QUANTUM - UNIVERSALITY

Quantum operators can be decomposed into a combination of CNOT, H, S and T Gates

$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ creates equal superposition of z basis states

$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$ $\pi/2$ or 90° rotation about z axis

$T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1+i}{\sqrt{2}} \end{bmatrix}$ $\pi/4$ or 45° rotation about z axis

$CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ It is a 2 qubit gate. It is explained in multi qubit gate.

Decomposing Pauli Gates

Z Gate

It rotates a qubit by 180° about the z axis on the bloch sphere.

$\boxed{Z = SS}$ (2 90° rotation about z axis is equivalent to z gate) $SS = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & i^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = Z$

$\boxed{Z = TTTT}$ (4 45° rotation about z axis is equivalent to z gate) $TT = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} = S$

X Gate

We know

$\boxed{X = HZH}$, let's work this out. If initial state is $|0\rangle$

Application of H gate gives $|+\rangle$

Application of Z gate on $|+\rangle$ gives $|-\rangle$

Application of H gate on $|-\rangle$ gives $|1\rangle$

Similarly if initial state is $|1\rangle$ we get $|0\rangle$ as output.

therefore This combination of gates gives X Gate.

A more mathematical proof is as follows

$$HZH = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = X$$

So $X = HZH$ holds

we know $Z = SS$

so $\boxed{X = HSSH}$

Y Gate

We know $\boxed{Y = SXS^\dagger}$

Mathematical proof

$$SXS^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ i & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = Y$$

so $Y = SXS^\dagger$

$Y = S(HSSH)S^\dagger$

$\boxed{Y = SHSSHS^\dagger}$