3

$$\bar{E} = -\nabla\times\bar{F} = -\nabla\times\bar{u}_z\psi$$

$$= -\nabla\times\bar{u}_z\left(e^{-jk_x x} e^{-jk_y y} e^{-jk_z z}\right)$$

$$= -\bar{u}_x \psi(-jk_y) + \bar{u}_y \psi(-jk_x)$$

$$= jk_y \psi \bar{u}_x - jk_x \psi \bar{u}_y$$

$$\bar{k}\times\bar{u}_z = (k_x\bar{u}_x + k_y\bar{u}_y + k_z\bar{u}_z)\times\bar{u}_z$$

$$= k_y\bar{u}_x - k_x\bar{u}_y$$

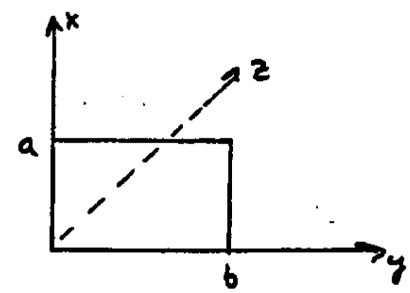
$$\bar{E} = j\psi\,\bar{k}\times\bar{u}_z$$

$$\hat{z}\bar{H} = -\nabla\times\bar{E}$$

$$= (k_x^2\psi + k_y^2\psi)\bar{u}_z - k_y k_z \psi \bar{u}_y - k_x k_z \psi \bar{u}_x$$

$$= (k^2\bar{u}_z - k_z\bar{k})\psi = \hat{z}\bar{H}$$

4-4



For TM_{mn} mode:

$$E_z = C \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$E_y = \frac{-j\beta C}{h^2}\frac{n\pi}{b}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$$

$$E_x = \frac{-j\beta C}{h^2}\frac{m\pi}{a}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$H_x = \frac{j\omega\varepsilon C}{h^2}\frac{n\pi}{b}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$$

$$H_y = -\frac{j\omega\varepsilon C}{h^2}\frac{m\pi}{a}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

4-4 (cont.)

$$h^2 = \omega^2\mu\varepsilon - \beta^2$$

$$\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \beta^2 = \omega^2\mu\varepsilon$$

$$\alpha = \frac{\text{Power loss on walls}}{2 \cdot \text{Power flow}}$$

$$P_f = \int_0^a\int_0^b \bar{E}\times\bar{H}^* dx\,dy$$

$$= \int_0^a\int_0^b (E_x H_y^* - E_y H_x^*)\,dx\,dy$$

$$E_x H_y^* = \frac{C^2\beta\omega\varepsilon}{h^2 \cdot h^2}\left(\frac{m\pi}{a}\right)^2 \cos^2\frac{m\pi x}{a}\sin^2\frac{n\pi y}{b}$$

$$E_y H^* = \frac{-C^2\beta\omega\varepsilon}{h^2 \cdot h^2}\left(\frac{n\pi}{b}\right)^2 \sin^2\frac{m\pi x}{a}\cos^2\frac{n\pi y}{b}$$

$$P_f = \frac{C^2\beta\omega\varepsilon}{h^4}\,\frac{a}{2}\cdot\frac{b}{2}$$

$$P_L = R\int\left[(H_x)^2 + (H_y)^2\right]dl$$

$$= 2R\left[\int_0^a \frac{\omega^2\varepsilon^2C^2}{h^4}\left(\frac{n\pi}{b}\right)^2 \sin^2\frac{m\pi x}{a}\,dx + \int_0^b\left(\frac{\omega\varepsilon C\, m\pi}{h^2 a}\right)^2\sin^2\frac{n\pi y}{b}\,dy\right]$$

$$= \frac{2R\omega^2\varepsilon^2C^2}{h^4}\left[\left(\frac{n\pi}{b}\right)^2\frac{a}{2} + \left(\frac{m\pi}{a}\right)^2\frac{b}{2}\right]$$

$$\alpha = \frac{R\omega^2\varepsilon^2C^2\pi^2\left[n^2a^3 + m^2b^3\right]h^2 \cdot 2}{h \cdot 2a^2b^2C^2\beta\omega\varepsilon\, ab}$$

$$= \frac{2R\,(m^2b^3 + n^2a^3)\,\omega\varepsilon}{(m^2b^2 + n^2a^2)\,ab\,\beta}$$

4-4 (cont.)

$$\beta = k\sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$

$$\eta = \frac{k}{\omega\varepsilon}$$

$$\alpha_{TM} = \frac{2R}{\eta\sqrt{1-\left(\frac{f_c}{f}\right)^2}\,ab}\cdot\frac{m^2b^3 + n^2a^3}{m^2b^2 + n^2a^2}$$

For TE_{mn} modes,

$$H_z = C\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$$

$$H_x = \frac{j\beta}{h^2}C\,\frac{m\pi}{a}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$$

$$H_y = \frac{j\beta}{h^2}C\,\frac{n\pi}{b}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$E_x = \frac{j\omega\mu}{h^2}C\,\frac{n\pi}{b}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$E_y = \frac{-j\omega\mu}{h^2}C\,\frac{m\pi}{a}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$$

$$h^2 = \omega^2\mu\varepsilon - \beta^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

$$E_x H_y^* - E_y H_x^* =$$

$$= \frac{\omega\mu\beta C^2}{h^4}\left(\frac{n\pi}{b}\right)^2\cos^2\frac{m\pi x}{a}\sin^2\frac{n\pi y}{b} + \frac{\beta\omega\mu}{h^4}\left(\frac{m\pi}{a}\right)^2\sin^2\frac{m\pi x}{a}\cos^2\frac{n\pi y}{b}$$

$$2P_f = \frac{\omega\mu\beta C^2}{h^4}\,\frac{ab}{2}$$

$$P_L = 2R\left[\int_0^a\left[C^2\cos^2\frac{m\pi x}{a} + \left(\frac{\beta C}{h^2}\frac{m\pi}{a}\right)^2\sin^2\frac{m\pi x}{a}\right]dx\right.$$

$$+ 2R\left[\int_0^b\left[C^2\cos^2\frac{n\pi y}{b} + \right.\right.$$

4-4 (cont.)

$$+\left(\frac{\beta C}{h^2}\frac{n\pi}{b}\right)^2 \sin^2\frac{n\pi y}{b}\Big] dy$$

$$= \frac{2RC^2}{2}\left[a + \frac{\beta^2}{h^4}\left(\frac{m\pi}{a}\right)^2 a + b + \frac{\beta^2}{h^4}\left(\frac{n\pi}{b}\right)^2 b\right]$$

$$= RC^2\left[(a+b) + \frac{\beta^2\pi^2}{h^4}\cdot\frac{m^2b+n^2a}{ab}\right]$$

$$\alpha = \frac{P_L}{2P_f}$$

$$= \frac{2R(a+b)h^2}{\omega\mu\beta ab} + \frac{RC^2\beta^2\pi^2}{h^4}\cdot\frac{2(m^2b+n^2a)}{ab\cdot ab}\cdot\frac{h^2}{\omega^2\mu^2\beta^2C^2}$$

$$\frac{h^2}{\omega\mu\beta} = \frac{\omega^2\mu\varepsilon-\beta^2}{\omega\mu\beta} = \frac{\omega\varepsilon}{\beta} - \frac{\beta}{\omega\mu}$$

$$= \frac{\omega\varepsilon}{k\sqrt{1-(f_c/f)^2}} - \frac{k\sqrt{1-(f/f_c)^2}}{\omega\mu}$$

$$= \frac{1}{\eta\sqrt{1-(f_c/f)^2}} - \frac{\sqrt{1-(f_c/f)^2}}{\eta}$$

$$= \frac{(f_c/f)^2}{\eta\sqrt{1-(f_c/f)^2}}$$

$$\alpha_{TE} = \frac{2R(a+b)(f_c/f)^2}{ab\eta\sqrt{1-(f_c/f)^2}} + \frac{m^2b+n^2a}{m^2b^2+n^2a^2}\cdot 2R\frac{\beta}{\omega\mu}$$

4-4 (cont.)

$$\alpha_{TE} = \frac{2R}{\eta}\left[\frac{a+b}{ab}\frac{(f_c/f)^2}{\sqrt{1-(f_c/f)^2}} + \frac{m^2b+n^2a}{m^2b^2+n^2a^2}\sqrt{1-(f_c/f)^2}\right]$$

For TE_{0n} modes,

$$E_x = E_0 \sin\frac{n\pi y}{b} e^{-\gamma z}$$

$$H_y = \frac{\beta E_0}{\omega\mu}\sin\frac{n\pi y}{b} e^{-\gamma z}$$

$$H_z = \frac{E_0}{j\eta}\frac{f_c}{f}\cos\frac{n\pi y}{b} e^{-\gamma z}$$

$$P_d\Big|_{y=0} = R\int_0^a |H_z|^2 dx$$

$$= R\int_0^a \frac{E_0^2}{\eta^2}\left(\frac{f_c}{f}\right)^2 \cos^2\frac{n\pi 0}{b}\, dx$$

$$= \frac{RE_0^2 a}{\eta^2}\left(\frac{f_c}{f}\right)^2$$

$$P_d\Big|_{x=0} = R\int_0^b |H_y|^2 + |H_z|^2 dy$$

$$= R\int_0^b \frac{\beta^2}{\omega^2\mu^2}E_0^2 \sin^2\frac{n\pi y}{b} dy + \frac{RE_0^2}{\eta^2}\int_0^b\left(\frac{f_c}{f}\right)^2\cos^2\frac{n\pi y}{b} dy$$

$$= R\left[\left(\frac{\beta E_0}{\omega\mu}\right)^2\frac{b}{2} + \left(\frac{E_0}{\eta}\right)^2\left(\frac{f_c}{f}\right)^2\frac{b}{2}\right]$$

$$= \frac{1}{2}P_d\ (\text{total})$$

$$P_f = \iint E_x H_y^* = \frac{\beta E_0^2}{\omega\mu}\frac{ba}{2}$$

$$\therefore \alpha = \frac{kR}{a\eta\beta}\left[1 + \frac{2a}{b}\left(\frac{f_c}{f}\right)^2\right]$$

4-5 X

For single mode operation over a 2:1 frequency range, $b/a = 2$

$f = 10^{10}$ Hz $\Rightarrow \lambda = 3$ cm.

If $b = 2$ cm, $\lambda_c = 4$ cm for TE_{01} mode

$$\alpha_c = \frac{2.61\times10^{-7}\sqrt{10^{10}}}{a(377)\sqrt{1-\left(\frac{.75}{1}\right)^2}}\left[1+\frac{29}{6}(.75)^2\right]$$

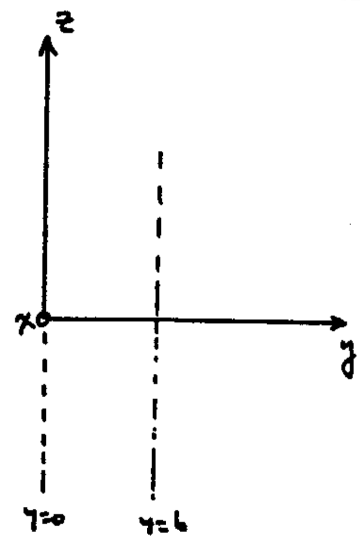
$$= 1.64\times10^{-4}\ \frac{\text{nepers}}{\text{cm.}}$$

If $b = 3$ cm, $a = 1.5$ cm,

then $\lambda_c = 6$ cm or $f_c = 5\times10^9$ Hz which is way above cutoff so we no longer have single mode propagation since TE_{10} mode starts at 10^{10} Hz.

4-6

Assume no variation in x-direction, i.e., $\frac{\partial}{\partial x} = 0$



The equation to be satisfied is:

$$\frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} + k^2\psi = 0$$

Let $\frac{\partial^2\psi_y}{\partial y^2} + k_y^2\psi_y = 0$ and $\frac{\partial^2\psi_z}{\partial z^2} + k_z^2\psi_z = 0$

$$\psi_y = A'\cos k_y y + B'\sin k_y y$$

4-6 (cont.)

$$\psi_z = C'e^{-jk_z z} + D'e^{jk_z z}$$

Assume the wave travels only in the +z direction and form the product solution $\psi = \psi_y\psi_z$:

$$\psi = (A\cos k_y y + B\sin k_y y)e^{-jk_z z}.$$

The ψ for TM and TE cases are obtained by matching the fields at the boundaries. For the TEM mode the Helmholtz equation separates into the following form assuming a variation of $e^{-jk_z z}$ in the z-direction.

$$\frac{\partial^2\psi_y}{\partial y^2} = 0 \qquad \frac{\partial^2\psi_z}{\partial z^2} + k_z^2\psi_z = 0$$

$$\psi_y = A'y \qquad \psi_z = Ce^{-jk_z z}$$

Total solution is:

$$\psi = Ay\,e^{-jk_z z}$$

4-7

From Eqn 4-33 and 4-34:

$$\psi_{mn}^{TM_x} = \cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}e^{-jk_z z}$$

$$\psi_{mn}^{TE_x} = \sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}e^{-jk_z z}$$

Since there is no x-variation for the parallel plate guide these reduce to the equations given in the problem statement.

4-8 X

$E_{mn}^{TE_x} = j\psi\bar{k}\times\bar{u}_x$ since $\bar{F} = \psi\bar{u}_x$

$E_{mn}^{TE} = j\psi\bar{k}\times\bar{u}_z$ since $\bar{F} = \psi\bar{u}_z$

4-8 (cont.)

$$E_{mn}^{TM} = \frac{1}{\hat{y}}\left(-k_z\bar{k} + \bar{u}_z k^2\right)\psi \quad \text{since } \bar{A} = \psi\bar{u}_z$$

$$E_{mn}^{TE_x} = A E_{mn}^{TE} + B E_{mn}^{TM}$$

$$j\psi\bar{k}\times\bar{u}_x = Aj\psi\bar{k}\times\bar{u}_z + \frac{B}{\hat{y}}\left(-k_z\bar{k} + \bar{u}_z k^2\right)\psi$$

$$j\psi(k_z\bar{u}_y - k_y\bar{u}_z) = (k_y\bar{u}_x - k_x\bar{u}_y)jA\psi + \frac{B}{\hat{y}}\left(-k_zk_x\bar{u}_x - k_zk_y\bar{u}_y + k_x^2\bar{u}_z + k_y^2\bar{u}_z\right)\psi$$

so $0 = -\frac{B}{\hat{y}}k_zk_x + Ak_yj$

$$jk_z = -Ak_xj - \frac{B}{\hat{y}}k_zk_y$$

$$-jk_y = (k_x^2 + k_y^2)\frac{B}{\hat{y}} \Rightarrow B = \frac{-\hat{y}jk_y}{k_x^2 + k_y^2}$$

$$\therefore A = \frac{-k_zk_x}{k_x^2 + k_y^2}$$

Restating $E_{mn}^{TE_x}$ in problem statement form:

$$E_{mn}^{TE_x} = A'\left(E_{mn}^{TE} + B'E_{mn}^{TM}\right)$$

$$A' = \frac{-k_zk_x}{k_x^2 + k_y^2}, \quad B' = \frac{j\hat{y}k_y}{k_zk_x}$$

$$H_{mn}^{TM_x} = j\psi\bar{u}_x\times\bar{k} \quad \text{since } \bar{A} = \psi\bar{u}_x$$

$$H_{mn}^{TE} = \frac{1}{\hat{z}}\left(-k_z\bar{k} + \bar{u}_z k^2\right)\psi \quad \text{since } \bar{F} = \psi\bar{u}_x$$

$$H_{mn}^{TM} = j\psi\bar{u}_z\times\bar{k} \quad \text{since } \bar{A} = \psi\bar{u}_z$$

$$H_{mn}^{TM_x} = CH_{mn}^{TE} + DH_{mn}^{TM}$$

4-8 (cont.)

$$j\psi\bar{u}_x\times\bar{k} = \frac{C}{\hat{z}}\left(-k_z\bar{k} + \bar{u}_z k^2\right)\psi + Dj\psi\,\bar{u}_x\times\bar{k}$$

$$j\psi(k_y\bar{u}_z - k_z\bar{u}_y) = \frac{C}{\hat{z}}\left(-k_zk_x\bar{u}_x - k_zk_y\bar{u}_y + k_x^2\bar{u}_z + k_y^2\bar{u}_z\right)\psi + jD\psi(k_x\bar{u}_y - k_y\bar{u}_x)$$

$$0 = -\frac{Ck_zk_x}{\hat{z}} - jDk_y$$

$$-jk_z = -\frac{Ck_zk_y}{\hat{z}} + jDk_x$$

$$-jk_y = \frac{C}{\hat{z}}(k_x^2 + k_y^2) \Rightarrow C = \frac{-j\hat{z}k_y}{k_x^2 + k_y^2}$$

$$\therefore D = \frac{-k_xk_z}{k_x^2 + k_y^2}$$

Restating $H_{mn}^{TE_x}$ in problem statement form:

$$H_{mn}^{TM_x} = C'\left(H_{mn}^{TE} + DH_{mn}^{TM}\right)$$

$$C' = \frac{-j\hat{z}k_y}{k_x^2 + k_y^2}, \quad D' = \frac{k_xk_z}{j\hat{z}k_y}$$

4-9

$$\psi_{mnp}^{TM} = \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\cos\frac{p\pi z}{c}$$

for a rectangular cavity. For this particular resonator there are no zeros in the z direction so

$$\psi_{mn}^{TM} = \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}.$$

The resonant frequencies are

$$f_r = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

which are the cutoff frequencies

18

<u>4-9 (cont.)</u>

of the rectangular guide given by Eqn. 4-24. These cutoff frequencies were found by setting $k^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \cancel{\left(\frac{p\pi}{c}\right)^2}\,0$

<u>4-10</u>

$$Q = \frac{\omega \times \text{energy stored}}{\text{average power lost}} = \frac{\omega W}{\bar{P}_d}$$

Only the general mnp cases will be computed.

$$A_z = \psi^{TM}_{mnp} = \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\cos\frac{p\pi z}{c}$$

$$E_x = \frac{1}{\hat{y}}\frac{\partial^2\psi}{\partial x\partial z} = -\frac{1}{\hat{y}}\frac{m\pi}{a}\frac{p\pi}{c}\sin\frac{n\pi y}{b}\cos\frac{m\pi x}{a}\sin\frac{p\pi z}{c}$$

$$E_y = \frac{1}{\hat{y}}\frac{\partial^2\psi}{\partial y\partial z} = -\frac{1}{\hat{y}}\frac{n\pi}{b}\frac{p\pi}{c}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\sin\frac{p\pi z}{c}$$

$$E_z = \frac{1}{\hat{y}}\left(\frac{\partial^2}{\partial z^2}+k^2\right)\psi$$

$$= \frac{1}{\hat{y}}\left(-\frac{p^2\pi^2}{c^2}+k^2\right)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\cos\frac{p\pi z}{c}$$

$$H_x = \frac{\partial\psi}{\partial y} = \frac{n\pi}{b}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\cos\frac{p\pi z}{c}$$

$$H_y = -\frac{\partial\psi}{\partial x} = \frac{m\pi}{a}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\cos\frac{p\pi z}{c}$$

$$P_d\big|_{y=0} = 2R\int_0^a\int_0^c\left(\frac{n\pi}{b}\right)^2\sin^2\frac{m\pi x}{a}\cos^2\frac{p\pi z}{c}\,dx\,dz$$

$$P_d\big|_{x=0} = 2R\int_0^b\int_0^c\left(\frac{m\pi}{a}\right)^2\sin^2\frac{n\pi y}{b}\cos^2\frac{p\pi z}{c}\,dy\,dz$$

$$P_d\big|_{z=0} = 2R\int_0^a\int_0^b\left(\frac{n\pi}{b}\right)^2\sin^2\frac{m\pi x}{a}\cos^2\frac{n\pi y}{b}$$

$$+\left(\frac{m\pi}{a}\right)^2\cos^2\frac{m\pi x}{a}\sin^2\frac{n\pi y}{b}\,dx\,dy$$

$$P_d^{\text{total}} = 2R\left[\left(\frac{n\pi}{b}\right)^2\frac{ac}{4}+\left(\frac{m\pi}{a}\right)^2\frac{bc}{4} + \right.$$

<u>4-10 (cont.)</u>

$$\left. +\left(\frac{n\pi}{b}\right)^2\frac{ab}{4}+\left(\frac{m\pi}{a}\right)^2\frac{ab}{4}\right]$$

$$P_d^{\text{total}} = \frac{R\pi^2}{2a^2b^2}\left[n^2a(b+c)+m^2b(c+a)\right]$$

$$W_m = \frac{\mu}{2}\int_0^c\int_0^b\int_0^a\left[(H_x)^2+(H_y)^2\right]dx\,dy\,dz$$

$$= \frac{\mu}{2}\left[\left(\frac{n\pi}{b}\right)^2\frac{abc}{8}+\left(\frac{m\pi}{a}\right)^2\frac{abc}{8}\right]$$

$$W = 2W_m = \frac{\mu abc\pi^2}{8a^2b^2}\left[n^2a^2+m^2b^2\right]$$

$$Q_{TM} = \frac{\omega abc\pi^2(n^2a^2+m^2b^2)\,2a^2b^2\mu}{8R\pi^2\left[n^2a^3(b+c)+m^2b^3(c+a)\right]a^2b^2}$$

$$= \frac{\eta abc\,k_{xy}^2\,k_r}{4R\left[b(a+c)k_x^2+a(b+c)k_y^2\right]}$$

Note $\omega\mu = \eta k$.

$$\psi^{TE}_{mnp} = \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\sin\frac{p\pi z}{c}$$

$$H_x = \frac{1}{\hat{z}}\frac{\partial^2\psi}{\partial x\partial z} = -\frac{1}{\hat{z}}\frac{p\pi}{c}\frac{m\pi}{a}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\cdot\sin\frac{p\pi z}{c}$$

$$H_y = \frac{1}{\hat{z}}\frac{\partial^2\psi}{\partial y\partial z} = -\frac{1}{\hat{z}}\frac{n\pi}{b}\frac{p\pi}{c}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\cos\frac{p\pi z}{c}$$

$$H_z = \frac{1}{\hat{z}}\left(\frac{\partial^2}{\partial z^2}+k^2\right)\psi$$

$$= \frac{1}{\hat{z}}\left(k^2-\frac{p^2\pi^2}{c^2}\right)\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\sin\frac{p\pi z}{c}$$

$$P_d\big|_{x=0} = \frac{2R}{\hat{z}^2}\left[\left(\frac{n\pi}{b}\right)^2\left(\frac{p\pi}{c}\right)^2\frac{bc}{4}\right.$$

$$\left.+\left(k^2-\frac{p^2\pi^2}{c^2}\right)^2\frac{bc}{4}\right]$$

$$P_d\big|_{y=0} = \frac{2R}{\hat{z}^2}\left[\left(\frac{p\pi}{c}\right)^2\left(\frac{m\pi}{a}\right)^2\frac{ac}{4} + \right.$$

4-10 (cont.)

$$+\left(k^2-\frac{p^2\pi^2}{c^2}\right)^2\frac{ac}{4}\Bigg]$$

$$P_d\Big|_{z=0}=\frac{2R}{\hat{z}^2}\left[\left(\frac{p\pi}{c}\right)^2\left(\frac{m\pi}{a}\right)^2\frac{ab}{4}+\left(\frac{p\pi}{c}\right)^2\left(\frac{n\pi}{b}\right)^2\frac{ab}{4}\right]$$

$$W_e=\frac{\varepsilon}{2}\int_0^a\int_0^b\int_0^c |E_x|^2+|E_y|^2\,dx\,dy\,dz$$

$$=\frac{\varepsilon}{2}\frac{abc}{8}\left[\left(\frac{n\pi}{b}\right)^2+\left(\frac{m\pi}{a}\right)^2\right]$$

$$W=2W_e=\frac{\varepsilon abc}{8}\pi^2\frac{(m^2b^2+n^2a^2)}{a^2b^2}$$

$$Q_{TE}=\frac{\varepsilon abc\,\pi^2(m^2b^2+n^2a^2)}{8a^2b^2P_d^{total}}$$

$$P_d^{total}=\frac{2R}{\hat{z}^2 4}\left\{\frac{p^2n^2}{bc}\pi^4+\left(k_r^2-\frac{p^2\pi^2}{c^2}\right)^2\left[bc+ca\right]+\frac{p^2m^2\pi^4}{ac}+\frac{p^2m^2\pi^4}{a^2c^2}ab+\frac{p^2n^2\pi^4}{b^2c^2}ab\right\}$$

$$=\frac{2R}{4\hat{z}^2}\left[k_z^2k_y^2bc+ack_z^2k_x^2+abk_z^2k_x^2+abk_z^2k_y^2+(bc+ca)k_{xy}^4\right]$$

$$=\frac{R}{2\hat{z}^2}\left[bc\left(k_{xy}^4+k_y^2k_z^2\right)+ca\left(k_{xy}^4+k_x^2k_z^2\right)+ab\,k_{xy}^2k_z^2\right]$$

$$Q=\frac{\omega\varepsilon\hat{z}^2}{4R}\left\{abck_{xy}^2\Big/\left[bc(k_{xy}^4+k_y^2k_z^2)+ac(k_{xy}^4+k_x^2k_z^2)+ab(k_{xy}^2k_z^2)\right]\right.$$

which gives the required expression.

4-11

$(f_r)_{011}=10^9$ Hz.

Abstract from Table 4-3 when $\frac{b}{a}=\frac{c}{a}=2$:

$$\left.\begin{aligned}(f_r)^{TE}_{101}&=1.58\times10^9\\(f_r)^{TM}_{110}&=1.58\times10^9\\(f_r)^{TE}_{012}&=1.58\times10^9\\(f_r)^{TE}_{021}&=1.58\times10^9\end{aligned}\right\}\text{ 4 degenerate modes}$$

$$\left.\begin{aligned}(f_r)^{TM}_{111}&=1.73\times10^9\\(f_r)^{TE}_{111}&=1.73\times10^9\end{aligned}\right\}\text{ 2 degenerate modes}$$

$$\left.\begin{aligned}(f_r)^{TE}_{102}&=2\times10^9\\(f_r)^{TM}_{120}&=2\times10^9\end{aligned}\right\}\text{ 2 degenerate modes}$$

$$\left.\begin{aligned}(f_r)^{TM}_{112}&=2.12\times10^9\\(f_r)^{TE}_{112}&=2.12\times10^9\end{aligned}\right\}\text{ 2 degenerate modes}$$

4-12 No variation in y-direction for this problem.

$$\psi_1^{TM}=\cos k_{1x}x\,e^{-jk_zz}\qquad 0\le x\le d$$

$$\psi_2^{TM}=\cos k_{2x}(a-x)\,e^{-jk_zz}\qquad d\le x\le a$$

$$k_{1x}^2+k_z^2=k_1^2=\omega^2\mu_1\varepsilon_1$$

$$k_{2x}^2+k_z^2=k_2^2=\omega^2\mu_2\varepsilon_2$$

$$E_{z1}=\frac{1}{\hat{y}_1}\frac{\partial^2\psi_1}{\partial x_1\partial z_1}=\frac{jk_zk_{1x}}{\hat{y}_1}\sin k_{1x}x\,e^{-jk_zz}$$

$$E_{z2}=-\frac{j}{\hat{y}_2}k_zk_{2x}\sin k_{2x}(a-x)\,e^{-jk_zz}$$

4-12 (cont.)

$$H_{y_1} = \frac{\partial \psi}{\partial z} = -jk_z \cos k_{1x} e^{-jk_z z}$$

$$H_{y_2} = -jk_z \cos k_{x2}(a-x)\, e^{-jk_z z}$$

$$E_{z_1}\Big|_{\substack{x=d\\z=0}} = E_{z2}\Big|_{\substack{x=d\\z=0}}$$

$$\frac{k_{1x}\sin k_{1x} d}{\varepsilon_1} - \frac{k_{2x}\sin k_{2x}(a-d)}{\varepsilon_2}$$

$$H_{y_1}\Big|_{\substack{x=d\\z=0}} = H_{y2}\Big|_{\substack{x=d\\z=0}}$$

$$\cos k_{1x} d = \cos k_{2x}(a-d)$$

$$\frac{k_{1x}}{\varepsilon_1}\tan k_{1x} d = -\frac{k_{2x}}{\varepsilon_2}\tan k_{2x}(a-d)$$

From Eqn. 4-42,

$$k_{x_1}^2 = \omega^2 \varepsilon_1 \mu_1 - k_z^2$$

and $k_{x2}^2 = \omega^2 \varepsilon_2 \mu_2 - k_z^2$

$$\psi_1^{TE} = \sin k_{1x} x\, e^{-jk_z z}$$

$$\psi_2^{TE} = \sin k_{2x}(a-x)\, e^{-jk_z z}$$

$$E_y = -\frac{\partial \psi}{\partial z}$$

$$E_{y_1} = -jk_z \sin k_{1x} x\, e^{-jk_z z}$$

$$E_{y_2} = -jk_z \sin k_{2x}(a-x)\, e^{-jk_z z}$$

$$H_{z_1} = \frac{k_1 x}{\hat{z}_1}\cos k_{1x} x\, e^{-jk_z z}$$

$$H_{z2} = -\frac{k_2 x}{\hat{z}_2}\cos k_{2x}(a-x)\, e^{-jk_z z}$$

$$E_{y_1}\Big|_{\substack{x=d\\z=0}} = E_{y2}\Big|_{\substack{x=d\\z=0}}$$

4-12 (cont.)

$$\sin k_{1x} d = \sin k_{2x}(a-d)$$

$$H_{z1}\Big|_{\substack{x=d\\z=0}} = H_{z2}\Big|_{\substack{x=d\\z=0}}$$

$$\frac{k_{1x}\cos k_{1x} d}{\mu_1} = -\frac{k_{2x}\cos k_{2x}}{\mu_2}(a-d)$$

$$\therefore \frac{k_{1x}}{\mu_1}\cot k_{1x} d = -\frac{k_{2x}}{\mu_2}\cot k_{2x}(a-d)$$

4-13

$$\frac{k_{1x}}{\varepsilon_1}\tan k_{1x} d = -\frac{k_{2x}}{\varepsilon_2}\tan k_{2x}(a-d)$$

$$k_{1x}^2 + k_z^2 = \omega^2 \mu_1 \varepsilon_1$$

$$k_{2x}^2 + k_z^2 = \omega^2 \mu_2 \varepsilon_2$$

as $\varepsilon_1 \to \varepsilon_2$ and $\mu_1 \to \mu_2$

and $d \to 0$,

$$\frac{k_{1x}^2 d}{\varepsilon_1} = -\frac{k_{2x}^2(a-d)}{\varepsilon_2}$$

If $a << \lambda_2$,

$$(-k_z^2 + \omega^2\mu_1\varepsilon_1)\frac{d}{\varepsilon_1} = -(\omega^2\mu_2\varepsilon_2 - k_z^2)\frac{(a-d)}{\varepsilon_2}$$

$$k_z^2\left[\frac{d}{\varepsilon_1} + \frac{a-d}{\varepsilon_2}\right] = \omega^2\left[\mu_2(a-d) + \mu_1 d\right]$$

$$k_z^2\frac{\left[\varepsilon_2 d + \varepsilon_1(a-d)\right]}{\varepsilon_1\varepsilon_2} = \omega^2\left[\mu_2(a-d) + \mu_1 d\right]$$

$$\therefore k_z \approx \omega\sqrt{\frac{\varepsilon_1\varepsilon_2\left[\mu_1 d + \mu_2(a-d)\right]}{\varepsilon_2 d + \varepsilon_1(a-d)}}$$

Using same method as done in ~~pro~~ obtaining Eqn 4-50:

$$L = \mu_1 d + \mu_2(a-d)$$

4-13 (cont.)

$$C = \frac{\varepsilon_2 d + \varepsilon_1(a-d)}{\varepsilon_1 \varepsilon_2}$$

$$k_z = \omega\sqrt{LC}$$

For dominant mode, $k_{x1} = k_{x2} = 0$

$\therefore k_z^2 = \omega^2 \mu \varepsilon$ which is the transmission line mode.

4-14 $\quad \frac{k_{x1}^2 d}{\varepsilon_1} \simeq -\frac{k_{x2}^2(a-d)}{\varepsilon_2}$

$$f(k_z, d) = \frac{k_{x1}^2 d}{\varepsilon_1} + \frac{k_{x2}^2(a-d)}{\varepsilon_2} = 0$$

$$k_{x1}^2 = k_1^2 - \left(\frac{\pi}{b}\right)^2 - k_z^2$$

$$= \beta_0^2 + k_1^2 - k_2^2 - k_z^2$$

$$k_{x2}^2 = k_2^2 - \left(\frac{\pi}{b}\right)^2 - k_z^2$$

$$= \beta_0^2 - k_z^2$$

$$f(k_z, d) = \frac{d}{\varepsilon_1}\left[\beta_0^2 + k_1^2 - k_2^2 - k_z^2\right] + \frac{(a-d)}{\varepsilon_2}\left[\beta_0^2 - k_z^2\right]$$

$$f_d = \frac{\partial f}{\partial d} = \frac{1}{\varepsilon_1}\left(\beta_0 + k_1^2 - k_2^2 - k_z^2\right) - \frac{1}{\varepsilon_2}\left(\beta_0^2 - k_z^2\right)$$

$$f_z = \frac{\partial f}{\partial k_z} = \frac{-2k_z d}{\varepsilon_1} - \frac{2k_z(a-d)}{\varepsilon_2}$$

Neglecting second order and higher terms, we write:

$$f(k_z, d) \simeq f(\beta_0, 0) + f_d(\beta_0, 0)d + f_z(\beta_0, 0)(k_z - \beta_0)$$

4-14 (cont.)

$$f(k_z, d) = 0 + (k_1^2 - k_2^2)\frac{d}{\varepsilon_1} + \frac{1}{\varepsilon_2}(-2\beta_0)(a)(k_z - \beta_0)$$

$$\frac{(k_1^2 - k_2^2)d}{\varepsilon_1} = \frac{2\beta_0}{\varepsilon_2}(k_z - \beta_0)a$$

$$\therefore k_z = \beta_0 + \frac{\varepsilon_2}{\varepsilon_1}\frac{(k_1^2 - k_2^2)}{2\beta_0}\frac{d}{a}$$

4-15

$$k_{x1}^2 = k_1^2 - k_z^2$$

$$k_{x2}^2 = k_2^2 - k_z^2 = \left(\frac{\pi}{a}\right)^2 + \beta_0^2 - k_z^2$$

$$\beta_0^2 = k_2^2 - \left(\frac{\pi}{a}\right)^2$$

Reciprocal of Eqn. 4-47:

$$0 = \frac{\mu_1}{k_{x1}}\tan k_{x1}d + \frac{\mu_2}{k_{x2}}\tan\left[k_{x2}(a-d)\right]$$

$$\mu_1\left(d + \frac{d^3}{3}k_{x1}^2\right) = -\frac{\mu_2}{k_{x2}}\left(k_{x2}(a-d) - \pi + \frac{(k_{x2}(a-d)-\pi)^3}{3}\right)$$

$$\mu_1 k_{x2}\left[d + \frac{d^3}{3}\left(k_{x2}^2 + k_1^2 - k_2^2\right)\right] = -\mu_2\left[k_{x2}(a-d) - \pi + \frac{(k_{x2}(a-d)-\pi)^3}{3}\right]$$

Let $k_{x2}a - \pi = C_0 + C_1 d + C_2 d^2 + C_3 d^3$

Neglecting fourth order and higher terms and solve for coefficients by setting terms of like powers equal.

<u>4-15 (cont.)</u>

$$c_0 = 0 \qquad c_1 = \frac{\mu_2-\mu_1}{\mu_2}\frac{\pi}{a} \qquad c_2 = \frac{c_1^2}{\pi} = \left(\frac{\mu_2-\mu_1}{\mu_2}\right)^2\frac{\pi}{a^2}$$

$$c_3 = -\frac{\left(c_1-\frac{\pi}{a}\right)^3}{3} + \frac{c_2}{a}\frac{(\mu_2-\mu_1)}{\mu_2} - \frac{\mu_1\pi}{3a\mu_2}\left[\left(\frac{\pi}{a}\right)^2+k_1^2-k_2^2\right]$$

$$c_3 = \left(\frac{\pi}{a}\right)^3\frac{\mu_1^3}{3\mu_2^3} + \frac{(\mu_2-\mu_1)^3}{\mu_2^3}\frac{\pi}{a^3} - \frac{\mu_1\pi}{3a\mu_2}\left(\left(\frac{\pi}{a}\right)^2+k_1^2-k_2^2\right)$$

$$k_{x2} = \frac{\pi + c_1 d + c_2 d^2 + c_3 d^3}{a} \quad \text{and} \quad k_z = \sqrt{k_2^2-k_{x2}^2}$$

$$k_z = \left[k_2^2 - \left(\frac{\pi}{a}\right)^2 - \frac{2c_1\pi d + (c_1^2+2c_2\pi)d^2 + (2c_3\pi+2c_1c_2)d^3}{a^2}\right]^{1/2}$$

$$= \beta_0\left[1 - \frac{2c_1\pi d + (c_1^2+2c_2\pi)d^2 + (2c_3\pi+2c_1c_2)d^3}{\beta_0^2a^2}\right]^{1/2}$$

$$c_1^2 + 2c_2\pi = 3c_1^2, \quad 2c_3\pi + 2c_1c_2 = \frac{4c_1^3}{\pi} + 2\pi\left[\left(\frac{\pi}{a}\right)^3\frac{\mu_1^3}{3\mu_2^3} - \frac{\mu_1\pi}{3a\mu_2}\left(\frac{\pi^2}{a^2}+k_1^2-k_2^2\right)\right]$$

Simplifying and using Binomial expansion,

$$k_z = \beta_0 + \frac{(\mu_1-\mu_2)}{\mu_2\beta_0}\left(\frac{\pi}{a}\right)^2\frac{d}{a} - \frac{(\mu_1-\mu_2)^2}{\mu_2^2}\left(\frac{3\beta_0^2a^2+\pi^2}{2\beta_0^3a^2}\right)\frac{\pi^2}{a^2}\left(\frac{d^2}{a^2}\right)$$

$$+\left[\frac{(\mu_1-\mu_2)^3}{2\mu_2^3\beta_0^5a^3}\left(\frac{\pi}{a}\right)^3\left(\frac{4}{\pi}\beta_0^4a^4 + 3\pi\beta_0^2a^2+\pi^3\right) - \frac{\pi^4}{3\beta_0a^2}\left(\frac{\mu_1}{\mu_2}\right)\left(\frac{\mu_1^2}{\mu_2^2}-1\right) + \frac{\pi^2\mu_1}{3\mu_2\beta_0}(k_1^2-k_2^2)\right]\left(\frac{d}{a}\right)^3$$

which reduces to the result in the book if $(\mu_1-\mu_2)=0$ in coefficients of higher powers of d. This problem was also done by straight differentiation thus obtaining derivatives of order 3. The results obtained were exactly the same.

4-16

$$\psi_1^{TM} = C_1 \cos k_{x1}x \sin\frac{n\pi y}{b}\sin\frac{p\pi z}{c}$$

$$\psi_2^{TM} = C_2 \cos[k_{x2}(a-x)] \sin\frac{n\pi y}{b}\sin\frac{p\pi z}{c}$$

$$E_{y1} = \frac{-1}{j\omega\varepsilon_1} C_1 k_{x1}\frac{n\pi}{b}\sin k_{x1}x \cos\frac{n\pi y}{b}\sin\frac{p\pi z}{c}$$

$$E_{y2} = \frac{1}{j\omega\varepsilon_2} C_2 k_{x2}\frac{n\pi}{b}\sin[k_{x2}(a-x)] \cdot \cos\frac{n\pi y}{b}\sin\frac{p\pi z}{c}$$

$$E_{z1} = \frac{-1}{j\omega\varepsilon_1} C_1 k_{x1}\frac{p\pi}{c}\sin k_{x1}x \sin\frac{n\pi y}{b}\cos\frac{p\pi z}{c}$$

$$E_{z2} = \frac{1}{j\omega\varepsilon_2} C_2 k_{x2}\frac{p\pi}{c}\sin[k_{x2}(a-x)] \cdot \sin\frac{n\pi y}{b}\cos\frac{p\pi z}{c}$$

From continuity of field quantities:

$$\frac{1}{\varepsilon_1} C_1 k_{x1}\sin k_{x1}d = \frac{-1}{\varepsilon_2} C_2 k_{x2}\sin[k_{x2}(a-d)]$$

$$H_{y1} = \frac{p\pi C_1}{c}\cos k_{x1}x \sin\frac{n\pi y}{b}\cos\frac{p\pi z}{c}$$

$$H_{y2} = \frac{p\pi}{c}C_2 \cos[k_{x2}(a-x)]\sin\frac{n\pi y}{b}\cos\frac{p\pi z}{c}$$

$$H_{z1} = \frac{n\pi C_1}{b}\cos k_{x1}x \cos\frac{n\pi y}{b}\sin\frac{p\pi z}{c}$$

$$H_{z2} = \frac{n\pi C_2}{b}\cos[k_{x2}(a-x)]\cos\frac{n\pi y}{b}\sin\frac{p\pi z}{c}$$

Again, from continuity,

$$C_1\cos k_{x1}d = C_2\cos[k_{x2}(a-d)]$$

Eqn. 4-45 is now easily obtained. Similarly for the TE case, variation in the z-direction cancels out giving Eqn. 4-47.

4-17

From prob. 4-14:

$$k_z \approx \beta_0 + \frac{\varepsilon_2}{\varepsilon_1}\left(\frac{k_1^2-k_2^2}{2\beta_0}\right)\frac{d}{a}$$

$$k_z^2 \approx k_2^2 - \left(\frac{\pi}{b}\right)^2 + \frac{d\,\varepsilon_2}{a\,\varepsilon_1}(k_1^2-k_2^2)$$

$$k_z^2 = \left(\frac{\pi}{c}\right)^2$$

$$k_1^2 - k_2^2 = \omega^2(\mu_1\varepsilon_1 - \mu_2\varepsilon_2)$$

$$\left(\frac{\pi}{b}\right)^2 + \left(\frac{\pi}{c}\right)^2 = \omega^2\mu_2\varepsilon_2 + \frac{d\,\varepsilon_2}{a\,\varepsilon_1}\omega^2(\mu_1\varepsilon_1 - \mu_2\varepsilon_2)$$

$$\omega^2\left[\mu_2\varepsilon_2 a\varepsilon_1 + d\varepsilon_2\mu_1\varepsilon_1 - d\varepsilon_2\mu_2\varepsilon_2\right] = a\varepsilon_1\left[\left(\frac{\pi}{b}\right)^2 + \left(\frac{\pi}{c}\right)^2\right]$$

$$\omega^2\mu_2\varepsilon_2 = \frac{a\varepsilon_1}{a\varepsilon_1 + d\left[\frac{\mu_1\varepsilon_1 - \mu_2\varepsilon_2}{\mu_2}\right]}\left[\left(\frac{\pi}{b}\right)^2 + \left(\frac{\pi}{c}\right)^2\right]$$

$$\sqrt{\omega^2\mu_2\varepsilon_2} = \left[1 + \frac{d}{a}\left(\frac{\mu_1\varepsilon_1 - \mu_2\varepsilon_2}{\mu_2\varepsilon_1}\right)\right]^{-1/2}\left[\left(\frac{\pi}{b}\right)^2 + \left(\frac{\pi}{c}\right)^2\right]$$

Using binomial theorem:

$$\sqrt{\omega^2\mu_2\varepsilon_2} = \left[1 - \frac{1}{2}\left(\frac{\mu_1}{\mu_2} - \frac{\varepsilon_2}{\varepsilon_1}\right)\frac{d}{a}\right]\left[\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{c}\right)^2\right]$$

$$\omega_r = \omega_0\left[1 - \frac{1}{2}\left(\frac{\mu_1}{\mu_2} - \frac{\varepsilon_2}{\varepsilon_1}\right)\frac{d}{a}\right]$$

where $\omega_0 = \frac{1}{\sqrt{\mu_2\varepsilon_2}}\left[\left(\frac{\pi}{b}\right)^2 + \left(\frac{\pi}{c}\right)^2\right]^{1/2}$

54 **4-18** Using results we obtained for prob. 4-15:

$$\frac{\pi}{c} = \beta_0 + \left[\frac{(\mu_1-\mu_2)}{\mu_2}\frac{\pi^2}{a^3}d - 3\frac{(\mu_1-\mu_2)^2}{2\mu_2^2}\frac{\pi^2}{a^4}d^2 + \left(\frac{2(\mu_1-\mu_2)^3\pi^2}{\mu_2^3 a^5} - \frac{\pi^4}{3a^5}\frac{\mu_1}{\mu_2}\left(\frac{\mu_1^2}{\mu_2^2}-1\right)\right)d^3\right]\frac{1}{\beta_0}$$

$$+\left[-\frac{(\mu_1-\mu_2)^2}{2\mu_2^2}\frac{\pi^4}{a^6}d^2 + \frac{3(\mu_1-\mu_2)^3\pi^4}{2\mu_2^3 a^7}d^3\right]\frac{1}{\beta_0^3} + \left[\frac{(\mu_1-\mu_2)^3\pi^6}{2\mu_2^3 a^9}d^3\right]\frac{1}{\beta_0^5}$$

$$+\left[\frac{\pi^2}{3}\frac{\mu_1\varepsilon_2}{a^3}\left(\frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}-1\right)d^3\right]\frac{\omega^2}{\beta_0}. \qquad \omega^2 = \frac{\beta_0^2}{\mu_2\varepsilon_2} + \frac{\pi^2}{a^2\mu_2\varepsilon_2}$$

Expand β_0 in powers of d:

$$\beta_0 = C_0 + C_1 d + C_2 d^2 + C_3 d^3$$

Substituting for β_0 in above expression we equate equal powers of d and solve for constants:

$$C_0 = \frac{\pi}{c}, \quad C_1 = -\frac{(\mu_1-\mu_2)}{\mu_2}\frac{\pi c}{a^3}, \quad C_2 = \frac{(\mu_1-\mu_2)^2}{\mu_2^2}\pi\left(\frac{-c^3}{2a^6}+\frac{3c}{2a^4}\right)$$

$$C_3 = -\frac{(\mu_1-\mu_2)^3}{\mu_2^3 2}\frac{\pi c^5}{a^4}\left[\frac{1}{a^5}-\frac{3}{c^2a^3}+\frac{4}{ac^4}\right] - \frac{\pi^3\mu_1 c}{3\mu_2 a^3}\left[\frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}\left(\frac{\omega_0^2}{\pi^2}\right) - \frac{\mu_1^2}{a^2\mu_2^2} - \frac{1}{c^2}\right]$$

$$\omega^2 = \omega_0^2 - 2\frac{(\mu_1-\mu_2)}{\mu_2}\frac{\pi^2}{a^2}\left(\frac{d}{a}\right) + \frac{(\mu_1-\mu_2)^2}{\mu_2^2 2}\left(\frac{\pi}{a}\right)^2\left[\frac{c^2+3a^2}{a^2}\right]\left(\frac{d}{a}\right)^2$$

$$+\left[-4\frac{(\mu_1-\mu_2)^3}{\mu_2^3}\left(\frac{\pi}{a}\right)^2 - \frac{2\pi^4\mu_1}{3\mu_2}\left[\frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}\left(\frac{\omega_0^2}{\pi^2}\right) - \frac{\mu_1^2}{a^2\mu_2^2} - \frac{1}{c^2} - \frac{1}{a^2} + \frac{1}{a^2}\right]\right]\left(\frac{d}{a}\right)^3$$

$$\frac{\pi^2}{a^2} = \omega_0^2\mu_2\varepsilon_2\left(\frac{c^2}{a^2+c^2}\right).$$ use binomial exp. for square root:

$$\omega \approx \omega_0\left[1 - \frac{(\mu_1-\mu_2)}{\mu_2}\frac{c^2}{(a^2+c^2)}\left(\frac{d}{a}\right) - \frac{\pi^2}{3}\frac{\mu_1}{\mu_2}\left(\frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}-1\right)\left(\frac{d}{a}\right)^3\right.$$

$$+\frac{(\mu_1-\mu_2)^2}{4\mu_2^2}\frac{(c^2+3a^2)}{a^2(a^2+c^2)}c^2\left(\frac{d}{a}\right)^2 - 2\frac{(\mu_1-\mu_2)^3}{\mu_2^3}\frac{c^2}{(a^2+c^2)}\left(\frac{d}{a}\right)^3$$

$$\left. - \frac{\pi^2\mu_1}{3\mu_2}\frac{c^2}{(a^2+c^2)}\left[1-\frac{\mu_1^2}{\mu_2^2}\right]\left(\frac{d}{a}\right)^3\right]$$

4-19 (cont.) modes TM to x:

$$\psi_0^{TM_x} = A\cos k_{x0}x \sin\frac{n\pi y}{b}\, e^{-jk_z z}$$

For regions ① & ③,

$$\psi_1^{TM_x} = B\sin k_{x1}\left|x-\frac{a}{2}\right| \sin\frac{n\pi y}{b}\, e^{-jk_z z}$$

For region ②,

$$E_y = \frac{1}{j\omega\varepsilon}\frac{\partial^2\psi}{\partial x\,\partial y}, \quad H_z = -\frac{\partial\psi}{\partial y}$$

$$\left.\begin{aligned} E_y &= -\frac{k_{x0}A}{j\omega\varepsilon_0}\frac{n\pi}{b}\sin k_{x0}x\cos\frac{n\pi y}{b}\,e^{-jk_z z} \\ H_z &= -\frac{n\pi}{b}B\cos k_{x0}x\cos\frac{n\pi y}{b}\,e^{-jk_z z}\end{aligned}\right\} ①\ \&\ ③$$

$$\left.\begin{aligned} E_y &= -\frac{Ak_{x1}}{j\omega\varepsilon_d}\frac{n\pi}{b}\cos k_{x1}\left|x-\frac{a}{2}\right|\cos\frac{n\pi y}{b}\,e^{-jk_z z} \\ H_z &= \frac{n\pi}{b}B\sin k_{x1}\left|x-\frac{a}{2}\right|\cos\frac{n\pi y}{b}\,e^{-jk_z z}\end{aligned}\right\} ②$$

at $x = \frac{a-d}{2}$ and $\frac{a+d}{2}$

$$-\frac{k_{x0}}{\varepsilon_0}\sin\left[k_{x0}\frac{(a-d)}{2}\right] = -\frac{k_{x1}}{\varepsilon_d}\cos\frac{k_{x1}d}{2}$$

and

$$-\cos\left[k_{x0}\frac{(a-d)}{2}\right] = -\sin k_{x1}\frac{d}{2}$$

dividing eqns. gives:

$$\frac{k_{x0}}{\varepsilon_0}\tan\left(k_{x0}\frac{(a-d)}{2}\right) = \frac{k_{x1}}{\varepsilon_d}\cot\left(\frac{k_{x1}d}{2}\right)$$

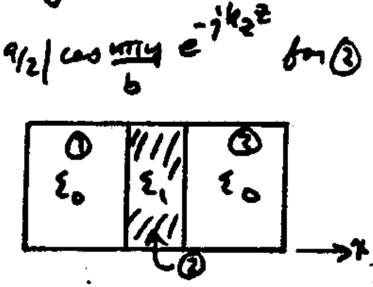
Mode functions for waves TE to x are:

$$\psi_0^{TE_x} = A\sin k_{x0}x\cos\frac{n\pi y}{b}\,e^{-jk_z z} \text{ for ① \& ③}$$

$$\psi_1^{TE_x} = B\cos k_{x1}|x-a/2|\cos\frac{n\pi y}{b}\,e^{-jk_z z} \text{ for ②}$$

$$H_y = \frac{1}{j\omega\mu}\frac{\partial^2\psi}{\partial x\,\partial y}$$

$$E_z = \frac{\partial\psi}{\partial y}$$



4-19 (cont.)

$$\left.\begin{aligned} E_z &= -\frac{n\pi A}{b}\sin k_{x0}x\sin\frac{n\pi y}{b}\,e^{-jk_z z} \\ H_y &= -\frac{k_{x0}A}{j\omega\mu_0}\frac{n\pi}{b}\cos k_{x0}x\sin\frac{n\pi y}{b}\,e^{-jk_z z}\end{aligned}\right\} \text{for ① \& ③}$$

$$\left.\begin{aligned} E_z &= -\frac{n\pi}{b}B\cos k_{x1}\left|x-\frac{a}{2}\right|\sin\frac{n\pi y}{b}\,e^{-jk_z z} \\ H_y &= -\frac{k_{x1}B}{j\omega\mu_1}\frac{n\pi}{b}\sin k_{x1}\left|x-\frac{a}{2}\right|\sin\frac{n\pi y}{b}\,e^{-jk_z z}\end{aligned}\right\} \text{for ②}$$

For continuity at $x = \frac{a-d}{2}$, $x = \frac{a+d}{2}$:

$$-\sin k_{x0}\left(\frac{a-d}{2}\right) = -\cos k_{x1}\frac{d}{2}$$

$$-\frac{k_{x0}}{\mu_0}\cos k_{x0}\left(\frac{a-d}{2}\right) = -\frac{k_{x1}}{\mu_1}\sin\frac{k_{x1}d}{2}$$

Dividing eqns. gives:

$$\frac{k_{x0}}{\mu_0}\cot\left(k_{x0}\frac{(a-d)}{2}\right) = \frac{k_{x1}}{\mu_1}\tan\left(\frac{k_{x1}d}{2}\right)$$

4-20

$$u^2 + k_z^2 = k_d^2 = \omega^2\mu_d\varepsilon_d$$

$$-v^2 + k_z^2 = k_0^2 = \omega^2\mu_0\varepsilon_0$$

$$\left.\begin{aligned} E_z &= \frac{u^2}{j\omega\varepsilon_d}A\cos ux\,e^{-jk_z z} \\ H_y &= Au\sin ux\,e^{-jk_z z}\end{aligned}\right\} |x| < \frac{a}{2}$$

$$\left.\begin{aligned} H_y &= vB\,e^{-v|x|}e^{-jk_z z} \\ E_z &= -\frac{v^2B}{j\omega\varepsilon_a}e^{-v|x|}e^{-jk_z z}\end{aligned}\right\} |x| > \frac{a}{2}$$

Continuity of E_z & H_y at $x = \pm\frac{a}{2}$ gives:

$$\frac{Au^2}{\varepsilon_d}\cos\frac{ua}{2} = -\frac{v^2B}{\varepsilon_a}e^{-va/2}$$

$$Au\sin\frac{ua}{2} = vB\,e^{-va/2}$$

Dividing eqns.:

$$\frac{a}{2}\frac{u}{\varepsilon_d}\cot\frac{ua}{2} = -\frac{v}{\varepsilon_a}\frac{a}{2}$$

or $-\frac{ua}{2}\cot\frac{ua}{2} = \frac{\varepsilon_d}{\varepsilon_a}\frac{va}{2}$ which is Eqn. 4-58.

4-21 From Eqn. 4-63,

$$f_c = \frac{n}{2a\sqrt{\varepsilon_d\mu_0 - \varepsilon_0\mu_0}} = \frac{n}{2a\sqrt{\varepsilon_0\mu_0}\sqrt{\varepsilon_r - 1}}$$

$$= n(1.6\times 10^{10})\ Hz \qquad \varepsilon_r = 2.56$$

$a = 3/4$

$\mu_d = \mu_0$

modes which occurs unattenuated are TM_0, TM_1, TE_0, TE_1.

Refer to Figure 4-11.

$$\frac{\omega a}{2}\sqrt{\varepsilon_d\mu_d - \varepsilon_0\mu_0} = 2.94 \approx .94\pi$$

$$\frac{u_1 a}{2} = \frac{3\pi}{8}, \quad \frac{u_2 a}{2} = .72\pi$$

$$k_z = \sqrt{k_d^2 - u^2}$$

$u_1 = \pi$, $u_2 = 6.03$

$k_0 = 2\pi$

$k_d = 3.2\pi$

$k_{z1} = 9.55$, $k_{z2} = 8.10$

Note: $k_0 < k_z < k_d$

4-22 A Taylor series expansion of the left side of Eqn. 4-56 about $a = 0$ and the right side about $v = 0$ gives:

$$\frac{u^2a^2}{4} = \frac{\varepsilon_d}{\varepsilon_0}\frac{va}{2}$$

$$\text{or } v = \frac{\varepsilon_0}{\varepsilon_d}\frac{a}{2}u^2$$

From Eqn 4-55, $u^2 = k_d^2 - k_z^2$

But $k_z = k_0 + \frac{v^2}{2k_0}$

and $k_z^2 \to k_0^2$ as $v \to 0$

$$\text{so } v = \frac{\varepsilon_0}{\varepsilon_d}\frac{a}{2}(k_d^2 - k_0^2)$$

4-22 (cont.)

Similarly for TE case Eqn. 4-59 becomes:

$$\frac{u^2a^2}{4} = \frac{\mu_d}{\mu_0}\frac{va}{2}$$

$$\text{or } v = \frac{\mu_0}{\mu_d}\frac{a}{2}u^2$$

and $u^2 \to k_d^2 - k_0^2$

$$\text{so } v = \frac{\mu_0}{\mu_d}(k_d^2 - k_0^2)\frac{a}{2}$$

4-23 $f = 3\times 10^{10}$ Hz

$t = .001'' = .0254$ cm.

$\lambda_d = \frac{1}{\sqrt{3}}$ cm $= .577$ cm.

$t = .044\lambda_d \qquad k_0 = 2\pi$

$v \approx 2\pi k_0\left(1 - \frac{1}{3}\right)(.044)$ (using formula for t small).

$v \approx 1.158$ = attenuation constant in the x direction. At .86 wavelengths the field decays to 36.8% of its value at the surface. It would be reasonable to say a tightly bound surface wave is possible.

4-24 Field attenuates as e^{-vx}.

For 36.8% attenuation, $vx = 1$

$$x = 1\lambda, \quad v = \frac{1}{\lambda}$$

From Eqn 4-70,

$$v = k_0 \tan k_0 d \qquad k_0 = \frac{2\pi}{\lambda}$$

$d = \frac{\lambda}{2\pi}\tan^{-1}\frac{1}{2\pi} = .025\lambda$ = minimum value of d for this attenuation.

4-25 For a dielectric filled, corrugated slot wave guide,

$$\left.\begin{aligned} E_x &= \frac{k_z}{\omega \varepsilon_d} H_y \\ E_z &= \frac{-B}{j\omega \varepsilon_d} v^2 e^{-vx} e^{-jk_z z} \\ H_y &= B v e^{-vx} e^{-jk_z z} \end{aligned}\right\} x > d$$

$$-v^2 + k_z^2 = k_d^2 = \omega^2 \mu_d \varepsilon_d$$

$$Z_{(-x)} = \frac{E_z}{H_y} = \frac{jv}{\omega \varepsilon_d} = j\eta_d \tan k_d d$$

$$\therefore v = \omega \varepsilon_d \eta_d \tan k_d d$$

$$\omega \varepsilon_d = \frac{k_d}{\eta_d}$$

$$v = \frac{\varepsilon_0}{\varepsilon_d} k_d \tan k_d d$$

$$k_z^2 = \omega^2 \varepsilon_0 \mu_0 + \frac{\varepsilon_0^2}{\varepsilon_d^2} k_d^2 \tan^2 k_d d$$

$$= k_0^2 \left[1 + \frac{\varepsilon_0^2 \omega^2 \varepsilon_d \mu_d}{\varepsilon_d^2 \omega^2 \varepsilon_0 \mu_0} \tan^2 k_d d \right]$$

$$k_z = k_0 \left[1 + \frac{\varepsilon_0 \mu_d}{\varepsilon_d \varepsilon_0} \tan^2 k_d d \right]^{1/2}$$

4-26 Taking a superposition of the modes given in problem 4-7:

$$\psi = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi y}{b} e^{-\gamma_n z}$$

$$E_y = \frac{-\partial \psi}{\partial z} = -\sum_{n=0}^{\infty} \gamma_n A_n \cos\frac{n\pi y}{b} e^{-\gamma_n z}$$

$$\int_0^b E_y\Big|_{z=0} \cos\frac{m\pi y}{b}\, dy$$

$$= \sum_{n=0}^{\infty} \gamma_n A_n \int_0^b \cos\frac{n\pi y}{b} \cos\frac{m\pi y}{b}\, dy$$

$$= \frac{\gamma_n A_n b}{\epsilon_n}$$

4-26 (cont.)

ϵ_n = Neumann number

$= 1, \; n = 0$

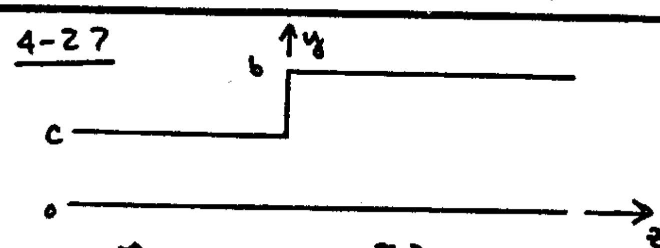
$= 2, \; n \geq 1$

$$\therefore A_n = \frac{\epsilon_n}{b \gamma_n} \int_0^b E_y\Big|_{z=0} \cos\frac{n\pi y}{b}\, dy$$

$$H_x = \frac{1}{\hat{z}} \left(k^2 - \frac{n^2 \pi^2}{b^2} \right) A_n \cos\frac{n\pi y}{b} e^{-\gamma_n z}$$

$$Z_0 = \frac{-E_y}{H_x} = \frac{\gamma_n}{j\omega\mu \left(k^2 - \frac{n^2\pi^2}{b^2} \right)}$$

4-27



$$\psi = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi y}{b} e^{-\gamma_n z}, \quad z > 0$$

$$E_y\Big|_{z=0} = \begin{cases} E_0 & ; 0 < y < c \\ 0 & , y > c \end{cases}$$

Using Eqns. 4-32,

$$E_y = \frac{-\partial \psi}{\partial x} = \gamma_n \sum_{n=0}^{\infty} A_n \gamma_n \cos\frac{n\pi y}{b} e^{-\gamma_n z}$$

$$H_x = \frac{1}{\hat{z}} k^2 \psi = \frac{1}{\hat{z}} k^2 \sum_{n=0}^{\infty} A_n \gamma_n \cos\frac{n\pi y}{b} e^{-\gamma_n z}$$

Power per unit width,

$$P = \int E \times H^* dy = \int_0^b \left[E_y H_x^* \right]\Big|_{z=0} dy$$

From Eqn 4-73,

$$\gamma_n A_n = \frac{2\epsilon_n}{b} \int_0^c E_0 \cos\frac{n\pi y}{b}\, dy$$

$$A_n \gamma_n = \frac{\epsilon_n}{n\pi} \sin\frac{n\pi c}{b}$$

For $n = 0$, $A_0 \gamma_0 = \dfrac{c E_0 \epsilon_n}{b}$

4-27 (cont.)

$$P = -\int_0^b \sum_{n=0}^{\infty} \gamma_n^2 A_n \cos\frac{n\pi y}{b} \cdot \frac{1}{\gamma} k^2 \sum_{p=0}^{\infty} \frac{A_p^* \cos\frac{p\pi y}{b}}{\gamma_p}$$

$$= \sum_{n=0}^{\infty} \frac{\gamma_n^2}{\gamma} A_n^2 \frac{b}{\epsilon_n}$$

From Eqn. 4-74,

$$P = \frac{c^2}{b}\left\{ (Y_0)_0 + 2\sum_{n=1}^{\infty} (Y_0)_n \left[\frac{\sin\frac{n\pi c}{b}}{\frac{n\pi c}{b}}\right]^2 \right\}$$

Voltage across center of aperture is:

$$V = E_0 c$$

$$Y_a = \frac{P^*}{|V|^2}$$

$$P = \frac{c^2}{b}(Y_0)_0 \left\{ 1 + 2\sum_{n=1}^{\infty} \frac{j2b\,(Y_0)_0}{\lambda_g\sqrt{n^2-(2b/\lambda_g)^2}} \cdot \left[\frac{\sin\frac{n\pi c}{b}}{(n\pi c/b)}\right]^2 \right\}$$

$$Y_a = G_a + jB_a$$

$$B_a = \frac{4}{\eta\lambda_g} \sum_{n=1}^{\infty} \frac{\sin^2\frac{n\pi c}{b}}{\left(\frac{n\pi c}{b}\right)^2 \sqrt{n^2-\left(\frac{2b}{\lambda_g}\right)^2}}$$

This susceptance is twice that given by Eqn. 4-78.

4-28

$$E_y\big|_{z=0} = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \gamma_{mn} A_{mn} \sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$$

$$\gamma_{mn} A_{mn} = \frac{2\epsilon_n}{ab}\int_0^a dx \int_{\frac{b}{2}-\frac{c}{2}}^{\frac{b}{2}+\frac{c}{2}} dy\, E_y\big|_{z=0} \sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$$

$$E_y\big|_{z=0} = \sin\frac{\pi x}{a}$$

$$\gamma_{10} A_{10} = \frac{2}{ab} c \frac{a}{2} = \frac{c}{b}$$

4-28 (cont.)

$$\gamma_{1n} A_{1n} = \frac{4}{ab}\frac{a}{2}\left[\frac{\sin\frac{n\pi y}{b}}{n\pi/b}\right]\Bigg|_{\frac{b-c}{2}}^{\frac{b+c}{2}}$$

$$= \frac{2}{n\pi}\left[\sin\left(\frac{n\pi}{2}+\frac{n\pi c}{2b}\right) - \sin\left(\frac{n\pi}{2}-\frac{n\pi c}{b}\right)\right]$$

$$= \frac{2}{n\pi}\left[\sin\frac{n\pi}{2}\cos\frac{n\pi c}{2b} + \cos\frac{n\pi}{2}\sin\frac{n\pi c}{2b} - \cos\frac{n\pi}{2}\sin\frac{n\pi c}{2b} - \sin\frac{n\pi}{2}\cos\frac{n\pi c}{2b}\right]$$

$$= \frac{4}{n\pi}\cos\frac{n\pi}{2}\sin\frac{n\pi c}{2b}$$

which are nonzero only for even values of n.

$$\gamma_{1n} A_{1n} = \frac{2}{n\pi}\sin\frac{n\pi c}{b}$$

$$P = \iint E \times H^* \, dS$$

$$= \frac{ac^2}{2b}\left\{ (Y_0)_{10} + 2\sum_{n=1}^{\infty} \frac{j2b\,(Y_0)_{10}}{\lambda_g\sqrt{(2n)^2-\left(\frac{2b}{\lambda_g}\right)^2}} \cdot \left[\frac{\sin\frac{n\pi c}{b}}{(n\pi c/b)}\right]^2 \right\}$$

$$\frac{P^*}{|V|^2} = Y_a = jB_a + G_a$$

$$B_a = \frac{4a}{2\eta\lambda_g}\sum_{n=1}^{\infty} \frac{1}{\left[(2n)^2-\left(\frac{2b}{\lambda_g}\right)^2\right]^{1/2}} \left[\frac{\sin\frac{n\pi c}{b}}{\left(\frac{n\pi c}{b}\right)}\right]^2$$

$$= \frac{2a}{\eta(2\lambda_g)}\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2-\left(\frac{b}{2\lambda_g}\right)^2}} \left[\frac{\sin\frac{n\pi c}{b}}{\left(\frac{n\pi c}{b}\right)}\right]^2$$

4-29

$$\psi = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b} e^{-\gamma_{mn} z}$$

$$E_y\big|_{z=0} = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \gamma_{mn} A_{mn} \sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$$

$$\gamma_{mn} A_{mn} \frac{ab}{2\epsilon_n} = \int_{\frac{a-c}{2}}^{\frac{a+c}{2}} dx \int_0^b dy\, E_y\big|_{z=0} \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

4-29 (cont.)

$$\frac{\gamma_{mn}A_{mn}ab}{2\epsilon_n} = \int_{-\frac{c}{2}}^{c/2} \sin\frac{\pi x}{c}\sin\frac{m\pi x}{a}dx \int_b^b \cos\frac{n\pi y}{b}dy$$

$$= \frac{b}{2}\int_{-c/2}^{c/2}\left[\cos\left(\frac{m\pi x}{a}-\frac{\pi x}{c}\right)-\cos\left(\frac{m\pi x}{a}+\frac{\pi x}{c}\right)\right]dx$$

$$= \frac{b}{2\pi}\left[\frac{\sin\left(\frac{m\pi c}{2a}-\frac{\pi}{2}\right)}{\frac{m}{a}-\frac{1}{c}} - \frac{\sin\left(\frac{m\pi c}{2a}+\frac{\pi}{2}\right)}{\frac{m}{a}+\frac{1}{c}}\right.$$

$$\left. - \frac{\sin\left(\frac{-m\pi c}{2a}+\frac{\pi}{2}\right)}{\frac{m}{a}-\frac{1}{c}} + \frac{\sin\left(\frac{-m\pi c}{2a}-\frac{\pi}{2}\right)}{\frac{m}{a}+\frac{1}{c}}\right]$$

$$= \frac{b}{2\pi}\left[\frac{-2\cos\left(\frac{m\pi c}{2a}\right)}{\frac{m}{a}-\frac{1}{c}} + \frac{2\cos\left(\frac{m\pi c}{2a}\right)}{\frac{m}{a}+\frac{1}{c}}\right]$$

$$= \frac{b}{\pi}\cos\left(\frac{m\pi c}{2a}\right)\left[\frac{2/c}{\frac{1}{c^2}-\frac{m^2}{a^2}}\right]$$

$$= \frac{2cb}{\pi}\;\frac{\cos\frac{m\pi c}{a}}{1-\left(\frac{mc}{a}\right)^2}$$

$$\gamma_{m0}A_{m0} = \frac{4c}{\pi a}\;\frac{\cos\frac{m\pi c}{2a}}{1-\left(\frac{mc}{a}\right)^2}$$

$$P = \int_{-\frac{c}{2}}^{\frac{c}{2}}\int_0^b (Y_0^+)_{m0}|E_{mn}|^2 dy\,dx$$

$$= \sum_{m=1}^{\infty}(Y_0^+)_{m0}\left[\frac{\cos\frac{m\pi c}{2a}}{1-\left(\frac{mc}{a}\right)^2}\right]^2\frac{16c^2ab}{\pi^2a^2\,2}$$

$$(Y_0)_{m0} = \frac{-j}{\eta}\sqrt{\left(\frac{m\lambda}{2a}\right)^2-1} \quad \text{for } m>1$$

$$Y_a = \frac{P}{|V|^2} = \frac{P}{b^2} = G_a + jB_a$$

4-29 (cont.)

$$\therefore B_a = \sum_{m=2}^{\infty}\frac{-8c^2\lambda}{\pi^2ba^2\eta}\sqrt{\left(\frac{m}{2}\right)^2-\left(\frac{a}{\lambda}\right)^2}\cdot\left[\frac{\cos\frac{m\pi c}{2a}}{1-\left(\frac{mc}{a}\right)^2}\right]^2$$

4-30

$$\frac{-b\eta B_a}{\lambda} = \frac{2}{\pi^2}\left(\frac{c}{a}\right)^2\sum_{m=2}^{\infty}\left[\frac{\sin\frac{m\pi c}{a}}{1-\left(\frac{mc}{a}\right)^2}\right]^2\cdot\sqrt{\left(\frac{m}{2}\right)^2-\left(\frac{a}{\lambda}\right)^2}$$

$$f = \frac{-b\eta B_a}{\lambda} \qquad \frac{mc}{a}\sim x$$

$$\frac{c}{a}\sim dx$$

$$dx = \frac{x}{m}$$

$$\lim_{\frac{c}{a}\to 0} f = \lim_{\frac{c}{a}\to 0}\left(\frac{-b\eta}{\lambda}B_a\right)$$

$$= \lim_{\frac{c}{a}\to 0}\frac{2}{\pi^2}\sum_{m=2}^{\infty}\left[\frac{(\sin\pi x)^2}{(1-x^2)^2}\right]\sqrt{\left(\frac{mc}{2a}\right)^2-\left(\frac{c}{\lambda}\right)^2}$$

$$\lim_{\frac{c}{a}\to 0}\sqrt{\left(\frac{mc}{2a}\right)^2-\left(\frac{c}{\lambda}\right)^2} = \frac{x}{2}$$

$$\therefore \lim_{\frac{c}{a}\to 0} f = \frac{1}{\pi^2}\int_0^{\infty}\left[\frac{\sin\pi x}{1-x^2}\right]^2 x\,dx$$

$$\text{let } u = \sin^2\pi x \quad dv = \frac{x\,dx}{(1-x^2)^2}$$

$$du = 2\sin\pi x\cos\pi x\,\pi\,dx \qquad v = \frac{(1-x^2)^{-1}}{2}$$

$$\int_0^{\infty}\left(\frac{\sin\pi x}{1-x^2}\right)^2 x\,dx$$

$$= \left.\frac{2\sin^2\pi x}{(1-x^2)}\right|_0^{\infty} - \frac{\pi}{2}\int_0^{\infty}\frac{\sin 2\pi x}{(1-x^2)}dx$$

$$= 0 + \frac{\pi}{2}Si(2\pi)$$

10

4-30 (cont.)

$$\therefore -\frac{b\gamma G_a}{\lambda} \to \frac{\pi}{2\pi^2} Si(2\pi) = \frac{Si(2\pi)}{2\pi}$$

4-31

Choose mode functions as:

$$\psi^+ = \sum_{n=0}^{\infty} B_n^+ \cos\frac{n\pi y}{b} e^{-\gamma_n z}, \quad z>0$$

$$\psi^- = \sum_{n=0}^{\infty} B_n^- \cos\frac{n\pi y}{b} e^{\gamma_n z}, \quad z<0$$

Continuity of E_x and E_y at $z=0$

$\Rightarrow B_n^+ = B_n^- = B_n$.

$$J_y = \left[H_x^- - H_x^+\right]\Big|_{z=0}$$

$$H_x = \frac{\partial\psi}{\partial z} \Rightarrow J_y = \sum_{n=0}^{\infty} 2\gamma_n B_n \cos\frac{n\pi y}{b} \quad (1)$$

Let $A_n = \gamma_n B_n$

then $\sum_{n=0}^{\infty} A_n \cos\frac{n\pi y}{b} e^{-\gamma_n|z|} = H_x, \quad z>0$

$= -H_x, \quad z<0$

and $A_n = \frac{\epsilon_n}{2b}\int_0^b J_y \cos\frac{n\pi y}{b}dy$ from (1).

4-32

$$\psi^+ = \sum_{n=1}^{\infty} A_n^+ \sin\frac{n\pi y}{b} e^{-\gamma_n z}, \quad z>0$$

$$\psi^- = \sum_{n=1}^{\infty} A_n^- \sin\frac{n\pi y}{b} e^{\gamma_n z}, \quad z<0$$

$$H_y = \frac{\partial}{\partial z} A_x$$

$$J_x = \left[H_y^- - H_y^+\right] = \sum_{n=1}^{\infty} 2\gamma_n A_n \sin\frac{n\pi y}{b}$$

$$E_x = j\omega\mu A_x$$

$$A_n = \frac{1}{\gamma_n b}\int_0^b J_x \sin\frac{n\pi y}{b} e^{-\gamma_n|z|}, \quad \forall z$$

or if $B_n = j\omega\mu A_n$ then,

4-32 (cont.)

$$B_n = \frac{j\omega\mu}{\gamma_n b}\int_0^b J_x \sin\frac{n\pi y}{b}dy$$

and $E_x = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi y}{b} e^{-\gamma_n|z|}$

4-33

For $z>0$

$$\psi^+ = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} B_{mn}^+ \cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b} e^{-\gamma_{mn} z}$$

For $z<0$,

$$\psi^- = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} B_{mn}^- \cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b} e^{\gamma_{mn} z}$$

At $z=0$, $B_{mn}^+ = B_{mn}^- = B_{mn}$

$$J_x = \left[H_y^- - H_y^+\right]_{z=0}$$

$$J_y = \left[H_z^+ - H_z^-\right]_{y=0}$$

$$H_y^+ = \frac{\partial\psi}{\partial z} = -\gamma_{01}B_{01}\sin\frac{\pi y}{b}$$

$$H_z^+ = -\frac{\partial\psi}{\partial y} = \frac{\pi}{b}B_{01}\cos\frac{\pi y}{b}$$

$$J_x = 2\gamma_{01}B_{01}\sin\frac{\pi y}{b}$$

$$J_y = 0$$

$$J_x = \cos kx\,\delta(y-c)$$

$$\int_0^a dx\int_0^b \sin\frac{\pi y}{b} J_x dy = \gamma_{01}B_{01}ba$$

$$\gamma_{01}B_{01} = \frac{1}{ab}\int_0^d \cos kx\,dx\int_0^b \sin\frac{\pi y}{b}\delta(y-c)dy$$

$$= \frac{1}{ab}\left[\frac{\sin kd}{k}\sin\frac{\pi c}{b}\right]$$

$$P = -\iint_{z=0} E\cdot J_s^* ds$$

$$= -\int_0^a dx\int_0^b dy\, J_x^* E_x\big|_{z=0}$$

4-33 (cont.)

$$E_x = \frac{1}{\hat{y}} k^2 \psi$$

$$= \frac{k^2}{\hat{y}} B_{mn} \sin \frac{\pi y}{b}$$

$$P = -\left| \int_0^a dx \int_b^b dy \; 2\, \gamma_{01}^2 B_{01}^2 \sin^2 \frac{\pi y}{b} \right| \frac{k^2}{\hat{y}\gamma_{01}}$$

$$\doteq -\frac{k^2}{\hat{y}\gamma_{01}} \, 2 \, \frac{b}{2} \, a \, \frac{1}{a^2 b^2} \frac{\sin^2 kd}{k^2} \sin^2 \frac{\pi c}{b}$$

$$= \frac{Z_0}{ab} \left(\frac{\sin kd}{k} \sin \frac{\pi c}{b} \right)^2$$

where $Z_0 = -\frac{k^2}{\hat{y}\gamma_{01}}$

$$R_i = \frac{P^2}{|I_{in}|^2} \qquad I_{in} = \cos k(d+c)$$

$$R_i = \frac{Z_0}{ab} \left[\frac{\sin kd \sin \frac{\pi c}{b}}{k \cos k(d+c)} \right]^2$$

4-34

$$J_x = \cos k(d+c-x)\, \delta(y-c)$$

$$J_y = \cos ky \, \delta(x-d)$$

$$\gamma_{01} B_{01} = \frac{1}{ab} \int_0^d \cos k(d+c-x)\, dx \, \sin \frac{\pi c}{b}$$

$$= -\frac{\sin \frac{\pi c}{b}}{abk} \left[\sin kc - \sin k(d+c) \right]$$

$$P = -\frac{\pi^2}{b^2 \hat{y} \gamma_{01}} \int_0^a dx \int_0^b dy \; 2\, \gamma_{01}^2 B_{01}^2 \sin^2 \frac{\pi y}{b}$$

$$= \frac{Z_0}{ab} \left[\frac{\sin \frac{\pi c}{b}}{k} \left(\sin k(d+c) - \sin kc \right) \right]^2$$

4-34 (cont.)

$$R_{in} = \frac{P^2}{I_{in}} \qquad I_{in} = \cos ky \big|_{y=0} = 1$$

$$\therefore R_i = \frac{Z_0}{ab} \left[\frac{\sin \frac{\pi c}{b}}{k} \left[\sin k(d+c) - \sin kc \right] \right]^2$$

4-35

$$\sin x \approx x^3 - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040}$$

$$\sin^2 x \approx x^2 - \frac{x^4}{3} + \frac{2x^6}{45} - \frac{x^8}{315}$$

$$\frac{\sin^2 x}{x^2} \approx 1 - \frac{x^2}{3} + \frac{2x^4}{45} - \frac{x^6}{315}$$

$$\lambda \eta G_a = 2 \int_0^\alpha \frac{1 - \frac{x^2}{3} + \frac{2x^4}{45} - \frac{x^6}{315}}{\sqrt{\alpha^2 - x^2}} \, dx$$

where $\alpha = \frac{ka}{2}$. This can be integrated term by term.

$$\int_0^\alpha \frac{1}{\sqrt{\alpha^2 - x^2}} dx = \frac{\pi}{2}$$

$$-\int_0^\alpha \frac{x^2}{3\sqrt{\alpha^2 - x^2}} dx = \frac{\pi}{2} \left[\left(\frac{ka}{2} \right)^2 \frac{1}{2 \cdot 3} \right]$$

$$\int_0^\alpha \frac{2x^4}{45\sqrt{\alpha^2 - x^2}} dx = \frac{\pi}{2} \left[\left(\frac{ka}{2} \right)^4 \frac{1}{60} \right]$$

$$-\int_0^\alpha \frac{x^6}{315\sqrt{\alpha^2 - x^2}} dx = \frac{\pi}{2} \left[\left(\frac{ka}{2} \right)^6 \frac{1}{1008} \right]$$

$\therefore$

$$\lambda \eta G_a = \pi \left[1 - \frac{1}{6} \left(\frac{ka}{2} \right)^2 + \frac{1}{60} \left(\frac{ka}{2} \right)^4 - \frac{1}{1008} \left(\frac{ka}{2} \right)^6 \right]$$

4-36

$$\lambda \eta B_a = I = \int_b^\infty \frac{dw}{w^2\sqrt{w^2-b^2}} - Re\int_{C_1} \frac{e^{2jw}\,dw}{w^2\sqrt{w^2-b^2}}$$

where $b = \frac{ka}{2}$

Using Dwight 282.01,

$$I = \frac{1}{b^2} - Re\int_{C_1} \frac{e^{j2w}\,dw}{w^2\sqrt{w^2-b^2}}$$

Integrals around C_0 and C_∞ vanish so

$$\int_{C_1} = -\int_{C_2}$$

$$\therefore I = \frac{1}{b^2} - Re\int_0^\infty \frac{e^{j2(b+jx)}\,j\,dx}{\sqrt{jx(2b+jx)}(b+jx)^2}$$

where $w = b + jx$

Now as b gets very large

$$I \approx \frac{1}{b^2} - Re\frac{e^{j2b}}{b^2}\int_0^\infty \frac{e^{-2x}\,j\,dx}{\sqrt{jx(2b)}}$$

$$I = \frac{1}{b^2} - Re\frac{e^{j2b}\,j}{b^{5/2}\sqrt{2}}\left(\frac{1-j}{\sqrt{2}}\right)\int_0^\infty \frac{e^{-2x}}{\sqrt{x}}dx$$

Using Dwight 860.05,

$$I = \frac{1}{b^2} - \frac{\sqrt{\pi}}{(2b)^{5/2}}(\cos 2b - \sin 2b)$$

$$I = \frac{1}{b^2} - \frac{\sqrt{\pi}}{2b^{5/2}}\cos(2b + \pi/4)$$

$$\therefore = \left(\frac{\lambda}{\pi a}\right)^2\left[1 - \frac{1}{2}\sqrt{\frac{\lambda}{a}}\cos\left(\frac{2a}{\lambda} + \frac{1}{4}\right)\pi\right]$$

4-37

Let $a = \frac{2}{\pi}$

Expanding $\left[\left(\frac{\pi}{2}\right)^2 - w^2\right]^2$ about $w = 0$, we get:

$$f = a^4\left[1 + 2a^2w^2 + 3a^4w^4 + 4a^6w^6 + 5a^8w^8 + \ldots\right]$$

$$\cos^2 w \approx 1 - w^2 + \frac{w^4}{3} - \frac{2w^6}{45} + \frac{w^8}{315}$$

$$f\cos^2 w = a^4\Big\{1 + w^2[2a^2 - 1] + w^4\left[\tfrac{1}{3} - 2a^2 + 3a^4\right] + w^6\left[\tfrac{2a^2}{3} - \tfrac{2}{45} - 3a^4 + 4a^6\right] + w^8\left[\tfrac{1}{315} - \tfrac{4a^2}{45} + a^4 - 4a^6 + 5a^8\right]\Big\}$$

$$= a^4\Big\{1 + w^2(-.1894305) + w^4(.01553101) + w^6(-.000740432) + w^8(.0000233736)\Big\}$$

Let $b = \frac{ka}{2}$, $t = \sqrt{t^2 - w^2}$

$$\int_0^b t\,dw = \frac{b^2}{2}\left(\frac{\pi}{2}\right) = \left(\frac{a}{\lambda}\right)^2\frac{\pi^3}{4}$$

$$\int_0^b tw^2\,dw = \frac{b^4}{8}\left(\frac{\pi}{2}\right) = \left(\frac{a}{\lambda}\right)^4\frac{\pi^5}{16}$$

$$\int_0^b tw^4\,dw = \frac{b^6}{16}\left(\frac{\pi}{2}\right) = \left(\frac{a}{\lambda}\right)^6\frac{\pi^7}{32}$$

$$\int_0^b tw^6\,dw = \frac{5b^8}{128}\left(\frac{\pi}{2}\right)$$

To find coefficients the above identities are used to integrate

$$\frac{1}{2}\int_0^b \sqrt{b^2 - w^2}\, f\cos^2 w\,dw$$

4-37 (cont.)

Factor out $\frac{2}{\pi}$ and compute first four coefficients,

$$b_1 = \frac{1}{2}\left(\frac{2}{\pi}\right)^3 \frac{\pi^3}{4} = 1.0$$

$$b_2 = \frac{1}{2}\left(\frac{2}{\pi}\right)^3 \frac{\pi^5}{16}(-.1894305)$$

$$= -.467407$$

$$b_3 = \frac{1}{2}\left(\frac{2}{\pi}\right)^3 \frac{\pi^7}{32}(.01553101)$$

$$= .189108$$

$$b_4 = \frac{1}{2}\left(\frac{2}{\pi}\right)^3 \frac{5\pi^9}{(128)2}(-.000740432)$$

$$= -.0556128$$

... etc.

4-38

$$\lim_{a\to 0} \frac{-\eta B_a}{\lambda} = \frac{1}{2}\int_0^\infty \frac{\omega \cos^2 \omega}{[(\pi/2)^2-\omega^2]^2}\, d\omega$$

Let $u = \cos^2 \omega$ $\qquad dv = \dfrac{\omega\, d\omega}{[(\frac{\pi}{2})^2-\omega^2]^2}$

$du = -2\cos\omega \sin\omega\, d\omega$

$= -2\sin 2\omega\, d\omega \qquad v = \dfrac{1}{2[(\frac{\pi}{2})^2-\omega^2]}$

$$\frac{-\eta B_a}{\lambda} = \frac{\cos^2\omega}{2[(\frac{\pi}{2})^2-\omega^2]}\Bigg|_0^\infty + \frac{1}{4}\int_0^\infty \frac{\sin 2\omega}{[(\frac{\pi}{2})^2-\omega^2]}\, d\omega$$

Use identity:

$$\int_0^\infty \frac{\sin 2x}{(\frac{\pi}{2})^2-x^2}\, dx = \frac{2}{\pi}\, Si(\pi)$$

$$\lim_{\omega\to\infty}\left[\frac{\cos^2\omega}{2[(\frac{\pi}{2})^2-\omega^2]}\right] = 0$$

4-38 (cont.)

$$-\lim_{\omega\to 0}\left[\frac{\cos^2\omega}{2[(\frac{\pi}{2})^2-\omega^2]}\right] = -\frac{2}{\pi^2}$$

$$\therefore \lim_{a\to 0} \frac{-\eta B_a}{\lambda} = \frac{1}{2\pi} Si(\pi) - \frac{2}{\pi^2}$$

4-39

$$\frac{\eta G_a}{\lambda} = \frac{1}{2}\int_0^{\frac{ka}{2}} \frac{\sqrt{(\frac{ka}{2})^2-\omega^2}}{[(\frac{\pi}{2})^2-\omega^2]^2}\cos^2\omega\, d\omega$$

$$Re\left[1+e^{2j\omega}\right] = 2\cos^2\omega$$

$\sqrt{(\frac{ka}{2})^2-\omega^2} \to \frac{ka}{2}$ as ka becomes large even when ω gets close to $\frac{ka}{2}$ because the denominator is increasing much faster than the numerator in the integrand.

$$\therefore \frac{\eta G_a}{\lambda} \xrightarrow[\frac{ka}{2}\to\infty]{} \frac{ka}{8}\, Re \int_{C_1} \frac{(1+e^{2j\omega})}{[(\frac{\pi}{2})^2-\omega^2]^2}\, d\omega$$

$\int_{C_\infty} \to 0$ because $|1+e^{2j\omega}| \le 2$ as $R\to\infty$

$|(\frac{\pi}{2})^2-\omega^2| \to \infty$ at order of R^2.

$\dfrac{1+e^{2j\omega}}{[(\frac{\pi}{2}+\omega)(\frac{\pi}{2}-\omega)]^2}$ has a pole of order 1 at $\omega = \frac{\pi}{2}$

$$\text{Residue} = \lim_{\omega\to\frac{\pi}{2}} \frac{1+e^{2j\omega}}{(\frac{\pi}{2}+\omega)^2(\frac{\pi}{2}-\omega)(-1)}$$

$$= -\lim_{\omega\to\frac{\pi}{2}} \frac{1}{(\frac{\pi}{2}+\omega)^2} \cdot \lim_{\omega\to\frac{\pi}{2}} \frac{(1+e^{2j\omega})}{(\frac{\pi}{2}-\omega)}$$

$$= -\frac{2j}{\pi^2}$$

4-39 (cont.)

$$\int_{C_0} = -\pi j\,[\text{Residue}]$$

$$= -\pi j\left(-\frac{2j}{\pi^2}\right) = -\frac{2}{\pi}$$

Over C_2, $w = 0 + jx$

$$\text{Re}\int_{C_2} = -\frac{ka}{8}\,\text{Re}\int_0^\infty \frac{\left(1+e^{2j(jx)}\right) j\,dx}{\left[(\pi/2)^2-(jx)^2\right]^2}$$

$$= -\frac{ka}{8}\,\text{Re}\, j\int_0^\infty \frac{1+e^{-2x}}{\left[(\pi/2)^2+x^2\right]^2}\,dx$$

Integrand is a real no.

$$\therefore\ \text{Re}\int_{C_2} = 0.$$

Now $\int_{C_0+C_1+C_2+C_\infty} = 0$ because the integrand is analytic inside the contour.

$$\therefore\ \text{Re}\int_{C_1} = -\int_{C_0} = \frac{2}{\pi}$$

And $\dfrac{\eta G_a}{\lambda} = \dfrac{ka}{8}\left(\dfrac{2}{\pi}\right) = \dfrac{ka}{4\pi}$

4-40 At $y=0$ assume $E_x = 1$, $0<x<a$

$E_x = -1$, $-a<x<0$

$$\bar{E}_x = \int_{-\infty}^{\infty} E_x(x,0)\,e^{-jk_x x}\,dx$$

(bar above quantity indicates transform)

$$\bar{E}_x = \int_{-a}^{0} -e^{-jk_x x}\,dx + \int_0^a e^{-jk_x x}\,dx$$

$$= \frac{2}{jk_x}\left[1-\cos k_x a\right] = \frac{4}{jk_x}\sin^2\frac{k_x a}{2}$$

4-40 (cont.)

$$P = -\int_{-\infty}^{\infty}\left[\bar{E}_x \bar{H}_z^*\right]_{y=0}\frac{dk_x}{2\pi}$$

$$= -\frac{16}{\lambda\eta}\int_{-\infty}^{\infty}\frac{\sin^4\frac{k_x a}{2}}{k_y^* k_x^2}\,dk_x$$

Voltage of one line is $V = \int_0^a E_x\,dx = a$

$$Y_a = \frac{P^*}{|V|^2} = \frac{-16}{\lambda\eta a^2}\int_{-\infty}^{\infty}\frac{\sin^4\frac{k_x a}{2}}{k_y k_x^2}\,dk_x$$

$$Y_a = G_a + jB_a$$

$$G_a = \frac{-16}{\lambda\eta a^2}\int_{-k}^{k}\frac{\sin^4\frac{k_x a}{2}}{-\left(\sqrt{k^2-k_x^2}\right)k_x^2}\,dk_x$$

Let $w = k_x a$

$$G_a = \frac{32}{\lambda\eta}\int_0^{ka}\frac{\sin^4\frac{w}{2}}{w^2\sqrt{(ka)^2-w^2}}\,dw$$

$$B_a = \frac{-16}{\lambda\eta a^2}\left(\int_{-\infty}^{k}+\int_k^{\infty}\right)\frac{\sin^4\frac{k_x a}{2}}{j\sqrt{k_x^2-k^2}\,k_x^2}\,dk_x$$

$$B_a = \frac{32}{\lambda\eta}\int_{ka}^{\infty}\frac{\sin^4\frac{w}{2}}{w^2\sqrt{w^2-(ka)^2}}\,dw$$

4-41 It is not clear to the authors how Green's Identity can be used to show this equivalence of the two ψ's. The extension to the far field as stated in the problem, however, is trivial as follows.

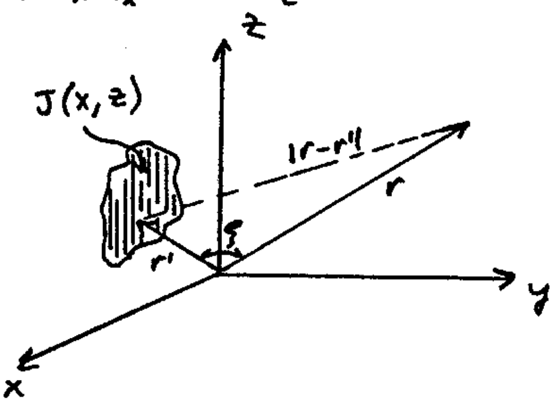
4-41 (cont.)

ψ by potential integral method gives:

$$\psi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{J(x',z')\, e^{-jk|r-r'|}}{4\pi |r-r'|}\, dx'dz'$$

$$r = x\bar{u}_x + y\bar{u}_y + z\bar{u}_z$$

$$r' = x'\bar{u}_x + z'\bar{u}_z$$



Specializing to far field,

$$|r-r'| \simeq r + r'\cos\xi$$

$$\simeq r + x'\sin\theta\cos\phi + z'\cos\theta$$

$$\psi = \frac{e^{-jkr}}{4\pi r}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J(x,z)\, e^{jk(x\sin\theta\cos\phi + z\cos\theta)}\, dx\,dz$$

$$\bar{J}_z = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J(x,z)\, e^{-jk_x x}\, e^{-jk_z z}\, dx\,dz = \bar{J}_z(k_x,k_z)$$

$$k_x = k\sin\theta\cos\phi$$

$$k_z = k\cos\theta$$

$$\therefore\ \psi = \frac{e^{-jkr}}{4\pi r}\,\bar{J}_z(-k\cos\phi\sin\theta, -k\cos\theta)$$

4-42

$$Z = \frac{P}{|I|^2}$$

$$J_x = \cos\frac{\pi x}{a}, \quad |x| < \frac{a}{2}$$

$$|I| = 1$$

4-42 (cont.)

$$Z_{\text{elec. ribb.}} = \frac{\eta^2}{2} Y_{\text{apert}}$$

Y_{apert} is given by Eqn. 4-114 because the $\bar{E}$ field (in Fig. 4-23) in the aperture is in the same direction as the current in the ribbon. (dual to aperture).

$\boxed{5.1}$ given $\psi = \sum_n \int_{k_\rho} g_n(k_\rho) B_n(k_\rho \rho) h(n\phi) h(k_z z)\, dk_\rho$

Hence

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \psi}{\partial \rho}\right) = \sum_n \int_{k_\rho} g_n(k_\rho)\frac{k_\rho}{\rho} B_n'(k_\rho \rho) h(n\phi) h(k_z z)\, dk_\rho + \sum_n \int_{k_\rho} k_\rho^2 g_n(k_\rho) B_n''(k_\rho \rho) h(n\phi) h(k_z z)\, dk_\rho ,$$

$$\frac{1}{\rho}\frac{\partial^2 \psi}{\partial \phi^2} = \sum_n \frac{n^2}{\rho^2}\int_{k_\rho} g_n(k_\rho) B_n(k_\rho \rho) h''(n\phi) h(k_z z)\, dk_\rho , \quad \text{and}$$

$$\frac{\partial^2 \psi}{\partial z^2} = \sum_n \int_{k_\rho} k_z^2 g_n(k_\rho) B_n(k_\rho \rho) h(n\phi) h''(k_z z) . \quad \text{So,}$$

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \psi}{\partial \rho}\right) + \frac{1}{\rho}\frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2} + k^2 \psi$$

$$= \sum_n \int_{k_\rho} g_n(k_\rho)\left[\frac{k_\rho}{\rho}\frac{B_n'(k_\rho \rho)}{B_n(k_\rho \rho)} + k_\rho^2 \frac{B_n''(k_\rho \rho)}{B_n(k_\rho \rho)} + \frac{n^2}{\rho^2}\frac{h''(n\phi)}{h(n\phi)} + k_z^2 \frac{h''(k_z z)}{h(k_z z)} + k^2\right]$$

$$\times\ B_n(k_\rho \rho) h(n\phi) h(k_z z)$$

The above equation satisfies Helmholtz equation because the component equations satisfy equation (5-7).

$\boxed{5.2}$ Since $\psi = (\log \rho) e^{-jkz}$ hence $\frac{\partial \psi}{\partial \rho} = \frac{1}{\rho} e^{-jkz}$; $\frac{\partial \psi}{\partial z} = -jk e^{-jkz}(\log \rho)$

and so $\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \psi}{\partial \rho}\right) = 0$; $\frac{\partial^2 \psi}{\partial z^2} + k^2 \psi = 0$.

Hence ψ satisfies the scalar Helmholtz equation.

<u>For TM case</u>

$H_\rho = 0$ $\qquad E_\rho = \frac{1}{\hat{y}}\left(\frac{-jk}{\rho}\right) e^{-jkz}$

$H_\phi = -\frac{e^{-jkz}}{\rho}$ $\qquad E_\phi = 0$

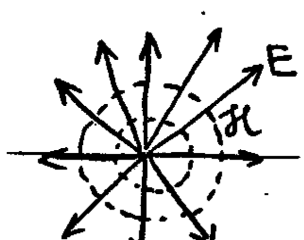
$H_z = 0$ $\qquad E_z = 0$

<u>For TE case</u>

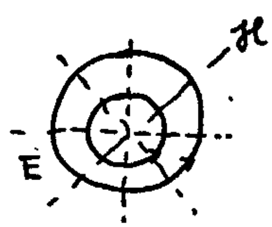
$E_\rho = 0$ $\qquad H_\rho = \frac{1}{\hat{z}}\left(-\frac{jk}{\rho}\right) e^{-jkz}$

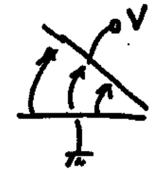
$E_\phi = \frac{1}{\rho} e^{-jkz}$ $\qquad H_\phi = 0$

$E_z = 0$ $\qquad H_z = 0$



<u>Sketch of E & $\mathcal{H}$ lines</u>





The coaxial cable supports the above fields

This system would support such fields

$\boxed{5-3}$ By the terms of the problem, if

$$\bar{A} = \bar{u}_\rho \, \rho \, \psi_1 \quad \text{and} \quad \bar{F} = \bar{u}_\rho \, \rho \, \psi_2 \quad \text{and} \quad \frac{\partial \psi_1}{\partial z} = \frac{\partial \psi_2}{\partial z} = 0$$

equations (3-79) become

$$\bar{E} = \bar{u}_z \frac{\partial \psi_2}{\partial \phi} + \frac{1}{\hat{y}}\left[-\bar{u}_\rho \frac{1}{\rho} \frac{\partial^2 \psi_1}{\partial \phi^2} + \bar{u}_\phi \frac{\partial^2 \psi_1}{\partial \rho \partial \phi} \right] \quad \cdots (1)$$

$$\bar{H} = -\bar{u}_z \frac{\partial \psi_1}{\partial \phi} + \frac{1}{\hat{z}}\left[-\bar{u}_\rho \frac{1}{\rho} \frac{\partial^2 \psi_2}{\partial \phi^2} + \bar{u}_\phi \frac{\partial^2 \psi_2}{\partial \rho \, \partial \phi} \right] \quad \cdots (2)$$

Since ψ_1 and ψ_2 must be solutions to

$$\nabla^2 \psi + k^2 \psi = 0 , \quad \text{thus}$$

$$\nabla^2 \left(\frac{A_\rho}{\rho} \right) + k^2 \left(\frac{A_\rho}{\rho} \right) = 0 \quad \text{and}$$

$$\nabla^2 \left(\frac{F_\rho}{\rho} \right) + k^2 \left(\frac{F_\rho}{\rho} \right) = 0 .$$

To verify gauge conditions, take $\hat{\rho}$ and $\hat{\phi}$ component of equation (3-78).

For $\bar{A}$, $\quad -\frac{1}{\rho} \frac{\partial^2 \psi_1}{\partial \phi^2} - k^2 \rho \psi_1 = -\hat{y} \frac{\partial \phi^a}{\partial \rho} \quad \cdots (3)$

$$\text{and} \quad \frac{\partial^2 \psi_1}{\partial \rho \partial \phi_1} = -\frac{\hat{y}}{\rho} \frac{\partial}{\partial \phi} \phi^a \quad \text{or} \quad \frac{\partial \psi_1}{\partial \rho} = -\frac{\hat{y}}{\rho} \phi^a + c(\rho)$$

[after integrating once with respect to ϕ]

$$\text{Now } \nabla^2 \psi_1 = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \psi_1}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \psi_1}{\partial \phi^2}$$

$$\text{or } \frac{1}{\rho} \frac{\partial^2 \psi_1}{\partial \phi^2} = \rho \nabla^2 \psi_1 - \frac{\partial}{\partial \rho} \rho \left(\frac{\partial \psi_1}{\partial \rho} \right) \quad \cdots (4)$$

Substitution of (4) into (3).

$$-\rho \nabla^2 \psi_1 + \frac{\partial}{\partial \rho} \rho \left(\frac{\partial \psi_1}{\partial \rho} \right) - k^2 \rho \psi_1 = -\hat{y} \frac{\partial}{\partial \rho} \phi^a .$$

Integrating with respect to ρ

 $\boxed{5\text{-}3}$ continued $\quad \rho\left(\frac{\partial \psi_1}{\partial \rho}\right) = -\hat{y}\,\phi^a + c_1(\phi)$; $\therefore \frac{c_1(\phi)}{\rho} = c(\rho)$

Hence $c_1(\phi) = c(\rho) = 0$; and $\phi^a = -\frac{1}{\hat{y}}\rho\frac{\partial \psi_1}{\partial \rho} = -\frac{1}{\hat{y}}\rho\frac{\partial}{\partial \rho}\left(\frac{A\rho}{\rho}\right)$

We must show that $\frac{\partial \psi}{\partial \phi}$ is an arbitrary solution to the wave equation.

$$\nabla^2\left[\frac{\partial \psi}{\partial \phi}\right] + k^2\left[\frac{\partial \psi}{\partial \phi}\right] = 0 \quad \text{for either } \psi_1 \text{ or } \psi_2$$

or $\frac{\partial}{\partial \phi}\left[\nabla^2\psi + k^2\psi\right] = 0$; hence $\nabla^2\psi + k^2\psi = f(\rho)$

or $\frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\left(\frac{\partial \psi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \psi}{\partial \phi^2} + k^2\psi = f(\rho)$

The above equation has a solution consisting of two parts (1) the homogeneous equation (2) particular integral

$\therefore \psi = R_p(\rho) + \psi_{homogeneous}$

$\boxed{5\text{-}4}$ $f_c = 9000$ MHz . Hence $\lambda_c^{TE} = 10/3$ cm $= 2\pi a/1.841$

$\therefore a = 18.41/(6\pi) = 0.977$

The cut off frequencies for the next ten lowest order modes are 14.3, 19.1, 20.6, 23.8, 26.2, 28.3, 31.4, 32.3, 32.7 and 36.4 GHz. For $\varepsilon_r = 4$ all the f_c's are divided by 2

$\boxed{5\text{-}5}$ Given $\psi = B_n(k_\rho \rho)\, h(n\phi)\, e^{\pm jk_z z}$

For TM modes,

$E_\rho = \frac{1}{\hat{y}\rho}(\pm jk_z) B_n(k_\rho \rho) e^{\pm jk_z z}\, n\, h'(n\phi)$

$= 0|_{\rho = a}$. Hence $B_n(k_\rho a) = 0$.

Now since the fields do not go to zero inside the waveguide

$B_n(k_\rho \rho) = A J_n(k_\rho \rho) + B Y_n(k_\rho \rho)$

Hence $B_n(k_\rho \rho) = \left[Y_n(k_\rho a) J_n(k_\rho \rho) - J_n(k_\rho a) Y_n(k_\rho \rho)\right]$

times a constant.

The ϕ variation could be $\sin n\phi$ or $\cos n\phi$ or could be a combination of both depending on the boundary conditions.

For TE modes

$E_\phi = B_n'(k_\rho \rho)\, k_\rho\, h(n\phi) e^{\pm jk_z z}$

$= 0|_{\rho = a}$

Hence $B_n'(k_\rho a) = 0$. Following arguments of the TM case

$B_n(k_\rho \rho) = \left[Y_n'(k_\rho a) J_n(k_\rho \rho) - J_n'(k_\rho a) Y_n(k_\rho \rho)\right]$

times a constant.

The ϕ variation could be $\sin n\phi$ or $\cos n\phi$ or a combination of both, depending on the boundary conditions.

$\boxed{5\text{-}5}$ contd.

Since also $B_n(k_\rho b) = 0$ or $E_\phi = 0$

Hence k_ρ is a root of the equation

$Y_n(k_\rho b) J_n(k_\rho a) - J_n(k_\rho b) Y_n(k_\rho a) = 0.$

Similarly as $E_\phi = 0|_{b} = 0$

Hence k_ρ is a root of the equation

$J_n'(k_\rho b) Y_n'(k_\rho a) - Y_n'(k_\rho b) J_n'(k_\rho a) = 0$

$\boxed{5\text{-}6}$ For the coaxial waveguide with a baffle $h(n\phi)$ is assumed to be $h(n\phi) = A\cos n\phi + B\sin n\phi$

For TM modes,

$E_\rho = 0$ at $\phi = 0$ & $\phi = 2\pi$

Hence $A\cos n\phi + B\sin n\phi = 0$ at $\phi = 0$

$\therefore A = 0.$

and $B\sin 2\pi n = 0$, so

$\sin 2n\pi = \sin k\pi$

$\therefore n = \frac{k}{2}$ for $k = \frac{1}{2}, \frac{2}{2}, \frac{3}{2}, \frac{4}{2} \cdots$

hence $h(n\phi) = \sin n\phi$

For TE modes,

$E_\rho = 0$ at $\phi = 0$ & $\phi = 2\pi$

So $h'(n\phi) = 0.$

or $n[A\sin n\phi + B\cos n\phi] = 0$

$\therefore B = 0$

and $\sin n2\pi = 0 = \sin \frac{k\pi}{2\pi}$

$n = \frac{k}{2}$ for $k = 1, 2, 3, \cdots$

hence $h(n\phi) = \cos n\phi$

$\boxed{5\text{-}7}$ For the wedge waveguide the boundary conditions are

for TM modes i) $E_\rho = 0$ at $\rho = 0$

& hence ρ variation is $J_n(k_\rho \rho)$

ii) $E_\rho = 0$ at $\phi = 0$; $A\cos n\phi + B\sin \phi n = 0 \therefore A = 0$

at $\phi = \phi_0 \quad \therefore n = \frac{k\pi}{\phi_0}$

Hence $\psi^{TM} = J_n(k_\rho \rho)\sin n\phi\, e^{\pm jk_z z}$

when $n = \frac{k\pi}{\phi_0}$ for $k = 1, 2, 3, \cdots$

for TE modes

i) $E_\rho = 0$ at $\rho = 0$ hence

$J_n'(k_\rho a) = 0$

and $\frac{d}{d\phi}[A\cos n\phi + B\sin n\phi] = 0$ at $\begin{cases}\phi = \\ \phi = \end{cases}$

Hence $B = 0$ and $\sin n\phi_0 = \sin k\pi$

so $\psi^{TE} = J_n(k_\rho \rho)\cos n\phi\, e^{\mp jk_z z}$

for $n = 0, \frac{\pi}{\phi_0}, \frac{2\pi}{\phi_0}, \cdots$

$\boxed{5\text{-}8}$ Since the dominate mode is the TE mode

$\psi^{TE} = J_n(k_\rho \rho)\, e^{-jk_z z}\,[A\sin n\phi + B\sin n\phi]$

$E_\rho = -\frac{1}{\rho} J_n(k_\rho \rho) e^{-jk_z z}[nA\cos n\phi - Bn\sin n\phi] = 0$ at $\begin{cases}\phi = 0 \\ \phi = 2\pi\end{cases}$

Hence $n = \frac{k}{2}$; $A = 0$. Again $E_\phi = 0$ at $\rho = 0$ and $\rho = a$

$\therefore J_n'(k_\rho a) = 0 = J_{k/2}'(k_\rho a)$

$j_0'(x) = \sqrt{\frac{\pi}{2x}}\, J_{n+1/2}'(x) = 0$ for $x = 1.16$; $\therefore \lambda_c = \frac{2\pi a}{1.16}$

$\boxed{5-9}$ For TM modes

$$\psi^{TM} = J_n(k_\rho \rho)\cos n\phi\, e^{-jk_z z}$$

$$E_\rho = \frac{1}{\hat{y}} J_n'(k_\rho \rho)\, k_\rho (-jk_z)\cos n\phi\, e^{-jk_z z}$$

$$E_\phi = -\frac{n}{\hat{y}\rho}\sin n\phi\, J_n(k_\rho \rho)(-jk_z)\, e^{-jk_z z}$$

$$H_\rho = -\frac{n}{\rho} J_n(k_\rho \rho)\sin n\phi\, e^{-jk_z z}$$

$$H_\phi = -J_n'(k_\rho \rho)\, k_\rho \cos n\phi\, e^{-jk_z z}$$

$$P_d = \mathcal{R}\int_0^{2\pi} \rho\, d\phi \left\{(H_\rho)^2 + (H_\phi)^2\right\}_{\rho=a;\, z=0}$$

$$= k_\rho^2\, a\, \mathcal{R}\, \pi J_n'^2(k_\rho a)$$

$$P_f = \int_0^a \rho\, d\rho \int_0^{2\pi} d\phi\, \left| E_\rho H_\phi^* - E_\phi H_\rho^* \right|$$

$$= k_z \pi k_\rho \int_0^a \left\{(k_\rho \rho) J_n'^2(k_\rho \rho) + \frac{n^2}{k_\rho \rho} J_n^2(k_\rho \rho)\right\} d\rho$$

$$= \frac{\pi k_z}{\hat{y}} \frac{k_\rho^2 a^2}{2} J_n'^2(k_\rho a)$$

$$\therefore \alpha = \frac{P_d}{2P_f} = \frac{\mathcal{R}\hat{y}}{a k_z} = \frac{\mathcal{R}}{\eta a\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

For TE modes

$$\psi^{TE} = J_n(k_\rho \rho)\cos n\phi\, e^{-jk_z z}$$

$$E_\rho = \frac{n}{\rho} J_n(k_\rho \rho)\sin n\phi\, e^{-jk_z z}$$

$$E_\phi = J_n'(k_\rho \rho)\, k_\rho \cos n\phi\, e^{-jk_z z}$$

$$H_\rho = \frac{1}{\hat{z}} k_\rho J_n'(k_\rho \rho)(-jk_z)\cos n\phi\, e^{-jk_z z}$$

$$H_\phi = -\frac{n}{\hat{z}\rho}\sin n\phi\, J_n(k_\rho \rho)(-jk_z)\, e^{-jk_z z}$$

$$H_z = \frac{1}{\hat{z}} J_n(k_\rho \rho)\, k_\rho^2 \cos n\phi\, e^{-jk_z z}$$

$$P_d = \mathcal{R}\left\{\int_0^{2\pi} \rho\, d\phi \left([H_\phi]^2 + [H_z]^2\right)\right\}_{z=0,\, \rho=a}$$

$$= \frac{\pi \mathcal{R} a}{|\hat{z}|^2}\left[J_n^2(k_\rho a)\, k_\rho^4 - \frac{n^2 k_z^2}{a^2} J_n^2(k_\rho a)\right]$$

$$P_f = \int_0^a \rho\, d\rho \int_0^{2\pi} d\phi\, \left| E_\rho H_\phi^* - E_\phi H_\rho^* \right|$$

$$= \frac{\pi k_z}{\hat{z}} \left| \int_0^a \left[\frac{n^2}{\rho} J_n^2(k_\rho a) + \rho J_n'^2(k_\rho \rho) k_\rho^2\right] d\rho \right|$$

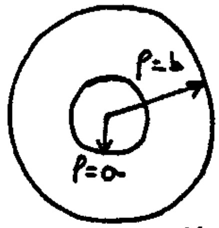
$$= \frac{\pi k_z}{\hat{z}} \frac{k_\rho^2 a^2}{2}\left(1 - \frac{n^2}{k_\rho^2 a^2}\right) J_n^2(k_\rho a)$$

$$\therefore \alpha = \frac{P_d}{2P_f} = \frac{\mathcal{R}}{\eta a\sqrt{1-\left(\frac{f_c}{f}\right)^2}}\left[\frac{k_\rho^2}{k^2} + \frac{n^2}{k_\rho^2 a^2 - n^2}\right]$$

since $k^2 = k_\rho^2 + k_z^2$ & $k_\rho a = x'_{np}$

$$\alpha = \frac{\mathcal{R}}{\eta a\sqrt{1-\left(\frac{f_c}{f}\right)^2}}\left[\left(\frac{f_c}{f}\right)^2 + \frac{n^2}{(x'_{np})^2 - n^2}\right]$$

$\boxed{5-10}$



$$\Psi = \left[A J_n(k\rho) + B Y_n(k\rho)\right] e^{-jn\phi}$$

For TM to z

$E_\phi = 0$ at $\rho = a$ & $\rho = b$.

$$\therefore\ A J_n(ka) + B Y_n(ka) = 0$$

$$A J_n(kb) + B Y_n(kb) = 0.$$

Hence $$-\frac{B}{A} = \frac{J_n(ka)}{Y_n(ka)} = \frac{J_n(kb)}{Y_n(kb)}$$

For TE to z

$E_\phi = \frac{\partial \Psi}{\partial \rho} = 0$ at $\rho = a$ & $\rho = b$.

$$\therefore\ A J_n'(ka) + B Y_n'(ka) = 0$$

$$A J_n'(kb) + B Y_n'(kb) = 0$$

Hence $$-\frac{B}{A} = \frac{J_n'(ka)}{Y_n'(ka)} = \frac{J_n'(kb)}{Y_n'(kb)}$$

$\boxed{5-11}$ By the terms of the problem

$$\Psi_{mn}^{TM} = \cos\left(\frac{m\pi z}{a}\right)\cos n\phi \; H_n^{(1)}(k_\rho \rho)$$

from (5.33)

$$\therefore E_z = \frac{\left[k^2 - \frac{m^2\pi^2}{a^2}\right]}{\hat{y}} \cos\left(\frac{m\pi z}{a}\right)\cos n\phi \; H_n^{(1)}(k_\rho \rho)$$

$$H_\phi = -\cos\left(\frac{m\pi z}{a}\right)\cos n\phi \; H_n^{(1)\prime}(k_\rho \rho)$$

From (5.36)

$$\beta_\rho \text{ of } E_z = \frac{\partial}{\partial \rho}\left[\tan^{-1}\frac{Y_n(k_\rho\rho)}{J_n(k_\rho\rho)}\right]$$

$$= \frac{Y_n' J_n - J_n' Y_n}{J^2\left[1 + \frac{Y_n^2}{J_n^2}\right]} \cdot k_\rho$$

using the Wronskians

$$\beta_\rho \text{ of } E_z = \frac{2}{\pi\rho} \frac{1}{J_n^2(k_\rho\rho) + Y_n^2(k_\rho\rho)}$$

$$\beta_\rho \text{ of } H_\phi = \frac{Y_n'' J_n' - J_n'' Y_n'}{\left[J_n'^2 + Y_n'^2\right]} \cdot k_\rho$$

from p. 76, Watson

$$= \frac{2}{\pi\rho}\left[1 - \frac{n^2}{(k_\rho\rho)^2}\right] \frac{1}{[J_n'(k_\rho\rho)]^2 + [Y_n'(k_\rho\rho)]^2}$$

$\boxed{5-13}$ given $V(\rho) = -aE_z$; $I(\rho) = 2\pi\rho H_\phi$

$$\therefore j\omega\mu H_\phi = \frac{\partial E_z}{\partial \rho} = -\frac{1}{a}\frac{\partial V}{\partial \rho}$$

$$\frac{j\omega\mu}{2\pi\rho} I(\rho) = -\frac{1}{a}\frac{\partial V}{\partial\rho} \quad \therefore \frac{\partial V}{\partial \rho} = -\frac{j\omega\mu a}{2\pi\rho} I(\rho) \triangleq -j\omega L I(\rho)$$

Also,

$$j\omega\epsilon E_z = \frac{1}{\rho}\frac{\partial}{\partial\rho}[\rho H_\phi] = \frac{1}{2\pi\rho}\frac{\partial I}{\partial \rho}$$

$$-j\omega\epsilon\frac{V}{a} = \frac{1}{2\pi\rho}\frac{\partial I}{\partial\rho}$$

$$\therefore \frac{\partial I}{\partial \rho} = -j\omega\frac{2\pi\rho\epsilon V}{a} \triangleq -j\omega C V$$

$$\therefore L = \frac{\mu a}{2\pi\rho} \quad \& \quad C = \frac{2\pi\rho\epsilon}{a}$$

$\boxed{5-12}$ We know,

$$\lim_{x\to\infty} \frac{H_n^{(1)}(x)}{H_n'^{(1)}(x)} = -j$$

$$\lim_{x\to\infty} \frac{H_n^{(2)}(x)}{H_n'^{(2)}(x)} = j$$

$$\lim_{k_\rho\rho\to\infty}\left[Z_{+\rho}^{TM} = -\frac{E_z}{H_\phi} = \frac{k_\rho}{j\omega\epsilon}\frac{H_n^{(2)}(k_\rho\rho)}{H_n^{(2)\prime}(k_\rho\rho)}\right]$$

$$= \eta$$

Similarly $Z_{-\rho}^{TM} = \eta$

$$Z_{+\rho}^{TM} = \frac{\eta}{j}\frac{H_0^{(2)}(k_\rho\rho)}{H_0^{(2)\prime}(k_\rho\rho)} \quad \text{for } n = 0$$

For small arguments,

$$H_0^{(2)}(k_\rho\rho) = 1 - \frac{2j}{\pi}\log\frac{\gamma k_\rho\rho}{2}$$

$$H_0^{(2)\prime}(k_\rho\rho) = -\frac{2j}{\pi}\frac{2\gamma}{\gamma k_\rho \rho}$$

$$\therefore Z_{+\rho}^{TM} = \frac{\eta}{j}\frac{\pi k_\rho\rho}{(-2j)}\left[1 + \frac{j2}{\pi}\log\frac{2}{\gamma k_\rho\rho}\right]$$

$$= \frac{\eta k_\rho\rho}{2}\left[\pi + 2j\log\frac{2}{\gamma k_\rho\rho}\right]$$

[note a factor of 2 is missing]

for $n \neq 0$

$$H_\nu^{(2)}(x) = \frac{1}{\nu!}\left(\frac{x}{2}\right)^\nu + j\frac{(\nu-1)!}{\pi}\left(\frac{2}{x}\right)^\nu$$

$$H_\nu'^{(2)}(x) = \frac{1}{\nu!}\frac{\nu x^{\nu-1}}{2^\nu} + j\frac{(\nu-1)!}{\pi}\frac{2^\nu(-\nu)}{x^{\nu+1}}$$

$$\therefore \frac{H_n^{(2)}(x)}{H_n'^{(2)}(x)} = \frac{x}{n}\frac{\left(\frac{x}{2}\right)^{2n}\frac{\pi}{(n!)^2} + \frac{j}{n}}{\left(\frac{x}{2}\right)^{2n}\frac{\pi}{(n!)^2} - \frac{j}{n}}$$

$$\therefore Z_{+\rho}^{TM} = \frac{\eta k_\rho\rho}{jn}\left[\frac{\left(\frac{k_\rho\rho}{2}\right)^{2n}\frac{\pi}{(n!)^2} + \frac{j}{n}}{\left(\frac{k_\rho\rho}{2}\right)^{2n}\frac{\pi}{(n!)^2} - \frac{j}{n}}\right]$$

$\boxed{5\text{-}14}$ $V(\rho) = E_\phi \rho \Phi_0$ & $I(\rho) = H_z a$

so

$$j\omega\epsilon E_\phi = -\frac{\partial H_z}{\partial \rho} \qquad -j\omega\mu H_z = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho E_\rho)$$

$$-\frac{j\omega\epsilon V(\rho)}{\rho\Phi_0} = a\frac{\partial I(\rho)}{\partial\rho} \qquad = \frac{1}{\rho\Phi_0}\frac{\partial V}{\partial\rho}$$

$$\therefore \frac{\partial I}{\partial\rho} = -j\omega\frac{\epsilon a}{\rho\Phi_0}V(\rho) \qquad \therefore \frac{\partial V}{\partial \rho} = -j\omega\mu\frac{I(\rho)}{a}\rho\Phi_0$$

$$= -j\omega C V \qquad = -j\omega L I(\rho)$$

Hence $L = \frac{\mu\rho\Phi_0}{a}$ and so $C = \frac{\epsilon a}{\rho\Phi_0}$

$\boxed{5\text{-}15}$ we know,

$$(f_r)^{TM} = \frac{1}{2\pi a\sqrt{\mu\epsilon}}\, x_{np}$$

(∵ of no z variation)

$$= (f_c)^{TM}_{np}$$

Similarly

$$(f_r)^{TE}_{npq} = \frac{1}{2\pi a\sqrt{\mu\epsilon}}\, x'_{np}$$

$$= (f_c)^{TE}_{np}$$

$\boxed{5\text{-}16}$ TM modes

$$\psi^{TM}_{npq} = J_n(k_\rho\rho)\left\{\begin{matrix}\sin n\phi\\ \cancel{\cos n\phi}\end{matrix}\right\}\cos\left(\frac{q\pi z}{d}\right)$$

and $J_n(k_\rho a) = 0$.

$$H_\rho = \frac{n}{\rho}\cos n\phi\, J_n(k_\rho\rho)e^{-jk_z z}$$

$$H_\phi = -k_\rho J_n'(k_\rho\rho)\sin n\phi\cos\left(\frac{q\pi z}{d}\right)$$

P_d = walls + two faces ($z=0$ and $z=d$)

$$= R\left[\int_0^d dz\int_0^{2\pi}\rho\, d\phi\,|H_\phi|^2 + \int_0^a\left[\begin{matrix}(H_\rho)^2_{z=0}+(H_\rho)^2_{z=d}\\ +(H_\phi)^2_{z=0}+(H_\phi)^2_{z=d}\end{matrix}\right]d\rho\int_0^{2\pi}\rho\, d\phi\right.$$

$$= R\left\{k_\rho^2 a J_n'^2(k_\rho a)\frac{\pi d}{2} + 2\pi\int_0^a\left[k_\rho\rho J_n'^2(k_\rho\rho) + \frac{n^2}{k_\rho\rho}J_n^2(k_\rho\rho)\right]k_\rho\, d\rho\right\}$$

$$= R k_\rho^2 a^2\pi\left[\frac{d}{2a}+1\right]J_n'^2(k_\rho a)$$

$$W = 2W_m = \mu\int_0^a\rho\, d\rho\int_0^{2\pi}d\phi\{(H_\rho)^2+(H_\phi)^2\}\int_0^d dz$$

$$= \frac{\mu\pi d}{2}\,\frac{(k_\rho a)^2}{2}J_n'^2(k_\rho a)$$

$$\therefore Q^{TM}_{npq} = \frac{\omega W}{P_d} = \frac{\omega\mu\, J_n'^2(k_\rho a)}{2R\left[\frac{1}{2a}+\frac{1}{d}\right]J_n'^2(k_\rho a)}$$

$$= \frac{\omega\mu a}{2R\left[1+\frac{2a}{d}\right]}$$

$$= \eta\sqrt{(x_{np})^2+\left(\frac{q\pi a}{d}\right)^2}\Big/\left[2R\left(1+\frac{2a}{d}\right)\right]$$

TE modes

$$\psi^{TE}_{npq} = J_n(k_\rho\rho)\sin n\phi\sin\left(\frac{q\pi z}{d}\right)$$

and $J_n'(k_\rho a) = 0$.

$$H_\rho = \frac{1}{\hat{z}}\frac{q\pi}{d}k_\rho J_n'(k_\rho\rho)\sin n\phi\cos\left(\frac{q\pi z}{d}\right)$$

$$H_\phi = \frac{1}{\hat{z}\rho}\frac{nq\pi}{d}J_n(k_\rho\rho)\cos n\phi\cos\left(\frac{q\pi z}{d}\right)$$

$$E_\rho = -\frac{n}{\rho}J_n(k_\rho\rho)\cos n\phi\sin\left(\frac{q\pi z}{d}\right)$$

$$E_\phi = k_\rho J_n'(k_\rho\rho)\sin n\phi\sin\left(\frac{q\pi z}{d}\right)$$

$$W = 2W_e = \epsilon\int_0^a d\rho\int_0^{2\pi}\rho\, d\phi\int_0^d dz\left(|E_\rho|^2+|E_\phi|^2\right)$$

$$= \frac{\epsilon\pi d}{2}\left[\frac{(k_\rho a)^2}{2}\left(1-\frac{n^2}{(k_\rho a)^2}J_n^2(k_\rho a)\right)\right]$$

P_d (on side walls)

$$= R\int_0^{2\pi}a\, d\phi\int_0^d dz\left[H_\phi^2+H_z^2\right]$$

$$= \frac{Rd}{2}\,\frac{\pi a}{(\hat{z})^2}\left[\frac{n^2q^2\pi^2}{a^2d^2}+k_\rho^4\right]J_n^2(k_\rho a)$$

$$P_d\text{(end plates)} = 2R\pi\left(\frac{q\pi}{d}\right)^2\frac{1}{(\hat{z})^2}\left[\frac{(k_\rho a)^2}{2}\left(1-\frac{n^2}{(k_\rho a)^2}\right)J_n^2(k_\rho a)\right]$$

$$\therefore Q^{TE}_{npq} = \frac{\eta}{2R}\,\frac{\left[(x'_{np})^2+\left(\frac{q\pi a}{d}\right)^2\right]\left[(x'_{np})^2-n^2\right]}{\left[(x'_{np})^4+(nq\pi a/d)^2+\frac{2a}{d}\left(\frac{q\pi a}{d}\right)^2(x'^2_{np}-n^2)\right.}$$

5-17 $a = d = 3$ cm.

$$(f_r)^{TM_{010}} = \frac{1}{2\pi a\sqrt{\varepsilon\mu}} \sqrt{\chi_{np}^2 + q^2\pi^2}$$

$$= \frac{10^{10}}{2\pi} \sqrt{(2.405)^2} = 3.94 \text{ GHz}$$

The other 9 resonant frequencies are obtained by multiplying f_r of the dominant mode by 1.5, 1.59, 1.63, 1.80, 2.05, 2.05, 2.13, 2.29 and 2.73.

$$Q_{010}^{TM} = \frac{\eta\, \chi_{01}}{4R} = 1.37 \times 10^4$$

5-19 given $\varepsilon_1 = 4\varepsilon_0$ and $\mu_1 = \mu_0$, $a = \lambda_0$

The cut off frequencies are given by (4.63)

$$f_c = \frac{n}{2\lambda_0\sqrt{\varepsilon_d\mu_d - \varepsilon_0\mu_0}} = \frac{nc}{2\lambda_0\sqrt{3}}$$

$$= \frac{nf_0}{2\sqrt{3}} \quad \text{where } f_0 = \frac{c}{\lambda_0} = \frac{c}{a}$$

for $n = 0, 1, 2, \ldots$

Hence the TE_0 and TM_0 modes propagate unattenuated, no matter how thin/thick the slab is.

For fig. 5-9c, the characteristic equation is (from 4.64)

$$f_c = \frac{n}{4t\sqrt{\varepsilon_d\mu_d - \varepsilon_0\mu_0}} = \frac{nf_0}{2\sqrt{3}}$$

where $f_0 = c/a$.

for TM modes $n = 0, 2, 4, \ldots$

TE modes $n = 1, 3, 5, \ldots$

5-18 we know

$$k_\rho^2 + k_{z_1}^2 = \omega^2\mu_1\varepsilon_1 = k_1^2$$

$$k_\rho^2 + k_z^2 = \omega^2\mu_2\varepsilon_2 = k_2^2$$

and $\dfrac{k_{z_1}}{\varepsilon_1}\tan k_{z_1} d = -\dfrac{k_{z_2}}{\varepsilon_2}\tan[k_{z_2}(a-d)]$

$$f(k_z, d) = \frac{k_{z_1}}{\varepsilon_1}\tan k_{z_1} d + \frac{k_{z_2}}{\varepsilon_2}\tan[k_{z_2}(a-d)]$$

$$\approx \frac{k_{z_1}^2 d}{\varepsilon_1} + \frac{k_{z_2}^2(a-d)}{\varepsilon_2}$$

$$k_{z_1}^2 = \omega^2\mu_1\varepsilon_1 - k_\rho^2$$

$$k_{z_2}^2 = \omega^2\mu_2\varepsilon_2 - k_\rho^2$$

$$f(\beta, 0) \approx \frac{k_{z_2}^2 a}{\varepsilon_2} = \frac{(\omega^2\mu_2\varepsilon_2 - \beta^2)a}{\varepsilon_2}$$

$$f_d'(\beta, 0) = \frac{k_{z_1}^2}{\varepsilon_1} - \frac{k_{z_2}^2}{\varepsilon_2}$$

also $f(k_z, d) = f(\beta, 0) + d\, f_d'(\beta, 0)$

$$0 = \frac{(k_2^2 - \beta^2)a}{\varepsilon_2} + d\left(\frac{k_1^2 - \beta^2}{\varepsilon_1} - \frac{k_2^2 - \beta^2}{\varepsilon_2}\right)$$

$$\frac{(\beta^2 - k_2^2)a}{\varepsilon_2} = \frac{d}{a\varepsilon_2}\left(\omega^2\mu_1\varepsilon_2 - \beta^2\frac{\varepsilon_2}{\varepsilon_1} - \omega^2\mu_2\varepsilon_2 + \beta^2\right)$$

$$\therefore \beta^2 = k_2^2 \frac{\left[1 + \left(\frac{\mu_1}{\mu_2} - 1\right)\frac{d}{a}\right]}{\left[1 + \left(\frac{\varepsilon_2}{\varepsilon_1} - 1\right)\frac{d}{a}\right]}$$

$$\therefore \beta \approx k_2\sqrt{\frac{1 + (\mu_1/\mu_2 - 1)d/a}{1 + (\varepsilon_2/\varepsilon_1 - 1)d/a}}$$

Hence if

$$LC = \frac{(\mu_1 - \mu_2)d + \mu_2 a}{\varepsilon_1 a + d(\varepsilon_2 - \varepsilon_1)}\, \varepsilon_1\varepsilon_2$$

& $k_2 = \omega\sqrt{\mu_2\varepsilon_2}$

So $\beta \approx \omega\sqrt{LC}$

The dominant mode is then the TM_0 mode which propagates unattenuated at all frequencies.

5-20 For $n=0$, Eqns. 5-75 and 5-76 should be:

$$\varepsilon_2 k_{e_1} F_1 F_3' - \varepsilon_1 k_{e_2} F_1' F_3 = 0$$

$$\mu_2 k_{e_1} F_2 F_4' - \mu_1 k_{e_2} F_2' F_4 = 0$$

For $n=1$, Eqn 5-74 becomes:

$$\varepsilon_2 k_{e_1}^2 F_1 \begin{vmatrix} \mu_2 k_{e_1}^2 F_2 & 0 & \mu_1 k_{e_2}^2 F_4 \\ \frac{k_z}{\omega\mu_1 a} F_2 & k_2 F_3' & \frac{k_z}{\omega\mu_2 a} F_4 \\ k_{e_1} F_2' & \frac{k_z}{\omega\varepsilon_2 a} F_3 & k_{e_2} F_4' \end{vmatrix}$$

$$+ \varepsilon_2 k_{e_2}^2 F_3 \begin{vmatrix} 0 & \mu_2 k_{e_1}^2 F_2 & \mu_1 k_{e_2}^2 F_4 \\ k_{e_1} F_1' & \frac{k_z F_2}{\omega\mu_1 a} & \frac{k_z F_4}{\omega\mu_2 a} \\ \frac{k_z F_1}{\omega\varepsilon_1 a} & k_{e_1} F_2' & k_{e_2} F_4' \end{vmatrix} = 0$$

$$\varepsilon_2 k_{e_1}^2 F_1 \Big[\mu_2 k_{e_1}^2 k_{e_2}^2 F_2 F_3' F_4' + \frac{k_z^2 k_{e_2}^2}{\omega^2 a^2 \varepsilon_2} F_2 F_3 F_4 - \mu_1 k_{e_1} k_{e_2}^3 F_2' F_3' F_4$$

$$- \frac{k_z^2 k_{e_1}^2}{\omega^2 \varepsilon_2 a^2} F_2 F_3 F_4 \Big] + \varepsilon_1 k_{e_2}^2 F_3 \Big[\mu_1 k_{e_1}^2 k_{e_2}^2 F_1' F_2' F_4$$

$$+ \frac{k_{e_1}^2 k_z^2}{\omega^2 \varepsilon_1 a^2} F_1 F_2 F_4 - \frac{k_z^2 k_{e_2}^2}{\omega^2 \varepsilon_1 a^2} F_1 F_2 F_4 - \mu_2 k_{e_2} k_{e_1}^3 F_1' F_2 F_4' \Big] = 0$$

$$\left[\varepsilon_1 k_{e_1} k_{e_2}^2 F_1' F_3 - \varepsilon_2 k_{e_1}^2 k_{e_2} F_1 F_3' \right] \left[\mu_1 k_{e_1} k_{e_2}^2 F_1' F_4 - \mu_2 k_{e_1}^2 k_{e_2} F_1 F_4' \right]$$

$$+ \frac{k_z^2}{\omega^2 a^2} (k_{e_2}^2 - k_{e_1}^2)(k_{e_1}^2 - k_{e_2}^2)(F_1)^2 F_3 F_4 = 0$$

$$(F_1 = F_2) \qquad k_1^2 = \omega^2 \mu_1 \varepsilon_1 = k_z^2 + k_{e_1}^2$$

$$k_2^2 = \omega^2 \mu_2 \varepsilon_2 = k_z^2 + k_{e_2}^2$$

$$\left[\varepsilon_1 k_{e_2} F_1' F_3 - \varepsilon_2 k_1 F_1 F_3' \right] \left[\mu_1 k_{e_2} F_1' F_4 - \mu_2 k_{e_1} F_1 F_4' \right]$$

$$- \frac{k_z^2 (k_1^2 - k_2^2)^2}{\ldots} (F_1)^2 F_3 F_4 = 0$$

5-20 (cont.)

$$F_1 = F_2 = J_1(k_{e_1}a)$$

$$F_3 = J_1(k_{e_2}a)N_1(k_{e_2}b) - N_1(k_{e_2}a)J_1(k_{e_2}b)$$

$$F_3' = J_1'(k_{e_2}a)N_1(k_{e_2}b) - N_1'(k_{e_2}a)J_1(k_{e_2}b)$$

$$F_4 = J_1(k_{e_2}a)N_1'(k_{e_2}b) - N_1(k_{e_2}a)J_1'(k_{e_2}b)$$

$$F_4' = J_1'(k_{e_2}a)N_1'(k_{e_2}b) - N_1'(k_{e_2}a)J_1'(k_{e_2}b)$$

Now charact. Eqn. becomes:

$$\Big\{ \varepsilon_1 k_{e_2} J_1'(k_{e_1}a)\big[J_1(k_{e_2}a)N_1(k_{e_2}b) - N_1(k_{e_2}a)J_1(k_{e_2}b)\big]$$
$$- \varepsilon_2 k_{e_1} J_1(k_{e_1}a)\big[J_1'(k_{e_2}a)N_1(k_{e_2}b) - N_1'(k_{e_2}a)J_1(k_{e_2}b)\big]\Big\} \times$$

$$\times \Big\{ \mu_1 k_{e_2} J_1'(k_{e_1}a)\big[J_1(k_{e_2}a)N_1'(k_{e_2}b) - N_1(k_{e_2}a)J_1'(k_{e_2}b)\big]$$
$$- \mu_2 k_{e_1} J_1(k_{e_1}a)\big[J_1'(k_{e_2}a)N_1'(k_{e_2}b) - N_1'(k_{e_2}a)J_1'(k_{e_2}b)\big]\Big\}$$
$$- \frac{k_z^2(k_1^2-k_2^2)^2}{k_{e_1}^2 k_{e_2}^2 \omega^2 a^2} F_1^2 F_3 F_4 = 0$$

$$\Big\{ N_1(k_{e_2}b)\big[\overbrace{\varepsilon_1 k_{e_2} J_1'(k_{e_1}a)J_1(k_{e_2}a) - \varepsilon_2 k_{e_1} J_1(k_{e_1}a)J_1'(k_{e_2}a)}^{A}\big]$$
$$+ J_1(k_{e_2}b)\big[\underbrace{\varepsilon_2 k_{e_1} J_1(k_{e_1}a)N_1'(k_{e_2}a) - \varepsilon_1 k_{e_2} J_1'(k_{e_1}a)N_1(k_{e_2}a)}_{B}\big]\Big\} \times$$

$$\times \Big\{ N_1'(k_{e_2}b)\big[\underbrace{\mu_1 k_{e_2} J_1'(k_{e_1}a)J_1(k_{e_2}a) - \mu_2 k_{e_1} J_1(k_{e_1}a)J_1'(k_{e_2}a)}_{C}\big]$$
$$+ J_1'(k_{e_2}b)\big[\underbrace{\mu_2 k_{e_1} J_1(k_{e_1}a)N_1'(k_{e_2}a) - \mu_1 k_{e_2} J_1'(k_{e_1}a)N_1(k_{e_2}a)}_{D}\big]\Big\}$$
$$- \frac{k_z^2(k_1^2-k_2^2)^2}{k_{e_1}^2 k_{e_2}^2 \omega^2 a^2} F_1^2 F_3 F_4 = 0$$

Charact. Eqn. is:

$$\big[A N_1(k_{e_2}b) + B J_1(k_{e_2}b)\big]\big[C N_1'(k_{e_2}b) + D J_1'(k_{e_2}b)\big]$$
$$- \frac{k_z^2(k_{z_2}^2 - k_{z_1}^2)^2}{k_{e_1}^2 k_{e_2}^2 \omega^2 a^2} J_1^2(k_{e_1}a)\big[J_1(k_{e_2}a)N_1(k_{e_2}b) - N_1(k_{e_2}a)J_1(k_{e_2}b)\big] \times$$
$$\times \big[J_1(k_{e_2}a)N_1'(k_{e_2}b) - N_1(k_{e_2}a)J_1'(k_{e_2}b)\big] = 0$$

$\boxed{5\text{-}21}$ See Schelkunoff, "Electromagnetic Waves", pp. 425-428, D. Van Nostrand Company, Inc., Princeton, N.J., 1943.

$\boxed{5\text{-}22}$ using small argument approximation for $K_0(z) \approx \ln \frac{2}{\gamma z}$

From $\boxed{5.21}$ — (problem nos.)

$$u^2 a^2 = \frac{\varepsilon_1 + \varepsilon_2}{\varepsilon_2 K_0(va)} = \frac{\varepsilon_1 + \varepsilon_2}{\varepsilon_2 \ln \frac{2}{\gamma v a}}$$

$$\therefore \ln \frac{2}{\gamma v a} \simeq \frac{\varepsilon_1 + \varepsilon_2}{\varepsilon_1 - \varepsilon_2} \frac{1}{(k_2 a)^2}$$

For $\varepsilon_1 = 9\varepsilon_2$, $a = 0.1\lambda_1$

$$\therefore \ln \frac{2}{\gamma v a} = 3.125 \qquad \therefore v = \frac{1.5}{\lambda_1}$$

$$\frac{K_1(v\rho)}{K_1(va)} \simeq .1 \text{ when } v\rho = 0.9$$

$$\therefore \rho \simeq 0.6\lambda_1$$

$\boxed{5\text{-}24}$ For the dominant mode

$$\psi_{opq}^{TM} = A J_0(k_1 \rho) \cos(k_{z_1} z) \text{ in medium (1)}$$

$$\psi_{opq}^{TM} = B J_0(k_2 \rho) \cos[k_{z_2}(d-z)] \text{ in medium (2)}$$

Hence continuity of E_ϕ at $z = b$ implies

$$-k_{z_1} \frac{A}{\varepsilon_1} k_1 J_0'(k_1 \rho) \sin(k_{z_1} b) = \frac{k_{z_2}}{\varepsilon_2} B k_2 J_0'(k_2 \rho) \sin[k_{z_2}(d-b)] \quad \text{— (I)}$$

Similarly continuity of H_ϕ at $z = b$ implies

$$A k_1 J_0'(k_1 \rho) \cos(k_{z_1} b) = B k_2 J_0'(k_2 \rho) \cos[k_{z_2}(d-b)] \quad \text{— (II)}$$

Hence dividing (I) by (II)

$$-\frac{k_{z_1}}{\varepsilon_1} \tan[k_{z_1} b] = \frac{k_{z_2}}{\varepsilon_2} \tan[k_{z_2}(d-b)]$$

for small d (See problem 5-18) it is very similar to that.

$\boxed{5.23}$ The fields are completely specified by the magnetic vector potentials

$$\left(\psi_{010}^{TM}\right)^{air} = \left[J_0(k_0 \rho) - \frac{J_0(k_0 a) Y_0(k_0 \rho)}{Y_0(k_0 a)}\right] \cos \frac{\pi z}{d}$$

$$\left(\psi_{010}^{TM}\right)^{dielectric} = J_0(k\rho) \cos \frac{\pi z}{d}$$

Hence from the continuity of E_z,

$$\frac{k_{\rho_1}^2}{\hat{y}_1} \psi^{air} = \frac{k_{\rho_2}^2}{\hat{y}_2} \psi^{d}$$

and from the continuity of H_ϕ

$$k_{\rho_1} \psi'^{\,air} = k_{\rho_2} \psi'^{\,d}$$

Hence
$$\frac{\psi'^{\,d}}{\psi^{d}} \frac{\hat{y}_2}{k_{\rho_2}} = \frac{\hat{y}_1}{k_\rho} \frac{\psi'^{\,air}}{\psi^{air}}$$

or,
$$\frac{1}{\eta} \frac{J_0'(kc)}{J_0(kc)} = \frac{1}{\eta_0} \frac{J_0'(k_0 c) Y_0(k_0 a) - J_0(k_0 a) Y_0'(k_0 c)}{J_0(k_0 c) Y_0(k_0 a) - J_0(k_0 a) Y_0(k_0 c)}$$

Hence

$$J_0'(k_1 c)\left[J_0(kc) Y_0(ka) - J_0(ka) Y_0(kc)\right] = \sqrt{\frac{\mu_0 \varepsilon_0}{\varepsilon \mu_0}} J_0(k_1 c)\left[J_0'(kc) Y_0(ka) - J_0(ka) Y_0'(kc)\right]$$

where
$$k_1 = \omega_0 \sqrt{\mu_0 \varepsilon_r}$$
$$k = \omega \sqrt{\mu_0 \varepsilon_r}$$
$$k_0 = \omega_0 \sqrt{\mu_0 \varepsilon_0}$$

Also
$$(k - k_0) a J_0'(k_0 a) \approx J_0(ka)$$

and using the small argument approximations for the ~~Bessel~~ functions one obtains

$$Y_0(k_0 a) \frac{\pi}{4} c^2 (k_1^2 - k_0^2) = (k - k_0) a J_0'(k_0 a)$$

[Here all the functions have been expanded which has a c in the argument]

$$\therefore \frac{\omega - \omega_0}{\omega_0} = \frac{\pi}{4} \frac{Y_0(k_0 a)}{J_0'(k_0 a)} \frac{c^2}{a^2} x_{01} \left(\frac{\varepsilon_r}{\varepsilon_0} - 1\right)$$

$\boxed{5.25}$ For the circular cavity with a conducting wedge the general solution is of the form

$$J_n(k_\rho \rho)\begin{Bmatrix}\sin n\phi \\ \cos n\phi\end{Bmatrix} e^{\pm jk_z z}$$

for $E_\rho = 0$,

$$A \sin n\phi + B\cos n\phi = 0 \text{ for } \begin{cases}\phi = 0 \\ \phi = 2\pi - \phi_0\end{cases}$$

$$2n\pi - n\phi_0 = k\pi$$

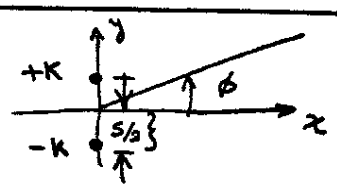
$$\therefore n = \frac{k\pi}{2\pi - \phi_0}$$

and for TM modes

$$J_n(k_\rho a) = 0, \text{ where } n = \frac{\pi}{2\pi - \phi_0}$$

for the dominant $k = 1$ mode.

$\boxed{5.27}$



$$F_z = F_z'(x, y - \tfrac{s}{2}) - F_z'(x, y + \tfrac{s}{2})$$

$$= -s\frac{\partial F_z'}{\partial y} = -\frac{Ks}{4j}\frac{\partial}{\partial y} H_0^{(2)}(k\rho)$$

$$= \frac{kKs}{4j} H_1^{(2)}(k\rho)\frac{\partial \rho}{\partial y} = \frac{kKs}{4j} H_1^{(2)}(k\rho)\sin\phi$$

and $\vec{E} = -\vec{\nabla} \times \hat{z} F_z$

$\boxed{5.26}$ Let $A_x = C H_0^{(2)}(k\rho)$ where C is a constant.

and by the terms of the problem $H_z \cdot 2\Delta z = I$ and

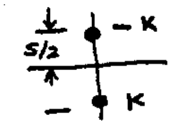
$$H_z = -\frac{\partial A_x}{\partial y} = -C\frac{\partial}{\partial y}\left[1 - \frac{2j}{\pi}\log\frac{\gamma x}{2}\right]$$

$$= \frac{2jC}{x\pi}$$

$$\therefore C = \frac{I}{4j} = \frac{J_x l}{4j}$$

$$\therefore A_x = \frac{I}{4j} H_0^{(2)}(k\rho)\ ;\ \therefore H_z = -\frac{I}{4j}\frac{\partial}{\partial y}\left[H_0^{(2)}(k\rho)\right]$$

$$F_z = F_z'(x, y + \tfrac{s}{2}) - F_z'(x, y - \tfrac{s}{2})$$



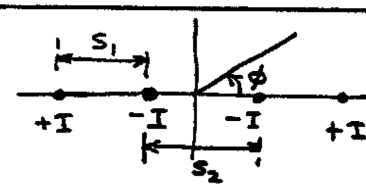
$$\therefore F_z = \lim_{s\to 0} s\frac{\partial F_z}{\partial y} = -\frac{Ks}{4j}\frac{\partial}{\partial y}\left\{H_0^{(2)}(k\rho)\right\}$$

$$\therefore H_z = -\frac{k^2}{\hat{z}}\frac{Ks}{4j}\frac{\partial}{\partial y}\left\{H_0^{(2)}(k\rho)\right\}$$

They are identical when

$$I = \frac{k^2}{j\omega\mu} Ks = -j\omega\varepsilon Ks$$

$\boxed{5.28}$



$$A_z^1 = -S_1\frac{\partial A_z}{\partial x}$$

$$= -\frac{S_1 I}{4j}\left[H_1^{(2)}(k\rho)\, k\frac{x}{\sqrt{x^2+y^2}}\right]$$

$$A_z^2 = S_2\frac{\partial A_z^1}{\partial x}$$

$$= -\frac{kS_1S_2 I}{4j}\left[k H_1^{(2)\prime}(k\rho)\cos^2\phi + \frac{H_1^{(2)}(k\rho)}{\rho}\frac{y^2}{x^2+y^2}\right]$$

$$2\frac{d}{k\,d\rho}\left[H_1^{(2)}(k\rho)\right] = H_0^{(2)}(k\rho) - H_2^{(2)}(k\rho)$$

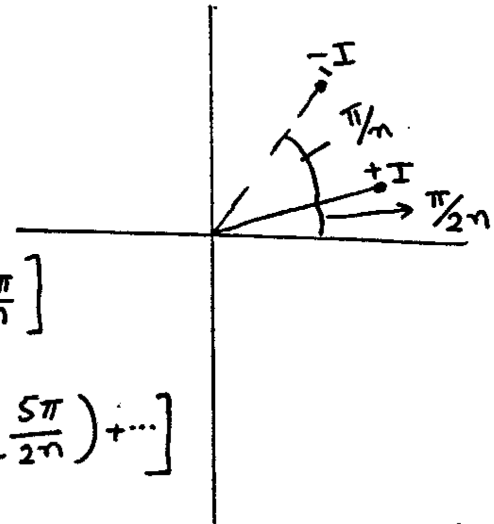
$$kH_1^{\prime(2)}(k\rho) = \frac{k}{2}\left[H_0^{(2)}(k\rho) - H_2^{(2)}(k\rho)\right] \text{ and}$$

$$H_1^{(2)}(k\rho) = \frac{k\rho}{2}\left[H_0^{(2)}(k\rho) + H_2^{(2)}(k\rho)\right]$$

$$\therefore A_z^2 = \frac{k^2 S_1 S_2 I}{8j}\left[-H_0^{(2)}(k\rho) + H_2^{(2)}(k\rho)\cos 2\phi\right]$$

$\boxed{5-29}$

Let us assume that the current filaments are represented as



$$J_z = \frac{I}{2\pi a}\sum_{p=1}^{2n}(-1)^{p+1}\,\delta\left[\phi-\frac{\pi}{2n}-(p-1)\frac{\pi}{n}\right]$$

$$= \frac{I}{2\pi a}\left[\delta\left(\frac{\pi}{2n}\right)-\delta\left(\frac{3\pi}{2n}\right)+\delta\left(\frac{5\pi}{2n}\right)+\cdots\right]$$

Because of symmetry, the ψ potentials can be represented as

$\psi^- = c_1 J_n(k\rho)\sin n\phi$ (Inside region)

$\psi^+ = c_2 H_n^{(2)}(k\rho)\sin n\phi$ (external to the current elements)

From the continuity of E_z (from 5.18) at $\rho = a$

$c_1 J_n(ka) = c_2 H_n^{(2)}(ka)$, and from the discontinuity of

H_ϕ; $\quad H_\phi^+ - H_\phi^- = J_z$.

Hence $\quad -c_1 k J_n'(ka)\sin n\phi + c_2 k H_n'^{(2)}(ka)\sin n\phi = J_z$

$$\therefore J_z = k\cdot c_2\cdot \sin n\phi\left[H_n'^{(2)}(ka) - \frac{H_n^{(2)}(ka)\,J_n'(ka)}{J_n(ka)}\right]$$

$$J_z = -\frac{2j\cancel{k}}{\pi\cancel{k}a}\sin n\phi\,\frac{c_2}{J_n(ka)}$$

Multiplying each side by $\sin n\phi$ and integrating from

0 to 2π

$$L.H.S = \frac{I}{2\pi a}\int_0^{2\pi}\sum_{p=1}^{2n}(-1)^{p+1}\,\delta\left[\phi-\frac{\pi}{2n}-(p-1)\frac{\pi}{n}\right]\sin n\phi\,d\phi$$

$$= \frac{I}{2\pi a}\left[\sin\frac{\pi}{2} - \sin\frac{3\pi}{2} + \sin\frac{5\pi}{2} - \cdots\right] = \frac{I\,n}{\pi a}$$

$R.H.S. \quad = -\dfrac{2j}{\cancel{\pi}a}\cdot\dfrac{c_2}{J_n(ka)}\cdot\cancel{\pi}$. Expanding $J_n(ka)$ for small argument approximations. $\quad c_2 = -\dfrac{nI}{2j\pi}\,\dfrac{1}{n!}\left(\dfrac{ka}{2}\right)^n$ [as $a \to 0$]

$$\therefore \psi_2 = -\frac{I}{2j\pi(n-1)!}\left(\frac{ka}{2}\right)^n H_n^{(2)}(k\rho)\sin n\phi$$

$\boxed{5.30}$ For the cylinder of current of fig 5.15,

$$\psi = \begin{cases} \frac{\pi a}{2j} \sum_{n=-\infty}^{\infty} A_n J_n(ka) H_n^{(2)}(k\rho) e^{jn\phi} \; ; \; \rho > a \\ \frac{\pi a}{2j} \sum_{n=-\infty}^{\infty} A_n H_n^{(2)}(ka) J_n(k\rho) e^{jn\phi} \; ; \; \rho < a \end{cases}$$

Since $H_\phi^+ - H_\phi^- = J$ and $H_\phi = -\frac{\partial A_z}{\partial \rho}$

$$\therefore H_\phi^+ = -\frac{\pi a}{2j} \sum_{n=-\infty}^{\infty} A_n J_n(ka) k H_n'^{(2)}(k\rho) e^{jn\phi}$$

$$H_\phi^- = -\frac{\pi a}{2j} \sum_{n=-\infty}^{\infty} A_n k J_n'(k\rho) H_n^{(2)}(ka) e^{jn\phi}$$

$$\therefore J_z = H_\phi^+ \Big|_{\rho=a} - H_\phi^- \Big|_{\rho=a}$$

$$= \frac{\pi a k}{2j} \sum_{n=-\infty}^{\infty} A_n e^{jn\phi} \left[J_n'(ka) H_n^{(2)}(ka) - J_n(ka) H_n'^{(2)}(ka) \right]$$

Using the Wronskians

$$J_z = \frac{\pi a k}{2j} \cdot \frac{2j}{\pi k a} \sum_{n=-\infty}^{\infty} A_n e^{jn\phi} = \sum_{n=-\infty}^{\infty} A_n e^{jn\phi}$$

$$\therefore A_n = \frac{1}{2\pi} \int_0^{2\pi} J_z e^{-jn\phi} \, d\phi .$$

$\boxed{5\text{-}33}$ Since $e^{-j\rho\cos\phi} = \sum_{n=-\infty}^{\infty} j^{-n} J_n(\rho) e^{jn\phi}$

and since $J_{-n}(\rho) = (-1)^n J_n(\rho)$

$$\cos(\rho\cos\phi) - j\sin(\rho\cos\phi)$$
$$= J_0(\rho) - J_2(\rho)\left[e^{j2\phi} + e^{-j2\phi}\right] - jJ_1(\rho)\left[e^{j\phi} + e^{-j\phi}\right] + \cdots\cdots$$

Let $\theta = \pi/2 - \phi$ and equating real and imaginary parts

$$\cos(\rho\sin\theta) = J_0(\rho) + 2J_2(\rho)\cos 2\theta + \cdots$$

$$\sin(\rho\sin\theta) = 2\left[J_1(\rho)\sin\theta + \cdots\cdots\right]$$

$\boxed{5.31}$ The vector potential due to a ribbon of ~~magnetic~~ el current is given by (5-98)

$$A_z = \frac{e^{-jkr}}{\sqrt{8j\pi kr}} \int_{-\frac{a}{2}}^{a/2} J_z e^{jkx'\cos\phi} \, dx'$$

$$= \frac{e^{-jkr}}{\sqrt{8j\pi kr}} J_z \, a \frac{\sin\left(\frac{ka}{2}\cos\phi\right)}{\left(\frac{ka}{2}\cos\phi\right)}$$

Since $E_z = -j\omega\mu A_z$

$$\therefore E_z = \frac{-j\omega\mu a e^{-jkr}}{\sqrt{8\pi jkr}} \frac{\sin\left(\frac{ka}{2}\cos\phi\right)}{\left(\frac{ka}{2}\cos\phi\right)}$$

and $H_\phi = -\frac{E_z}{\eta}$ in the far zone.

$\boxed{5.32}$ Assuming that tangential E in the slot is $\hat{u}_x E_0$, a constant.

$$\bar{M}_s = \bar{E}^i \times \bar{n} = (\hat{x} E_0) \times \hat{y} = \hat{z} E_0$$

The field is produced by $2\bar{M}_s$ and the electric vector potential is given by (5-98)

$$F_z = \frac{e^{-jk\rho}}{\sqrt{8j\pi k\rho}} \int_{x'} M(\rho') e^{jkx'\cos\phi} dx'$$

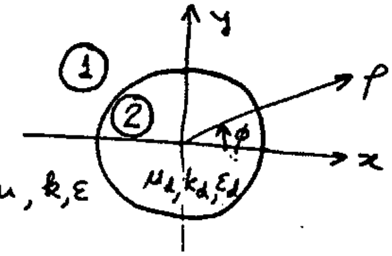
$$\therefore F_z = \frac{e^{-jk\rho}}{\sqrt{8j\pi k\rho}} E_0 \int_{-a/2}^{a/2} e^{jkx'\cos\phi} dx'$$

$$= \frac{e^{-jk\rho}}{\sqrt{8j\pi k\rho}} 2E_0 a \frac{\sin\left(\frac{ka}{2}\cos\phi\right)}{\left(\frac{ka}{2}\cos\phi\right)}$$

Since $H_z = -j\omega\varepsilon F_z$

$$\therefore H_z = \frac{-j\omega\varepsilon a e^{-jk\rho}}{\sqrt{8j\pi k\rho}} E_0 \frac{\sin\left(\frac{ka}{2}\cos\phi\right)}{\left(\frac{ka}{2}\cos\phi\right)}$$

$\boxed{5-34}$ Assuming the incident plane wave to be



$$E_z^i = E_0 e^{-jk\rho\cos\varphi} = \sum_{n=-\infty}^{\infty} E_0 j^{-n} J_n(k\rho) e^{jn\phi}$$

and $$E_{z_2} = E_0 \sum_{n=-\infty}^{\infty} j^{-n} C_n J_n(k_d \rho) e^{jn\phi}$$

$$E_{z_1} = E_z^i + E_z^s = E_0 \sum_{n=-\infty}^{\infty} j^{-n} \left[J_n(k\rho) + a_n H_n^{(2)}(k\rho) \right] e^{jn\phi}$$

Since $E_{z_1} = E_{z_2}$ at $\rho = a$

$$\therefore C_n = \frac{J_n(ka) + a_n H_n^{(2)}(ka)}{J_n(k_d a)}$$

Also $$H_{\phi_1} = E_0 \sum_{n=-\infty}^{\infty} j^{-n} \frac{1}{j\omega\mu} k \left[J_n'(k\rho) + a_n H_n'^{(2)}(k\rho) \right] e^{jn\phi}$$

and $$H_{\phi_2} = E_0 \sum_{n=-\infty}^{\infty} j^{-n} C_n \frac{1}{j\omega\mu_d} k_d J_n'(k_d\rho) e^{jn\phi}$$

Since $H_{\phi_1} = H_{\phi_2}$ at $\rho = a$

$$\frac{k}{\mu}\left[J_n'(ka) + a_n H_n'^{(2)}(ka) \right] = \frac{J_n(ka) + a_n H_n^{(2)}(ka)}{J_n(k_d a)} \frac{k_d}{\mu_d} J_n'(k_d a)$$

$$\therefore a_n = -\frac{J_n(ka)}{H_n^{(2)}(ka)} \frac{\dfrac{\mu}{k\mu_d} \dfrac{J_n'(k_d a)}{J_n(k_d a)} - \dfrac{1}{k_d} \dfrac{J_n'(ka)}{J_n(ka)}}{\dfrac{\mu}{k\mu_2} \dfrac{J_n'(k_d a)}{J_n(k_d a)} - \dfrac{1}{k_d} \dfrac{H_n'^{(2)}(ka)}{H_n^{(2)}(ka)}}$$

Since $k_d = \omega\sqrt{\mu_d \varepsilon_d}$ $\quad \therefore \dfrac{\varepsilon_d}{\varepsilon k_d} = \dfrac{k_d \mu}{k^2 \mu_d}$

$$a_n = -\frac{J_n(ka)}{H_n^{(2)}(ka)} \cdot \frac{\dfrac{\varepsilon_d}{\varepsilon k_d a} \dfrac{J_n'(k_d a)}{J_n(k_d a)} - \dfrac{1}{ka} \dfrac{J_n'(ka)}{J_n(ka)}}{\dfrac{\varepsilon_d}{\varepsilon k_d a} \dfrac{J_n'(k_d a)}{J_n(k_d a)} - \dfrac{1}{ka} \dfrac{H_n'^{(2)}(ka)}{H_n^{(2)}(ka)}}$$

The fields internal to the cylinder is given by E_{z_2}.

As $\varepsilon_d \to 0$, $E_{z_2} \to 0$ and $a_n \to -\dfrac{J_n(ka)}{H_n^{(2)\prime}(ka)}$. Hence this reduces to the solution of a conducting cylinder.

$\boxed{5-35}$ Since the problem is dual to the problem of $\boxed{5-34}$ the answer is similar.

5-36 By the terms of the problem, a_n of (5-34) $\to 0$ as $ka \to 0$ for all n except $n = 0$. So

$$a_0 = -\frac{1}{H_0^{(2)}(ka)}\left[\frac{-\frac{\varepsilon_d k_d a}{2\varepsilon k_d a} + \frac{ka}{2ka}}{-\frac{\varepsilon_d}{2\varepsilon} + \left(\frac{2}{ka}\right)^2}\right]$$

$$= \frac{\pi j}{2}\,\frac{\left[1 - \frac{\varepsilon_d}{\varepsilon}\right](ka)^2}{2}$$

Since

$$E_z^s = E_0 \sum_{n=-\infty}^{\infty} j^{-n} a_n H_n^{(2)}(k\rho) e^{jn\phi}$$

$$\therefore E_z^s = -\frac{j\pi}{4} E_0 (\varepsilon_r - 1) H_0^{(2)}(k\rho)\,(ka)^2$$

where $\varepsilon_r = \varepsilon_d/\varepsilon$.

5-38 H_z as given by (5.134)

$$= H_0 \sum_{n=0}^{\infty} \varepsilon_n j^{n/2} J_{n/2}(k\rho)\cos\frac{n\phi'}{2}\cos\frac{n\phi}{2}$$

Since

$J_\rho = H_z^{0+} - H_z^{2\pi^-}$; hence

$$= 2H_0 \sum_{n=0}^{\infty} \varepsilon_n j^{n/2} J_{n/2}(k\rho)\cos\frac{n\phi'}{2}$$

$J_\rho \xrightarrow[k\rho\to 0]{} 2H_0$ ($\because J_{n/2}(k\rho) = 1$ for $n = 0$).

Now, $E_\rho = \frac{1}{j\omega\varepsilon}\frac{1}{\rho}\frac{\partial H_z}{\partial\phi}$

$$= \frac{-H_0}{j\omega\varepsilon\rho}\sum_{n=0}^{\infty}\varepsilon_n j^{n/2} J_{n/2}(k\rho)\cos\frac{n\phi'}{2}\sin\frac{n\phi}{2}\cdot n$$

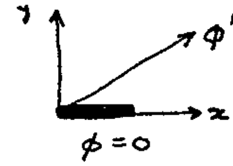
[This is nonzero for $n = 1$ when $k\rho \to 0$]

$$\therefore E_\rho \xrightarrow[k\rho\to 0]{} -\eta H_0\sqrt{\frac{2}{\pi j k\rho}}\cos\frac{\phi'}{2}\sin\frac{\phi}{2}$$

as $J_{1/2}(k\rho) \xrightarrow[k\rho\to 0]{Lt} \sqrt{\frac{2k\rho}{\pi}}$

and $\eta = \frac{k}{\omega\varepsilon}$

5-37



In this case

$$E_z^i = E_0 e^{jk\rho\cos(\phi-\phi')}$$

$$= 2E_0 \sum_{n=1}^{\infty} j^{n/2} J_{n/2}(k\rho)\sin\frac{n\phi'}{2}\sin\frac{n\phi}{2}$$

from (5.129)

From p. 233

$$H_\phi = \frac{1}{j\omega\mu}\frac{\partial E_z}{\partial\rho} \quad\text{and}\quad H_\rho = -\frac{1}{j\omega\mu}\frac{\partial E_z}{\partial\rho}$$

$$\therefore H_\phi = \frac{2E_0}{j\omega\mu} k \sum_{n=1}^{\infty} j^{n/2} J_{n/2}'(k\rho)\sin\frac{n\phi'}{2}\sin\frac{n\phi}{2}$$

$$H_\rho = \frac{2E_0}{-j\omega\mu\rho}\sum_{n=1}^{\infty} j^{n/2} J_{n/2}(k\rho)\, n\sin\frac{n\phi'}{2}\cdot\cos\frac{n\phi}{2}$$

Since $H_\rho^+ - H_\rho^-\big|_{\phi=0} = -J_z$

$$\therefore J_z = \frac{2E_0}{j\omega\mu\rho}\sum_{n=1}^{\infty} n j^{n/2} J_{n/2}(k\rho)\sin\frac{n\phi'}{2}$$

for $k\rho \to 0$

$$J_z = \frac{2E_0}{j\omega\mu\rho} j^{1/2}\sin\frac{\phi'}{2} J_{1/2}(k\rho)$$

$\underset{k\rho\to 0}{Lt}\; J_{1/2}(k\rho) = \sqrt{\frac{2k\rho}{\pi}}$

$$J_z \xrightarrow[k\rho\to 0]{} \frac{2E_0}{\eta}\sqrt{\frac{2}{jk\rho\pi}}\sin\phi'/2 \quad\text{and}$$

$$E_z^i = 2E_0\sqrt{\frac{2k\rho j}{\pi}}\sin\frac{\phi'}{2}\sin\frac{\phi}{2}$$

$\boxed{5-39}$ As explained in $\boxed{5-40}$ F^i = dual of Ψ_i and so

$$F_z = F^s + F^i = \frac{Kl\, e^{-jkr}}{4\pi r} e^{jkz'\cos\theta} \sum_{n=0}^{\infty} \varepsilon_n (j)^n \left[J_n(k\rho'\sin\theta) + b_n H_n^{(2)}(k\rho'\sin\theta) \right] \cos n\phi$$

By the terms of the problem $E_\phi = 0 \big|_{\rho=a} = \frac{\partial F_z}{\partial \rho}\Big|_{\rho=a} = 0$

Hence $b_n = -\dfrac{J_n'(ka\sin\theta)}{H_n'^{(2)}(ka\sin\theta)}$. For far field,

$E_\phi = -jk F_z \sin\theta$ (from 3.97). So for this case

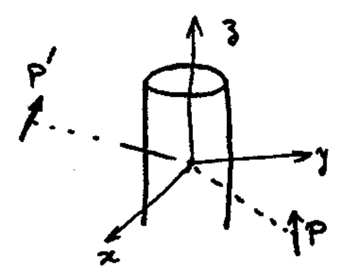
$$E_\phi \Big|_{z'=0,\, \rho'=a} = -jk \frac{Kl\, e^{-jkr}}{4\pi r} \sin\theta \sum_{n=0}^{\infty} \varepsilon_n j^n \cos n\phi \cdot \left[J_n(ka\sin\theta) - \frac{J_n'(ka\sin\theta) H_n^{(2)}(ka\sin\theta)}{H_n'^{(2)}(ka\sin\theta)} \right]$$

Since $J_n(x) N_n'(x) - N_n(x) J_n'(x) = \frac{2}{\pi x}$, hence

$$E_\phi = \frac{-jkKl\, e^{-jkr}}{4\pi r} \sin\theta \sum_{n=0}^{\infty} \varepsilon_n j^n \cos n\phi \; \frac{-2j}{\pi ka\sin\theta} \cdot \frac{1}{H_n'^{(2)}(ka\sin\theta)} \Big]$$

$$= -\frac{Kl\, e^{-jkr}}{2\pi^2 a} \sum_{n=0}^{\infty} \varepsilon_n (j)^n \frac{\cos n\phi}{H_n^{(2)'}(ka\sin\theta)}$$

$\boxed{5-40}$ The radiation pattern of a dipole at P near a conducting cylinder is obtained as a receiving antenna instead of as a transmitting antenna. The wave received from a distance source will be essentially a plane wave that is easily expanded into a sum of standing cylindrical waves. Equating the sum of the incident and the scattered tangential electric fields to zero at the cylinder surface gives the magnitude of the secondary or re-radiated wave.



So, the incident field is

$$\psi^i = \frac{Il\, e^{-jkr}}{4\pi r} e^{jkz'\cos\theta} e^{jk\rho'\cos\phi\sin\theta}$$

$$= \frac{Il\, e^{-jkr}}{4\pi r} e^{jkz'\cos\theta} \sum_{n=0}^{\infty} \varepsilon_n (j)^n J_n(k\rho'\sin\theta) \cos n\phi$$

$\boxed{5\text{-}40}$ contd.

$$\psi^s = \frac{I\ell e^{-jkr}}{4\pi r} e^{jkz'\cos\theta} \sum_{n=0}^{\infty} b_n \varepsilon_n (j)^n H_n^{(2)}(k\rho' \sin\theta)\cos n\phi$$

Since $E_z = 0$ at $\rho = a$, $\therefore\ b_n = \dfrac{J_n(ka\sin\theta)}{H_n^{(2)}(ka\sin\theta)}$.

$$\therefore E_\theta = \left[E_\rho \cos\theta' - E_z \sin\theta'\right]_{\rho'=b,\ \theta'=90°,\ z'=0}$$

$$= -E_z\big|_{\rho'=b} = -j\omega\mu\sin\theta\, A_z\big|_{\rho'=b} \quad \text{(in the far fields)}$$

$$\therefore E_\theta = j\omega\mu \frac{I\ell e^{-jkr}}{4\pi r}\sin\theta \sum_{n=0}^{\infty}\left[J_n(kb\sin\theta) - \frac{J_n(ka\sin\theta)}{H_n^{(2)}(ka\sin\theta)} H_n^{(2)}(kb\sin\theta)\right] \cdot \varepsilon_n j^n \cos n\phi$$

$$= f(r)\sin\theta \sum_{n=-\infty}^{\infty} j^n e^{jn\phi}\left[J_n(kb\sin\theta) - \frac{J_n(ka\sin\theta)}{H_n^{(2)}(ka\sin\theta)} H_n^{(2)}(kb\sin\theta)\right]$$

$$= j f(r)\sin\theta \sum_{n=0}^{\infty} j^n e^{jn\phi}\left[\frac{J_n(ka\sin\theta)N_n(kb\sin\theta) - J_n(kb\sin\theta)N_n(ka\sin\theta)}{H_n^{(2)}(ka\sin\theta)}\right]$$

Since $J_{-n}(z) = (-1)^n J_n(z)$; $N_{-n}(z) = (-1)^n N_n(z)$ and

$H_{-n}^{(2)}(z) = e^{-n\pi i}(z)$; hence

$$E_\theta = \overset{\swarrow \text{a constant}}{A} \cdot f(r)\sin\theta \sum_{n=0}^{\infty} j^n e^{jn\phi}\left[\frac{J_n(\alpha)N_n(\beta) - J_n(\beta)N_n(\alpha)}{H_n^{(2)}(\alpha)}\right]$$

$\boxed{5\text{-}41}$ For this configuration.

$$\psi_x^i = \frac{I\ell e^{-jkr}}{4\pi r} e^{jk\rho'\cos\phi\sin\theta}$$

$$= \frac{I\ell e^{-jkr}}{4\pi r}\sum_{n=0}^{\infty}\varepsilon_n (j)^n J_n(k\rho'\sin\theta)\cos n\phi$$

Since $\psi_x = \psi_x^i + \psi_x^s = \dfrac{I\ell e^{-jkr}}{4\pi r}\displaystyle\sum_{n=0}^{\infty}\varepsilon_n (j)^n \left[J_n(k\rho'\sin\theta) + b_n H_n^{(2)}(k\rho'\sin\theta)\right] \cdot \cos n\phi$

and $E_z = 0$ at $\rho = a$ implies $\dfrac{1}{\hat{y}}\dfrac{\partial^2\psi}{\partial x\,\partial z}\Big|_{\rho=a} = -\dfrac{1}{\hat{y}}\dfrac{\partial^2\psi}{\rho\cos\phi\,\partial\phi\,\partial z}$

$$\therefore b_n = -\frac{J_n'(ka\sin\theta)}{H_n'^{(2)}(ka\sin\theta)}$$

$\boxed{5-41}$ contd. In the case, $\theta' = 90°$, $\theta = 90°$

$$\psi_x = \frac{Il e^{-jkr}}{4\pi r} \sum_{n=0}^{\infty} \varepsilon_n (j)^n \left[J_n(k\rho') - \frac{J_n'(ka)}{H_n^{(2)\prime}(ka)} H_n^{(2)}(k\rho') \right] \cos n\phi$$

$$\therefore E_\phi = E_y \cos\phi' - E_x \sin\phi' \Big|_{\substack{\rho'=a \\ \phi'=0}} = E_y = \frac{1}{\hat{y}\,\rho \cos\varphi' d\varphi} \frac{d\psi_x}{}$$

$$= -\frac{Il e^{-jkr}}{4\pi r} \sum_{n=0}^{\infty} \varepsilon_n (j)^n \frac{n \sin n\phi}{H_n^{(2)\prime}(ka)} \cdot \frac{2}{\pi ka}$$

$$= f(\rho) \sum_{n=1}^{\infty} \frac{n (j)^n \sin n\phi}{H_n^{(2)\prime}(ka)}$$

$\boxed{5-42}$ The field from a magnetic dipole at angle θ lying in x-z plane is

$$F^z = \frac{Kl e^{-jkr}}{4\pi r} e^{jkz'\cos\theta} e^{jk\rho'\sin\theta\cos(\phi-\phi')}$$

Since the solution has the same form as (5-133)

$$F^z = \frac{Kl e^{-jkr}}{4\pi r} e^{jkz'\cos\theta} \sum_{n=0}^{\infty} \varepsilon_n j^{n/2} J_{n/2}(k\rho\sin\theta) \cos\frac{n\phi}{2}$$

In the far field

$$E_\phi \Big|_{z'=0, \rho=a} = -jk F_z^3 \sin\theta \quad \text{(from 3.97)}. \quad \text{Hence}$$

$$E_\phi = -\frac{jkKl e^{-jkr}}{4\pi r} \sin\theta \sum_{n=0}^{\infty} \varepsilon_n j^{n/2} J_{n/2}(ka\sin\theta) \cos\frac{n\phi}{2}$$

$\boxed{6-1}$ If $\bar{A} = \bar{u}_z \psi = \bar{u}_r \psi\cos\theta - \bar{u}_\theta \psi\sin\theta \triangleq \bar{u}_r A_r + \bar{u}_\theta A_\theta$ where $A_r = \psi\cos\theta$, $A_\theta = -\psi\sin\theta$

$$\therefore \bar{\nabla}\cdot\bar{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\theta}$$

$$= \frac{\cos\theta}{r^2}\frac{\partial}{\partial r}(r^2\psi) - \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\psi\sin^2\theta)$$

$$\bar{\nabla}(\bar{\nabla}\cdot\bar{A}) = \bar{u}_r \frac{\partial}{\partial r}(\bar{\nabla}\cdot\bar{A}) + \bar{u}_\theta \frac{1}{r}\frac{\partial}{\partial\theta}(\bar{\nabla}\cdot\bar{A}) + \bar{u}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(\bar{\nabla}\cdot\bar{A})$$

$$= \bar{u}_r \frac{\partial}{\partial r}\left[\frac{\cos\theta}{r^2}\frac{\partial}{\partial r}(r^2\psi) - \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\psi\sin^2\theta)\right]$$

$$+ \bar{u}_\theta \frac{1}{r}\frac{\partial}{\partial\theta}\left[\frac{\cos\theta}{r^2}\frac{\partial}{\partial r}(r^2\psi) - \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\psi\sin^2\theta)\right] + \bar{u}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left[\frac{\cos\theta}{r^2}\frac{\partial}{\partial r}(r^2\psi) - \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\psi\sin^2\theta)\right]$$

$$\therefore E_r = -j\omega\mu\psi\cos\theta + \frac{1}{j\omega\varepsilon}\frac{\partial}{\partial r}\left[\frac{\cos\theta}{r^2}\frac{\partial}{\partial r}(r^2\psi) - \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\psi\sin^2\theta)\right]$$

$$E_\theta = j\omega\mu\psi\sin\theta + \frac{1}{j\omega\varepsilon r}\frac{\partial}{\partial\theta}\left[\frac{\cos\theta}{r^2}\frac{\partial}{\partial r}(r^2\psi) - \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\psi\sin^2\theta)\right]$$

$$E_\phi = \frac{1}{j\omega\varepsilon r\sin\theta}\frac{\partial}{\partial\phi}\left[\frac{\cos\theta}{r^2}\frac{\partial}{\partial r}(r^2\psi) - \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\psi\sin^2\theta)\right]$$

Now $H = \bar{\nabla}\times\bar{A}$

$$\therefore H_r = \frac{1}{r}\frac{\partial\psi}{\partial\phi}$$

$$H_\theta = \frac{1}{r\sin\theta}\left[\frac{\partial\psi}{\partial r}\cos\theta\right] = \frac{\cot\theta}{r}\frac{\partial\psi}{\partial r}$$

$$H_\phi = \frac{1}{r}\left[r\frac{\partial A_\theta}{\partial r} - \frac{\partial A_r}{\partial\theta}\right] + A_\theta = \cos\theta\frac{\partial\psi}{\partial r} - \frac{\psi\sin\theta}{r} - \frac{1}{r}\frac{\partial}{\partial\theta}(\psi\cos\theta)$$

$\boxed{6-2}$

$$(\nabla^2 + k^2)\frac{A_r}{r} = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\left(\frac{A_r}{r}\right)\right] + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{r}\frac{\partial A_r}{\partial\theta}\right) + \frac{1}{r^2\sin\theta}\cdot\frac{1}{r}\frac{\partial^2 A_r}{\partial\phi^2} + k^2\frac{A_r}{r} = 0.$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial A_r}{\partial r} - A_r\right] + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial A_r}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 A_r}{\partial\phi^2} + k^2 A_r = 0$$

$$\therefore \frac{\partial^2 A_r}{\partial r^2} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial A_r}{\partial\theta}\right) + \frac{1}{r^2\sin\theta}\frac{\partial^2 A_r}{\partial\phi^2} + k^2 A_r = 0.$$

$\boxed{6-3}$ $r = 5$ cm. The dominant mode is the TM_{011} mode.

The first ten resonant frequencies are

$$(f_r)^{TM}_{mnp} = \frac{u'_{np}}{2\pi a\sqrt{\varepsilon\mu}} = \frac{3\times10^9}{\pi} u'_{np}$$

in GHz, 2.62 ($TM_{m,1,1}$) ; 3.7 ($TM_{m,2,1}$) ; 4.3 ($TE_{m,1,1}$) ; 4.75 ($TM_{m,3,1}$) ; 5.5 ($TE_{m,2,1}$) ; 5.8 ($TM_{m,4,1}$) ;

5.85 ($TM_{m,1,2}$) ; 6.67 ($TE_{m,3,1}$) ; 6.83 ($TM_{m,5,1}$) ; 7.1 ($TM_{m,4,2}$)

$$Q = 1.01\frac{\eta}{R} = \frac{1.01\times120\pi}{2.61\times10^{-7}\times\sqrt{2.62\times10^{-7}}} = 2.85\times10^5$$

$\boxed{6-4}$ for TM to r modes

$$A_r = \hat{J}_n(kr)P_n^m(\cos\theta)\cos m\phi$$

$$E_\theta = -\frac{1}{\hat{y}r}k\hat{J}_n'(kr)\sin\theta\, P_n^{m\prime}(\cos\theta)\cos m\phi$$

$$E_\phi = -\frac{mk}{\hat{y}r\sin\theta}\hat{J}_n'(kr)P_n^m(\cos\theta)\sin m\phi$$

$$H_\theta = -\frac{m}{r\sin\theta}\hat{J}_n(kr)P_n^m(\cos\theta)\sin m\phi$$

$$H_\phi = \frac{1}{r}\hat{J}_n(kr)P_n^{m\prime}(\cos\theta)\sin m\phi\sin\theta$$

$H_r = 0$ and $\hat{J}_n'(ka) = 0$.

$$W = 2W_m = \mu\int_0^{2\pi}d\varphi\int_0^{\pi}d\theta\int_0^a r^2\sin\theta\, dr\left[H_\theta^2 + H_\phi^2\right]$$

since $$\int_0^\pi\left\{\left[\frac{P_n^m(\cos\theta)\right]^2 m^2}{\sin^2\theta} + P_n^{m\prime}(\cos\theta)\right\}^2\sin\theta\, d\theta = \frac{2n(n+1)}{2n+1}\frac{(n+m)!}{(n-m)!}$$

and $$\int_0^a \hat{J}_n^2(kr)\,dr = \frac{\pi}{2k}\frac{(kr)^2}{2}\left\{J_n^2(kr) - J_{n-1}(kr)J_{n+1}(kr)\right\}$$

$$\therefore W = \frac{\mu\pi}{k}\left[\frac{2n(n+1)}{2n+1}\frac{(n+m)!}{(n-m)!}\right]\left[\frac{ka}{2}\right]\left[1-\frac{n(n+1)}{(ka)^2}\right]$$

$$P_d = R\int_0^{2\pi}d\varphi\int_0^\pi d\theta\; a^2\sin\theta\left[H_\theta^2 + H_\phi^2\right]_{r=a}$$

$$= R\pi\hat{J}_n^2(ka)\frac{2n(n+1)}{2n+1}\frac{(n+m)!}{(n-m)!}$$

$$Q = \frac{\omega W}{P_d} = \frac{\omega\mu}{2kR}\left(=\frac{\eta}{2R}\right)\left[u'_{np} - \frac{(n+1)n}{u'_{np}}\right]$$

where $u'_{np} = ka$

for TE to r modes

$$F_r = \hat{J}_n(kr)P_n^m(\cos\theta)\cos m\phi$$

$E_r = 0$ and $\hat{J}_n(ka) = 0$

$$E_\theta = \frac{m}{r\sin\theta}\hat{J}_n(kr)P_n^m(\cos\theta)\sin m\varphi$$

$$E_\phi = -\frac{1}{r}\hat{J}_n(kr)P_n^{m\prime}(\cos\theta)\sin\theta\cos m\varphi$$

$$H_\theta = -\frac{k}{\hat{z}r}\hat{J}_n'(kr)P_n^{m\prime}(\cos\theta)\sin\theta\cos m\varphi$$

$$H_\phi = -\frac{km}{\hat{z}r\sin\theta}\hat{J}_n'(kr)P_n^m(\cos\theta)\sin m\varphi$$

$$W = 2W_e = \varepsilon\int_0^{2\pi}d\varphi\int_0^\pi d\theta\int_0^a r^2\sin\theta\, dr\left[E_\theta^2 + E_\varphi^2\right]$$

$$W = \frac{\varepsilon\pi}{k}\frac{2n(n+1)}{2n+1}\frac{(n+m)!}{(n-m)!}\frac{ka}{2}\left\{J_n'(ka)\right\}^2$$

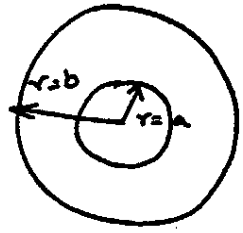
$$P_d = R\pi\hat{J}_n'^2(ka)\frac{2n(n+1)}{2n+1}\frac{(n+m)!}{(n-m)!}$$

$$Q = \frac{\omega\varepsilon}{2kR}\frac{ka}{2} = \frac{\eta}{2R}u_{np}$$

where $u_{np} = ka$

$\boxed{6-5}$ For TE modes

$$E_\theta = \frac{m A}{r \sin\theta}\left[\hat{J}_n(kr) + B\,\hat{N}_n(kr)\right] P_n^m(\cos\theta)\sin m\phi$$



Since $E_\theta = 0$ at $r = a$ & $r = b$, Hence

$$B = -\frac{\hat{J}_n(ka)}{\hat{N}_n(ka)} = -\frac{\hat{J}_n(kb)}{\hat{N}_n(kb)}$$ & so the characteristic eq^n is

$$\frac{\hat{J}_n(ka)}{\hat{J}_n(kb)} = \frac{\hat{N}_n(ka)}{\hat{N}_n(kb)}$$

For TM modes

$$E_\theta = -\frac{k\sin\theta\, A}{j\omega\varepsilon r}\left[\hat{J}_n'(kr) + B\,\hat{N}_n'(kr)\right] P_n^{m\,\prime}(\cos\theta)\cos m\phi$$

Again since $E_\theta = 0$ at $r = a$ & $r = b$, hence

$$B = -\frac{\hat{J}_n'(ka)}{\hat{N}_n'(ka)} = -\frac{\hat{J}_n'(kb)}{\hat{N}_n'(kb)}$$ & so the characteristic eq^n is

$$\frac{\hat{J}_n'(ka)}{\hat{J}_n'(kb)} = \frac{\hat{N}_n'(ka)}{\hat{N}_n'(kb)}$$

$\boxed{6-6}$ Since $\hat{J}_1'(kb)\,\hat{N}_1'(ka) - \hat{J}_1'(ka)\,\hat{N}_1'(kb) = 0$ and

$$\hat{J}_1(x) = \frac{\sin x}{x} - \cos x \qquad \hat{N}_1(x) = -\frac{\cos x}{x} - \sin x$$

$$\hat{J}_1(x) \approx \frac{x^3}{3} \qquad \hat{N}_1(x) \approx -\frac{1}{x}$$

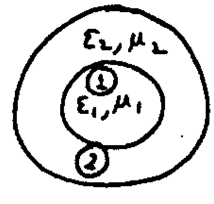
$$\hat{J}_1'(x) \approx \frac{2x}{3} \qquad \hat{N}_1'(x) \approx \frac{1}{x^2}$$

Also, ~~Hence~~ $J_1'(kb) \approx (k - k_0)\, b\, J_1''(k_0 b)$

$$(k - k_0)\, b\, J_1''(k_0 b)\,\frac{1}{k_0^2 a^2} = \frac{2}{3}\, k_0 a\, \hat{N}_1'(k_0 b)$$

$$\frac{\omega - \omega_0}{\omega_0} = \frac{2}{3}(k_0 b)^2\,\frac{\hat{N}'(k_0 b)}{\hat{J}_1''(k_0 b)}\left(\frac{a}{b}\right)^3$$ where $k_0 b = 2.744$

$\boxed{6-7}$ For the dominant mode ; $n=1$, $m=0$.



Hence,

$$E_{\theta_1} = -\frac{C_1 k_1 \sin\theta}{j\omega\varepsilon_1 r} \hat{J}_n'(k_1 r) P_n^{m\prime}(\cos\theta)\cos m\phi$$

$$E_{\theta_2} = -\frac{C_2 k_2 \sin\theta}{j\omega\varepsilon_2 r}\left[\hat{J}_n'(k_1 r) - \frac{\hat{J}_n'(k_2 b)}{\hat{N}_n'(k_2 b)}\hat{N}_n'(k_1 r)\right] \cdot P_n^{m\prime}(\cos\theta)\cos m\phi$$

and

$$H_{\phi_1} = \frac{C_1 \sin\theta}{r} \hat{J}_n(k_1 r) P_n^{m\prime}(\cos\theta)\cos m\phi$$

$$H_{\phi_2} = \frac{C_2 \sin\theta}{r}\left[\hat{J}_n(k_2 r) - \frac{\hat{J}_n'(k_2 b)}{\hat{N}_n'(k_2 b)} N_n(k_2 r)\right] P_n^{m\prime}(\cos\theta)\cos m\phi$$

Appling continuity of E_θ and H_ϕ at $r=a$

$$\frac{C_1 k_1}{\varepsilon_1}\hat{J}_n'(k_1 a) = \frac{C_2 k_2}{\varepsilon_2}\left[\hat{J}_n'(k_2 a) - \frac{\hat{J}_n'(k_2 b)}{\hat{N}_n'(k_2 b)}\hat{N}_n'(k_2 a)\right]$$

$$C_1 \hat{J}_n(k_1 a) = C_2\left[\hat{J}_n(k_2 a) - \frac{\hat{J}_n'(k_2 b)}{\hat{N}_n'(k_2 b)}\hat{N}_n(k_2 a)\right]$$

Eliminiation of C_1 & C_2 leads to (for $m=0$, $n=1$)

$$\frac{\hat{N}_1'(k_2 b)\hat{J}_1'(k_2 a) - \hat{J}_1'(k_2 b)\hat{N}_1'(k_2 a)}{\hat{N}_1'(k_2 b)\hat{J}_1(k_2 a) - \hat{J}_1'(k_2 b) N_1(k_2 a)} = \frac{\hat{J}_1'(k_2 a)}{\hat{J}_1(k_2 a)}\cdot\frac{\eta_1}{\eta_2}$$

$\boxed{6.8}$ $\hat{J}_1(k_1 a)\left[\hat{N}_1'(kb)\hat{J}_1'(ka) - \hat{J}_1'(kb)\hat{N}_1'(ka)\right]$

$$= \sqrt{\frac{\mu_1\varepsilon_0}{\varepsilon_1\mu_0}}\hat{J}_1'(k_1 a)\left[\hat{N}_1'(kb)\hat{J}_1(ka) - \hat{J}_1'(kb)\hat{N}_1(ka)\right]$$

Since $\hat{J}_1(x) \approx \frac{x^2}{3}$; $\hat{J}_1'(x) \approx \frac{2x}{3}$; $\hat{N}_1(x) = -\frac{1}{x}$, $\hat{N}_1'(x) \approx \frac{1}{x^2}$

and $\hat{J}_1'(kb) \approx (k-k_0)\, b\, J_1''(k_0 b)$ where $k = \omega\sqrt{\mu_0\varepsilon_0}$ and $k_0 = \omega_0\sqrt{\mu_0\varepsilon_0}$

then $$\left(\frac{k_1 a}{3}\right)^2\left[\hat{N}_1'(k_0 b)\frac{2}{3}k_0 a - (k-k_0)\, b\,\frac{\hat{J}_1''(k_0 b)}{(k_0 a)^2}\right]$$

$$= \sqrt{\frac{\mu_1\varepsilon_0}{\varepsilon_1\mu_0}}\,\frac{2}{3}(k_1 a)\left[\hat{N}_1'(k_0 b)\frac{(k_0 a)^2}{3} + \frac{(k-k_0)\, b\,\hat{J}_1''(k_0 b)}{k_0 a}\right]$$

Hence

$$\frac{\omega-\omega_0}{\omega_0} = \frac{2}{3}(k_0 b)^2\,\frac{\hat{N}_1'(k_0 b)}{\hat{J}_1''(k_0 b)}\,\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\left(\frac{a}{b}\right)^3 \quad \text{where } k_0 b = 2.744$$

$\boxed{6-9}$ given $f(\theta,\phi) = \begin{cases} 1 & 0<\theta<\pi/2 \\ 0 & \pi/2<\theta<\pi \end{cases}$

A Fourier Legendre series for a function $f(\theta,\phi)$ on a spherical surface is given as

$$f(\theta,\phi) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} (a_{mn}\cos m\phi + b_{mn}\sin m\phi)\, P_n^m(\cos\theta)$$

where $a_{on} = \frac{2n+1}{4\pi}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, f(\theta,\phi)\, P_n(\cos\theta)$

$$= \frac{2n+1}{2}\int_0^{\pi/2} P_n(\cos\theta)\, d\theta$$

$$a_{mn} = \frac{2n+1}{2\pi}\,\frac{(n-m)!}{(n+m)!}\int_0^{2\pi} d\phi\int_0^{\pi} d\theta\, f(\theta,\phi)\, P_n^m(\cos\theta)\cos m\phi\,\sin\theta$$

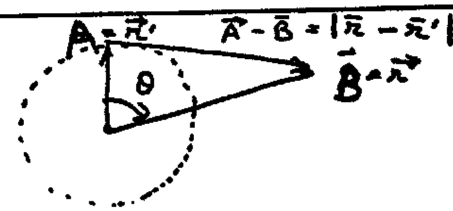
$$= \frac{2n+1}{2\pi}\,\frac{(n-m)!}{(n+m)!}\int_0^{2\pi}\cos m\phi\, d\phi\int_0^{\pi/2} P_n{}^m(\cos\theta)\sin\theta\, d\theta = 0$$

$$b_{mn} = \frac{2n+1}{2\pi}\,\frac{(n-m)!}{(n+m)!}\int_0^{2\pi} d\phi\int_0^{\pi} d\theta\, f(\theta,\phi)\, P_n^m(\cos\theta)\sin m\phi\,\sin\theta$$

$$= \frac{2n+1}{2\pi}\,\frac{(n-m)!}{(n+m)!}\int_0^{2\pi}\sin m\phi\, d\phi\int_0^{\pi/2} P_n^m(\cos\theta)\sin\theta\, d\theta = 0.$$

Hence $a_{00} = \frac{1}{2}\cdot\frac{\pi}{2}$ $\quad a_{01} = \frac{3}{2}$ & so on ....

$\boxed{6-10}$ Consider a point charge at $\bar{A}$ and field at $\bar{B}$, hence



$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{\sqrt{A^2+B^2-2AB\cos\theta}} \quad \cdots (1)$$

outside sphere of radius A,

$$\frac{1}{|\vec{r}-\vec{r}'|} = \sum c_n r^n P_n(\cos\theta) = \sum_{n=0}^{\infty} c_n B^{-(n+1)} P_n(\cos\theta) \quad \cdots (2)$$

Let $\theta = 0$ in (1) and (2) expand (1) in Taylor series in $\frac{B}{A}$, i.e.

(1) $\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{B-A} = \frac{1}{B}\left[1+\frac{A}{B}+\frac{A^2}{B^2}+\cdots\right]$

(2) $\frac{1}{|\vec{r}-\vec{r}'|} = \frac{c_0}{B}+\frac{c_1}{B^2}+\frac{c_2}{B^3}+\cdots$ $\quad$ Equating (1) to (2)

$c_0 = 1,\ c_1 = A,\ c_2 = A^2,\ \cdots\cdots$

$$\therefore \frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{\sqrt{A^2+B^2-2AB\cos\theta}} = \frac{1}{B}\sum^{\infty}\left(\frac{A}{B}\right)^n P_n(\cos\theta)$$

To $\boxed{6-11}$ $Z_{+r}^{TM} = j\eta \dfrac{\hat{H}_n^{(2)\prime}(kr)}{\hat{H}_n^{(2)}(kr)}$

$$= j\eta \frac{\hat{J}_n'(kr) - j\hat{N}_n'(kr)}{\hat{J}_n(kr) - j\hat{N}_n(kr)}$$

$$Z_{-r}^{TM} = -j\eta \frac{\hat{H}_n^{(1)\prime}(kr)}{\hat{H}_n^{(1)}(kr)}$$

$$= -j\eta \frac{\hat{J}_n'(kr) + j\hat{N}_n'(kr)}{\hat{J}_n(kr) + jN_n(kr)}$$

& hence $Z_{+r}^{TM} = \left(Z_{-r}^{TM}\right)^*$

assuming η & k to be real.

$$\lim_{kr\to\infty} Z_{+r}^{TM} = j\eta \lim_{kr\to\infty} \frac{H_n^{(2)\prime}(kr)}{H_n^{(2)}(kr)}$$

if $z = kr$

$$= j\eta \frac{\sqrt{2/(\pi z)}\,(-j)\,e^{-jz - \frac{n\pi}{2} - \pi/4}}{\sqrt{2/(\pi z)}\; e^{-jz - n\pi/2 - \pi/4}}$$

$$= \eta$$

$$\therefore \lim_{kr\to\infty} Z_{+r}^{TM} = \eta$$

Since, $\lim_{z\to 0} H_\nu^{(2)}(z) \Rightarrow \dfrac{j}{\pi}\Gamma(\nu)\left(\dfrac{z}{2}\right)^{-\nu}$

$$\lim_{z\to 0} H_\nu^{(2)\prime}(z) \Rightarrow \frac{j}{\pi}\Gamma(\nu)(-\nu)\left(\frac{z}{2}\right)^{-\nu-1}$$

$$\therefore \lim_{kr\to 0} Z_{+r}^{TM} = j\eta \frac{(kr)^{-\nu-1}}{(kr)^{-\nu}}(-\nu)$$

$$= -j\nu\eta/(kr)$$

Also since

$$Z_{+r}^{TE} = -j\eta \frac{H_n^{(2)}(kr)}{H_n^{(2)\prime}(kr)}$$

& $Z_{+r}^{TM} = j\eta \dfrac{H_n^{(2)\prime}(kr)}{H_n^{(2)}(kr)}$

$$\therefore Z_{+r}^{TE} \cdot Z_{+r}^{TM} = \eta^2$$

$\boxed{6.12}$ The TM to r modes are given by

$$(A_r)_{m\nu} = P_\nu^m(\cos\theta)\begin{Bmatrix}\cos m\phi \\ \sin m\phi\end{Bmatrix}\hat{H}_\nu^{(2)}(kr)$$

[from (6.61)]

For the dominant mode $m = 0$ and for $\nu = 1$ the spatial TM mode is the field of an electric current element as shown in (6.85) on p. 287.

Similarly for TE to r modes the potential is given by

$$(F_r)_{m\nu} = P_\nu^m(\cos\theta)\begin{Bmatrix}\cos m\phi \\ \sin m\phi\end{Bmatrix}\hat{H}_1^{(2)}(kr)$$

[from (6.63)]

For the dominant mode $m = 0$ and for $\nu = 1$, the spatial TE mode which is the field of a magnetic current element (This is dual of (6.85) on p. 287.

6-13 The field components are given by

$$E_\theta^{\mp} = \frac{jk}{\omega\varepsilon r \sin\theta} e^{\pm jkr}$$

$$H_\phi^{\mp} = \mp \frac{j}{r\sin\theta} e^{\pm jkr}$$

where are the upper signs refer to inward-travelling waves and the lower signs to outward-travelling waves.

$$\therefore P_f = \int_0^{2\pi} r\sin\theta\, d\phi \int_{\theta_1}^{\theta_2} r\,d\theta \frac{jk}{\omega\varepsilon r\sin\theta} e^{(\pm jkr)} \frac{(-j)}{r\sin\theta} e^{(\pm jkr)}$$

$$= \frac{2\pi k}{\omega\varepsilon}\int_{\theta_1}^{\theta_2}\frac{d\theta}{\sin\theta} = \frac{2\pi k}{\omega\varepsilon}\left(\log\tan\frac{\theta}{2}\right)_{\theta_1}^{\theta_2} = 2\pi\eta\log\frac{\tan\frac{\theta_2}{2}}{\tan\frac{\theta_1}{2}}$$

$$P_d = \int_0^{2\pi} r\sin\theta_1\, d\phi \frac{R}{r^2\sin^2\theta_1} + \int_0^{2\pi} r\sin\theta_2\, d\phi\frac{R}{\sin^2\theta_2\, r^2} = \frac{2\pi R}{r}\left[\operatorname{cosec}\theta_1 + \operatorname{cosec}\theta_2\right]$$

$$\therefore \alpha = \frac{P_d}{2P_f} = \frac{R}{2\eta r}\frac{[\operatorname{cosec}\theta_1 + \operatorname{cosec}\theta_2]}{\log\frac{\cot\theta_1/2}{\cot\theta_2/2}}$$

6-14 The dominant TE mode for spherical wedge wave guide is given by $(F_r)_{n\omega} = P_n^{\omega}(\cos\theta)\cos\omega\phi\,\hat{H}_n^{(2)}(kr)$, where $\omega = \frac{p\pi}{\phi_1}$ and $p=0$ and $n=1$ is the dominant mode. Hence,

$$(F_r)_{n\omega} = \cos\theta\,\hat{H}_1^{(2)}(kr)\,; \quad \therefore E_\phi = -\frac{\sin\theta}{r}\hat{H}_1^{(2)}(kr)$$

$$H_\theta = -\frac{k}{\hat{z} r}\hat{H}_1^{(2)\prime}(kr)\sin\theta.$$

For the large argument approximation of $\hat{H}_1^{(2)}(kr)$ and $\hat{H}_1'(kr)$ one ~~ohe~~ obtains

$$E_\phi \propto \frac{\sin\theta}{r}e^{-jkr}$$

$$H_\theta \propto \eta\frac{\sin\theta}{r}e^{-jkr}$$

which are the dual fields of an electric dipole given by (2-114).

6-15 The field components for the spherical horn guide may be written as $H_r = f(\theta)\left[\hat{H}_\nu^{(1)}(kr) + B\hat{H}_\nu^{(2)}(kr)\right]\cos\nu\phi$. If we think of the lowest order mode propagating radially inward it would be quite similar to the TE_{10} mode of a rectangular guide, although modified by the convergence of the sides. We would consequently expect a cut-off phenomenon at such a radius r_c that the width $r_2\phi_1$ becomes half a wavelength. $\therefore \phi_1 r_c = \lambda/2$. A very effective cut-off phenomenon at about the radius r_c the reactive energy for a given power transfer becomes very great for radii less than this.

The radial impedances $\quad Z_r^{+}(kr) = j\eta\dfrac{H_\nu^{(2)}(kr)}{H_\nu^{(2)\prime}(kr)}\quad$ and

$Z_r^-(kr) = -j\eta \dfrac{H_\nu^{(2)\prime}(kr)}{H_\nu^{(1)\prime}(kr)}$ become predominantly reactive at a value $kr \approx \nu$ which is compatible with $\phi_1 r_c = \lambda/2$. Circumferential modes might exist for radial lines greater than a wavelength in maximum circumference.

$\boxed{6-16}$ The first ten resonant frequencies are

$$(f_r)_{mnp} = c\,\frac{u_{np} \text{ or } u'_{np}}{2\pi a} = \frac{3\times10^{10}}{20\pi}\,u_{np} \text{ or } u'_{np}$$

$= 1.31$ GHz ; 1.85 ; 2.15 ; 2.38 ; 2.76 ; 2.9 ; 2.92 and 3.34.

$$Q_{\text{dominant mode}} = \frac{0.573\times120\pi}{2.61\times10^{-7}\times\sqrt{1.31\times10^{9}}} = 2.29\times10^{4}$$

$\boxed{6-17}$ Now, $\omega_r = \dfrac{2\pi c}{2a} = \dfrac{\pi c}{a}$ $\quad\therefore \lambda_c = 2a$

$\therefore k = \dfrac{2\pi}{\lambda} = \pi/a$ and $Q = 0.35\,\dfrac{\eta}{\mathcal{R}} = 1.4\times10^{4}$

$\cdot Z_{in} = jZ_0 \tan ka = jZ_0\tan\pi = 0$

$\boxed{\ 18}$ The following results can be obtained from (6-26), (6-30) and ~~problem~~ Equations (6-33).

$\boxed{6-19}$ The result for this case is the dual of eq^n (6-86) and the field is given by $\bar{E} = -\bar{\nabla}\times\bar{u}_z F_z$

$\boxed{6-20}$ Since the potential from a current element ~~dipole~~ source is $\left[\dfrac{kI\ell}{4\pi j}\,h_0^{(2)}(kr) = A_z\right]$

Hence A'_z for the quadrupole source $= s_1 s_2 \dfrac{\partial^2}{\partial z^2}(A_z)$

$$A'_z = \frac{k^3 s_1 s_2 I\ell}{4\pi j}\left[h_0''^{(2)}(kr)\cos^2\theta + \frac{h_0'(kr)}{k}(kr)\;\frac{r - z^2/r}{r^2}\right]$$

Since $h_0'^{(2)} = h_1^{(2)}$ $\quad\left[-h_1'^{(2)}(kr)\right] = \frac{1}{3}\left[h_0^{(2)}(kr) - 2h_2^{(2)}(kr)\right]$

$$\frac{h_1^{(2)}(kr)}{kr} = \frac{1}{3}\left[h_0^{(2)}(kr) + h_2^{(2)}(kr)\right]$$

$$\therefore A'_z = -\frac{k^3 s_1 s_2 I\ell}{12\pi j}\left[h_0^{(2)}(kr) + h_2^{(2)}(kr)\{\sin^2\theta - 2\cos^2\theta\}\right]$$

Also $P_2(\cos\theta) = \frac{1}{4}[3\cos2\theta + 1] = [\cos^2\theta - \sin^2\theta/2]$

$$\therefore A'_z = \frac{k^3 s_1 s_2 I\ell}{12\pi j}\left[-h_0^{(2)}(kr) + 2P_2(\cos\theta)\,h_2^{(2)}(kr)\right]$$

$\boxed{6-21}$ From eq^n (6-84) hence

$$\frac{e^{-j|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|} = \frac{h_0^{(2)}(|\bar{r}-\bar{r}'|)}{j} \quad \text{and.}$$

$$h_0^{(2)}(|\bar{r}-\bar{r}'|) = \begin{cases} \sum_{n=0}^{\infty} (2n+1) h_n^{(2)}(r') j_n(r) P_n(\cos\xi) & r < r' \\ \sum_{n=0}^{\infty} (2n+1) j_n(r') h_n^{(2)}(r) P_n(\cos\xi) & r > r' \end{cases}$$

Also, $j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2x}} \sqrt{\frac{2}{\pi x}} \hat{J}_n(x)$ from (D-20) & (D-21)

and $h_n^{(2)}(x) = \frac{\hat{H}_n^{(2)}(x)}{x}$

$$\therefore \frac{e^{-j|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|} = \frac{1}{jrr'} \sum_{n=0}^{\infty} (2n+1) \hat{J}_n(r') H_n^{(2)}(r) P_n(\cos\xi), \quad r > r'$$

where ξ is the angle between r and r'

$\boxed{6-22}$ The integral representation of wave functions that are finite on the axis is

$$\left.\begin{matrix}\cos\\ \sin\end{matrix}\right\} m\phi \, J_m(\lambda r) e^{-jhz} = \frac{j^m}{2\pi} \int_0^{2\pi} e^{-j\lambda r\cos(\beta-\phi) - jhz} \left.\begin{matrix}\cos\\ \sin\end{matrix}\right\} m\beta \, d\beta$$

[Stratton p. 412, Eq^n 80]

Now $\bar{k}.\bar{R} = \lambda\cos(\beta-\phi) - jhz = kR[\sin\alpha\sin\theta\cos(\phi-\beta) + \cos\alpha\cos\theta]$

Also $e^{-jkR\cos\gamma} = \sum_{n=0}^{\infty} (-j)^n (2n+1) j_n(kR) \Big[P_n(\cos\alpha) P_n(\cos\theta)$

$$+ 2 \sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} P_n^m(\cos\alpha) P_n^m(\cos\theta) \cos m(\phi-\beta)\Big]$$

[Stratton p. 409, Eq^n 57]

Upon integration with respect to β

$$\left.\begin{matrix}\cos\\ \sin\end{matrix}\right\} m\phi \, J_m(kr) e^{-jhz} = \sum_{n=0}^{\infty} (-j)^{n-m} (2n+1) \frac{(n-m)!}{(n+m)!} \left.\begin{matrix}\cos\\ \sin\end{matrix}\right\} m\phi \, P_n^m(\cos\alpha) P_n^m(\cos\theta) j_n(kR).$$

Since P_n^m vanishes if $m > n$ the first m terms of the above equation are zero and the expansion can be written

$$J_m(\lambda r) e^{-jhz} = \sum_{n=0}^{\infty} (-j)^n (2n+2m+1) \frac{n!}{(n+2m)!} P_{n+m}^m(\cos\alpha) P_{n+m}^m(\cos\theta) j_{n+m}(kR)$$

then $\alpha = \pi/2$, $h=0$, $\lambda = k$ and

$$P_{n+m}^m(0) = 0, \text{ if } n \text{ is odd}$$

$$= \frac{(n+2m-1)!}{2^{n+m-1} (\frac{n}{2})! (\frac{n}{2}+m-1)!}; \text{ if } n \text{ is even}$$

The result is an expansion of a cylindrical Bessel function in a series of

94 of spherical Bessel function

$$J_m(kr) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{2^{2\ell+m-1}} \frac{(4\ell+2m-1)}{(2\ell+2m)} \frac{(2\ell)!}{\ell!\,(\ell+m-1)!} P_{2\ell+m}^{m}(\cos\theta)\, j_{2\ell+m}(kR)$$

$\boxed{6-23}$ Since

$$h_0^{(2)}(|\bar{r}-\bar{r}'|) = \begin{cases} \sum_{n=0}^{\infty} (2n+1) h_n^{(2)}(r') j_n(r) P_n(\cos\xi) & r < r' \\ \sum_{n=0}^{\infty} (2n+1) j_n(r') h_n^{(2)}(r) P_n(\cos\xi) & r > r' \end{cases}$$

[p.292, (6.94) Harrington]

Then multiplying both sides by $[P_0(\cos\xi) = 1]$ and integrating from -1 to $+1$.

$$\int_{-1}^{+1} h_0^{(2)}(|r-r'|)\, d(\cos\xi) = \begin{cases} \sum_{n=0}^{\infty} (2n+1)\frac{2}{(2n+1)} h_0^{(2)}(r') j_0(r) & ; r < r' \\ \sum_{n=0}^{\infty} 2\, j_0(r')\, h_0^{(2)}(r) & ; r > r' \end{cases}$$

Since $\int_0^{\pi} P_n(\cos\theta) P_\ell(\cos\theta) \sin\theta\, d\theta = 0$ for $n \neq \ell$
$= \frac{2}{2n+1}$ for $n = \ell$

$\boxed{6-24}$ As seen from equation (6.109) on p-296 for the dominant $n=1$ mode and for $\theta = 60°$, $E_\theta^s = 0$, and $E_\phi^s \neq 0$ and hence the wave is <u>linearly polarized</u>.

For a general situation, when $n > 1$, the scattering of a plane-polarized wave by a small conducting sphere no longer produces a linearly polarized wave at $\theta = 60°$, since E_θ^S is no longer zero.

$\boxed{6-25}$

"Scattering of Electromagnetic Waves from Two Concentric Spheres", by A.L. Aden and M. Kerker, Journal of Applied Physics, Volume 22, Number 10, October 1951

$\boxed{6-26}$ The incident wave is $H_x^i = H_0 e^{-jkr'\cos\theta'}$

$$\therefore\ H_r^i = H_0 \frac{\cos\varphi'}{jkr} \frac{\partial}{\partial\theta'}\left(e^{-jkr\cos\theta'}\right)$$

where $H_0 = -\dfrac{j\omega\varepsilon Kl}{4\pi r} e^{-jkr}$ (dual of 6-116)

$$\text{or } H_r^i = -\frac{jH_0\cos\varphi'}{(kr')^2}\sum_{n=1}^{\infty} j^{-n}(2n+1)\hat{J}_n(kr')P_n^1(\cos\theta') \quad \text{(from p. 293)}$$

Hence the electric vector potential is

$$F_r^i = \frac{H_0}{\omega\varepsilon}\cos\varphi'\sum_{n=1}^{\infty} a_n \hat{J}_n(kr')P_n^1(\cos\theta')$$

and $F_r^s = \dfrac{H_0}{\omega\varepsilon}\cos\varphi'\displaystyle\sum_{n=1}^{\infty} c_n \hat{H}_n^{(2)}(kr')P_n^1(\cos\theta')$. Hence the total potential

$$F = \frac{H_0}{\omega\varepsilon}\cos\varphi'\sum_{n=1}^{\infty}\left[a_n\hat{J}_n(kr') + c_n H_n^{(2)}(kr')\right]P_n^1(\cos\theta')$$

where $c_n = -a_n \dfrac{\hat{J}_n(ka)}{\hat{H}_n^{(2)}(ka)}$ since $E_\varphi = E_\theta = 0$ at $r = a$.

$\therefore\ \bar{H} = \dfrac{1}{j\omega\mu}\left(\dfrac{\partial^2}{\partial r'^2} + k^2\right)\bar{F}$. Since, $\left(\dfrac{\partial^2}{\partial r'^2} + k^2\right)\hat{B}_n(kr') = \dfrac{n(n+1)}{b^2}\hat{B}_n(kr')$

$$\therefore\ H_{r'b} = \frac{H_0\cos\varphi'}{j\,k^2b^2}\sum_{n=1}^{\infty} n(n+1)\left[a_n\hat{J}_n(kr') + c_n\hat{H}_n^{(2)}(kr')\right]P_n^1(\cos\theta')$$

(similar to 6-117)

at $r' = b$, $\theta' = \pi - \theta$, $\varphi' = 0$, $H_{r'b}$ equals $-H_\theta^a$.

Hence

$$H_{r'b} = \frac{jH_0}{k^2b^2}\sum_{n=1}^{\infty} n(n+1)\left[a_n\hat{J}_n(kb) + c_n\hat{H}_n^{(2)}(kb)\right](-1)^n P_n'(\cos\theta)$$

$$= \frac{Kl}{4\pi k\eta b^2}\frac{e^{-jkr}}{r}\sum_{n=1}^{\infty} n(n+1)\left[a_n\hat{J}_n(kb) + c_n\hat{H}_n^{(2)}(kb)\right](-1)^n P_n'(\cos\theta)$$

$\boxed{6-27}$ For the case of a dielectric sphere there are two regions, one region $r<a$ characterized by ε_d, μ_d and the region $r>a$ by ε_0, μ_0.

From the continuity of the tangential electric and magnetic fields we find that the fields external to the dielectric sphere will be given by (6-117) ~~by b_n, c_n~~. The potentials internal and external to the spheres are given by (6-101) and (6.112). The coefficients then are given by (6-113). Then the radiation fields are given by (6-117) and b_n by (6-113).

$\boxed{6-28}$ As $R \to 0$ of problem (6-29)

$$F_r = \sum_n b_n J_n(kr) \quad \text{for } r < a.$$

& hence $A_n^{-1} = \left[\hat{H}_n^{(2)\prime}(ka) - \dfrac{\hat{J}_n'(ka)}{\hat{J}_n(ka)} \hat{H}_n^{(2)}(ka) \right]$

$$= -\frac{j}{\hat{J}_n(ka)}$$

and the results of (6.29) applies with A_n replaced by $\hat{J}_n(ka)$.

$$\therefore A_n = \frac{\hat{J}_n(ka)}{-j}$$

$$\therefore E_\phi = \frac{\eta I e^{-jkr}}{jr} \sum_n j^n \hat{J}_n(ka) P_n^{(1)}(0) P_n^{(1)}(\cos\theta) \frac{2n+1}{2n(n+1)}$$

$\boxed{6-29}$ The electric vector potential F can be written as

$$F = \begin{cases} \sum_n a_n \hat{H}_n^{(2)}(kr) P_n(\cos\theta) & \text{for } r > a \\ \sum_n b_n \left[\hat{J}_n(kr) - \dfrac{\hat{J}_n(kR)}{\hat{N}_n(kR)} \hat{N}_n(kr) \right] P_n(\cos\theta) & \text{for } r < a \end{cases}$$

This takes into account the boundary conditions at $r = R$

From the continuity of E_ϕ at $r = a$.

$$a_n H_n^{(2)}(ka) = b_n \left[\hat{J}_n(ka) - \frac{\hat{J}_n(kR)}{\hat{Y}_n(kR)} \hat{Y}_n(ka) \right] \quad \cdots\cdots (A)$$

Finally H_θ at $r = a$ must be discontinuous by an amount equal to the surface-current density. Thus from (A) & the equation of current density

$$J_\phi = \frac{k}{j\omega\mu a} \sum_n \frac{\partial}{\partial\theta} P_n(\cos\theta) a_n \left[\hat{H}_n^{(2)\prime}(ka) - \frac{\hat{J}_n'(ka)\hat{N}_n(kR) - \hat{J}_n(kR)\hat{N}_n'(ka)}{\hat{J}_n(ka)\hat{N}_n(kR) - \hat{J}_n(kR) N_n(ka)} \hat{H}_n^{(2)}(ka) \right]$$

$$\triangleq \frac{1}{j\eta a} \sum_n \frac{\partial}{\partial\theta} P_n(\cos\theta) \frac{a_n}{A_n} \qquad \text{Since } J_\phi = \frac{I}{a} \delta(\theta - \pi/2) \text{ and}$$

$$\int_0^\pi J_\phi P_n^{(1)}(\cos\theta) \sin\theta \, d\theta = \frac{1}{j\eta a} \int_0^\pi \left[P_n^{(1)}(\cos\theta)\right]^2 \sin\theta \, d\theta \cdot \frac{a_n}{A_n}$$

$$a_n = \frac{j A_n \eta I P_n^{(1)}(0)}{\dfrac{2n(n+1)}{(2n+1)}} \quad \text{and} \quad F_r = \sum_n \frac{2n+1}{2n(n+1)} A_n \eta I P_n^{(1)}(0) P_n(\cos\theta) \hat{H}_n^{(2)}(kr) \text{ for } r > a$$

$E_\phi = \dfrac{1}{r} \dfrac{\partial F_r}{\partial\theta}$ and using the large argument approximation of $\hat{H}_n^{(2)}(kr)$ $\left[= j^{n+1} e^{-jkr} \right]$

$$= -\eta I e^{-jkr}/r \sum_n j^n A_n P_n^{(1)}(0) P_n^{(1)}(\cos\theta) \frac{2n+1}{2n(n+1)}.$$

$\boxed{6-30}$ For a magnetic current the potential is given by

$$A_r = \sum_u c_u P_u(\cos\theta)\,\hat{H}_u^{(2)}(kr) \quad \text{for } r>a \qquad (6.137)$$

where u are ordered solutions to $P_u(\cos\theta_1)=0$ (6.138)

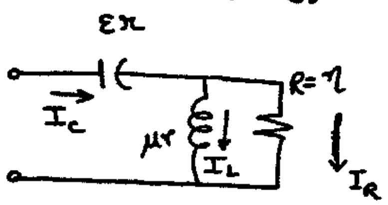
and $c_u = -\dfrac{k}{\eta M_u}\hat{J}_u(ka)\sin\theta_2 \dfrac{\partial}{\partial\theta_2}P_u(\cos\theta_2)$ . . . (6.145)

and $M_u = \dfrac{u(u+1)}{2u+1}\left[\sin\theta \dfrac{\partial P_u}{\partial\theta}\ \dfrac{\partial P_u}{\partial u}\right]_{\theta=\theta_1}$ (6.142)

In the limit as $a\to 0$, $\hat{J}_u(ka)\to 1$ only for the first u and for the rest u's $\hat{J}_u(ka)\to 0$ and hence

$E_\theta = \dfrac{1}{\hat{y}r}\dfrac{\partial^2 A_r}{\partial r\partial\theta} = \eta f(r)\sin\theta\, P_u'(\cos\theta)$ only for the first u of $P_u(\cos\theta_1)=0$

$\boxed{6-31}$ For TM modes the equivalent circuit is for the $n=1$ mode



$I_c = I_L + I_R$ and $j\omega\mu r I_L = \eta I_R$

$\therefore I_R = \dfrac{j\omega\mu r}{\eta} I_L$

$\therefore I_R = \dfrac{I_c}{1+\frac{1}{jkr}}$

since $\omega\mu = k\eta$ and $\omega\varepsilon\eta = k$

Now $\bar{P}_d = \eta I_R I_R^* = \dfrac{\eta\,|I_c|^2}{1+\frac{1}{k^2r^2}}$

and $\bar{W}_e = \frac{1}{2}CV_cV_c^* = \frac{1}{2}\varepsilon r\dfrac{|I_c|^2}{\omega^2\varepsilon^2r^2} = \dfrac{|I_c|^2}{2\omega^2\varepsilon r} = \dfrac{|I_c|^2\eta}{2kr\omega}$

$\therefore Q = \dfrac{2\omega\bar{W}_e}{\bar{P}_d}$ since $\bar{W}_e > \bar{W}_m$

$= \dfrac{2\omega|I_c|^2\eta}{2kr\omega\,\eta|I_c|^2}\left[1+\dfrac{1}{k^2r^2}\right] = \dfrac{1}{kr}+\dfrac{1}{k^3r^3}\Big|_{r=a}$

7-1

Since $\quad -\bar{\nabla}\times\bar{E}_0 = j\omega_0\mu\bar{H}_0 \quad - \text{I} \qquad -\bar{\nabla}\times\bar{E} = j\omega\mu\bar{H} \quad - \text{III}$

$\quad \bar{\nabla}\times\bar{H}_0 = (\sigma + j\omega_0\varepsilon)\bar{E}_0 \quad - \text{II} \qquad \bar{\nabla}\times\bar{H} = (\sigma + j\omega\varepsilon)\vec{E} \quad - \text{IV}$

Multiplying (IV) by $\bar{E}_0$ and conjugate of (I) by $\vec{H}$ we obtain

$$\vec{E}_0\cdot(\bar{\nabla}\times\bar{H}) = (\sigma + j\omega\varepsilon)\vec{E}\cdot\vec{E}_0 \quad \text{and}$$

$$-\bar{H}\cdot(\bar{\nabla}\times\bar{E}_0) = +j\omega_0\mu\bar{H}\cdot\bar{H}_0 \quad . \quad \text{Since}$$

$$\bar{\nabla}\cdot(\bar{H}\times\bar{E}_0) = \bar{E}_0\cdot\bar{\nabla}\times\bar{H} - \bar{H}\cdot\bar{\nabla}\times\bar{E}_0 = (\sigma + j\omega\varepsilon)\bar{E}\cdot\bar{E}_0 + j\omega_0\mu\bar{H}\cdot\bar{H}_0 \quad \cdots (A)$$

Similarly $\bar{\nabla}\cdot(\bar{H}_0\times\bar{E}) = \bar{E}\cdot\bar{\nabla}\times\bar{H}_0 - \bar{H}_0\cdot\bar{\nabla}\times\bar{E} = (\sigma + j\omega_0\varepsilon)\bar{E}\cdot\bar{E}_0 + j\omega\mu\bar{H}\cdot\bar{H}_0 \cdots (B)$

Subtracting (B) from (A)

$$\nabla\cdot(\bar{H}\times\bar{E}_0) - \bar{\nabla}\cdot(\bar{H}_0\times\bar{E}) = -j(\omega_0-\omega)\,\varepsilon\bar{E}\cdot\bar{E}_0 - j(\omega-\omega_0)\mu\,\bar{H}\cdot\bar{H}_0$$

Taking $\iiint_{\tau'}$ on both sides

$\iiint_{\tau'}\nabla\cdot(\bar{H}_0\times\bar{E})\,d\tau = 0$, since $n\times\bar{E} = 0$ on S' $\left(\because \iint_{S'}\bar{E}\times\bar{H}_0\,dS = 0\right)$

$$\iiint_{\tau'}\nabla\cdot(\bar{H}\times\bar{E}_0)\,d\tau = \iint_{S'}\bar{H}\times\bar{E}_0\,dS = \left(\iint_S - \iint_{\Delta S}\right)(\bar{H}\times\bar{E}_0\,dS) = -\iint_{\Delta S}(\bar{H}\times\bar{E}_0)\,dS \quad \left[\text{Since } (n\times\bar{E}_0) = 0 \text{ on } S\right]$$

So $\iint_{\Delta S}\bar{H}\times\bar{E}_0\,dS = \iiint_{\tau'} j(\omega-\omega_0)\left[\mu\bar{H}\cdot\bar{H}_0 - \varepsilon\bar{E}\cdot\bar{E}_0\right]d\tau$

$$\therefore \quad \omega - \omega_0 = \frac{j\iint_{\Delta S}\bar{H}\times\bar{E}_0\,dS}{\iiint_{\tau'}\left[\varepsilon\bar{E}\cdot\bar{E}_0 - \mu\bar{H}\cdot\bar{H}_0\right]d\tau}$$

7-2 from perfectly conducting wall $\Rightarrow$ having a wall impedance of Z

hence $\bar{n}\times\bar{E} = Z\bar{H}_t$. From previous problem

$$-\nabla\cdot(\bar{H}_0\times\bar{E}) + \bar{\nabla}\cdot(\bar{H}\times\bar{E}_0) = j(\omega-\omega_0)\mu\bar{H}\cdot\bar{H}_0 - j(\omega-\omega_0)\varepsilon\bar{E}\cdot\bar{E}_0$$

Taking the volume integral $\iiint_{\tau'}\nabla\cdot\left[-\bar{H}_0\times\bar{E} + \bar{H}\times\bar{E}_0\right]d\tau = \oiint_{S'}\bar{H}_0\times\bar{E}\,dS$

Since $\bar{n}\times\bar{E}_0 = 0$ on S. Since $\bar{n}\times\bar{E} = Z\bar{H}_t$ and

$\oiint_{S'}\bar{H}_0\times\bar{E}\,dS = -\oiint_{S'}Z\,\bar{H}\cdot\bar{H}_0\,dS$. Hence

$$(\omega-\omega_0) = \frac{-\oiint_{S'}Z\,\bar{H}\cdot\bar{H}_0\,dS}{\iiint\left[\varepsilon\bar{E}\cdot\bar{E}_0 - \mu\bar{H}\cdot\bar{H}_0\right]d\tau}$$

7-3 If $|\bar{E}| \approx |\bar{E}_0| = |\bar{E}_0|$

$|\bar{H}| \approx |\bar{H}_0| = j|\bar{H}_0|$

$$\therefore \omega-\omega_0 = \frac{-j\oint Z\, \bar{H}\cdot\bar{H}_0\, dS}{\iiint(\varepsilon\, \bar{E}\cdot\bar{E}_0 - \mu\, \bar{H}\cdot\bar{H}_0)\,d\tau}$$

By substituting $\bar{E}_0$ & $\bar{H}_0$; $\bar{H}$ & $\bar{E}$ in the above equation,

$$\omega-\omega_0 = \frac{+j\oint Z\,|H_0|^2\,dS}{\iiint(\varepsilon|E_0|^2+\mu|H_0|^2)\,d\tau} \quad \text{...(A)}$$

It is assumed $\bar{W}_m = \bar{W}_e$

If $\omega = \omega_r\left(1+\frac{j}{2Q}\right)$ and $Z = R + jX$

By taking real and imaginary parts of equation (A) it is clear

$$\omega_r-\omega_0 = \frac{-\oint X\,|H_0|^2\,dS}{2\iiint \mu|H_0|^2\,d\tau} \quad \text{and}$$

$$\frac{\omega_r}{2Q} = \frac{\oint R\,|H_0|^2\,dS}{2\iiint\mu|H_0|^2\,d\tau}$$

$$\therefore Q = \frac{\omega_0\iiint\mu|H_0|^2\,d\tau}{\oint R\,|H_0|^2\,dS}$$

[ω_r can be replaced by ω_0 since the difference is only a few per cent]

7-4 The impedance for a metal wall is $Z = R + jR$

Hence

$$\frac{\omega_r-\omega_0}{\omega_0} = -\frac{\oint R\,|H_0|^2\,dS}{2\iiint\mu|H_0|^2\,d\tau} = -\frac{1}{2Q} \quad \text{[from 7.3]}$$

7-5 E_0, H_0 ; n ; ε, μ, σ $\qquad$ E, H ; n ; $\varepsilon+\Delta\varepsilon, \mu+\Delta\mu, \sigma+\Delta\sigma$

Since

$-\bar{\nabla}\times\bar{E}_0 = j\omega_0\mu\bar{H}_0$ $\qquad$ $-\bar{\nabla}\times\bar{E} = j\omega(\mu+\Delta\mu)\bar{H}$

$\bar{\nabla}\times\bar{H}_0 = (\sigma+j\omega_0\varepsilon)\bar{E}_0$ $\qquad$ $\bar{\nabla}\times\bar{H} = [\sigma+\Delta\sigma + j\omega(\varepsilon+\Delta\varepsilon)]\bar{E}$

and so

$$\bar{\nabla}\cdot(\bar{H}\times\bar{E}_0) = \bar{E}_0\cdot\bar{\nabla}\times\bar{H} - \bar{H}\cdot\bar{\nabla}\times\bar{E}_0 = [\sigma+\Delta\sigma+j\omega(\varepsilon+\Delta\varepsilon)]\bar{E}\cdot\bar{E}_0 + j\omega_0\mu\,\bar{H}\cdot\bar{H}_0 \quad \text{....(A)}$$

$$\bar{\nabla}\cdot(\bar{H}_0\times\bar{E}) = \bar{E}\cdot\bar{\nabla}\times\bar{H}_0 - \bar{H}_0\cdot\bar{\nabla}\times\bar{E} = (\sigma+j\omega_0\varepsilon)\bar{E}\cdot\bar{E}_0 + j\omega(\mu+\Delta\mu)\bar{H}\cdot\bar{H}_0 \quad \text{...(B)}$$

Subtracting (A) from (B)

$$\bar{\nabla}\cdot(\bar{H}\times\bar{E}_0) - \bar{\nabla}\cdot(\bar{H}_0\times\bar{E}) = j(\omega-\omega_0)[\varepsilon\,\bar{E}\cdot\bar{E}_0 - \mu\,\bar{H}\cdot\bar{H}_0] - j\omega\Delta\mu\,\bar{H}\cdot\bar{H}_0 + j\omega\left(\Delta\varepsilon - \frac{j\Delta\sigma}{\omega}\right)\bar{E}\cdot\bar{E}_0$$

Taking the volume integral of the above equation

$$\iiint_\tau\left[\bar{\nabla}\cdot(\bar{H}\times\bar{E}_0) - \bar{\nabla}\cdot(\bar{H}_0\times\bar{E})\right]d\tau = \oint_S(\bar{H}\times\bar{E}_0 - \bar{H}_0\times\bar{E})\,dS$$

Since $n\times E_0 = 0 = n\times E$ on surface S

$$\therefore 0 = j(\omega-\omega_0)\iiint[\varepsilon\,\bar{E}\cdot\bar{E}_0 - \mu\,\bar{H}\cdot\bar{H}_0]\,d\tau + \iiint j\omega\left[\left\{\Delta\varepsilon-\frac{j\Delta\sigma}{\omega}\right\}\bar{E}\cdot\bar{E}_0 - \Delta\mu\,\bar{H}\cdot\bar{H}_0\right]d\tau$$

$$\therefore \frac{\omega-\omega_0}{\omega} = \frac{-\iiint\left[\left(\Delta\varepsilon-\frac{j\Delta\sigma}{\omega}\right)\bar{E}\cdot\bar{E}_0 - \Delta\mu\,\bar{H}\cdot\bar{H}_0\right]d\tau}{\iiint[\varepsilon\,\bar{E}\cdot\bar{E}_0-\mu\,\bar{H}\cdot\bar{H}_0]\,d\tau}$$

00

$\boxed{7-6}$ Let $\bar{E} \approx \bar{E}_0 = |E_0|$ and

$\bar{H} \approx \bar{H}_0 = j|H_0|$ and

$\omega \approx \omega_r + j\frac{\omega_0}{2Q}$ and from $\boxed{7-5}$

$$\omega_r - \omega_0 + j\frac{\omega_0}{2Q} = -\frac{\omega \iiint [\Delta\varepsilon |E_0|^2 + \Delta\mu |H_0|^2] d\tau}{\iiint [\varepsilon |E_0|^2 + \mu |H_0|^2] d\tau}$$

$$+ \frac{\iiint j\,\Delta\sigma |E_0|^2 d\tau}{\iiint [\varepsilon |E_0|^2 + \mu |H_0|^2] d\tau}$$

Equating the Imaginary parts and assuming $\bar{W}_m = \bar{W}_e$

$$\frac{\omega_0}{2Q} = \frac{\iiint \Delta\sigma |E_0|^2 d\tau}{2\iiint \varepsilon |E_0|^2 d\tau}$$

$$\therefore Q \doteq \omega_0 \frac{\iiint \varepsilon |E_0|^2 d\tau}{\iiint \Delta\sigma |E_0|^2 d\tau}$$

$\boxed{7-7}$ From $\boxed{7-5}$

$$\omega_r - \omega_0 + \frac{j\omega_0}{2Q} \approx -\frac{\omega \iiint (\varepsilon' - \varepsilon_0 + j\varepsilon'') |E_0|^2 d\tau}{\iiint \varepsilon_0 |E_0|^2 d\tau} \quad \cdots\cdots \text{(A)}$$

and since $\bar{W}_m = \bar{W}_e$, and $\Delta\sigma$ of prob $\boxed{7-6}$ $= \omega\varepsilon''$, hence equating imaginary parts of (A)

$$\therefore \omega Q \approx \frac{\omega_0 \iiint \varepsilon_0 |E_0|^2 d\tau}{\iiint \varepsilon'' |E_0|^2 d\tau}$$

Division of (A) by ω_0 leads to,

$$\frac{\omega_r - \omega_0}{\omega_0} + \frac{j}{2Q} = -\frac{\iiint (\varepsilon' - \varepsilon_0 - j\varepsilon'') |E_0|^2 d\tau}{Q \iiint \varepsilon'' |E_0|^2 d\tau}$$

and by equating the real parts of the above equation

$$\frac{\omega_0 - \omega_r}{\omega_0} = \frac{\iiint (\varepsilon' - \varepsilon_0) |E_0|^2 d\tau}{Q \iiint \varepsilon'' |E_0|^2 d\tau}$$

Hence $Q \cdot \frac{\omega_0 - \omega_r}{\omega_r} \approx \frac{\varepsilon' - \varepsilon_0}{\varepsilon''}$

$\boxed{7-8}$ From equation (7-18)

$$\frac{\omega - \omega_0}{\omega} = -\frac{\iiint_{\tau'} \Delta\varepsilon\, \bar{E}_{int} \cdot \bar{E}_0^* d\tau}{2\iiint_{\tau} \varepsilon |E_0|^2 d\tau}$$

By the terms of the problem $E_{int} = E_{ext}$ and the dominant mode :

$$\bar{E}_{ext} = E \sin\frac{\pi y}{b} \sin\frac{\pi z}{c} \cdot \hat{u}_x$$

$$\therefore \frac{\omega - \omega_0}{\omega} \approx -\frac{A_{rea} \cdot a \cdot E^2 (\varepsilon_0 \varepsilon_r - \varepsilon_0)}{2\varepsilon_0 a E^2 \int_0^b \int_0^c \sin^2\frac{\pi y}{b} \sin^2\frac{\pi z}{c}\, dy\, dz}$$

$$\approx \frac{2 A_{rea} (1 - \varepsilon_r)}{bc}$$

$\boxed{7-9}$ Same as in $\boxed{7-8}$ except

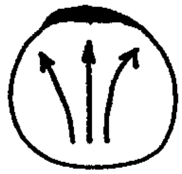
$$E_{int} = \frac{3}{2+\varepsilon_r} E_{ext}$$

$$\therefore \frac{\omega - \omega_0}{\omega} = -\frac{\frac{4}{3}\frac{\pi d^3}{8} \frac{3}{2+\varepsilon_r} (\varepsilon_r - 1) \varepsilon_0 \varepsilon_0^*}{2 \cdot a \cdot \frac{b}{2} \cdot \frac{c}{2}}$$

$$= -\frac{\pi d^3}{abc} \frac{\varepsilon_r - 1}{\varepsilon_r + 2}$$

$\boxed{7-10}$

$\phi = 90°$



$\phi = 0°$

$$\psi_{11}^{TE} = J_1(k_\rho \rho)\cos\varphi\, e^{-jk_z z} \quad \text{and } J_1'(k_\rho a) = 0$$

and

$$\frac{\Delta\omega_c}{\omega_c} = -\frac{\Delta W_e}{W}.$$

$$E_\rho = \frac{1}{\rho}\sin\varphi\, J_1(k_\rho \rho)\, e^{-jk_z z}$$

$$E_\phi = k_\rho J_1'(k_\rho \rho)\cos\varphi\, e^{-jk_z z}$$

$$H_\rho = \frac{1}{\hat{z}} k_\rho(-jk_z) J_1'(k_\rho \rho)\cos\phi\, e^{-jk_z z}$$

$$H_\phi = \frac{1}{\hat{z}\rho}\sin\phi\,(jk_z)\, J_1(k_\rho \rho)\, e^{-jk_z z}$$

$$H_z = \frac{k_\rho^2}{\hat{z}} J_1(k_\rho \rho)\cos\varphi\, e^{-jk_z z}$$

$$\Delta W_e = \frac{1}{2}\varepsilon\, |E_\rho|^2 A_{rea}\Big|_{\substack{\rho = a\\ \phi = 90}} = \frac{A\varepsilon}{2}\frac{J_1^2(k_\rho a)}{a^2}$$

$$W = 2W_e = k_\rho \varepsilon\pi \int_0^a \left\{ \frac{J_1^2(k_\rho \rho)}{k_\rho \rho} + (k_\rho \rho) J_1'^2(k_\rho \rho)\right\} d\rho$$

$$= \frac{\varepsilon\pi}{2}\left[k_\rho^2 a^2 - 1\right] J_1^2(k_\rho a)$$

$$\therefore \frac{\Delta\omega_c}{\omega_c} = -\frac{\Delta W_e}{W_e}$$

$$= -\frac{2A\varepsilon}{2a^2\,\varepsilon\pi\left[k_\rho^2 a^2 - 1\right]}$$

but $k_\rho a = 1.841$ and $k_\rho^2 a^2 = 3.389$

hence

$$\frac{\Delta\omega_c}{\omega_c} = -\frac{A}{\pi a^2}\,\frac{1}{2.389}$$

$$= -0.418\, A/(\pi a^2)$$

For the y-polarization

$$\Delta W_m = \frac{\mu}{2}|H_z|^2 A\,\Big|_{\phi = 0°}$$

$$= \frac{\mu A}{2}\,\frac{k_\rho^4}{\hat{z}^2}\, J_1^2(k_\rho a)$$

$$W_m = \frac{\varepsilon\pi}{2} \cdot 2.389 \cdot J_1^2(k_\rho a)$$

$$\therefore \frac{\Delta\omega_c}{\omega_c} = \frac{\Delta W_m}{W_m}$$

$$= \frac{\mu A}{2}\,\frac{k_\rho^4 \cdot 2}{\omega^2\mu\varepsilon\pi \cdot (2.389)}$$

$$= \frac{A\, k_\rho^4 a^4}{\pi a^4\, \omega^2\mu\varepsilon \times 2.389}$$

since $\omega^2\mu\varepsilon = \dfrac{(1.841)^2}{a^2}$.

and $k_\rho a = 1.841$

$$\therefore \frac{\Delta\omega_c}{\omega_c} = \frac{A}{\pi a^2}\,\frac{(k_\rho a)^4}{(1.841)^2 \times 2.389}$$

$$= \frac{A}{\pi a^2}\,\frac{(1.841)^2}{2.389}$$

$$= 1.42\,\frac{A}{\pi a^2}$$

.2 $\boxed{7-11}$ From equation (7-25)

$$\frac{\Delta\omega_c}{\omega_c} = -\frac{\iint_{S'} \Delta\varepsilon\, \bar{E}_{int}\cdot \bar{E}_0^{*}\, dS}{2\iint_S \varepsilon |E_0|^2\, dS}$$

By the terms of the problem

$$E_{int} = \frac{2}{1+\varepsilon_r} E_{ext} \quad \text{and}$$

$$E_{ext} = E_x \sin\frac{\pi y}{b} \text{ for the}$$

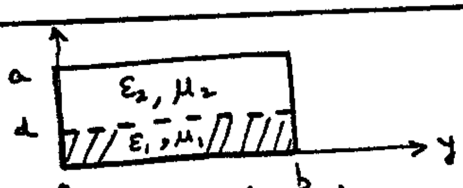
dominant mode.

$$\therefore \frac{\Delta\omega_c}{\omega_c} = -\frac{(\varepsilon_r - 1)\frac{\pi d^2}{4} E_0^{*}\Big|_{b/2} \frac{2}{1+\varepsilon_r} E_0\Big|_{b/2}}{2\int_0^a\int_0^b \varepsilon E_0^2 \sin^2\frac{\pi y}{b}\, dx\, dy}$$

$$= -(\varepsilon_r - 1)\frac{\pi d^2}{4}\,\frac{2}{1+\varepsilon_r}\,\frac{E_0^2}{ab\, E_0^2}$$

$$= -\frac{\varepsilon_r - 1}{\varepsilon_r + 1}\,\frac{\pi d^2}{2ab}$$

$\boxed{7-13}$



Also
$$\beta - \beta_0 \approx \frac{\omega \iint_S \Delta\varepsilon\, \hat{E}_0 E^{int} Z_0\, dS}{2\iint_S |E_0|^2 dS}$$

Since $E_0 = \hat{x}\sin\frac{\pi y}{b}$ for the dominant mode and $E_{int} = \frac{\varepsilon_2}{\varepsilon_1} E^{ext}$.

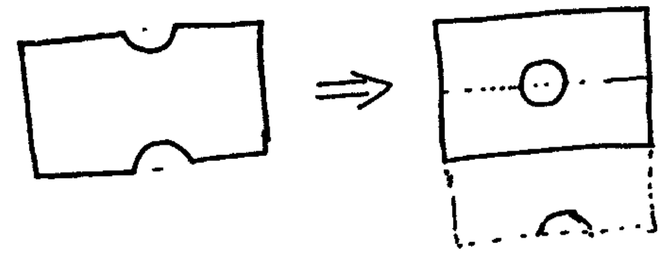
$$\therefore \beta - \beta_0 = \frac{\omega\left(\frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1}\right)\int_0^d\int_0^a \sin^2\frac{\pi y}{b}\, dy\, dx}{2\int_0^a\int_0^b \sin^2\frac{\pi y}{b}\, dy\, dx}\cdot Z_0$$

$$\approx \frac{\varepsilon_2}{\varepsilon_1}\,\frac{\omega(\varepsilon_1-\varepsilon_2)}{2}\,\frac{d}{a}\cdot Z_0 . \quad \text{Since}$$

$\omega\eta = \frac{\omega^2\mu_0}{k_2}$, Hence

$$\beta - \beta_0 = \frac{\varepsilon_2}{\varepsilon_1}\,\frac{(k_1^2 - k_2^2)\, d}{2a\, k_2 \left[1-(f_c/f)^2\right]^{1/2}}$$

$\boxed{7-12}$ The problem of a half cylinder in an infinite ground plane is equivalent to a metallic cylider



A material perturbation is assumed inside the cylinder and let $\varepsilon_r \to \infty$ and $\mu_r \to 0$ in the region $\Delta\tau$.

Hense as $\varepsilon_r \to \infty$ in $\boxed{7-11}$

then $\frac{\Delta\omega_c}{\omega_c} = -\frac{\pi d^2}{2ab}$

For TE_{02} mode

$$\frac{\Delta\omega_c}{\omega_c} = -\frac{\iint (\Delta\varepsilon\, \bar{E}\cdot\bar{E}_0^{*} + \Delta\mu\, \bar{H}\cdot\bar{H}_0^{*})\, dS}{\iint (\varepsilon|E_0|^2 + \mu|H_0|^2)\, dS}$$

and
$$E_y^{\circ} = -\frac{j\omega\mu b}{2\pi} C \sin\frac{2\pi y}{b}$$

$$H_x^{\circ} = \frac{j\beta b}{2\pi} C \sin\frac{2\pi y}{b}$$

$$H_z^{\circ} = C\cos\frac{2\pi y}{b}$$

$$\therefore \frac{\Delta\omega_c}{\omega_c} = -\frac{\mu_r H_z\big|_{b/2}\, \pi d^2/4}{\mu_r\int_0^b\int_0^a\left\{\frac{\omega^2\mu_0\varepsilon\mu_r b^2}{4\pi^2} C^2 \sin^2\frac{2\pi y}{b} + C^2\cos^2\frac{2\pi y}{b} + \frac{\beta^2 b^2 C^2}{(4\pi)^2}\sin^2\frac{2\pi y}{b}\right\} dx\, dy}$$

$$= \frac{\mu_r\, \pi d^2/4\, C^2}{C^2\, ab} = \frac{\pi d^2}{4ab}$$

$\boxed{7-14}$ By the terms of the problem,

$$\beta - \beta_0 = \omega \frac{\iint_S (\Delta\varepsilon \hat{E} \cdot E_0^*) dS}{2\iint_S (\bar{E}_0^* \times \bar{H}) \cdot \bar{u}_z dS}$$

Since for the dominant mode

$$\bar{E} = \hat{u}_x E_0 \sin\frac{\pi y}{b} \quad \text{and}$$

$$E_{ext} = \frac{2}{1+\varepsilon_r} E_{int} \quad . \text{ Hence}$$

$$\beta - \beta_0 \simeq \frac{\omega(\varepsilon_r - 1) E_0^2 \pi d^2/4 \cdot 2 \cdot \varepsilon_0}{2 \cdot \frac{ab}{2} \cdot \frac{E_0^2}{Z_0^{TE}} (1+\varepsilon_r)}$$

Also, $Z_0^{TE} = \dfrac{\eta}{\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}}$ and hence

$$\beta - \beta_0 = \frac{\pi d^2}{2ab} \cdot \frac{\varepsilon_r - 1}{\varepsilon_r + 1} \frac{\omega \eta \varepsilon_0}{[1-(\omega_c/\omega)^2]^{1/2}}$$

and $k_0 = \omega \varepsilon_0 \eta$ and therefore

$$\frac{\beta - \beta_0}{k_0} = \frac{\pi d^2}{2ab} \frac{\varepsilon_r - 1}{\varepsilon_r + 1} \frac{1}{\left[1 - \left(\frac{\omega_c}{\omega}\right)^2\right]^{1/2}}$$

$\boxed{7-16}$ By the terms of the problem

$$\bar{\nabla} \cdot (\bar{E}_0 \times \bar{H}) = -j\omega_0 \mu \bar{H} \cdot \bar{H}_0 - j\omega_0(\varepsilon + \Delta\varepsilon)\bar{E} \cdot \bar{E}_0 - (\sigma + \Delta\sigma)\bar{E} \cdot \bar{E}_0$$

and

$$\bar{\nabla} \cdot (\bar{E} \times \bar{H}_0) = -j\omega_0(\mu + \Delta\mu)\bar{H} \cdot \bar{H}_0 - j\omega_0 \varepsilon \bar{E} \cdot \bar{E}_0 - \sigma \bar{E} \cdot \bar{E}_0$$

Subtracting one equation from other and taking the volume integral.

$$\oiint_S (\bar{E}_0 \times \bar{H} - \bar{E} \times \bar{H}_0) = \iiint_\tau [\bar{\nabla} \cdot (\bar{E}_0 \times \bar{H}) - \bar{\nabla} \cdot (\bar{E} \times \bar{H}_0)] d\tau$$

$$= -j \iiint_\tau \{[\omega_0 \Delta\varepsilon - j\Delta\sigma] \bar{E} \cdot \bar{E}_0 - \omega_0 \Delta\mu \bar{H} \cdot \bar{H}_0\} d\tau$$

$$\oiint_S = \oiint \text{side walls} + \oiint \text{top} + \oiint \text{bottom} = \oiint \text{top} + \oiint \text{bottom}$$

& hence from $\boxed{7.15}$ and after simplification

$$\gamma - \gamma_0 = \frac{-j \iint_S \{(\omega_0 \Delta\varepsilon - j\Delta\sigma)\hat{E} \cdot \hat{E}_0 - \omega \Delta\mu \hat{H} \cdot \hat{H}_0\} dS}{\iint_S (\hat{E}_0 \times \hat{H} - \hat{E} \times \hat{H}_0) \cdot \hat{u}_z dS}$$

$\boxed{7-15}$ It is given

$$\bar{E}_0 = \hat{E}_0 e^{-\gamma_0 z} \quad \text{and} \quad \bar{E} = \hat{E} e^{\gamma z}$$

$$\bar{H}_0 = \hat{H}_0 e^{-\gamma_0 z} \qquad \bar{H} = \hat{H} e^{\gamma z}$$

Therefore,

$$\bar{\nabla} \cdot (\bar{E}_0 \times \bar{H}) - \bar{\nabla} \cdot (\bar{E} \times \bar{H}_0)$$

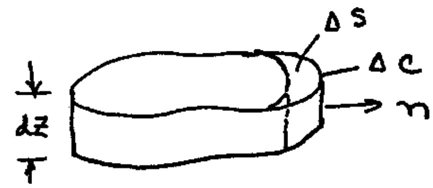
$$= \bar{H} \cdot \bar{\nabla} \times \bar{E}_0 - \bar{E}_0 \cdot \bar{\nabla} \times \bar{H} - \bar{H}_0 \cdot \bar{\nabla} \times \bar{E} + \bar{E} \cdot \bar{\nabla} \times \bar{H}_0$$

$$= -j\omega\mu \bar{H} \cdot \bar{H}_0 - j\omega\varepsilon \bar{E} \cdot \bar{E}_0 + j\omega\mu \bar{H} \cdot \bar{H}_0 + j\omega\varepsilon \bar{E} \cdot \bar{E}_0$$

$$= 0$$

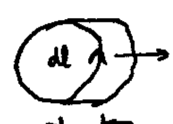
Taking the surface integral

$$\oiint_{S'} [(\bar{E}_0 \times \bar{H}) - (\bar{E} \times \bar{H}_0)] \cdot d\bar{S} = 0.$$



$$\oiint_{S'} (\bar{E}_0 \times \bar{H}) \cdot d\bar{S} = \oiint_{\substack{\text{top + bottom walls} \\ S}} - \oiint_{\Delta S} = 0$$

$$= \Delta z \frac{\partial}{\partial z} \iint_{S'} (\bar{E}_0 \times \bar{H}) \cdot d\bar{S} - \iint_{\Delta S} (\bar{E}_0 \times \bar{H}) dS$$



$$ds = n \, dl \, dz$$

$$= (\gamma - \gamma_0) \Delta z \iint_{S'} (\bar{E}_0 \times \bar{H}) \cdot d\bar{S} - \iint_{\Delta S} (\bar{E}_0 \times \bar{H}) d\bar{S}$$

Similarly $\oiint_{S'} (\bar{E} \times \bar{H}_0) \cdot d\bar{S} = (\gamma - \gamma_0) \Delta z \iint_{S'} (\bar{E} \times \bar{H}_0) \cdot d\bar{S} = 0$

$$\therefore (\gamma - \gamma_0) \Delta z \iint_{S'} (\bar{E}_0 \times \bar{H} - \bar{E} \times \bar{H}_0) \cdot d\bar{S} - \iint_{\Delta S} (\bar{E}_0 \times \bar{H}) \cdot d\bar{S} = 0$$

Since $\iint_{\Delta S} (\bar{E}_0 \times \bar{H}) \cdot dS = \Delta z \oint_{\Delta C} (\bar{E}_0 \times \bar{H}) \cdot \bar{n} \, dl$

$$\therefore \gamma - \gamma_0 = \frac{\oint_{\Delta C} (\bar{E}_0 \times \bar{H}) \cdot \bar{n} \, dl}{\iint_{S'} (\bar{E}_0 \times \bar{H} - \bar{E} \times \bar{H}_0) \cdot \bar{u}_z dS}$$

$\boxed{7-17}$ From $\boxed{7-16}$

$$-\gamma_0 = -j\,\frac{\iint_S\left[(\omega\Delta\varepsilon - j\Delta\sigma)\hat{E}\cdot\hat{E}_0 - \omega\Delta\mu\hat{H}\cdot\hat{H}_0\right]dS}{\iint_S(\hat{E}_0\times\hat{H} - \hat{E}\times\hat{H}_0)\cdot\hat{u}_z\,dS}$$

Let $\hat{E}\approx\hat{E}_0^*$ and $\hat{H}\approx-\hat{H}_0^*$

and $\gamma=\alpha+j\beta$ and substitution in the above equation and separation of the real and imaginary parts lead to

$$\alpha = \frac{\iint_S\Delta\sigma\,|\hat{E}_0|^2\,dS}{\iint_S(\hat{E}_0\times\hat{H}_0^* + \hat{E}_0^*\times\hat{H}_0)\cdot\hat{u}_z\,dS}$$

and

$$\beta-\beta_0 = \frac{\iint_S\omega\left[\Delta\varepsilon\,|\hat{E}_0|^2 - \Delta\mu\,|\hat{H}_0|^2\right]}{\iint_S(\hat{E}_0^*\times\hat{H}_0 + \hat{E}_0\times\hat{H}_0^*)\cdot\hat{u}_z\,dS}$$

$\boxed{7-18}$ Since

$$\bar{n}\times\bar{E} = \mathcal{Z}\,\bar{H}$$

and from $\boxed{7-15}$ we obtain

$$\gamma-\gamma_0 = \frac{\oint\hat{E}_0\times\hat{H}\cdot\bar{n}\,dl}{\iint_S(\hat{E}_0\times\hat{H} - \hat{E}\times\hat{H}_0)\cdot\bar{u}_z\,dS}$$

Since $\hat{E}_0\times\hat{H}\cdot\bar{n}$

$$= \bar{n}\times\bar{E}_0\cdot\bar{H}$$

$$= \mathcal{Z}\,\bar{H}\cdot\bar{H}_0$$

$$\therefore\gamma-\gamma_0 = \frac{\oint\mathcal{Z}\,\hat{H}\cdot\hat{H}_0\,dl}{\iint(\hat{E}_0\times\hat{H} - \hat{E}\times\hat{H}_0)\cdot\bar{u}_z\,dS}$$

$\boxed{7-19}$ Let $\gamma_0 = j\beta_0$

$$\mathcal{Z} = R + jX$$
$$\gamma = \alpha + j\beta$$
$$E = E_0^*$$
$$H = -H_0^*$$

Hence from $\boxed{7-18}$

$$\alpha + j(\beta-\beta_0) = \frac{\oint(R+jX)\,|\hat{H}_0|^2\,dl}{\iint(\hat{E}_0\times\hat{H}_0^* + \hat{E}_0^*\times\hat{H}_0)\cdot\hat{u}_z\,dS}$$

$$= \frac{\oint(R+jX)\,|\hat{H}_0|^2\,dl}{2\,\mathrm{Re}\iint(\hat{E}_0\times\hat{H}_0^*)\cdot\hat{u}_z\,dS}$$

$$\therefore\ \alpha = \frac{\oint R\,|\hat{H}_0|^2\,dl}{2\,\mathrm{Re}\iint(\hat{E}_0\times\hat{H}_0^*)\cdot\hat{u}_z\,dS}$$

$$\beta-\beta_0 = \frac{\oint X\,|\hat{H}_0|^2\,dl}{2\,\mathrm{Re}\iint(\hat{E}_0\times\hat{H}_0^*)\cdot\hat{u}_z\,dS}$$

$\boxed{7-20}$ Here,

$$\omega_r^2 = \frac{\iiint\bar{E}\cdot\bar{\nabla}\times\mu^{-1}(\bar{\nabla}\times\bar{E})\,d\tau}{\iiint\varepsilon|E|^2\,d\tau}$$

and

$$\nabla\cdot\left[(\mu^{-1}\bar{\nabla}\times\bar{E})\times\bar{E}\right] = \bar{E}\cdot\bar{\nabla}\times(\mu^{-1}\bar{\nabla}\times\bar{E}) - (\mu^{-1}\bar{\nabla}\times\bar{E})\cdot(\bar{\nabla}\times\bar{E})$$

Taking the volume integral of both sides

$$\iiint_V\bar{\nabla}\cdot\left[(\mu^{-1}\bar{\nabla}\times\bar{E})\times\bar{E}\right]dV = \oiint_S\left[\mu^{-1}(\bar{\nabla}\times\bar{E})\times\bar{E}\right]\cdot\bar{n}\,dS = 0$$

Since $\bar{n}\times\bar{E} = 0$ then

$$\iiint_V\bar{E}\cdot\bar{\nabla}\times(\mu^{-1}\bar{\nabla}\times\bar{E})\,dV = \iiint_V\mu^{-1}(\bar{\nabla}\times\bar{E})^2\,dV$$

then

$$\omega_r^2 = \frac{\iiint_V\mu^{-1}(\bar{\nabla}\times\bar{E})^2\,dV}{\iiint\varepsilon E^2\,dV}$$

$\boxed{7.21}$ $\omega_r^2 = \dfrac{\iiint \varepsilon^{-1}(\bar\nabla\times\bar H)^2\, d\tau}{\iiint \mu H^2\, d\tau} = \dfrac{N}{D}$

$$\delta\omega_r^2 = \frac{1}{D^2}\left[D\,\delta N - N\,\delta D\right]$$
$$= \frac{1}{D}\left[\delta N - \omega_r^2\,\delta D\right]$$
$$= \frac{1}{D}\Big[2\iiint \varepsilon^{-1}(\bar\nabla\times\bar H)\cdot(\bar\nabla\times\delta\bar H)\,d\tau - \omega_r^2\, 2\iiint \mu\,\bar H\cdot\delta\bar H\,\delta\tau\Big]$$
$$= \frac{1}{D}\Big\{2\iiint \delta\bar H\cdot\bar\nabla\times\varepsilon^{-1}(\bar\nabla\times\bar H)\,d\tau - \omega_r^2\, 2\iiint \mu\bar H\cdot\delta\bar H\,\delta\tau\Big]$$
$$= \frac{2}{D}\iiint \delta\bar H\cdot\left[\bar\nabla\times\varepsilon^{-1}(\bar\nabla\times\bar H) - \omega_r^2\mu\bar H\right]d\tau$$
$$= 0 \text{ for arbitrary } \delta H$$

$\boxed{7-22}$ Here $\omega_r^2 = \dfrac{\iiint \mu^{-1}(\bar\nabla\times\bar E)^2\,d\tau}{\iiint \varepsilon|E|^2\,d\tau}$

and $\bar E = \bar u_x\, y z (y-b)(z-c)$

$$\nabla\times\bar E = \bar u_y\left[y(y-b)(z-c) + yz(y-b)\right] - \bar u_z\left[z(y-b)(z-c) + yz(z-c)\right]$$

$$\therefore \omega_r^2 = \frac{1}{\mu\varepsilon}\,\frac{\int_0^b\int_0^c dz\,dy\ |\bar\nabla\times\bar E|^2}{\int_0^b\int_0^c dz\,dy\ y^2z^2(y-b)^2(z-c)^2}$$

$$= \frac{1}{\mu\varepsilon}\,\frac{\int_0^b\int_0^c dz\,dy\left[y^2(y-b)^2(2z-c)^2 + z^2(z-c)^2(2y-b)^2\right]}{\frac{b^5}{30}\cdot\frac{c^5}{30}}$$

$$= \frac{10}{\mu\varepsilon}\,\frac{\left[b^2+c^2\right]}{b^2c^2}$$

$$\therefore \omega_r = \sqrt{\frac{10}{\mu\varepsilon}}\ \frac{\sqrt{b^2+c^2}}{bc}$$

$\boxed{7-23}$ Since

$$\omega_r^2 = \frac{\iiint \mu^{-1}(\bar\nabla\times\bar E)^2\,d\tau + 2\oint\!\!\oint\left[(\mu^{-1}\,\bar\nabla\times\bar E)\times E)\right]\cdot dS}{\iiint_\tau \varepsilon\,|E|^2\,d\tau}$$

and $\iiint \bar E\cdot\bar\nabla\times\mu^{-1}(\bar\nabla\times\bar E)\,d\tau = \iiint \mu^{-1}(\bar\nabla\times\bar E)^2\,d\tau + \oint\!\!\oint\left[\mu^{-1}(\bar\nabla\times\bar E)\times\bar E\right]\cdot d\bar S$

hence $\omega_r^2 = \dfrac{\iiint \bar E\cdot\bar\nabla\times\mu^{-1}(\bar\nabla\times\bar E)\,d\tau + \oint\!\!\oint_{S'}\left[\mu^{-1}(\bar\nabla\times\bar E)\times\bar E)\right]\cdot d\bar S}{\iiint \varepsilon\,|E|^2\,d\tau}$

But $\bar E\cdot\bar\nabla\times(-j\omega\bar H) = \bar E\cdot(\omega^2\varepsilon)\bar E = \omega^2\varepsilon\,|E|^2$ and assuming $\tau' = \tau$ and $\omega = \omega_0$

we obtain

$$\therefore \omega_r^2 = \frac{\iiint_\tau \omega^2\varepsilon\,|E|^2\,d\tau + \oint\!\!\oint_{S'}\left[\mu^{-1}(\bar\nabla\times E)\times\bar E\right]\cdot d\bar S}{\iiint_\tau \varepsilon\,|E|^2\,d\tau}$$

Since $\omega\to\omega_0$ and $E\to E_0$ hence

$$\omega_r^2 - \omega_0^2 = \frac{\oint\!\!\oint_{S'}\left[\mu^{-1}(\bar\nabla\times\bar E_0)\times\bar E_0\right]\cdot d\bar S}{\iiint_\tau \varepsilon\,|E|^2\,d\tau}\ .\ \text{Also}$$

$$\oint\!\!\oint_{S'}\left[\mu^{-1}(\bar\nabla\times\bar E_0)\times\bar E_0\right]\cdot d\bar S = \oint\!\!\oint_{S'}\left[j\omega_0\,\bar H_0\times\bar E_0\right]\cdot d\bar S = \oint\!\!\oint_{S}\left[j\omega_0\,\bar H_0\times\bar E_0\cdot d\bar S\right] - \oint\!\!\oint_{\Delta S}\left[j\omega_0\,\bar H_0\times\bar E_0\right]\cdot d\bar S$$

$\boxed{7\text{-}23}$ continued.

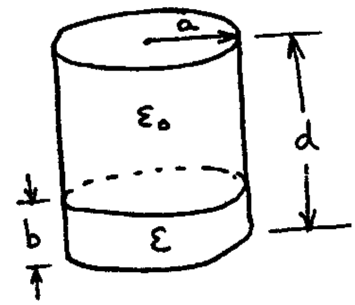
as $\bar{n} \times \bar{E}_0 = 0$ on S

$$\oint\!\!\oint_{S'} = -\oint\!\!\oint_{\Delta S} [j\omega \bar{H}_0 \times \bar{E}_0 \cdot ds] = -j\omega_0 \iiint_{\Delta\tau} \nabla\cdot(\bar{H}_0 \times \bar{E}_0)\, d\tau$$

$$= -j\omega_0 \iiint_{\Delta\tau} (\bar{E}_0 \cdot \bar{\nabla}\times \bar{H}_0 - \bar{H}_0 \cdot \bar{\nabla}\times \bar{E}_0)\, d\tau \quad = \omega_0^2 \iiint_{\Delta\tau} [\varepsilon|E_0|^2 - \mu|H_0|^2]\, d\tau$$

$$\therefore \frac{\omega_r^2 - \omega_0^2}{\omega_0^2} = \frac{\iiint_{\Delta\tau} (\varepsilon|E_0|^2 - \mu|H_0|^2)\, d\tau}{\iiint_{\tau} \varepsilon|E_0|^2\, d\tau}$$

$\boxed{7\text{-}24}$ By the terms of the problem



$\bar{H} = \hat{u}_\rho J_1(2.405\frac{\rho}{a})$ and from (7.46)

$$\omega_r^2 = \frac{\iiint \varepsilon^{-1} (\bar{\nabla}\times\bar{H})^2\, d\tau}{\iiint \mu |H|^2\, d\tau}$$

since $\bar{\nabla}\times\bar{H} = \hat{u}_z \left[\frac{J_1(2.405\rho/a)}{\rho} + \frac{2.405}{a} J_1'(2.405\rho/a)\right]$

and $d\tau = r\, dr\, d\phi\, dz$, hence

$$\mu \iiint_\tau |H|^2 d\tau = 2\pi d \int_0^a J_1^2(2.405\rho/a)\rho\, d\rho$$

$$= 2\pi d\left(\frac{a}{2.405}\right)^2 \left[\frac{(2.405)^2}{2}\left\{1 - \frac{1}{(2.405)^2}\right\} J_1^2(2.405) + J_1'^2(2.405)\right]$$

and

$$\int_0^a (\nabla\times H)^2 d\tau = \int_0^a \left\{\frac{J_1^2(2.405\frac{\rho}{a})}{\frac{2.405\rho}{a}} + \rho\left(\frac{2.405}{a}\right)\left[J_1'(2.405\tfrac{\rho}{a})\right]^2\right\} d\rho \left(\frac{2.405}{a}\right)$$

$$= \frac{1}{2}(2.405)^2\left\{\left(1 - \frac{1}{(2.405)^2}\right) J_1^2(2.405)\right\} + \{J_1'(2.405)\}^2$$
$$+ 2.405\, J_1(2.405)\, J_1'(2.405).$$

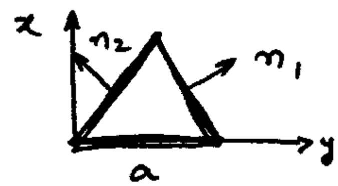
Since $J_1'(2.405) = -\frac{J_2}{2}(2.405)$

$$\therefore \omega_r^2 \approx \frac{2\pi b/(\varepsilon_0\varepsilon_r) + 2\pi(d-b)/\varepsilon_0}{2\pi\, \mu\, d\, (a/2.405)^2}$$ {neglecting the second term in the expansion}

$$\approx \left(\frac{2.405}{a}\right)^2 \frac{1}{\mu_0 \varepsilon_0}\left[1 - \frac{b}{d}\left(1 - \frac{1}{\varepsilon_r}\right)\right]$$

$\boxed{7-25}$ The E-field formula is given by equation (7.80) and

$$\omega_c^2 = \frac{\iint \mu^{-1}(\bar\nabla\times\bar E)^2 ds + 2\oint[\mu^{-1}(\bar\nabla\times\bar E)\times\bar E]\cdot\bar n\, dl}{\iint \varepsilon|\bar E|^2 ds}$$



Let $\bar E = \hat x \sin\frac{\pi y}{a}$, then

$$\iint \varepsilon|\bar E|^2 ds = 2\varepsilon\int_0^{a/2} y \sin^2\frac{\pi y}{a}\, dy = \frac{\varepsilon a^2}{8}$$

Now $\bar\nabla\times\bar E = -\hat z\,\frac{\pi}{a}\cos\frac{\pi y}{a}$ and so

$$\therefore \iint \mu^{-1}(\bar\nabla\times\bar E)^2 ds = 2\frac{\pi^2}{\mu a^2}\int_0^{a/2} y\cos^2\frac{\pi y}{a}\, dy = \pi^2/(8\mu)$$

Since $(\bar\nabla\times\bar E)\times\bar E = \hat y\,\frac{\pi}{2a}\sin\frac{2\pi y}{a}$ and $\bar n_1 = \frac{1}{\sqrt2}\hat x + \frac{1}{\sqrt2}\hat y = \frac{\hat x+\hat y}{\sqrt2}$

$\bar n_2 = \frac{\hat x - \hat y}{\sqrt2}$ hence

$$\oint[(\bar\nabla\times\bar E)\times\bar E]\cdot\bar n\, dl = \int_a^{a/2}\frac{\pi}{2\sqrt2 a}\sin\frac{2\pi y}{a}\,dy - \int_{a/2}^{0}\frac{\pi}{2\sqrt2 a}\sin 2\pi y\, dy = \frac{\pi}{2\sqrt2 a}\cdot 4\cdot\frac{a}{2\pi}$$

$$\therefore \omega_c^2 = \frac{\frac{\pi^2}{8\mu}+\frac{\sqrt2}{\mu}}{\frac{a^2\varepsilon}{8}} = \frac{\pi^2+8\sqrt2}{\mu\varepsilon a^2} \approx \frac{9.87+11.3}{\mu\varepsilon a^2} \approx \frac{21.17}{\mu\varepsilon a^2}$$

$$\therefore \omega_c = \frac{4.5}{a\sqrt{\mu\varepsilon}} \quad \left[\text{Exact Solution} = \frac{4.2}{a\sqrt{\mu\varepsilon}}\right]$$

$\boxed{7-26}$ By the terms of the problem

$$\bar\nabla\times\bar E = -\hat u_y\,\frac{\pi}{c}\sin\frac{\pi y}{b}\cos\frac{\pi z}{c} + \hat u_z\,\frac{\pi}{b}\cos\frac{\pi y}{b}\sin\frac{\pi z}{c} \quad \text{and}$$

$$\bar\nabla\times\bar H = -\hat u_x\left[\frac{A_2\pi}{b}\sin\frac{\pi y}{b}\sin\frac{\pi z}{c} - \frac{A_1\pi}{c}\sin\frac{\pi y}{b}\sin\frac{\pi z}{c}\right] \text{ and}$$

$$(\bar E\times\bar H)\cdot d\bar s = \frac{A_1}{2}\sin^2\frac{\pi y}{b}\sin\frac{2\pi z}{c}. \quad \text{Hence}$$

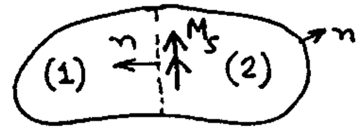
$$\iiint_\tau[\bar E\cdot\bar\nabla\times\bar H + \bar H\cdot\bar\nabla\times\bar E]\,d\tau = \frac{abc}{4}\left(-\frac{A_2\pi}{b}+\frac{A_1\pi}{c}\right) \;\&\; \iiint_\tau[\mu H^2 - \varepsilon E^2]\,d\tau$$
$$= \mu\left[A_1^2\frac{abc}{4} + A_2^2\frac{abc}{4}\right] - \frac{\varepsilon abc}{4}$$

and $\oint\!\!\oint_s (\bar E\times\bar H)\cdot ds = 0$

$$\therefore \omega = j\,\frac{\left(\frac{A_1}{c}-\frac{A_2}{b}\right)\pi}{\mu A_1^2+\mu A_2^2-\varepsilon}. \quad \text{Since } \frac{\partial\omega}{\partial A_1} = 0 = \frac{\partial\omega}{\partial A_2} \quad \therefore \frac{A_1}{A_2} = -\frac{b}{c} \text{ and}$$

$$A_1 = \frac{jb}{\eta\sqrt{b^2+c^2}}\;;\; A_2 = \frac{-jc}{\eta\sqrt{b^2+c^2}}\;;\text{ hence } \omega = \frac{\pi}{bc}\sqrt{\frac{b^2+c^2}{\varepsilon\mu}}$$

$\boxed{7-27}$ An assumed E-field can be supported by the electric currents



$$\bar{J} = -j\omega\varepsilon\bar{E} - \frac{1}{j\omega}\bar{\nabla}\times(\mu^{-1}\bar{\nabla}\times\bar{E}) \quad \text{--- [A]}$$

Here no magnetic surface currents are required on S because $\bar{n}\times\bar{E} = 0$. But an additional magnetic surface current $\bar{M}_s = \bar{n}\times(\bar{E}_2 - \bar{E}_1)$ --- (B) is required on s since $\bar{n}\times\bar{E} \neq 0$.

Hence from Harrington (7.67) and equations (A) and (B)

$$0 = -j\omega\iiint\varepsilon\,\bar{E}.\bar{E}\,d\tau + \frac{j}{\omega}\iiint\bar{E}.\bar{\nabla}\times(\mu^{-1}\bar{\nabla}\times\bar{E})\,d\tau - \frac{j}{\omega}\oiint_S[\bar{n}\times(\bar{E}_2-\bar{E}_1)].(\mu^{-1}\bar{\nabla}\times\bar{E})\,dS$$

Also $\iiint\bar{E}.\bar{\nabla}\times(\mu^{-1}\bar{\nabla}\times\bar{E})\,d\tau = \iiint\bar{\nabla}.(\mu^{-1}\bar{\nabla}\times\bar{E})\times\bar{E}\,d\tau + \iiint(\mu^{-1}\bar{\nabla}\times\bar{E}).(\bar{\nabla}\times\bar{E})\,d\tau$

$$= \iint_S(\mu^{-1}\bar{\nabla}\times\bar{E})\times\bar{E}.d\bar{S} + \iiint(\mu^{-1}\bar{\nabla}\times\bar{E}).(\bar{\nabla}\times\bar{E})\,d\tau$$

since $\bar{n}\times(\mu^{-1}\bar{\nabla}\times\bar{E})$ is continuous across s and $\bar{n}\times\bar{E}$ is discontinu

$$= \iint_s(\mu^{-1}\bar{\nabla}\times\bar{E})\times(\bar{E}_2-\bar{E}_1).d\bar{S} + \iiint(\mu^{-1}\bar{\nabla}\times\bar{E}).(\bar{\nabla}\times\bar{E})\,d\tau$$

Hence $j\omega\iiint\varepsilon E^2\,d\tau = \frac{j}{\omega}\iiint(\mu^{-1}\bar{\nabla}\times\bar{E}).(\bar{\nabla}\times\bar{E})\,d\tau + \frac{j}{\omega}\oiint_s 2(\mu^{-1}\bar{\nabla}\times\bar{E})\times(\bar{E}_2-\bar{E}_1)$

$$\therefore\ \omega^2 = \frac{\iiint(\mu^{-1}\bar{\nabla}\times\bar{E}).(\bar{\nabla}\times\bar{E})\,d\tau + 2\iint_s(\bar{E}_1-\bar{E}_2)\times(\mu^{-1}\bar{\nabla}\times\bar{E}).d\bar{s}}{\iiint\varepsilon E^2\,d\tau}$$

By the terms of the problem

$$\iiint\bar{E}.\bar{\nabla}\times(\mu^{-1}\bar{\nabla}\times\bar{E})\,d\tau = \iiint\bar{\nabla}.(\mu^{-1}\bar{\nabla}\times\bar{E})\times\bar{E}\,d\tau + \iiint(\mu^{-1}\bar{\nabla}\times\bar{E}).(\bar{\nabla}\times\bar{E})\,d\tau$$

$$= \iint_S(\mu^{-1}\bar{\nabla}\times\bar{E})\times\bar{E}.d\bar{S} + \iiint(\mu^{-1}\bar{\nabla}\times\bar{E}).(\bar{\nabla}\times\bar{E})\,d\tau$$

Since $\bar{n}\times\bar{E}$ is continuous at s and $\bar{n}\times\bar{E} = 0$ on S

$$= \iiint\mu^{-1}(\nabla\times E)^2\,d\tau \qquad \text{and also}$$

$$\frac{j}{\omega}\oiint_S\bar{n}\times(\bar{E}_2-\bar{E}_1).\mu^{-1}(\bar{\nabla}\times\bar{E})\,dS = 0$$

Hence $0 = -j\omega\iiint\varepsilon E^2\,d\tau + \frac{j}{\omega}\iiint\mu^{-1}(\bar{\nabla}\times\bar{E})^2\,d\tau$ and so

$$\omega^2 = \frac{\iiint\mu^{-1}(\nabla\times E)^2\,d\tau}{\iiint\varepsilon E^2\,d\tau}$$

$\boxed{7-29}$ From (7-72)

$$\omega\iiint(\mu H^2-\varepsilon E^2)\,d\tau = j\iiint(\bar{E}\cdot\nabla\times\bar{H}+\bar{H}\cdot\nabla\times\bar{E})\,d\tau \quad - j\oiint \bar{E}\times\bar{H}\cdot d\bar{s}$$

or, by terms of the problem

$$\omega\iiint(\mu H^2-\varepsilon E^2)\,d\tau + j\oiint \bar{E}\times\bar{H}\cdot dS = j\iiint\left[j\omega(\varepsilon+\Delta\varepsilon)|E|^2 - j\omega(\mu+\Delta\mu)|H|^2\right]d\tau$$

similarly for the unperturbed cavity

$$\omega_0\iiint(\mu H_0^2-\varepsilon E_0^2)\,d\tau + j\oiint \bar{E}_0\times\bar{H}_0\cdot dS = j\iiint\left[\,j\omega_0\varepsilon\,|E_0|^2 - j\omega_0\mu_0|H_0|^2\right]d\tau$$

Since $n\times E=0$ on S and $n\times E_0=0$ on S, and in the limit, as $\Delta\epsilon\to 0$, and $\Delta\mu\to 0$, we can approximate E, H, ω by E_0, H_0, ω and obtain

$$\frac{\omega-\omega_0}{\omega_0} = -\frac{\iiint\left(\Delta\varepsilon|E_0|^2+\Delta\mu\,|H_0|^2\right)d\tau}{\iiint\left(\varepsilon\,|E_0|^2+\mu\,|H_0|^2\right)d\tau}$$

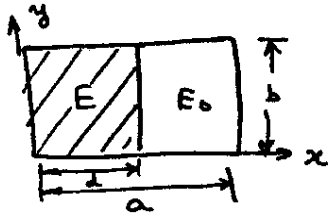
It is assumed in deriving the above formula that $\bar{E}$ and $\bar{H}$ are 90° out of phase in the loss free case.

$\boxed{7-30}$ chapter 9 –

Field Computation by Moment Methods

by R.F. Harrington.

$\boxed{7\text{-}31}$ By the terms of the problem, $\bar{E} = \bar{u}_y \sin\frac{\pi x}{a}$



and $$\omega_c^2 = \frac{\iint \mu^{-1}(\bar\nabla\times\bar E)^2 dS + 2\oint [(\mu^{-1}\bar\nabla\times\bar E)\times\bar E]\cdot\bar n\, dl}{\iint \varepsilon E^2 dS}$$

Since $\bar\nabla\times\bar E = \bar u_z \frac{\pi}{a}\cos\frac{\pi x}{a}$ and $(\bar\nabla\times\bar E)\times\bar E = -\hat u_x \frac{\pi}{2a}\sin\frac{2\pi x}{a}$

$$\therefore \iint \mu^{-1}(\bar\nabla\times\bar E)^2 dS = \frac{\pi^2}{a^2\mu}\frac{ab}{2} \quad\text{and}\quad \oint[(\mu^{-1}(\bar\nabla\times\bar E)\times\bar E]\cdot\bar n\,dl = \frac{\pi}{2a}\sin\frac{2\pi x}{a}\Big|_{x=a} = 0$$

Also $$\iint\varepsilon E^2 dS = \int_0^b dx\left[\int_0^d \sin^2\frac{\pi x}{a}\,\varepsilon_r\,dx + \int_d^a \sin^2\frac{\pi x}{a}\,dx\right]$$

$$= \frac{ab\varepsilon}{2}\left[1 + (\varepsilon_r - 1)\left\{\frac{d}{a} - \frac{1}{2\pi}\sin\frac{2\pi d}{a}\right\}\right]$$

$$\therefore \omega_c^2 = \frac{\pi^2}{a^2\mu\varepsilon}\,\frac{1}{\left[1+(\varepsilon_r-1)\left\{\frac{d}{a} - \frac{1}{2\pi}\sin\frac{2\pi d}{a}\right\}\right]}$$

$\boxed{7\text{-}32}$ By the terms of the problem,

$$\bar\nabla\times\bar E + j\omega\mu\bar H = j\beta\,\bar u_z\times\bar E \quad - (A)$$
$$\bar\nabla\times\bar H - j\omega\varepsilon\bar E = j\beta\,\bar u_z\times\bar H \quad - (B)$$

Multiplying (A) by $\bar H$ and (B) by $\bar E$ and subtracting one from the other we obtain

$$\bar\nabla\cdot(\bar E\times\bar H) + j\omega(\mu H^2 + \varepsilon E^2) = j\,2\beta\,\bar u_z\cdot\bar E\times\bar H$$

After taking the integral over the whole cross-section of the wave guide

$$\beta = \frac{\omega\iint(\mu H^2 + \varepsilon E^2)dS - j\oint \bar E\times\bar H\cdot\bar n\,dl}{2\iint \bar E\times\bar H\cdot\bar u_z\,dS}$$

$\boxed{7\text{-}33}$ From $\boxed{7\text{-}32}$

$$\beta = \frac{\omega\iint_{S'}(\mu H^2 + \varepsilon E^2)dS - j\oint_{c'}\bar E\times\bar H\cdot\bar n\,dl}{2\iint_{S'}(\hat E\times\hat H\cdot\hat u_z)dS} \quad\text{and}$$

$$\beta_0 = \frac{\omega\iint_S(\mu H_0^2 + \varepsilon E_0^2)dS - j\oint_c(\bar E_0\times\bar H_0)\cdot\bar n\,dl}{2\iint_S(\bar E_0\times\bar H_0)\cdot\hat u_z\,dS}$$

The two denominators can be assumed to be the same, because they represent the time average power flow. Hence if $\bar E \approx \bar E_0$ & $\bar H \approx \bar H_0$

$$\beta - \beta_0 \approx \frac{+j\oint_{\Delta c}(\bar E_0\times\bar H_0)\cdot\bar n\,dl}{2\iint(\bar E_0\times\bar H_0)\cdot\hat u_z\,dS}\ \left\{\begin{matrix}\text{Application of}\\ \text{1-62 leads to}\end{matrix}\right\} \approx -\omega\,\frac{\iint_{\Delta S}(\mu|\hat H_0|^2 - \varepsilon|\bar E_0|^2)dS}{2\iint_S(\bar E_0\times\bar H_0)\cdot\hat u_z\,dS}$$

$\boxed{7-34}$ If we assume the material is homogeneous then $\hat{E}^+$ and $\hat{E}^-$ of equation (7.86) becomes $\hat{E}^+ = \hat{E}^- = \hat{E}$ and similarly $\hat{H}^+ = \hat{H}^- = \hat{H}$. Since the denominator is twice the average power flow in the guide, it may be assumed to be the same both in the perturbed and unperturbed guide for shallow, smooth deformations of waveguide walls. Hence

$$\beta - \beta_0 = \frac{\iint \left[\omega\,\Delta\varepsilon(\bar{\bar{E}}_0)^2 - \omega\,\Delta\mu\,(\bar{H}_0)^2\right] dS}{2\iint \bar{E}_0 \times \bar{H}_0 \cdot \bar{u}_z\, dS}$$

Since in the loss free case

$\bar{E}_0 = \hat{z} E_0$

$\bar{H}_0 = \hat{y}(-j) H_0$

Hence

$$\beta - \beta_0 = \frac{\omega \iint \left[\Delta\varepsilon\, E_0^2 + \Delta\mu\, H_0^2\right] dS}{2\iint \bar{E}_0 \times \bar{H}_0 \cdot \bar{u}_z\, dS}$$

$\boxed{7-35}$ Assume a constant current $I_z \hat{z}$. Then

$$Z_{in} = -\frac{\langle u, u\rangle}{I^2} = \frac{-\iint \bar{E}_a \cdot \bar{I}_a\, dS}{\pi d\, I^2}. \text{ Now}$$

$$E_z^a = \frac{-k^2 I_s^a H_0^{(2)}(k\rho)}{4\omega\varepsilon} \quad \text{and} \quad H_\varphi = -\frac{k I_s^a}{4j} H_0^{(2)\prime}(k\rho) \quad \cdots (5.84)$$

Hence $$I = \int_0^{2\pi} H_\varphi\, \rho\, d\varphi \Big|_{\rho = d/2} = \frac{k}{4j} I_s^a H_1^{(2)}(k\rho)\, 2\pi\rho \Big|_{\rho = d/2}$$

Expanding $$H_1^{(2)}(k\rho) \xrightarrow[k\rho \to 0]{} \left(\frac{k\rho}{2}\right) - \frac{j}{\pi}\left(\frac{2}{k\rho}\right) \quad \cdots (D.10)$$

$$I = \left(\frac{k\rho}{4j}\right) I_s^a\, 2\pi \left[\frac{k\rho}{2} - \frac{j}{\pi}\frac{2}{k\rho}\right]\Big|_{\rho = d/2} = \frac{I_s^a}{4j}\left[\pi\left(\frac{k^2 d^2}{4}\right) - j4\right]$$

Hence $$E_z^a\Big|_{\substack{\text{Small argument} \\ \text{for Hankel fn.}}} = \frac{-k^2 I_s^a}{4\omega\varepsilon}\left[1 - \frac{j2}{\pi}\log\frac{\gamma k\rho}{2}\right] \quad -- (D.9)$$

(natural log)

$$\langle u^a \cdot u^a\rangle = \iint E^a \cdot I_s^a\, dS = -\frac{\pi d\, (I_s^a)^2 k^2 a}{\cancel{k}\, 4\omega\varepsilon\, \cancel{(I_s)}}\left[1 - j\frac{2}{\pi}\log\frac{\gamma k d}{4}\right] \text{ ~~for a << λ~~}$$

$$Z_{in} = \frac{k^2 a}{4\pi\varepsilon}\left[1 - j\frac{2}{\pi}\log\frac{\gamma k d}{4}\right] \quad \text{for } a \ll \lambda$$

$$= \frac{k\eta a}{4}\left(1 - j\frac{2}{\pi}\log\frac{\gamma k d}{4}\right)$$

2 $\boxed{7-36}$ Since $Z_{in} = -\dfrac{\langle a,a\rangle}{I^2} = \dfrac{-\iint \bar{E}^a \cdot \bar{I}_s^a \, dS}{\pi d\, I^2}$ knowing E^a, I_s^a & I will be sufficient to evaluate Z_{in}.

$$E_z^a = \frac{-k^2 I_s^a}{4\omega\varepsilon} H_0^{(2)}(k\rho) \quad \text{and} \quad H_\varphi^a = -\frac{k I_s^a}{4j} H_0^{(2)\prime}(k\rho) = \frac{k I_s^a}{4j} H_1^{(2)}(k\rho)$$

$$\text{Now } I = \int_0^{2\pi} H_\varphi^a \cdot \rho \, d\varphi \Big|_{\rho=\frac{d}{2}} = \frac{2\pi\rho k I_s^a}{4j} H_1^{(2)}(k\rho)\Big|_{\rho=\frac{d}{2}} = \frac{\pi d k I_s^a H_1^{(2)}(\frac{kd}{2})}{4j}$$

$$\langle E_z^a, I_s^a\rangle = -\frac{k^2}{4\omega\varepsilon} H_0^{(2)}\left(\frac{kd}{2}\right) \pi d \int_0^a \cos^2 k(a-z)\, dz$$

$$= -\frac{k^2}{4\omega\varepsilon} H_0^{(2)}\left(\frac{kd}{2}\right) \frac{\pi d}{2}\left[a + \frac{\sin 2ka}{2k}\right]$$

$$\text{Hence } Z_{in} = -\frac{k^2}{4\pi\varepsilon} \frac{H_0^{(2)}(\frac{kd}{2})}{\pi d} \frac{\pi d}{2} \frac{\left[a + \frac{\sin 2ka}{2k}\right] 16}{\pi^2 d^2 k^2 \cos^2 ka \left\{H_1^{(2)}(\frac{kd}{2})\right\}^2}$$

$$= -\frac{2}{\omega\varepsilon} \frac{\left(a + \frac{\sin 2ka}{2k}\right) H_0^{(2)}(\frac{kd}{2})}{\left[\pi d \cos ka \; H_1^{(2)}(\frac{kd}{2})\right]^2}$$

$\boxed{7-37}$

$$Z_{in} = \frac{\langle u,v\rangle^2 - \langle u,u\rangle\langle v,v\rangle}{I_v^2\langle u,u\rangle - 2 I_u I_v \langle u,v\rangle + I_u^2 \langle v,v\rangle} \qquad \text{from (7.99)}$$

Here $I^u = \cos k(a-z)$; $I^v = 1$; hence

$$\langle u,u\rangle = -\frac{k^2}{4\omega\varepsilon} H_0^{(2)}\left(\frac{kd}{2}\right) \frac{1}{2}\left(a + \frac{\sin 2ka}{2k}\right) \qquad \text{from } \boxed{7-36}$$

$$\langle v,v\rangle = -\frac{k^2}{4\omega\varepsilon}\, a\, H_0^{(2)}\left(\frac{kd}{2}\right) \qquad \text{from } \boxed{7-35}$$

$$\langle u,v\rangle = \iint \bar{E}^u \cdot \bar{J}^v \, ds = \int_0^a \frac{-k^2}{4\omega\varepsilon} H_0^{(2)}\left(\frac{kd}{2}\right) \cos k(a-z)\, dz$$

$$= -\frac{k^2}{4\omega\varepsilon} H_0^{(2)}\left(\frac{kd}{2}\right) \frac{\sin(ka)}{k}$$

$$\therefore Z_{in} = \frac{\left[\frac{k^2}{4\omega\varepsilon} H_0^{(2)}(\frac{kd}{2}) \frac{\sin ka}{k}\right]^2 - \left[\frac{k^2}{4\omega\varepsilon} H_0^{(2)}(\frac{kd}{2})\right]^2 \frac{a}{2}\left(a + \frac{\sin 2ka}{2k}\right)}{-\frac{k^2}{4\omega\varepsilon} H_0^{(2)}(\frac{kd}{2})\left[\frac{1}{2}\left(a + \frac{\sin k2a}{2k}\right) - 2\frac{\sin ka}{k}\cos ka + \cos^2 ka \cdot a\right]}$$

$$= -\frac{\frac{k^2}{4\omega\varepsilon} H_0^{(2)}(\frac{kd}{2})\left[\frac{\sin^2 ka}{k^2} - \frac{a^2}{2} - \frac{a \sin 2ka}{4k}\right]}{\frac{a}{2} + \frac{\sin 2ka}{4k} - \frac{\sin 2ka}{k} + \cos^2 ka \cdot a} \qquad \text{for } a = \frac{\lambda}{4},\ ka = \frac{\pi}{2}$$

$$= -\frac{k\eta}{4a} H_0^{(2)}\left(\frac{kd}{2}\right)\left[\frac{1}{k^2} - \frac{\lambda^2}{32}\right] = \eta\, H_0^{(2)}\left(\frac{kd}{2}\right) \times (\cdot 0386)$$

$\boxed{7-38}$ Given $J_y = \hat{u}_y \cos\frac{\pi y}{a}$ and

$$L_e = \frac{\pi}{2\lambda}\left|\eta \frac{\left(\oint J_y^a e^{jkx} dl\right)^2}{\oint \bar{E}_p^a \cdot \bar{J}^a \, dl}\right|^2$$

— (7.124) Harrington

Now $\oint J_y^a e^{jkx} dl \Big|_{x=0} = \int_{-a/2}^{a/2} \cos\frac{\pi y}{a} = \frac{2a}{\pi}$

and $\oint \bar{E}^a \cdot \bar{J}^a \, dl = \int_{-a/2}^{a/2} E_y^a J_y^a \, dy$

$$= \int_{-a/2}^{a/2} E_y^a J_y^{a*} \, dy = -P, \text{ where } P$$

is the complex power per unit length supplied by J_y^a. But the ribbon have already been analyzed in Sec 4.12 and $Z_{elec\ rib} = \frac{\eta^2}{2} Y_{apert}$.

In defining $Y_{apert} = \frac{P^*}{|V|^2}$ (p. 185)

when V is the voltage per unit length of the aperture. Hence

$P = |I_y|^2 Z_{elec\ rib}$ where I is the current per unit length & so

$$P = \frac{\eta^2}{2} Y_{apert} = Z_{elec\ rib} \quad (\text{since } |I_y| = 1)$$

$$\therefore L_e = \frac{\pi}{2\lambda}\left|\eta \frac{4a^2}{\pi^2} \frac{2}{\eta^2 Y_{apert}}\right|^2$$

$$= \frac{32a^4}{\pi^3 \lambda}\left|\frac{1}{\eta Y_{apert}}\right|^2$$

$\boxed{7-39}$ Here

$$A_e = \pi \left|\frac{\eta}{\lambda} \frac{\left(\oiint J_e^a e^{jkx} ds\right)^2}{\oiint \bar{E}^a \cdot \bar{J}^a \, ds}\right|^2$$

— (7.115) Harrington

Since $\int_{-L/2}^{L/2} \cos kz = \frac{2}{k}\Big|_{L=\lambda/2} = \frac{\lambda}{\pi}$

and $\langle a, a\rangle = 73 = \oiint \bar{E}^a \cdot \bar{J}^a ds$

$$\therefore A_e = \pi \left|\frac{\eta}{\lambda} \frac{\lambda^2}{\pi^2} \cdot \frac{1}{73}\right|$$

$$= 0.86\lambda^2$$

$\boxed{7-40}$

Same as $\boxed{7-41}$ with

$\theta = \theta'$

$\boxed{7-41}$ Let the E-field be of the form

$$\bar{E} = E_0 \left(\bar{u}_z \sin\theta + \bar{u}_\rho \cos\theta\right) e^{jkz\cos\theta}$$

The current induced on the plate then has a form $J_z = \sin\theta\, e^{jkz\cos\theta}$

Hence $\oint\!\!\oint \bar{u}_z \cdot \bar{J}^a_{transmitter}\, ds$

$$= \frac{2}{k} \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} = \frac{\lambda}{\pi} \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}$$

Similarly $\oint\!\!\oint \bar{u}_z \cdot \bar{J}^a_{receiver}\, ds$

$$= \frac{\lambda}{\pi} \frac{\cos(\pi/2 \cos\theta')}{\sin\theta'}$$

. Hence application of (7.135) yields

$$A_e = \pi \left| \frac{\eta}{\lambda} \frac{\lambda^2}{\pi^2} \frac{1}{73} \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \cdot \frac{\cos\left(\frac{\pi}{2}\cos\theta'\right)}{\sin\theta'} \right|^2$$

$$\approx 0.86\,\lambda^2 \left[\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \cdot \frac{\cos\left(\frac{\pi}{2}\cos\theta'\right)}{\sin\theta'} \right]^2$$

$\boxed{7-42}$ By the terms of the problem, the currents induced on the obstacle is given by

$$\bar{J}^d = j\omega(\varepsilon - \varepsilon_0)\bar{E} = K_e \bar{E}, \text{ and}$$

$$\bar{M}^d = j\omega(\mu - \mu_0)\bar{H} = K_m \bar{H}$$

(from 7-136). Hence (7-137) can be replaced by

$$-V_r^s = \iiint \left[(\bar{E}^i)^r \cdot (\bar{J}^d)^t - (\bar{H}^i)^r \cdot (\bar{M}^d)^t \right] d\tau$$

and (7-138) by

$$-V_r^s = \iiint \Big[K_e^{-1} (\bar{J}^d)^r (\bar{J}^d)^t - (\bar{E}^s)^r \cdot (\bar{J}^d)^t + K_m^{-1} (\bar{M}^d)^r (\bar{M}^d)^t - (\bar{H}^s)^r \cdot (\bar{M}^d)^t \Big] d\tau$$

$$= F(C_r, C_t) - \langle C_r, C_t \rangle$$

Hence (7.142) can be expressed as

$$-V_r^s = \frac{\left[\iiint \{ (\bar{E}^i)^r \cdot (\bar{J}^a)^t - (\bar{H}^i)^r \cdot (\bar{M}^a)^t \} d\tau\right]}{\iiint \left[K_e^{-1} (\bar{J}^a)^r \cdot (\bar{J}^a)^t - K_m^{-1} (\bar{M}^a)^r \cdot (\bar{M}^a)^t \right] d\tau}$$

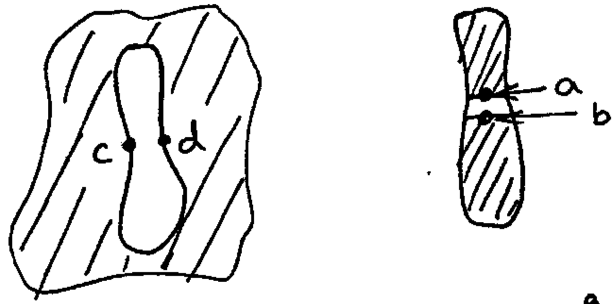
$$\times \frac{\left[\iiint \{ (\bar{E}^i)^t \cdot (\bar{J}^a)^r - (\bar{H}^i)^t \cdot (\bar{M}^a)^r \} d\tau\right]}{-\iiint \{ (\bar{E}^a)^r \cdot (\bar{J}^a)^t - (\bar{H}^a)^r \cdot (\bar{M}^a)^t \} d\tau}$$

when the transmitter and receiver are represented by the same source (7.143) would then be replaced by

$$\text{Echo} = \frac{\left(\frac{1}{I} \iiint \{ \bar{E}^i \cdot \bar{J}^a - \bar{H}^i \cdot \bar{M}^a \} d\tau\right)^2}{\iiint \{ K_e^{-1} (J^a)^2 - K_m^{-1} (M_a)^2 \} d\tau}$$

$$\overline{-\iiint \{ \bar{E}^a \cdot \bar{J}^a - \bar{H}^a \cdot \bar{M}^a \} d\tau}$$

$\boxed{7-43}$



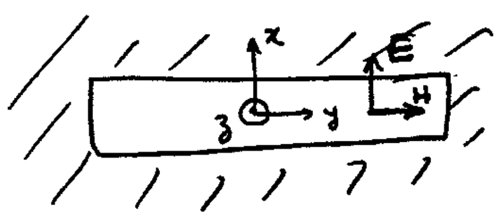
$$I = 2\int_a^b \bar{H}_s \cdot \bar{ds} \qquad V = -\int_b^a \bar{E}_d \cdot \bar{ds}$$

$$V = \int_c^d \bar{E}_s \cdot \bar{ds} \qquad I = \oint \bar{H}_d \cdot \bar{ds} = 2\int_c^d \bar{H}_d \cdot \bar{ds}$$

$$Y_s = \frac{2\int_a^b \bar{H}_s \cdot \bar{ds}}{\int_c^d \bar{E}_s \cdot \bar{ds}} \qquad Z_d = \frac{-\int_b^a \bar{E}_d \cdot \bar{ds}}{2 \cdot \int_c^d \bar{H}_d \cdot \bar{ds}}$$

At points, sufficiently distant away the fields are essentially plane waves hence $\bar{E}_s = \eta \bar{H}_s$ and $\bar{E}_d = -\eta \bar{H}_d$. Also it is apparent that, mathematically these two problems are identical. It is necessary only to interchange $\bar{E}$ and $\bar{H}$ to pass from one problem to the other. Therefore, except for a constant, the solution obtained for E for the slot will be the same as the solution for H for the dipole and it is possible to write for the fields at any corresponding points $\bar{E}_s = k_1 \bar{H}_d$ where the subscripts s and d refer to the slot and dipole respectively. Similarly, the magnetic field of the slot and the dipole field of the dipole are related by $\bar{H}_s = k_1 \bar{E}_d$. Hence $\frac{Z_d}{Y_s} = -\frac{k_1}{4k_2} = \eta^2/4$.

$\boxed{7-44}$



Let $\bar{E}^i = \bar{u}_x$; $H^i = \frac{\bar{u}_y}{\eta}$ $\therefore M_s = \bar{E} \times \bar{n} = -\bar{u}_y$

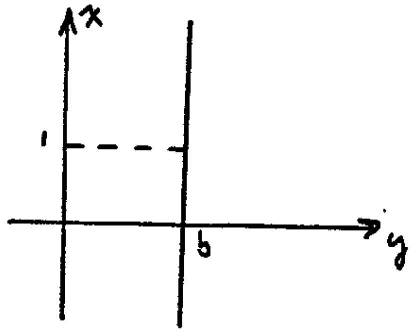
$$\left(\iint \bar{H}^i \cdot \bar{M}_s^a \, ds\right)^2 = \left(\frac{\lambda \omega}{2\eta}\right)^2 = \frac{\lambda^2 \omega^2}{4\eta^2}$$

and $P^i = \frac{\omega \lambda}{2\eta}$ also $\iint \bar{H}^a \cdot \bar{M}_s^a \, ds = |V|^2 Y_{apert} = \omega^2 Y_{apert}$.

$$\therefore T = \frac{2\eta}{\omega\lambda} \frac{\omega^2 \lambda^2}{4\eta^2 \omega^2 Y_{apert}} = \frac{\lambda}{2\omega\eta Y_{apert}} \qquad \text{But } Y_{apert} = \frac{4 \times 73}{120\pi \eta}$$

$$\therefore T = \frac{\lambda}{\omega} \frac{120\pi\, \eta}{2\eta \times 4 \times 73} = 0.645 \frac{\lambda}{\omega}$$

6 8-1



From prob. 2-28:

$$\psi_n^e = \cos\frac{n\pi y}{b}, \quad n = 1, 2, \ldots$$

$$\psi_n^m = \sin\frac{n\pi y}{b}, \quad n = 0, 1, 2, \ldots$$

For TM case:

$$e^m = -\nabla_t \psi_n^m = -\frac{n\pi}{b}\cos\frac{n\pi y}{b}\,\bar{u}_y$$

$$\int_0^1\int_0^b e^m \cdot e^m\, dx\, dy = \left(\frac{n\pi}{b}\right)^2 \frac{b}{2}$$

$\therefore$ Normalized eigenfunction is

$$\Psi_n^m = \frac{\sqrt{2b}}{n\pi}\sin\frac{n\pi y}{b}, \quad n = 1, 2, \ldots$$

For $n = 0$,

$$\psi_0^m = A y, \quad \int_0^1\int_0^b A\, dx\, dy = b$$

$$\Psi_0^m = \frac{y}{\sqrt{b}}$$

For TE case,

$$e^e = \bar{u}_z \times \nabla_t \psi_n^e = \frac{n\pi}{b}\sin\frac{n\pi y}{b}\,\bar{u}_z$$

$$\int_0^1\int_0^b e^m \cdot e^m\, dx\, dy = \left(\frac{n\pi}{b}\right)^2 \frac{b}{2}$$

$$\Psi_n^e = \frac{\sqrt{2b}}{n\pi}\cos\frac{n\pi y}{b}, \quad n = 1, 2, \ldots$$

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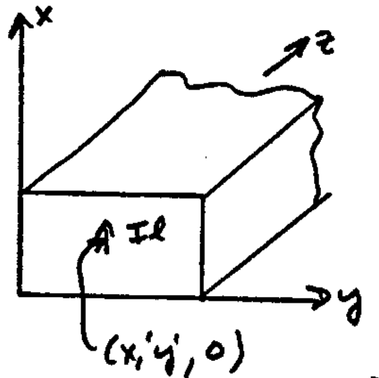
2 An arbitrary field inside a waveguide can be expressed as a sum over all possible modes.

8-2 (cont'd)

Take $z' = 0$



For $z < 0$,

$$\bar{E} = \sum_m\sum_n \left[V_{mn}^{m-}\,\bar{e}^m + V_{mn}^{e-}\,\bar{e}^e\right] e^{\gamma z}$$

$$\bar{H} = \sum_m\sum_n \left[\frac{V_{mn}^{m-}\,\bar{h}^m}{Z_0^m} + \frac{V_{mn}^{e-}\,\bar{h}^e}{Z_0^e}\right] e^{\gamma z}$$

and for $z > 0$,

$$\bar{E} = \sum_m\sum_n \left[V_{mn}^{m+}\,\bar{e}^m + V_{mn}^{e-}\,\bar{e}^e\right] e^{-\gamma z}$$

$$\bar{H} = \sum_m\sum_n \left[\frac{V_{mn}^{m+}\,\bar{h}^m}{Z_0^m} + \frac{V_{mn}^{e+}\,\bar{h}^e}{Z_0^e}\right] e^{-\gamma z}$$

The E field must be continuous at $z = 0$ so

$$V_{mn}^{m+} = V_{mn}^{m-} = V_{mn}^m$$

and $V_{mn}^{e+} = V_{mn}^{e-} = V_{mn}^e$

$$\psi_{mn}^e = \frac{1}{\pi}\sqrt{\frac{ab\,\varepsilon_n\varepsilon_m}{(mb)^2 + (na)^2}}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$$

$$\psi_{mn}^m = \frac{2}{\pi}\sqrt{\frac{ab}{(mb)^2 + (na)^2}}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$\psi_{0n}^e = \frac{1}{\pi n}\sqrt{\frac{2b}{a}}\cos\frac{n\pi y}{b}, \quad n = 1, 2, \ldots$$

$$\psi_{0n}^m = 0, \quad n = 1, 2, \ldots$$

$$\bar{e}^e = \bar{u}_y \frac{\partial}{\partial x}\psi_{mn}^e - \bar{u}_x \frac{\partial}{\partial y}\psi_{mn}^e$$

$$\bar{h}^e = -\bar{u}_x \frac{\partial}{\partial x}\psi_{mn}^e - \bar{u}_y \frac{\partial}{\partial y}\psi_{mn}^e$$

8-2 (cont.)

$$\bar{e}^m = -\bar{u}_x \frac{\partial}{\partial x} \psi_{mn}^m - \bar{u}_y \frac{\partial}{\partial y} \psi_{mn}^m$$

$$\bar{h}^m = -\bar{u}_y \frac{\partial}{\partial x} \psi_{mn}^m + \bar{u}_x \frac{\partial}{\partial y} \psi_{mn}^m$$

For $m, n, \neq 0$, $z > 0$:

$$H_x^+ = \sum_m \sum_n 2 \left[\frac{n V_{mn}^m}{Z_0^m b} + \frac{m V_{mn}^e}{a Z_0^e} \right] \sqrt{\frac{a b}{(mb)^2 + (na)^2}} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} e^{-\gamma z}$$

$$H_y^+ = \sum_m \sum_n 2 \left[\frac{n V_{mn}^e}{b Z_0^e} - \frac{m V_{mn}^m}{a Z_0^m} \right] \sqrt{\frac{a b}{(mb)^2 + (na)^2}} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} e^{-\gamma z}$$

$H_x^+ = 0$ at $z = 0$ so,

$$V_{mn}^m = -\frac{m b Z_0^m}{n a Z_0^e}$$

also $$-\bar{u}_x (H_y^+ - H_y^-)\Big|_{z=0} = I\ell\, \delta(x-x')\, \delta(y-y')\, \bar{u}_x$$

This gives $$4 \sum_m \sum_n \left[\frac{m V_{mn}^m}{a Z_0^m} - \frac{n V_{mn}^e}{b Z_0^e} \right] \sqrt{\frac{a b}{(mb)^2 + (na)^2}} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

$$= I\ell\, \delta(x-x')\, \delta(y-y')$$

multiply through by $\cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$ and integrate from 0 to a and 0 to b.

$$V_{mn}^m = \underbrace{m Z_0^m \sqrt{\frac{b}{a}} \frac{I\ell}{\sqrt{(mb)^2 + (na)^2}} \cos \frac{m\pi x'}{a} \sin \frac{n\pi y'}{b} e^{-\gamma z}}_{f_{mn}}$$

$$V_{mn}^e = -n Z_0^e \sqrt{\frac{a}{b}} f_{mn}$$

For $m = 0$, $n = 1, 2, \ldots$

Eqn. to be satisfied is:

$$-\sum_n \frac{2 V_{0n}^e}{Z_0^e} \sqrt{\frac{2}{ab}} \sin \frac{n\pi y}{b} = I\ell\, \delta(x-x')(y-y')$$

multiply through by $\sin \frac{n\pi y}{b}$ and integrate from 0 to b and 0 to a,

$$V_{0n}^e = -I\ell Z_0^e \sqrt{\frac{1}{2ab}} \sin \frac{n\pi y'}{b} e^{-\gamma z}$$

8-3 From eqns. 1-61:

$$w_e = \frac{1}{2}\iint_{cyl.} \varepsilon |E|^2 dS$$

$$w_m = \frac{1}{2}\iint_{cyl.} \mu |H|^2 dS$$

For each mode,

$$E = E^e + E_t^m + E_z^m$$

$$H = H^m + H_t^e + H_z^e$$

$$|E|^2 = \left| e^e V^e + e^m V^m + \frac{(k_c^m)^2}{j\omega\varepsilon}\Psi^m Z^m \right|^2$$

and $|e^e| = |e^m| = 1$

$$Z^m = I^m$$

$$\iint |\Psi_m|^2 = \iint \frac{|\nabla_t \Psi_m|^2}{(k_c^m)^2} = \frac{1}{(k_c^m)^2}$$

Because $e^m = -\nabla_t \psi^m$.

$$\therefore |\Psi^m| = \frac{1}{k_c^m}$$

So $|E|^2 = |V^e|^2 + |V^m|^2 + \left(\frac{k_c^m}{\omega\varepsilon}\right)^2 |I^m|^2$

Thus for all modes,

$$w_e = \frac{1}{2}\sum_i \varepsilon |V_i^e|^2 + \varepsilon |V_i^m|^2 + \left(\frac{k_{c_i}^m}{\omega\varepsilon}\right)^2 \varepsilon |I_i^m|^2$$

$$|H|^2 = \left| h^m I^m + h^e I^e + \frac{k_c^2}{j\omega\mu}\Psi^e Z^e \right|^2$$

$$|h^m| = |h^e| = 1$$

$$Z^e = V^e$$

$$|\Psi^e| = \frac{1}{k_c^e}$$

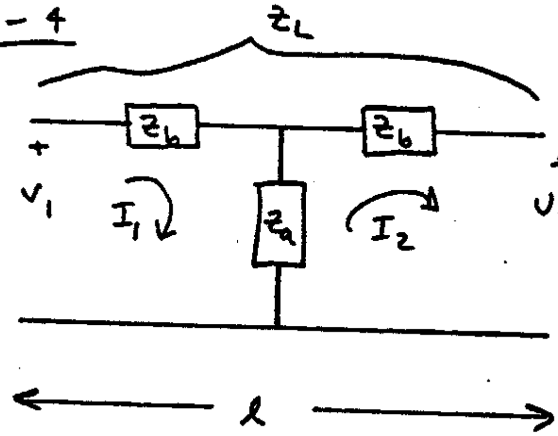
8-3 (cont.)

$$|H|^2 = |I^m|^2 + |I^e|^2 + \left(\frac{k_c^e}{\omega\mu}\right)^2 |V^e|^2$$

and

$$w_m = \frac{1}{2}\sum_i \mu |I_i^m|^2 + \mu |I_i^e|^2 + \mu\left(\frac{k_{c_i}^e}{\omega\mu}\right)^2 |V_i^e|^2$$

8-4



$$I_1 Z_b + (I_1 - I_2) Z_a = V_1$$

$$I_2 Z_b + (I_2 - I_1) Z_a = V_2$$

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_b + Z_a & -Z_a \\ -Z_a & Z_b + Z_a \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

From analogy with transmission line formulas,

$$Z_{11} = -jZ_0 \cot\beta l = Z_a + Z_b$$

$$Z_{12} = -jZ_0 \csc\beta l = Z_a$$

$$Z_b = jZ_0 \csc\beta l - jZ_0 \cot\beta l$$

$$= jZ_0 \tan\frac{\beta l}{2}$$

8-5 For TE modes,

$$P = \bar{u}_z \cdot \iint \bar{E}^e \times \bar{H}^{e*} dS$$

$$= \bar{u}_z \cdot \iint (\bar{u}_z \times \nabla_t \psi^e)\left(Z^e \frac{1}{j\omega\mu}\right) \times (\nabla_t \psi^e) \frac{dZ^{e*}}{dz}$$

8-5 (cont.)

$$P = -\bar{u}_z \cdot \iint (\nabla_t \psi^e)^2 \, \bar{u}_z \times \frac{Z^e}{j\omega\mu} \frac{dZ^{e*}}{dz} \, dS$$

If $Z^e = A e^{-jk_z z}$

$$\frac{dZ_e^*}{dz} = jk_z A e^{jk_z z}$$

$$P = -\bar{u}_z \cdot \iint (\nabla_t \psi_e)^2 \, \bar{u}_z \frac{k_z}{\omega\mu} \, dS$$

$$= \frac{k_z}{\omega\mu} \bar{u}_z$$

since $\iint (\nabla_t \psi^e)^2 = 1$

$$P_d = R\left[\oint (\bar{l} \cdot \bar{H}_e)^2 dl + \oint (\bar{u}_z \cdot \bar{H}_e)^2 dl\right]$$

$$= R\left[\oint \frac{k_z^2}{\omega^2\mu^2}\left(\frac{\partial \psi^e}{\partial l}\right)^2 dl + \oint \frac{k_c^4}{\omega^2\mu^2}(\psi_e)^2 dl\right]$$

$$\alpha_c = \frac{P_d}{2P} = \frac{R k_z}{2\omega\mu}\left[\oint \left(\frac{\partial \psi_e}{\partial l}\right)^2 dl + \frac{k_c^4}{k_z^2}\oint (\psi^e)^2 dl\right]$$

$$= \frac{R k_z}{2\eta k} \oint \left[\left(\frac{\partial \psi_e}{\partial l}\right)^2 + \frac{k_c^4}{k_z^2}(\psi_e)^2\right] dl$$

$(k_z = \beta)$.

For TM modes,

$$P_z = \frac{-\gamma}{j\omega\varepsilon} = \frac{\beta}{\omega\varepsilon}$$

$$P_d = \mathrm{Re}\left\{R \oint |H_{tan}|^2 dl\right\}$$

$$\bar{H}_{tan} = \bar{l} \cdot (-\bar{u}_z \times \nabla_t \psi^m)$$

$$= -\frac{\partial \psi^m}{\partial n}$$

8-5 (cont.)

$$\alpha = \frac{P_d}{2P_z} = \frac{R\omega\varepsilon}{2\beta} \oint \left(\frac{\partial \psi_m}{\partial n}\right)^2 dl$$

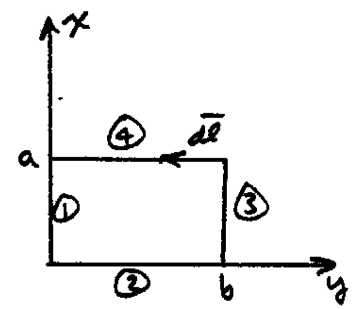
$$k = \omega\sqrt{\mu\varepsilon}$$

$$\alpha = \frac{Rk}{2\eta\beta} \oint \left(\frac{\partial \psi_m}{\partial l}\right)^2 dl$$

8-6 For rectangular guides:

$$\psi^e = \underbrace{\frac{1}{\pi}\sqrt{\frac{ab\,\varepsilon_m \varepsilon_n}{(mb)^2 + (na)^2}}}_{B} \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b}$$

$$\psi^m = \underbrace{\frac{2}{\pi}\sqrt{\frac{ab}{(mb)^2 + (na)^2}}}_{A} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$



at ③:

$$\frac{\partial \psi^m}{\partial n} = \frac{\partial \psi^m}{\partial y}\Big|_{y=b}$$

$$= A\frac{n\pi}{b} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} = -\frac{\partial \psi^m}{\partial y} \text{ at ①}$$

at ④

$$\frac{\partial \psi^m}{\partial n} = \frac{\partial \psi^m}{\partial x} = A\frac{m\pi}{a} \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

$$= -\frac{\partial \psi^m}{\partial y} \text{ at ②}$$

$$\alpha_c = \frac{Rk}{2\eta\beta}\left[2\int_0^a A^2 \left(\frac{n\pi}{b}\right)^2 \sin^2\frac{m\pi x}{a} \, dy + 2\int_0^b A^2 \left(\frac{m\pi}{a}\right)^2 \sin^2\frac{n\pi x}{a} \, dx\right]$$

$$= \frac{RkA^2}{2\eta\beta}\left[a\left(\frac{n\pi}{b}\right)^2 + b\left(\frac{m\pi}{a}\right)^2\right]$$

$$\alpha_c = \frac{Rk}{2\eta\beta}\frac{4}{\pi^2}\frac{ab\pi^2}{(mb)^2+(na)^2}\left[\frac{n^2a^3+m^2b^3}{a^2b^2}\right]$$

$$\beta = k\sqrt{1-(f_c/f)^2}$$

$\therefore$ so

$$\alpha_c = \frac{2R}{\eta ab\sqrt{1-(f_c/f)^2}}\left[\frac{m^2b^3+n^2a^3}{m^2b^2+n^2a^2}\right]$$

as in prob. 4-4.

For TM_{mn} modes.

For TE_{mn} modes:

at ③ $\frac{\partial\psi^e}{\partial l} = \frac{\partial\psi^e}{\partial x}\Big|_{y=b}$

$$= -B\frac{m\pi}{a}\sin\frac{\pi m x}{a}$$

$$= -\frac{\partial\psi}{\partial l} \text{ at ①}$$

at ②, $\frac{\partial\psi^e}{\partial l} = \frac{\partial\psi^e}{\partial y}\Big|_{x=0} = -B\frac{n\pi}{b}\sin\frac{n\pi y}{b}$

$$\alpha_c = \frac{R\beta}{2\eta k}\left[2\int_0^a B^2\left(\frac{m\pi}{a}\right)^2\sin^2\frac{m\pi x}{a}dx + 2\int_0^b B^2\left(\frac{n\pi}{b}\right)^2\sin^2\frac{n\pi y}{b}dy + \frac{k_c^4}{\beta^2}\left\{2\int_0^a B^2\cos^2\frac{m\pi x}{a}dx + 2\int_0^b B^2\sin^2\frac{n\pi y}{b}dy\right\}\right]$$

$$\alpha_c = \frac{1}{\pi^2}\frac{R\beta}{2\eta k}\left[\frac{4ab}{(mb)^2+(na)^2}\right]\times$$

8-6 (cont.)

$$\left[a\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2 b + \frac{k_c^4}{\beta^2}(a+b)\right]$$

$$\frac{\beta}{k} = \sqrt{1-(f_c/f)^2}$$

$$\alpha_c = \frac{R\beta}{2\eta k}\left[\frac{k_c^4}{\beta^2}(a+b) + \frac{bm^2+an^2}{ab}\right]\times\left[\frac{4ab}{(mb)^2+(na)^2}\right]$$

$$= \frac{2R}{\eta}\left[\frac{(a+b)(f_c/f)^2}{ab\sqrt{1-(f_c/f)^2}} + \sqrt{1-(f_c/f)^2}\,\frac{bm^2+an^2}{b^2m^2+a^2n^2}\right]$$

Now for TE_{0n} modes,

$$\psi^e = \underbrace{\frac{1}{\pi}\sqrt{\frac{2ab}{(na)^2}}}_{C}\cos\frac{n\pi y}{b}$$

at ③ & ①, $\frac{\partial\psi}{\partial l} = 0$

at ② & ④,

$$\left|\frac{\partial\psi^e}{\partial l}\right| = C\frac{n\pi}{b}\sin\frac{n\pi y}{b}$$

$$\alpha_c = \frac{R\beta}{2\eta k}\left[2\int_0^b C^2\left(\frac{n\pi}{b}\right)^2\sin^2\frac{n\pi y}{b}dy + \frac{k_c^4}{\beta^2}\left\{2\int_0^a C^2\cos^2\frac{n\pi y}{b}dx + 2\int_0^b C^2\left(\frac{n\pi}{b}\right)^2\cos^2\frac{n\pi y}{b}dy\right\}\right]$$

$$= \frac{R\beta}{2\eta k\pi^2}\frac{2ab}{(na)^2}\left[\left(\frac{n\pi}{b}\right)^2\frac{1}{b} + \right.$$

8-6 (cont.)

$$+ \frac{k_c^4}{\beta^2}\left[2a + b\left(\frac{n\pi}{b}\right)^2\right]\Bigg]$$

$$\alpha_c = \frac{R}{a\eta\sqrt{1-(f_c/f)^2}}\left[1+\frac{2a}{b}\left(\frac{f_c}{f}\right)^2\right]$$

Now for cylindrical case:

TM case,

$$\frac{\partial\psi_m}{\partial\rho} = \frac{\partial}{\partial\rho}\left[\sqrt{\frac{\varepsilon_n}{\pi}}\,\frac{J_n(x_{np}\rho/a)}{x_{np}J_{n+1}(x_{np})}\cos n\phi\right]$$

$$= \frac{1}{a}\sqrt{\frac{\varepsilon_n}{\pi}}\,\frac{J_n'(x_{np}\rho/a)}{J_{n+1}(x_{np})}\cos n\phi$$

$$\alpha_c = \frac{1}{2}\frac{R}{\eta}\frac{k}{\beta}\int_0^{2\pi}\frac{\varepsilon_n}{\pi a}\left[\frac{J_n'(x_{np}\rho/a)}{J_{n+1}(x_{np})}\right]^2_{\rho=a}\cos^2 n\phi\, d\phi$$

$$\frac{J_n'(x_{np})}{J_{n+1}(x_{np})} = \frac{-J_{n+1}(x_{np}) + \frac{n}{x_{np}}J_n(x_{np})}{J_{n+1}(x_{np})} = -1$$

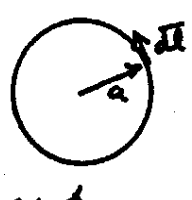
$$\int_0^{2\pi}\frac{\varepsilon_n}{2\pi}\cos^2 n\phi = 1,\ n\neq 0$$
$$= 1,\ n=0$$

$$\alpha_c = \frac{Rk}{a\eta\beta} = \frac{R}{a\eta}\frac{1}{\sqrt{1-(f_c/f)^2}}$$

TE case:

$$\psi^e = \sqrt{\frac{\varepsilon_n}{\pi[x_{np}'^2-n^2]}}\,\frac{J_n(x_{np}'\rho/a)}{J_n(x_{np}')}\sin n\phi$$

$dl = a\,d\phi$



$$\frac{\partial\psi^e}{\partial l} = \frac{1}{\rho}\frac{\partial\psi^e}{\partial\phi}\Big|_{\rho=a}$$

$$= \frac{n}{a}\sqrt{\frac{\varepsilon_n}{\pi[x_{np}'^2-n^2]}}\cos n\phi$$

8-6 (cont.)

$$\frac{1}{2}\oint_0^{2\pi}\left(\frac{\partial\psi^e}{\partial l}\right)^2 a\,d\phi = \frac{n^2}{a}\frac{1}{[x_{np}'^2-n^2]}$$

because $\int_0^{2\pi}\frac{\varepsilon_n\cos^2 n\phi}{2\pi}\,d\phi = 1$

$$\frac{1}{2}\int_0^{2\pi}(\psi^e)^2 a\,d\phi = \frac{a}{[x_{np}'^2-n^2]}$$

because $\int_0^{2\pi}\frac{\varepsilon_n\sin^2 n\phi}{2\pi}\,d\phi = 1$

$k_c = \frac{x_{np}'}{a}$, $\beta^2 = k^2[1-(f_c/f)^2]$

Formula for α_c is:

$$\alpha_c = \frac{1}{2}\frac{R}{\eta}\frac{\beta}{k}\left[\oint\left(\frac{\partial\psi^e}{\partial l}\right)^2 dl + \frac{k_c^4}{\beta^2}\oint(\psi^e)^2 dl\right]$$

$$\alpha_c = \frac{R\beta}{\eta k}\left[\frac{n^2}{a[x_{np}'^2-n^2]} + \frac{k_c^4}{\beta^2}\frac{a}{[x_{np}'^2-n^2]}\right]$$

$$= \frac{Rk}{a\eta\beta}\left[\frac{n^2\beta^2}{k^2[x_{np}'^2-n^2]} + \frac{k_c^4a^2}{k^2[x_{np}'^2-n^2]}\right]$$

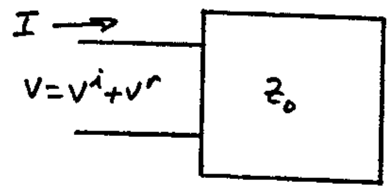
$$k_c^2a^2 = x_{np}'^2$$

$$= \frac{Rk}{a\eta\beta}\left[\frac{1}{[x_{np}'^2-n^2]}\left(n^2[1-(f_c/f)^2] + \frac{x_{np}'^2k_c^2}{k^2}\right)\right]$$

$$= \frac{Rk}{a\eta\beta}\left\{\frac{1}{[x_{np}'^2-n^2]}\left[n^2+[x_{np}'^2-n^2]\left(\frac{f_c}{f}\right)^2\right]\right\}$$

$$= \frac{R}{a\eta}\frac{1}{\sqrt{1-(f_c/f)^2}}\left[\frac{n^2}{[x_{np}'^2-n^2]}+\left(\frac{f_c}{f}\right)^2\right]$$

8-7



$$\Gamma = \frac{V^r}{V^i}, \quad V = V^i + \Gamma V^i$$
$$= V^i(1+\Gamma)$$

$$I = \frac{V}{Z_0} = \frac{V^i}{Z_0}(1+\Gamma)$$

$$I \cdot I^* = |I|^2 = \frac{|V^i|^2}{Z_0^2}(1+\Gamma)(1-\Gamma^*)$$

$$P = |I|^2 Z_0 = \frac{|V^i|^2}{Z_0}\left(1-|\Gamma|^2+2j\,\mathrm{Im}(\Gamma)\right)$$

$$= P_d + 2j\omega(W_m - W_e)$$

Equating real and imaginary parts:

$$P_d = \frac{|V^i|^2}{Z_0}(1-|\Gamma|^2)$$

$$W_m - W_e = \frac{1}{\omega}\frac{|V^i|^2}{Z_0}\mathrm{Im}(\Gamma)$$

8-8

$$[Z] = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \qquad I = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

From Eqn. 8-41,

$$P = \iint E \times H^* \cdot dS = \Sigma [V][I]^*$$

but $[V] = [I][Z]$

so

$$P = \Sigma [I][Z][I^*]$$
$$= P_d + j2\omega(W_m - W_e)$$

Take real part of both sides:

$$\mathrm{Re}(Z_{11}) I_1 I_1^* + \mathrm{Re}(Z_{12}) I_1 I_2^* +$$

8-8 (cont.)

$$\mathrm{Re}(Z_{21}) I_2 I_1^* + \mathrm{Re}(Z_{22}) I_2 I_2^* = P_d$$

If I_1 or $I_2 = 0$, since $P_d \geq 0$,

$$\left.\begin{aligned} \mathrm{Re}(Z_{11}) \geq 0 \\ \mathrm{Re}(Z_{22}) \geq 0 \end{aligned}\right\} \text{Eqn 8-70}$$

The coefficient of $\mathrm{Re}(Z_{12})$ and $\mathrm{Re}(Z_{21})$ is a minimum when:

$$I_1 = I_1 e^{j\theta}, \quad I_2 = I_2 e^{j\phi}$$

$$I_1 I_2 e^{j(\theta-\phi)}$$

$$\theta - \phi = 0 \text{ or } \pi .$$

$$\therefore \frac{I_1}{I_2} = \frac{I_1}{I_2} e^{j(\theta-\phi)} = \text{real no.}$$

Dividing through by $I_2 I_2^*$,

$$\mathrm{Re}(Z_{11})\left(\frac{I_1}{I_2}\right)^2 + \mathrm{Re}(Z_{12})\frac{I_1}{I_2}$$
$$+ \mathrm{Re}(Z_{21})\frac{I_1}{I_2} + \mathrm{Re}(Z_{22}) \geq 0$$

which is satisfied if $\left(\frac{I_1}{I_2}\right)$ is not a single root.

$$\therefore \mathrm{Re}(Z_{12})\mathrm{Re}(Z_{21}) - \mathrm{Re}(Z_{11})\mathrm{Re}(Z_{22}) \leq 0$$

or $\mathrm{Re}(Z_{11})\mathrm{Re}(Z_{22}) - \mathrm{Re}(Z_{12})\mathrm{Re}(Z_{21}) \geq 0$

which is one of Eqns. 8-72.

The Eqn. for $[Y]$ can be derived by using:

$$P = \Sigma [V][Y][V^*] .$$

8-9

$$Z_{01} = Z_{02} = 1$$

$$V_1 = V_1^i + V_1^r$$

$$I_1 = I_1^i + I_1^r = V_1^i - V_1^r$$

$$V_2 = V_2^i + V_2^r$$

$$I_2 = I_2^i + I_2^r = V_2^i - V_2^r$$

$$[V] = [Z][I] \Rightarrow V_1 = z_{11} I_1 + z_{12} I_2$$

$$V_2 = z_{21} I_1 + z_{22} I_2$$

$$V_1^i + V_1^r = z_{11}(V_1^i - V_1^r) + z_{12}(V_2^i - V_2^r)$$

$$V_2^i + V_2^r = z_{21}(V_1^i - V_1^r) + z_{22}(V_2^i - V_2^r)$$

$$V_2^i - V_2^r = \frac{V_1^i + V_1^r}{z_{12}} - \frac{z_{11}}{z_{12}}(V_1^i - V_1^r)$$

$$V_2^i + V_2^r = z_{21}(V_1^i - V_1^r) + \frac{z_{22}}{z_{12}}(V_1^i + V_1^r) - \frac{z_{11} z_{22}}{z_{12}}(V_1^i - V_1^r)$$

$$2V_2^i = \frac{V_1^i + V_1^r}{z_{12}} + \frac{z_{11}}{z_{12}}(V_1^i - V_1^r) + z_{21}(V_1^i - V_1^r) + \frac{z_{22}}{z_{12}}(V_1^i + V_1^r) - \frac{z_{11} z_{22}}{z_{12}}(V_1^i - V_1^r)$$

Also,

$$2V_2^r = -\frac{(V_1^i + V_1^r)}{z_{12}} - \frac{z_{11}}{z_{12}}(V_1^i - V_1^r) + z_{21}(V_1^i - V_1^r) + \frac{z_{22}}{z_{12}}(V_1^i + V_1^r) + \frac{z_{11} z_{22}}{z_{12}}(V_1^i - V_1^r)$$

Now
$$\begin{bmatrix} V_2^r \\ V_2^i \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} V_1^i \\ V_1^r \end{bmatrix}$$

8-9 (cont.)

$$V_2^r = T_{11} V_1^i + T_{12} V_1^r$$

$$V_2^i = T_{21} V_1^i + T_{22} V_1^r$$

$$2T_{11} = \frac{-1}{z_{12}} - \frac{z_{11}}{z_{12}} + z_{21} + \frac{z_{22}}{z_{12}} + \frac{z_{11} z_{22}}{z_{12}}$$

$$= z_{21} + \frac{1}{z_{12}}\left[z_{22} + z_{11} z_{22} - z_{11} - 1\right]$$

$$= z_{21} + \frac{1}{z_{12}}(1 + z_{11})(z_{22} - 1)$$

$$2T_{12} = \frac{-1}{z_{12}} + \frac{z_{11}}{z_{12}} - z_{21} + \frac{z_{22}}{z_{12}} - \frac{z_{11} z_{22}}{z_{12}}$$

$$= -z_{21} + \frac{1}{z_{12}}(1 - z_{11})(z_{22} - 1)$$

$$2T_{21} = \frac{1}{z_{12}} + \frac{z_{11}}{z_{12}} + z_{21} + \frac{z_{22}}{z_{12}} + \frac{z_{11} z_{22}}{z_{12}}$$

$$= z_{21} + \frac{1}{z_{12}}(1 + z_{11})(z_{22} + 1)$$

$$2T_{22} = \frac{1}{z_{12}} - \frac{z_{11}}{z_{12}} - z_{21} + \frac{z_{22}}{z_{12}} - \frac{z_{11} z_{22}}{z_{12}}$$

$$= -z_{21} + \frac{1}{z_{12}}(1 - z_{11})(z_{22} + 1)$$

For the lossless case, $[Z]$ is pure imaginary and for any complex no. z, $(-z)^* = z$

So
$$2T_{22}^* = (-z_{21})^* + \frac{1}{z_{12}^*}(1 + z_{11}^*)(z_{22}^* + 1)$$

$$= z_{21} - \frac{1}{z_{12}}(1 - z_{11})(-z_{22} + 1)$$

$$= z_{21} + \frac{1}{z_{12}}(1 - z_{11})(z_{22} - 1)$$

$$= 2T_{11}$$

$$T_{21}^* = z_{21}^* + \frac{1}{z_{12}^*}(1 - z_{11}^*)(z_{22}^* + 1)$$

8-9 (cont.)

$$= -z_{21} - \frac{1}{z_{12}}(1+z_{11})(-z_{22}+1)$$

$$= -z_{21} + \frac{1}{z_{12}}(1+z_{11})(z_{22}-1)$$

$$= 2T_{12}$$

8-10 $\bar{E}_t^1 = A(e^{-j\beta z} + e^{j\beta z})\bar{e}$

$$= 2A\cos\beta z\,\bar{e}$$

$$\bar{H}_t^1 = \frac{A}{z_0}(e^{-j\beta z} - e^{j\beta z})\bar{h}$$

$$= \frac{2A}{jz_0}\sin\beta z\,\bar{h}$$

$$E_t^2 = H_t^2 = 0$$

Boundary conditions at $z=-l$,

$$\bar{u}_z \times [H_t^1] = \bar{J}_s$$

$$\bar{E}_t^1 \times \bar{u}_z = \bar{m}_s$$

$$\bar{J}_s = \frac{2A}{jz_0}\sin\beta l\,\bar{e} \qquad (\bar{u}_z \times \bar{h} = -\bar{e})$$

$$\bar{m}_s = -2A\cos\beta l\,\bar{h} \qquad (\bar{e}\times\bar{u}_z) = -\bar{h}$$

$$\langle s,s\rangle = \iint \bar{E}\cdot\bar{J}_s + \bar{H}\cdot\bar{m}_s\, dS$$

$$= \frac{4A^2}{jz_0}\cos\beta l\sin\beta l + \frac{4A^2}{jz_0}\sin\beta l\cos\beta l$$

$$= \frac{4A^2}{jz_0}\sin 2\beta l$$ = self-reaction of unidirectional source.

8-11

8-11 (cont.)

8-13 $\quad f(x) = \sin\frac{\pi x}{c}$

$$\int_0^c \sin\frac{\pi x}{c}\sin\frac{n\pi x}{a}\,dx$$

$$= \int_0^c \left[\cos\pi x\left(\frac{1}{c}-\frac{n}{a}\right) - \cos\pi x\left(\frac{1}{c}+\frac{n}{a}\right)\right]dx$$

$$= \frac{\sin\pi\left(\frac{1}{c}-\frac{n}{a}\right)x}{\pi\left(\frac{1}{c}-\frac{n}{a}\right)}\Bigg|_0^c - \frac{\sin\pi\left(\frac{1}{c}+\frac{n}{a}\right)x}{\pi\left(\frac{1}{c}+\frac{n}{a}\right)}\Bigg|_0^c$$

$$= \frac{\sin\frac{n\pi c}{a}}{\pi\left(\frac{1}{c}-\frac{n}{a}\right)} + \frac{\sin\frac{n\pi c}{a}}{\pi\left(\frac{1}{c}+\frac{n}{a}\right)}$$

$$= \frac{4c}{\pi}\frac{\sin\frac{n\pi c}{a}}{\left[1-\left(\frac{nc}{a}\right)^2\right]}$$

$$\int_0^c \sin\frac{\pi x}{c}\sin\frac{\pi x}{a}\,dx = \frac{4\sin\frac{\pi c}{a}}{c\pi\left(\frac{1}{c^2}-\frac{1}{a^2}\right)}$$

$$= \frac{4c\sin\frac{\pi c}{a}}{\pi\left[1-\left(\frac{c}{a}\right)^2\right]}$$

$$B_a = \frac{-2\lambda_g}{\eta\pi^2 b}\left(\frac{c}{a}\right)^2\sum_{n=2}^{\infty}\left[\sqrt{\left(\frac{n}{2}\right)^2-\left(\frac{a}{\lambda_g}\right)^2}\right]\cdot$$

$$\cdot\left[\frac{\sin\frac{n\pi c}{a}}{1-\left(\frac{nc}{a}\right)^2}\right]^2$$

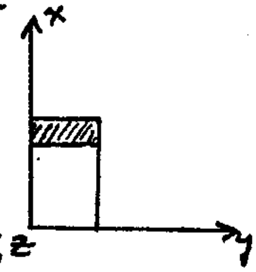
From Eqn 4-83,

$$\frac{B}{Y_0} = \frac{-B_a\eta\pi^2 ba\lambda_g}{c^2\lambda}\left[\frac{1-\left(\frac{c}{a}\right)^2}{\sin\frac{\pi c}{a}}\right]^2$$

$$= -\left(\frac{\eta B_a b}{a}\frac{\lambda_g}{\lambda}\right)\left[\frac{\pi a}{c}\,\frac{1-\left(\frac{c}{a}\right)^2}{\sin\frac{\pi c}{a}}\right]^2$$

8-14

$$\psi = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{a}\,e^{\gamma_n z}$$



$$\gamma_n = \sqrt{\left(\frac{n\pi}{a}\right)^2 - k^2}$$

$$\bar{H} = \nabla\times\psi\,\bar{u}_y$$

$$H_x = -\sum \gamma_n A_n \sin\frac{n\pi x}{a}\,e^{\gamma_n z}$$

$$H_y = 0\,,\quad H_z = \sum_n \frac{n\pi}{a}A_n\cos\frac{n\pi x}{a}\,e^{\gamma_n z}$$

The current on the diaphragm is backed by a magnetic conductor.

$$J_y = -H_x\big|_{z=0} = \sum \gamma_n A_n \sin\frac{n\pi x}{a}$$

$$\langle a,a\rangle_m = P^* = \frac{ab}{2}\sum_{n=0}^{\infty}(Y_0)_{n0}|E_{n0}|^2$$

where $E_{1n} = \frac{2}{a}\int_0^c dx\,E_y\big|_{z=0}\sin\frac{n\pi x}{a}$

$$= \frac{2}{a}\int_0^c f(x)\sin\frac{n\pi x}{a}\,dx$$

$$P = \frac{ab}{2}\left[(Z_0)_{10}(J_{10})^2\right] + \sum_{n=2}^{\infty}(Z_0)_{n0}|J_{n0}|^2$$

From Eqn 8-108,

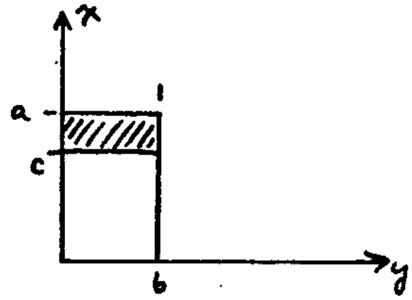
$$\frac{B}{2Y_0} = \frac{-I_m\langle a,a\rangle}{Re\langle a,a\rangle}$$

$$= -\frac{\sum_{n=2}^{\infty}\frac{\lambda_g\sqrt{n^2-\left(\frac{2a}{\lambda_g}\right)^2}}{2a}\left[\int_0^c f(x)\sin\frac{n\pi x}{a}dx\right]^2}{\left[\int_0^c f(x)\sin\frac{\pi x}{a}dx\right]^2}$$

$$\frac{B}{Y_0} = \frac{\frac{-2\lambda_g}{a}\sum_{n=2}^{\infty}\sqrt{\left(\frac{n}{2}\right)^2-\left(\frac{a}{\lambda_g}\right)^2}\left[\int_0^c f(x)\sin\frac{n\pi x}{a}dx\right]^2}{\left[\int_0^c f(x)\sin\frac{\pi x}{a}dx\right]^2}$$

8-15

$\bar{J}_s = \bar{u}_y\, g(x)$



On the window,

$$E_y\big|_{z=0} = -Z_0^{TE^x} \cdot H_x\big|_{z=0}$$

$$= Z_0^{TE^x} J_y\big|_{z=0}$$

$$= Z_0^{TE_x}\, \bar{u}_y\, g(x)$$

Hence E_{n0} are given by Eqn 4-73.

$$E_{n0} = \frac{2}{a}\int_c^a g(x)\sin\frac{n\pi x}{a}\,dx$$

Then as on p. 416,

$$\langle a,a\rangle = \frac{ab}{2}\sum_{n=1}^{\infty}(Y_0)_{n0}|E_{n0}|^2$$

and

$$(Y_0)_{n0} = \frac{j2a(Y_0)_{10}}{\lambda_g\sqrt{n^2-\left(\frac{2a}{\lambda_g}\right)^2}}$$

$\therefore$ Eqn 8-107 becomes:

$$\frac{2Y_0}{B} = -\frac{\sum_{n=2}^{\infty}|Y_0|_{n0}|E_{n0}|^2}{(Y_0)_{00}|E_{10}|^2}$$

$$\frac{Y_0}{B} = -\sum_{n=2}^{\infty}\frac{a}{\lambda_g\sqrt{n^2-\left(\frac{2a}{\lambda_g}\right)^2}}\frac{\left[\int_c^a \sin\frac{n\pi x}{a}g(x)\,dx\right]^2}{\left[\int_c^a \sin\frac{\pi x}{a}g(x)\,dx\right]^2}$$

$$\frac{Y_0}{B} = -\sum_{n=2}^{\infty}\frac{a}{2\lambda_g\sqrt{\left(\frac{n}{2}\right)^2-\left(\frac{a}{\lambda_g}\right)^2}}\frac{\left[\int_c^a \sin\frac{n\pi x}{a}g(x)\,dx\right]^2}{\left[\int_c^a \sin\frac{\pi x}{a}g(x)\,dx\right]^2}$$

8-16

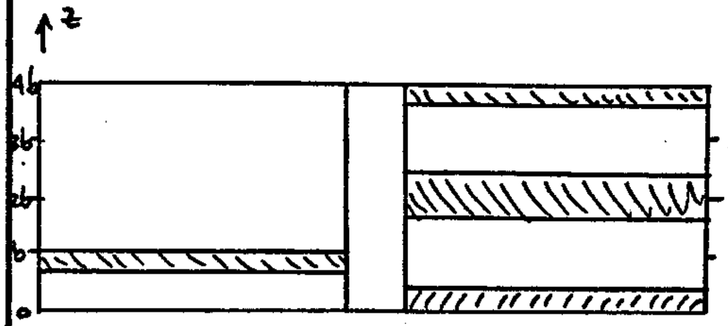
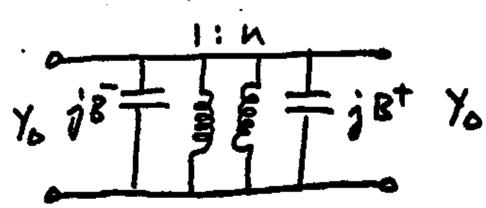


Fig. A Fig. B

Extend Fig. 8-22a into Fig A above and using the same techniques as in obtaining Figs. 8-22 b & c conducting plates are placed at $z = \pm nb$, $n=1,2,\ldots$ to form a system of images. The two configurations are identical and thus have the same equivalent circuit and shunt capacitance.

8-17 Using equivalent circuit of Fig. 8-20 b from which Eqn 8-127 was derived and changing slightly we have:



Replace jB in Fig 8-20 with two susceptances in parallel with ideal transformer.

Since capacitors in parallel add, the susceptances in the above circuit are $\frac{1}{2}$ that used in

8-17 (cont.)

Eqn. 8-120 where the signs used depend upon which way one looks into the circuit.

Thus,

$$\frac{B^+}{Y_0} \approx \frac{4b^+}{\lambda_g} \log \csc \frac{\pi c}{2b^+}$$

$$\frac{B^-}{Y_0} \approx \frac{4b^-}{\lambda_g} \log \csc \frac{\pi c}{2b^-}$$

using ideal transformer relationships

To find turns ratio:

$$\frac{1}{n^2} = \frac{B^+}{B^-} \Rightarrow n^2 = \frac{b^-}{b^+}$$

8-18 Replacing quantities in Eqn. 8-127 with their duals, we have:

$$\frac{1}{Y_0} + \hat{V}_0^2 + \sum_i \hat{z}_i \hat{I}_i = \frac{1+\Gamma}{1-\Gamma} Z_0^- I_0^2 - \sum_i z_i I_i^2$$

$$I_m\left[\frac{1-\Gamma}{1+\Gamma}\right] = \frac{jB}{Y_0} \Rightarrow I_m\left[\frac{1+\Gamma}{1-\Gamma}\right] = \frac{Y_0}{jB}$$

$$\frac{1+\Gamma}{1-\Gamma} = \frac{\frac{1}{Y_0} + \hat{I}_0^2 + \sum_i z_i I_i^2 + \sum_i \hat{z}_i \hat{I}_i^2}{Z_0^- I_0^2}$$

Z_0 are real and z_i, $i \neq 0$ are imaginary for real I_i, $\hat{I}_i$. Thus specializing above Eqn. for B^+ and B^-, we have:

$$\frac{Y_0^-}{jB^-} = \frac{\sum_i z_i I_i^2}{Z_0^- I_0^2}, \quad \frac{Y_0^+}{jB^+} = \frac{\sum_i \hat{z}_i \hat{I}_i^2}{Z_0^+ \hat{I}_0^2}$$

From the dual of equivalent cct:

8-18 (cont.)

$$\frac{1}{GZ_0^-} = \frac{1}{n^2}\frac{Z_0^+}{Z_0^-}$$ from above, $$GZ_0^- = \frac{Z_0^+ \hat{I}_0^2}{Z_0^- I_0}$$

$$\text{So } \frac{1}{n^2} = \frac{I_0^2}{\hat{I}_0^2}$$

8-19 Since we are in a source free region, the angle operator D behaves like a true angle in the sense that

$$(\cos^2 D + \sin^2 D)\psi = \psi.$$

Expressing ψ in the form of an integral over plane waves travelling in all directions:

$$\psi(x,y) = \int_0^{2\pi} A(\phi') e^{jk(x\cos\phi' + y\sin\phi')} d\phi'$$

$$= \int_0^{2\pi} A(\phi') e^{jk\rho\cos(\phi-\phi')} d\phi'$$

where $A(\phi)$ = plane wave function.

Then

$$\frac{1}{2\pi}\int_0^{2\pi} \psi(x,y) e^{jn\phi} d\phi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} d\phi' A(\phi') \int_0^{2\pi} d\phi\, e^{jk\rho\cos(\phi-\phi') + jn\phi}$$

$$= \frac{j^n}{2\pi}\int_0^{2\pi} d\phi' e^{jn\phi'} A(\phi') \int_{-\phi'-\frac{\pi}{2}}^{(2\pi-\phi'-\frac{\pi}{2})} d\phi'' e^{jn\phi'' - jk\rho\sin\phi''}$$

where $\phi'' = \phi - \phi' - \frac{\pi}{2}$

Thus

$$\frac{1}{2\pi}\int_0^{2\pi} \psi(x,y) e^{jn\phi} d\phi = j^n J_n(k\rho) \int_0^{2\pi} d\phi' e^{jn\phi'} A(\phi')$$

8-19 (cont.)

Now,

$$e^{jnD}\psi(x,y) =$$

$$= \left(\frac{1}{jk}\frac{\partial}{\partial x} + j\frac{1}{jk}\frac{\partial}{\partial y}\right)^n \int_0^{2\pi} A(\phi') e^{jk(x\cos\phi' + y\sin\phi')} d\phi'$$

$$= \int_0^{2\pi} A(\phi') e^{jk(x\cos\phi' + y\sin\phi')} (\cos\phi' + j\sin\phi')^n d\phi'$$

$$\therefore e^{jnD}\psi(0) = \int_0^{2\pi} A(\phi') e^{jn\phi'} d\phi'$$

$$\frac{1}{2\pi}\int_0^{2\pi} f(a,\phi) e^{jn\phi} d\phi = j^n J_n(ka)\left[e^{jnD}\psi(0)\right]$$

8-20

From Eqn. 8-152,

$$n^2 = \frac{2a}{b}\left(\frac{\tan ka}{ka}\right)^2 \sin^2\frac{\pi c}{b}$$

Since $a \ll \lambda$,

$$\frac{\tan ka}{ka} \approx 1$$

$$\therefore n^2 = \frac{2a}{b}\sin^2\frac{\pi c}{b}$$

Using the results of sec. 8-7 for a post in a waveguide,

$$E_x(\text{as } kd \to 0) \approx +\frac{k\eta}{4}\left[1 - j\frac{2}{\pi}\ln\frac{\gamma kd}{2}\right]$$

$$J_s^a = u_x \frac{I}{\pi d}$$

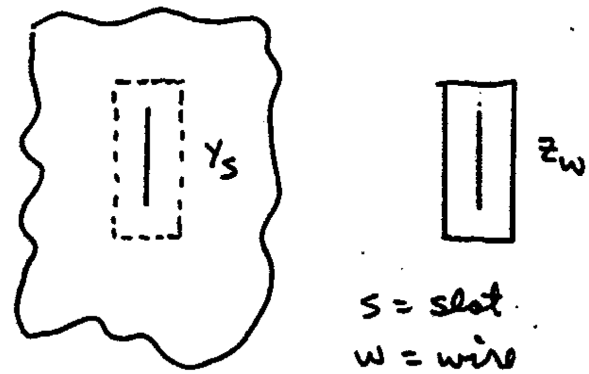
$$\langle a,a\rangle = \int_0^a\int_0^{2\pi} E_x^a \cdot J_s^a \, ds = -\frac{k\eta a}{2\pi}\ln\frac{\gamma kd}{2}$$

$$= -\frac{\eta a}{\lambda}\ln \gamma kd$$

8-21

From results of prob. 7-43,

$$Y_s = \frac{4}{\eta^2} Z_w$$



s = slot
w = wire

($\langle a,a\rangle$ = radiation resistance)

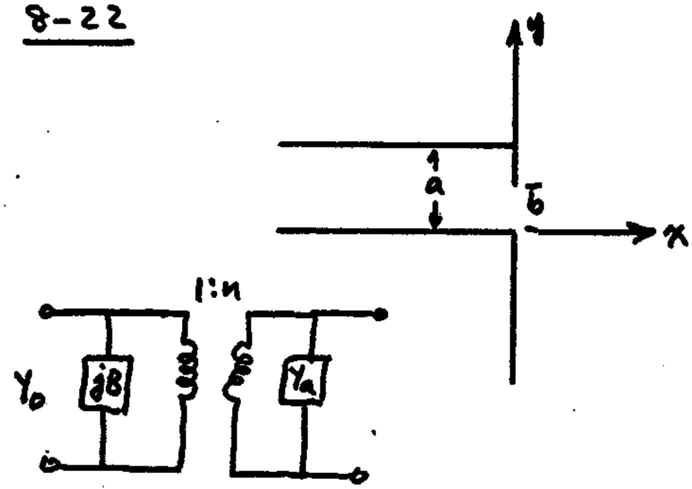
$Z_w = \langle a,a\rangle \frac{b}{a}$ because of impedance transformation by ideal transformer.

From results of prob. 7-39,

$$Y_s = \frac{4}{\eta^2}(73)(2)\frac{b}{\lambda}, \quad a = \frac{\lambda}{2}$$

$$= .004\frac{b}{\lambda}$$

8-22



Assume aperture field as:

$$E_y^a = 1, \quad 0 \le y \le b$$

8-22 (cont.) $\psi_0^m = \frac{y}{\sqrt{a}}$

$$e_0^m = \frac{-1}{\sqrt{a}}\,\bar{u}_y$$

using Eqn. 8-155:

$$V_0 = \int_0^1\int_0^b E_y^a \cdot e_0^m \,dy\,dz = -\frac{b}{\sqrt{a}}$$

Reference voltage $V = \sqrt{b}$

$$\therefore n^2 = \frac{V^2}{V_0^2} = \frac{a}{b}$$

The relationship for the ideal transformer holds as:

$$jB = n^2 Y_a$$

$Y_a = G_a + jB_a$ where Y_a is obtained from Fig. 4-22

$$B_a = \frac{2}{\lambda\eta}\int_{\frac{ka}{2}}^{\infty}\frac{\sin^2 w\,dw}{w^2\sqrt{\left(\frac{ka}{2}\right)^2 - w^2}}$$

$$\therefore B = \frac{2a}{b\lambda\eta}\int_{\frac{ka}{2}}^{\infty}\frac{\sin^2 w\,dw}{w^2\sqrt{\left(\frac{ka}{2}\right)^2 - w^2}}$$

and Y_0 is the characteristic admittance of the waveguide $= \frac{\omega\varepsilon}{\beta}$.

8-23 See "Radiation and Scattering of Waves" by Felsen & Marcuvitz, Prentice-Hall, 1973. pp. 237.

8-24 The normalized mode vector of the dominant mode is $\bar{E}_0 = \bar{u}_x \frac{2}{\sqrt{abc\varepsilon}} \sin\frac{\pi y}{b}\sin\frac{\pi z}{c}$

From Eqn. 8-162.

The current on the probe is assumed to be:

$$J_x^a = \begin{cases} I\,\delta(y-b')\,\delta(y-c') & x<d \\ 0 & x>d \end{cases}$$

$$a_0 = \iiint \bar{E}_0 \cdot \bar{J}^a \,d\tau$$

$$= \int_0^d \frac{2I}{\sqrt{\varepsilon abc}} \sin\frac{\pi b'}{b}\sin\frac{\pi c'}{c}\,dy$$

$$\frac{a_0}{I} = \frac{2d}{\sqrt{\varepsilon abc}} \sin\frac{\pi b'}{b}\sin\frac{\pi c'}{c}$$

when $c' \ll c$, $\sin\frac{\pi c'}{c} \approx \frac{\pi c'}{c}$

$$\frac{a_0}{I} \approx \frac{2}{\sqrt{\varepsilon abc}}\sin\left(\frac{\pi b'}{b}\right)\frac{\pi c' d}{c}$$

$$\approx \frac{2\pi A}{\sqrt{\varepsilon abc}}\sin\left(\frac{\pi b'}{b}\right) \quad \text{where } A = c'd$$

and R, L, and C of the equivalent circuit are given by Eqn. 8-181.

8-25 The dominant mode for the circular cavity is a TM_{010} mode and is given by:

$$E_z = \frac{k^2}{j\omega\varepsilon} J_0\left(\frac{x_{01}\rho}{a}\right)$$

The normalization constant is obtained from:

8-25 (cont.)

$$\varepsilon \iiint |E|^2 d\tau = 1$$

$$= \frac{\pi k^4 b a^2}{\omega^2 \varepsilon} J_1^2(x_{01})$$

$$\therefore E_0 = \frac{\bar{u}_z}{j} \frac{k^2}{j\omega\varepsilon} \sqrt{\frac{\omega^2 \varepsilon}{\pi k^4 b a^2 J_1^2(x_{01})}}\, J_0\left(x_{01}\frac{\rho}{a}\right)$$

$$= \frac{\bar{u}_z}{j} \frac{J_0(x_{01}\rho/a)}{a\sqrt{\varepsilon\pi b}\, J_1(x_{01})}$$

Where $x_{01} = 2.405$.

8-26 The assumed current distribution on the probe is:

$$J_z = I \frac{\sin k(d-z)}{\sin kd} \delta(\rho - c) \qquad z < d$$

$$= 0 \qquad z > d$$

E_0 is given by ~~Eq.~~ Prob. 8-25.

$$a_0 = \iiint \bar{E}_0 \cdot \bar{J}\, d\tau$$

$$= \int_0^d I \frac{J_0(x_{01}\frac{c}{a})}{a\sqrt{\varepsilon\pi b}\, J_1(x_{01})} \frac{\sin k(d-z)}{\sin kd} dz$$

$$\int_0^d \frac{\sin k(d-z)}{\sin kd} dz = \frac{\tan \frac{kd}{2}}{k}$$

$$\therefore \frac{a_0}{I} = \frac{J_0(x_{01}\frac{c}{a})}{a\sqrt{\varepsilon\pi b}\, J_1(x_{01})} \frac{\tan\left(\frac{kd}{2}\right)}{k}$$

8-27 The assumed current distribution is:

$$J_z = I\delta(\rho - c) \qquad z < d$$

$$0 \qquad z > d$$

E_0 is given by Prob. 8-25.

$$a_0 = \iiint \bar{E}_0 \cdot \bar{J}\, d\tau$$

$$= \int_0^d I\delta(\rho - c) \frac{1}{a\sqrt{\varepsilon\pi b}\, J_1(x_{01})} J_0\left(x_{01}\frac{\rho}{a}\right) dz$$

$$\frac{a_0}{I} = \frac{J_0(x_{01}\frac{c}{a})\, d}{a\sqrt{\pi\varepsilon b}\, J_1(x_{01})}$$

when $c \approx a$,

$$J_0\left(x_{01}\frac{c}{a}\right) = J_0\left[x_{01}\left(1 + \frac{c-a}{a}\right)\right]$$

Now expanding in a Taylor's series when $c = a$,

$$J_0\left(x_{01}\frac{c}{a}\right) = J_0(x_{01}) + x_{01}\frac{(c-a)}{a} J_0'(x_{01})$$

$$= x_{01}\frac{(c-a)}{a} J_0'(x_{01})$$

$$= x_{01}\frac{(a-c)}{a} J_1(x_{01})$$

$$\therefore \frac{a_0}{I} \approx \frac{d(a-c)x_{01}}{a^2\sqrt{\pi\varepsilon b}} \approx \frac{A x_{01}}{a^2\sqrt{\pi\varepsilon b}}$$

8-28 Replace loop with magnetic current element

$$\overline{K\ell} = -j\omega\mu A I\,\delta(x-d)\,\delta(y-b')\,\delta(z-c')\,\hat{u}_y$$

where A is the area of loop = c'd

From Eqn. 2-96,

$$H_y = \frac{jbE_0}{\eta\sqrt{b^2+c^2}}\sin\frac{\pi y}{b}\cos\frac{\pi z}{c}$$

to find E_0 for normalization:

$$w = 2\overline{W}_m = \mu\iiint |H|^2\,d\tau = 1$$

$$\text{as } E_0 = \frac{2\eta\sqrt{b^2+c^2}}{\sqrt{\mu}\,b\sqrt{abc}}$$

then normalized H is:

$$H_y = \frac{2j}{\sqrt{\mu abc}}$$

Now $$R = \frac{2\omega\mu A}{\sqrt{abc}\sqrt{\mu}}\sin\frac{\pi b'}{b}\cos\frac{\pi c'}{c}$$

in the cavity, $k = \beta = \omega\sqrt{\mu\varepsilon}$

$$\therefore \frac{\omega\mu}{\sqrt{\mu}} = \frac{\beta}{\sqrt{\varepsilon}}$$

but $\beta = \frac{\pi}{c}$ as in section 2-8.

also as $c' \ll c$, $\cos\frac{\pi c'}{c} \to 1$

$$\therefore R = \frac{2\pi A}{c\sqrt{abc\varepsilon}}\sin\frac{\pi b'}{b}$$

where the I has been suppressed during calculations.

8-29

$$H_\phi = \frac{x_{01}}{a}J_1\left(\frac{x_{01}\rho}{a}\right)$$

Using Eqn 5-56 and normalizing H_ϕ we have,

$$H_\phi = \frac{x_{01}}{a}\sqrt{\frac{\omega^2\varepsilon}{\pi k^4 b a^2}}\;\frac{J_1(x_{01}\rho/a)}{J_1(x_{01})}$$

$$\overline{K\ell} = -j\omega\mu A\,\delta(\rho-c)\,\hat{u}_\phi$$

$$\therefore R = \frac{x_{01}A}{a^2\sqrt{b\pi\varepsilon}}\;\frac{J_1\left(x_{01}\frac{c}{a}\right)}{J_1(x_{01})}$$

for $c \approx a$,

$$R = \frac{x_{01}A}{a^2\sqrt{\pi b\varepsilon}}$$

8-30 The magnetic field for dominant mode is given by:

$$H_\phi = \frac{1}{r}\hat{J}_1\left(2.744\frac{r}{a}\right)\sin\theta$$

For normalization constant,

$$w = \mu\iiint |H|^2\,d\tau = 1 = \frac{8\pi\mu}{3k}(1.14)$$

$$H_\phi = \sqrt{\frac{3k}{8\pi\mu(1.14)}}\;\frac{\hat{J}_1\left(2.744\frac{r}{a}\right)}{r}\sin\theta$$

where $ka = 2.744$

$$\therefore H_\phi = \frac{.535}{r\sqrt{a\mu}}\hat{J}_1\left(2.744\frac{r}{a}\right)\sin\theta$$