DSC TIET- Basics of Quantum Computing

CONTROLLED GATES

1. CNOT Gate

It is a conditional gate that performs X gate on target qubit if the control qubit is $|1\rangle$

TRUTH TABLE

Input (control, target)	Output (c,t)
0 0	0 0
0 1	0 1
1 0	1 1
1 1	1 0

$$CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & X \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X$$

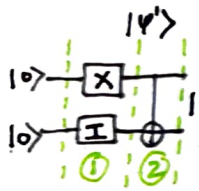
(I Gate, X Gate)

Examples

$$|00\rangle = |0\rangle \otimes |0\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad CNOT = \begin{bmatrix} I & 0 \\ 0 & X \end{bmatrix}$$

$$|\psi\rangle = CNOT\,|00\rangle = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow |00\rangle$$

So $|00\rangle$ remains the same.
As the control bit is not $|1\rangle$



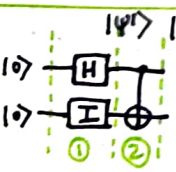
① $|00\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $\quad X \otimes I = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

$$|\psi'\rangle = XI\,|00\rangle = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \rightarrow |10\rangle$$

Now

② $|\psi\rangle = CNOT\,|\psi'\rangle = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \rightarrow |11\rangle$

control qubit is $|1\rangle$
So target qubit $|0\rangle$ becomes $|1\rangle$
Therefore $|10\rangle$ becomes $|11\rangle$



① $|00\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $\quad H \otimes I = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$

$$|\psi'\rangle = HI\,|00\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \rightarrow |00\rangle \\ 0 \\ 1 \rightarrow |10\rangle \\ 0 \end{bmatrix}$$

50% probability of being in $|00\rangle$
50% probability of being in $|10\rangle$

$$|\psi'\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |10\rangle)$$

② $|\psi\rangle = CNOT\,|\psi'\rangle = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \rightarrow |00\rangle \\ 0 \\ 0 \\ 1 \rightarrow |11\rangle \end{bmatrix} = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)$

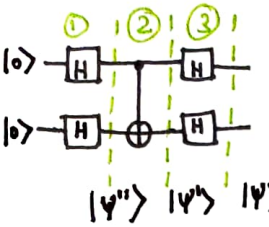
So $|\psi'\rangle \rightarrow |\psi\rangle$

$$\frac{1}{\sqrt{2}} (|00\rangle + |10\rangle) \rightarrow \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)$$

control qubit $|0\rangle$ — remains same
control qubit $|1\rangle$ — target qubit flips

50% probability of being in $|00\rangle$
50% probability of being in $|11\rangle$
This is a Bell State that represents Entanglement.

(More on Entanglement LATER)



① $|00\rangle = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$ $\quad H\otimes H = \frac{1}{2}\begin{bmatrix}1&1\\1&-1\end{bmatrix}\begin{bmatrix}1&1\\1&-1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1&1&1&1\\1&-1&1&-1\\1&1&-1&-1\\1&-1&-1&1\end{bmatrix}$ $\quad |\psi''\rangle = HH|00\rangle = \frac{1}{2}\begin{bmatrix}1&1&1&1\\1&-1&1&-1\\1&1&-1&-1\\1&-1&-1&1\end{bmatrix}\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$

② $|\psi'\rangle = CNOT|\psi''\rangle = \frac{1}{2}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$

③ $|\psi\rangle = HH|\psi'\rangle = \frac{1}{4}\begin{bmatrix}1&1&1&1\\1&-1&1&-1\\1&1&-1&-1\\1&-1&-1&1\end{bmatrix}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \frac{1}{4}\begin{bmatrix}4\\0\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} = |00\rangle$

So $|00\rangle \rightarrow |00\rangle$

Similarly

$|01\rangle \rightarrow |11\rangle$

$|10\rangle \rightarrow |10\rangle$

$|11\rangle \rightarrow |01\rangle$

It affects the state of control qubit, while target qubit remains unchanged.

Overall Gate = $(HH)\otimes CNOT\otimes(HH)$

$= \frac{1}{4}\begin{bmatrix}1&1&1&1\\1&-1&1&-1\\1&1&-1&-1\\1&-1&-1&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}\begin{bmatrix}1&1&1&1\\1&-1&1&-1\\1&1&-1&-1\\1&-1&-1&1\end{bmatrix}$

$= \frac{1}{4}\begin{bmatrix}1&1&1&1\\1&-1&-1&1\\1&1&-1&-1\\1&-1&1&-1\end{bmatrix}\begin{bmatrix}1&1&1&1\\1&-1&1&-1\\1&1&-1&-1\\1&-1&-1&1\end{bmatrix} = \frac{1}{4}\begin{bmatrix}4&0&0&0\\0&0&0&4\\0&0&4&0\\0&4&0&0\end{bmatrix}$

$= \begin{bmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{bmatrix}$

So, the given quantum circuit is equivalent to

$CNOT = \begin{bmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{bmatrix}$

This is example of Phase Kickback

Phase Kickback - operation of a gate on a qubit is 'kicked back' into another qubit via a controlled operation.

Like in the example above, X gate operation on the TARGET QUBIT is kicked on to the CONTROL QUBIT.

ADVANTAGE - allows CNOT in both directions using Hadamard Gate by wrapping them around CNOT gate.

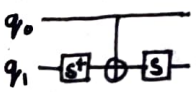
Controlled-Z (CZ)

We know $HXH = Z$

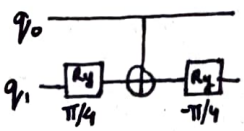
So CZ gate is

$CZ = H(CNOT)H$  or

Controlled Y Gate



Controlled H Gate



CNOT can be transformed to controlled version of any rotation about the Bloch Sphere, by preceding and succeeding CNOT with correct rotations.

These Rotations are of 180°.

T Gate

$T = \begin{bmatrix}1&0\\0&e^{i\pi/4}\end{bmatrix}$

Controlled-T

$= \begin{bmatrix}I&0\\0&T\end{bmatrix} = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&e^{i\pi/4}\end{bmatrix}$

More generally Controlled U operation for any gate

$U = \begin{bmatrix}U_{00}&U_{01}\\U_{10}&U_{11}\end{bmatrix}$ $\quad C\text{-}U = \begin{bmatrix}I&0\\0&U\end{bmatrix} = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&U_{00}&U_{01}\\0&0&U_{10}&U_{11}\end{bmatrix}$