Monte Carlo method

- Monte Carlo method is a set of algorithms which use randomness to solve problems either exactly or approximately.
- We will study Monte Carlo integration - a method used to approximate a definite integral using random point generation.
- Consider the problem of approximating the definite integral

$$\int_a^b f(x)dx$$

  where, f is some complicated function so that exact ingratiation is not possible with any existing method.
- Assume that $0 \leq f(x) \leq c$ and hence the integral is finite.
- We can generate and use random numbers to approximate this integral!

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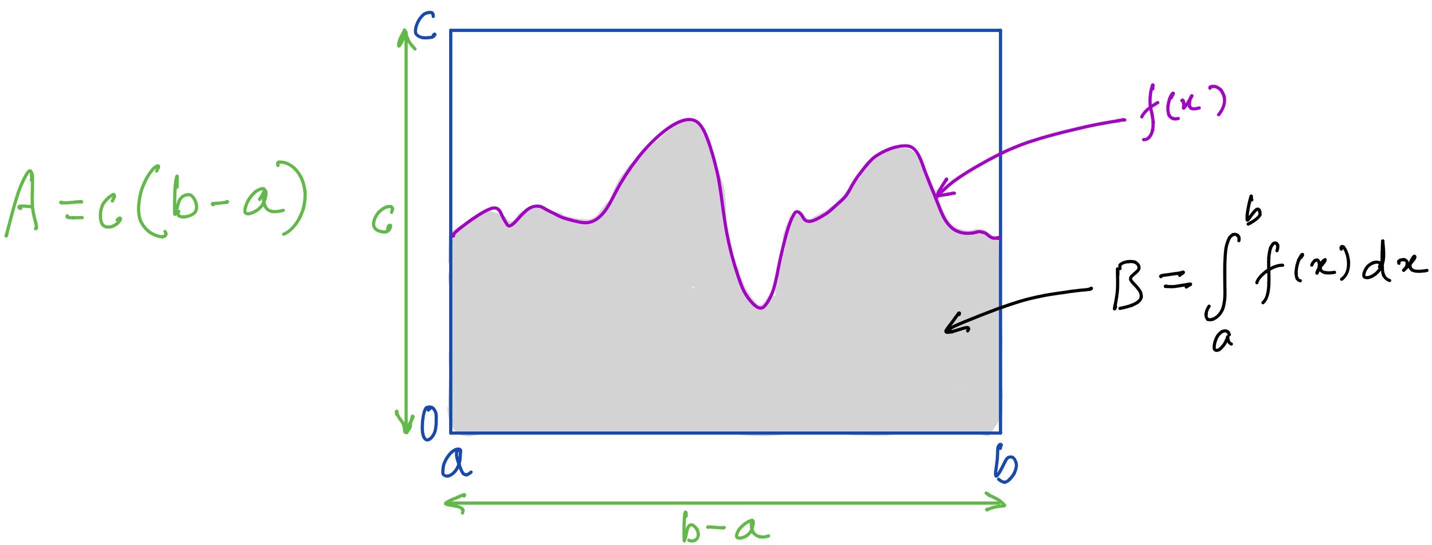
- We can generate and use random numbers to approximate this integral!
- Let A be the area of the rectangle in the (x, y)-plane defined by $a \leq x \leq b$ and $0 \leq y \leq c$.
- Let B be the region under the curve $y = f(x)$ for $a \leq x \leq b$,
- Thus, the area of B is the desired integral.



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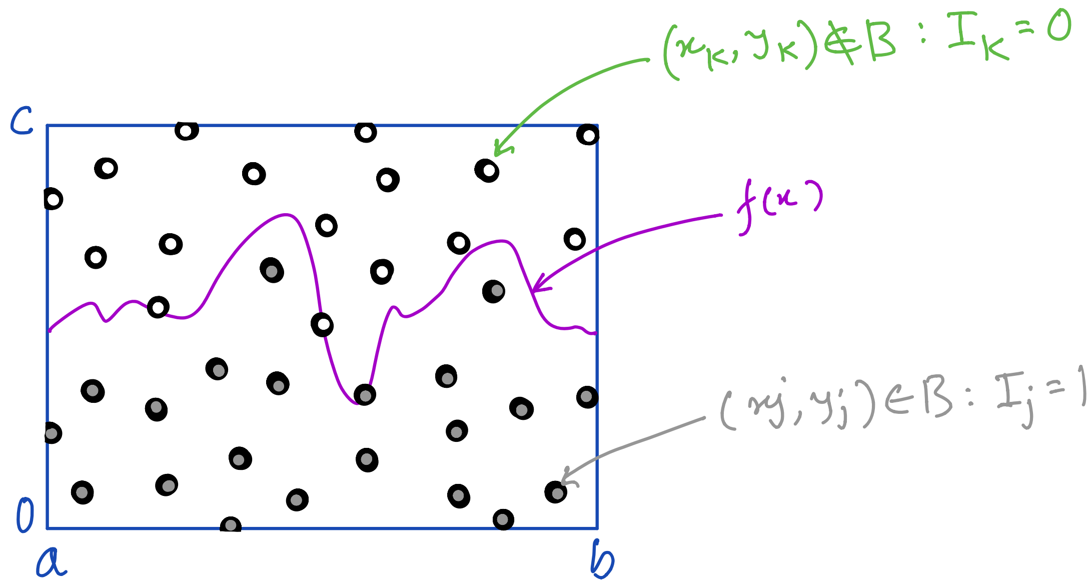
- Method: take random samples from A, then calculate the proportion of the samples that also fall into the area B.
- Generate iid points $(X_1, Y_1), \ldots, (X_n, Y_n)$ uniformly over A. Now let I_j be the Bernoulli r.v. such that $I_j = 1$ if $(X_j, Y_j) \in B$ and 0 otherwise.



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- Then
$$p = E(I_j) = P(I_j = 1) = \frac{B}{A} = \frac{\int_a^b f(x)dx}{c(b-a)}.$$
- By the WLLN, for large n we can approximate p as
$$\frac{1}{n}\sum_{j=1}^{n} I_j.$$
- Hence,

$$\frac{1}{n}\sum_{j=1}^{n} I_j \approx \frac{\int_a^b f(x)\,dx}{c(b-a)}$$

$$\Rightarrow \int_a^b f(x)\,dx \approx \frac{c(b-a)}{n}\sum_{j=1}^{n} I_j$$

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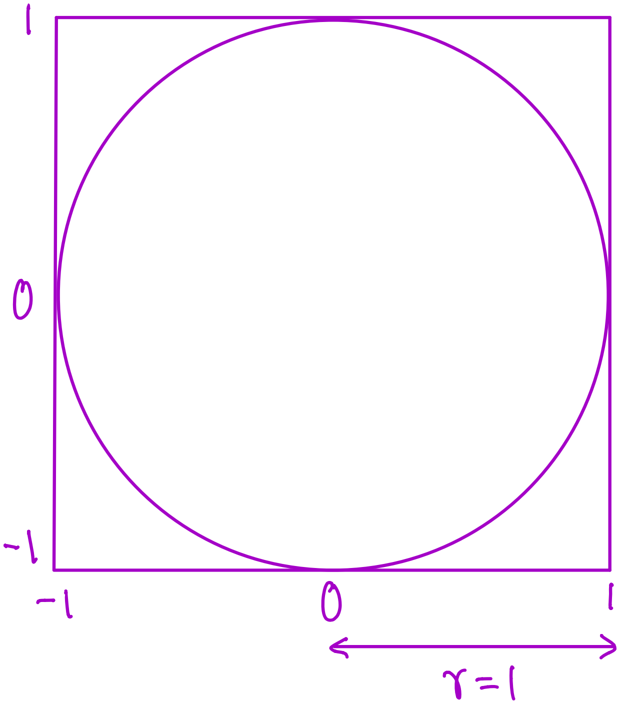
- Example: Approximate the value of the number π.



- Generate samples $(x_1, y_1), \dots, (x_n, y_n)$ iid uniformly in the square.

- Let $I_j = \begin{cases} 1 & \text{if } x_j^2 + y_j^2 \le 1 \\ 0 & \text{otherwise.} \end{cases}$

- Then, $\pi r^2 = \pi$

$$\approx \underbrace{4}_{\text{area of the square}} \cdot \frac{1}{n} \sum_{j=1}^{n} I_j$$