ME623 – Finite Element Methods in Engineering Mechanics

Computer Assignment – 2

Group – G2

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Problem Number – **(3) Question 2.1 (Plain Strain Problem)**

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1) Formulation of the Problem

a) The variational functional for the problem is given by

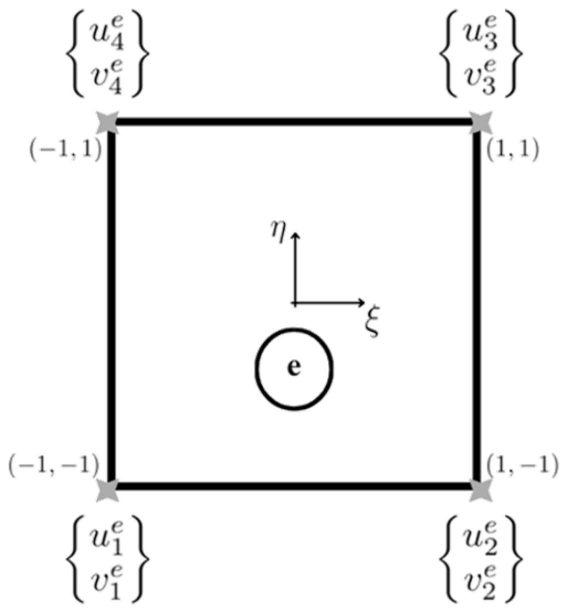
$$I(u,v) = \int_D \left[U - \left(b_x u + b_y v\right)\right] dx\, dy - \int_{C_2} \left(t_x^* u + t_y^* v\right) ds$$

where the strain energy density U is given by

$$U = \frac{1}{2}\left(\sigma_{xx}\epsilon_{xx} + \sigma_{yy}\epsilon_{yy} + 2\sigma_{xy}\epsilon_{xy}\right)$$

$$U = \frac{1}{2}\left\{\lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)^2 + 2\mu\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2\right]\right\}$$

b) The problem requires a lagrangian element because the completeness and compatibility criteria are to be satisfied and therefore a four-node element with displacement in X(u) and displacement in Y(v) as the degree of freedoms are used. The representation of a local element e is shown in the figure below,



The mapping function is given by

$$x = \alpha + \beta\xi$$
$$y = \gamma + \delta\eta$$

where, $\alpha = \frac{x_1^e + x_2^e}{2}$, $\beta = \frac{l^e}{2} = \frac{x_2^e - x_1^e}{2}$, $\gamma = \frac{y_1^e + y_2^e}{2}$ and $\delta = \frac{l^e}{2} = \frac{y_2^e - y_1^e}{2}$

therefore,

$$x = \frac{x_1^e + x_2^e}{2} + \frac{l^e}{2}\xi$$

$$y = \frac{y_1^e + y_2^e}{2} + \frac{l^e}{2}\eta$$

and consequently, $\frac{d\xi}{dx} = \frac{2}{l^e}$ and $dx = \frac{l^e}{2} d\xi$, $\frac{d\eta}{dy} = \frac{2}{l_e}$ and $dy = \frac{l^e}{2}\, d\eta$.

The Elemental shape functions $[B^e]_{3\times8}$:

$$\begin{bmatrix} \frac{\partial(N_1^e)}{\partial\xi} & 0 & \frac{\partial(N_2^e)}{\partial\xi} & 0 & \frac{\partial(N_3^e)}{\partial\xi} & 0 & \frac{\partial(N_4^e)}{\partial\xi} & 0 \\ 0 & \frac{\partial(N_1^e)}{\partial\eta} & 0 & \frac{\partial(N_2^e)}{\partial\eta} & 0 & \frac{\partial(N_3^e)}{\partial\eta} & 0 & \frac{\partial(N_4^e)}{\partial\eta} \\ \frac{\partial(N_1^e)}{\partial\eta} & \frac{\partial(N_1^e)}{\partial\xi} & \frac{\partial(N_2^e)}{\partial\eta} & \frac{\partial(N_2^e)}{\partial\xi} & \frac{\partial(N_3^e)}{\partial\eta} & \frac{\partial(N_3^e)}{\partial\xi} & \frac{\partial(N_4^e)}{\partial\eta} & \frac{\partial(N_4^e)}{\partial\xi} \end{bmatrix}$$

where,

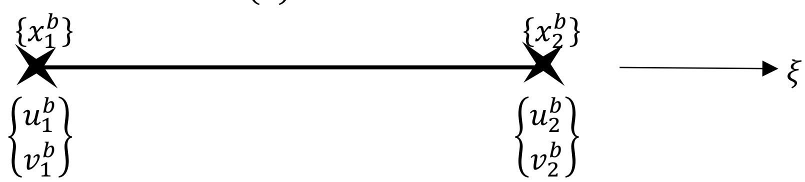
$$N_1^e = \frac{1}{4}(1-\xi)(1-\eta)$$
$$N_2^e = \frac{1}{4}(1+\xi)(1-\eta)$$
$$N_3^e = \frac{1}{4}(1+\xi)(1+\eta)$$
$$N_4^e = \frac{1}{4}(1-\xi)(1+\eta)$$

Thus, $[B^e]$ is given as,

$$\begin{bmatrix} -\frac{1}{4}(1-\eta) & 0 & \frac{1}{4}(1-\eta) & 0 & \frac{1}{4}(1+\eta) & 0 & -\frac{1}{4}(1+\eta) & 0 \\ 0 & -\frac{1}{4}(1-\xi) & 0 & -\frac{1}{4}(1+\xi) & 0 & \frac{1}{4}(1+\xi) & 0 & \frac{1}{4}(1-\xi) \\ -\frac{1}{4}(1-\xi) & -\frac{1}{4}(1-\eta) & -\frac{1}{4}(1+\xi) & \frac{1}{4}(1-\eta) & \frac{1}{4}(1+\xi) & \frac{1}{4}(1+\eta) & \frac{1}{4}(1-\xi) & -\frac{1}{4}(1+\eta) \end{bmatrix}$$

The boundary element e is shown in the figure below

(b)



The mapping function is given by

$$x = \alpha + \beta\xi$$

where, $\alpha = \frac{x_1^b + x_2^b}{2}$ and $\beta = \frac{l^b}{2} = \frac{x_2^b - x_1^b}{2}$.

The Elemental Boundary shape functions $[N^b]_{2\times4}$:

$$\begin{bmatrix} N_1^b & 0 & N_1^b & 0 \\ 0 & N_2^b & 0 & N_2^b \end{bmatrix}$$

where

$$N_1^b = \frac{1}{2}(1-\xi)$$
$$N_2^b = \frac{1}{2}(1+\xi)$$

Thus,

$$[N^b] = \begin{bmatrix} \frac{1}{2}(1-\xi) & 0 & \frac{1}{2}(1-\xi) & 0 \\ 0 & \frac{1}{2}(1+\xi) & 0 & \frac{1}{2}(1+\xi) \end{bmatrix}$$

c) The element coefficient matrix $[k]^e$ and the right side vector $\{q\}^b_{4\times1}$ is given by

$$[k]^e = \int_{D^e} [B]^{e^T}_{8\times3}[C]_{3\times3}[B]^e_{3\times8}dxdy$$

$$\{q\}^b_{4\times1} = \int_{l_b} [N]^{b^T}_{4\times2}\{t^*\}_{2\times1}ds$$

d) The simplified assembly relations for the coefficient matrix is given as:

$$\text{if } C_{ep} = r \text{ and } C_{eq} = s$$
$$K^e_{2r-1,2s-1} = k^e_{2p-1,2q-1}$$
$$K^e_{2r,2s-1} = k^e_{2p,2q-1}$$
$$K^e_{2r-1,2s} = k^e_{2p-1,2q}$$
$$K^e_{2r,2s} = k^e_{2p,2q}$$

And for the right-side vector

$$\text{if } C'_{bp} = r$$
$$Q^b_{2r-1,1} = q^b_{2p-1,1}$$
$$Q^b_{2r,1} = q^b_{2p,1}$$

e) The Gauss Legendre Integration scheme has been used, where number of Gauss points is taken as 2 as the degree of integrand turns out to be 2 ($2n_G - 1$). The coordinates and weights are tabulated as follows,

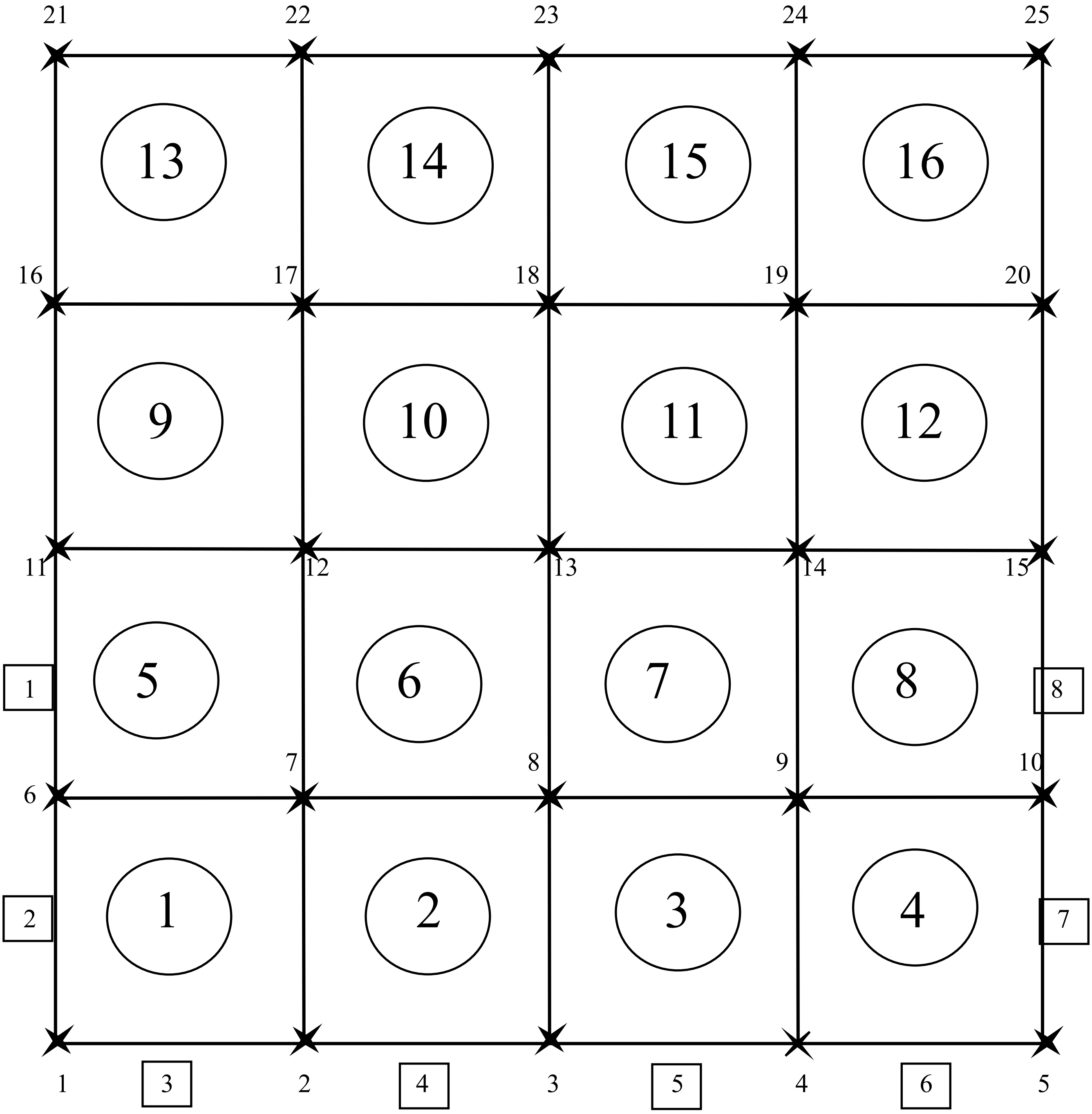
Gauss Point	**Coordinates (ξ_k)**	**Weights (w_k)**
ξ_1	-0.577350269189626	1.000000000000000
ξ_2	0. 577350269189626	1.000000000000000

The applied integration scheme is as follows,

$$[k]^e = \sum_{i=1}^{2}\sum_{j=1}^{2} w_i w_j \left([B]^{e^T}[C][B]^e\right)\Big|_{(\xi_i,\eta_j)}$$

$$\{q\}^b = \sum_{k=1}^{2} w_k \frac{l^b}{2}\, [N]^{b^T}\{t^*\}|_{\xi_k}$$

f)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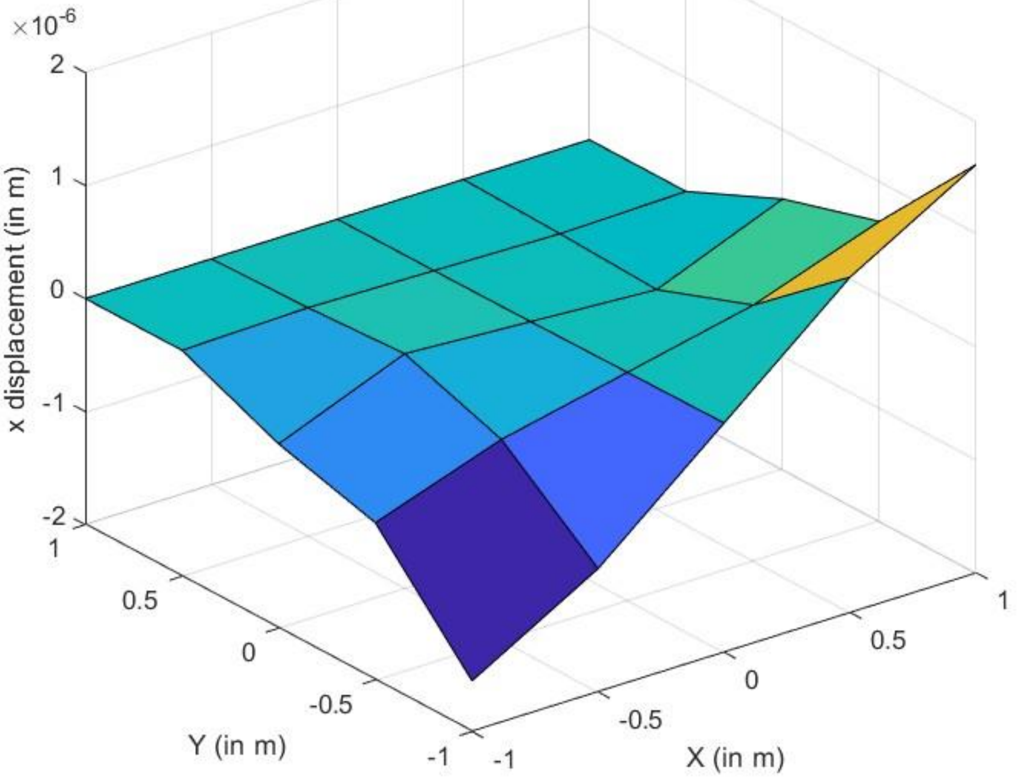
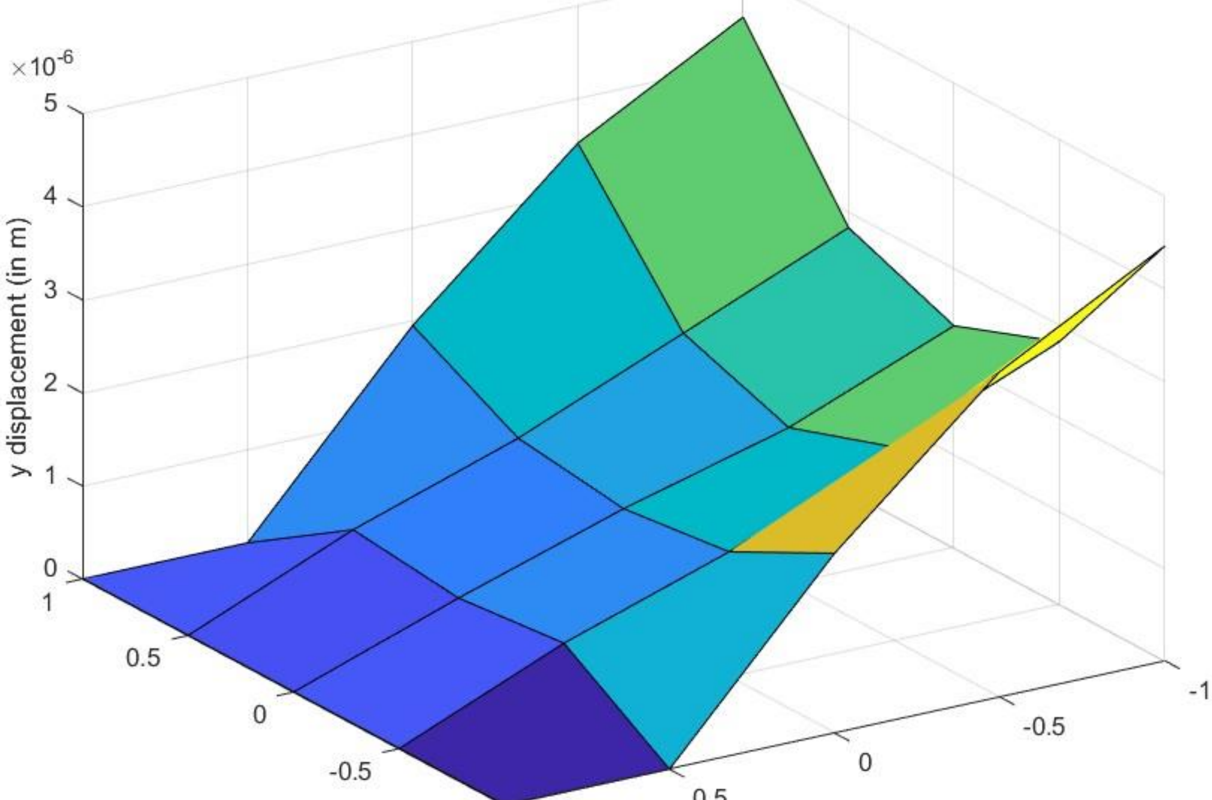
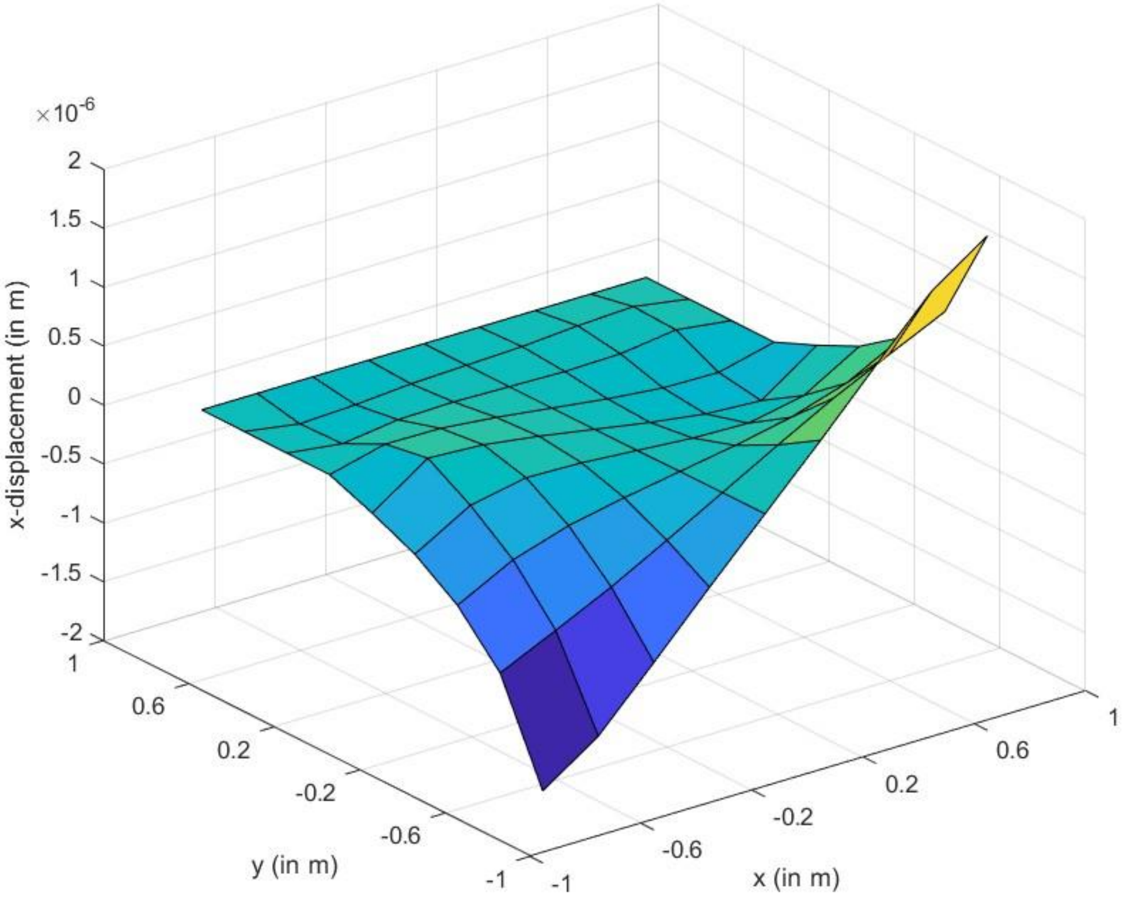
Variation of u over the domain for coarse discretization



Variation of v over the domain for coarse discretization



Variation of u over the domain for finer discretization



Variation of y-displacement over the domain for finer discretization