BELL STATES

Maximally entangled states on 2 qubits

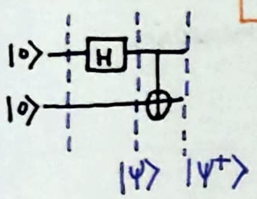
$$|\psi^{\pm}\rangle = (|00\rangle \pm |11\rangle)/\sqrt{2}$$

$$|\phi^{\pm}\rangle = (|01\rangle \pm |10\rangle)/\sqrt{2}$$

ENTANGLEMENT

- Let there be 2 qubits
- We know measuring both of them separately, can result in either $|0\rangle$ or $|1\rangle$
- But if the 2 qubits are entangled,
  determining either bit by measurement;
  exactly determines the other without even measuring. (that too at a distance)

BELL STATE 1: $\boxed{|\psi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}}$



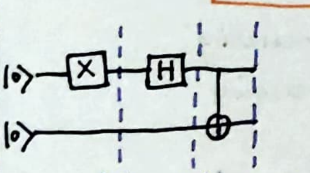
$$|00\rangle = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \qquad H \otimes I = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} \otimes \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0 & 1 & 0\\0 & 1 & 0 & 1\\1 & 0 & -1 & 0\\0 & 1 & 0 & -1\end{bmatrix}$$

$$|\psi\rangle = HI|00\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0 & 1 & 0\\0 & 1 & 0 & 1\\1 & 0 & -1 & 0\\0 & 1 & 0 & -1\end{bmatrix}\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\\0\end{bmatrix}$$

$$|\psi^+\rangle = CNOT|\psi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 1\\0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}1\\0\\1\\0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\1\end{bmatrix}\begin{matrix}\rightarrow|00\rangle\\ \\ \\ \rightarrow|11\rangle\end{matrix} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

If we measure first qubit to be $|0\rangle$, then we know second bit will be $|0\rangle$ and
If we measure first qubit to be $|1\rangle$, then we know second bit will be $|1\rangle$
without even measuring the second qubit. (and vice versa).

BELL STATE 2: $\boxed{|\psi^-\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}}$



$$|00\rangle = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \qquad X \otimes I = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} \otimes \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}0 & 0 & 1 & 0\\0 & 0 & 0 & 1\\1 & 0 & 0 & 0\\0 & 1 & 0 & 0\end{bmatrix}$$

$$|\psi\rangle = XI|00\rangle = \begin{bmatrix}0 & 0 & 1 & 0\\0 & 0 & 0 & 1\\1 & 0 & 0 & 0\\0 & 1 & 0 & 0\end{bmatrix}\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}$$

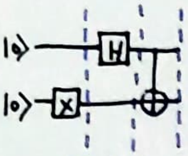
$$|\psi'\rangle = HI|\psi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0 & 1 & 0\\0 & 1 & 0 & 1\\1 & 0 & -1 & 0\\0 & 1 & 0 & -1\end{bmatrix}\begin{bmatrix}0\\0\\1\\0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\-1\\0\end{bmatrix}$$

$$|\psi^-\rangle = CNOT|\psi'\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 1\\0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}1\\0\\-1\\0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\-1\end{bmatrix} = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$

If qubit 1 is $|0\rangle$ on measuring, then qubit 2 is $|0\rangle$ without even measuring
If qubit 1 is $|1\rangle$ on measuring, then qubit 2 is $|1\rangle$ without even measuring
and vice versa.

For Bell State 1 and 2, both the qubits are same
If one qubit is $|0\rangle$ other will also be $|0\rangle$
If one qubit is $|1\rangle$ other will also inevitably be $|1\rangle$

BELL STATE 3 :- $|\phi^+\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$



$|\phi\rangle$ $|\phi'\rangle$ $|\phi^+\rangle$

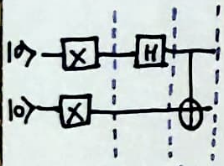
$|00\rangle = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$ $\quad I \otimes X = \begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix} = \begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}$ $\quad |\phi\rangle = IX|00\rangle = \begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}$

$$|\phi'\rangle = HI|\phi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1&0&1&0\\0&1&0&1\\1&0&-1&0\\0&1&0&-1\end{bmatrix}\begin{bmatrix}0\\1\\0\\0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\0\\1\end{bmatrix}$$

$$|\phi^+\rangle = CNOT\,|\phi'\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}\begin{bmatrix}0\\1\\0\\1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\1\\0\end{bmatrix} \begin{matrix}\rightarrow |01\rangle\\ \rightarrow |10\rangle\end{matrix} = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$

If first qubit $=|0\rangle$ on measurement, then second qubit $=|1\rangle$ without measuring
If first qubit $=|1\rangle$ on measurement, then second qubit $=|0\rangle$ without measuring
and vice versa.

BELL STATE 4 :- $|\phi^-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$



$|\phi\rangle$ $|\phi'\rangle$ $|\phi^-\rangle$

$|00\rangle = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$ $\quad X \otimes X = \begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix} = \begin{bmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{bmatrix}$ $\quad |\phi\rangle = \begin{bmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{bmatrix}\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$

$$|\phi'\rangle = HI\,|\phi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1&0&1&0\\0&1&0&1\\1&0&-1&0\\0&1&0&-1\end{bmatrix}\begin{bmatrix}0\\0\\0\\1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\0\\-1\end{bmatrix}$$

$$|\phi^-\rangle = CNOT\,|\phi'\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}\begin{bmatrix}0\\1\\0\\-1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\-1\\0\end{bmatrix} = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

If qubit 1 $= |0\rangle$ on measuring, then qubit 2 $= |1\rangle$ without measuring
If qubit 1 $= |1\rangle$ on measuring, then qubit 2 $= |0\rangle$ without measuring

OR

If qubit 2 $= |0\rangle$ on measuring, then qubit 1 $= |1\rangle$ without measuring
If qubit 2 $= |1\rangle$ on measuring, then qubit 1 $= |0\rangle$ without measuring

* This works no matter the distance between the 2 qubits.
* These qubits are not independent, they are entangled in such a way that information about one qubit gives information about the second one.

For Bell State 3 & 4
* If one qubit is $|0\rangle$ other is $|1\rangle$
* If one qubit is $|1\rangle$ other is $|0\rangle$

SUMMARY

Let qubits be A and B

① $|\psi^+\rangle = \frac{|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B}{\sqrt{2}} = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$ ② $|\psi^-\rangle = \frac{|0\rangle_A \otimes |0\rangle_B - |1\rangle_A \otimes |1\rangle_B}{\sqrt{2}} = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$

③ $|\phi^+\rangle = \frac{|0\rangle_A \otimes |1\rangle_B + |1\rangle_A \otimes |0\rangle_B}{\sqrt{2}} = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$ ④ $|\phi^-\rangle = \frac{|0\rangle_A \otimes |1\rangle_B - |1\rangle_A \otimes |0\rangle_B}{\sqrt{2}} = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$

* NOTE:- They can be represented visually using QSphere. (It is implemented in the Jupyter Notebook)