

Learning and Testing Convex Functions

Renato Ferreira Pinto Jr.

Columbia University

renato.ferreira@uwaterloo.ca

Cassandra Marcussen

Harvard University

cmarcussen@g.harvard.edu

Elchanan Mossel

MIT

elmos@mit.edu

Shivam Nadimpalli

MIT

shivamn@mit.edu

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Abstract

We consider the problems of *learning* and *testing* real-valued convex functions over Gaussian space. Despite the extensive study of function convexity across mathematics, statistics, and computer science, its learnability and testability have largely been examined only in discrete or restricted settings—typically with respect to the Hamming distance, which is ill-suited for real-valued functions.

In contrast, we study these problems in high dimensions under the standard Gaussian measure, assuming sample access to the function and a mild smoothness condition, namely Lipschitzness. A smoothness assumption is natural and, in fact, necessary even in one dimension: without it, convexity cannot be inferred from finitely many samples. As our main results, we give:

- **Learning Convex Functions:** An agnostic proper learning algorithm for Lipschitz convex functions that achieves error ε using $n^{O(1/\varepsilon^2)}$ samples, together with a complementary lower bound of $n^{\text{poly}(1/\varepsilon)}$ samples in the *correlational statistical query (CSQ)* model.
- **Testing Convex Functions:** A tolerant (two-sided) tester for convexity of Lipschitz functions with the same sample complexity (as a corollary of our learning result), and a one-sided tester (which never rejects convex functions) using $O(\sqrt{n}/\varepsilon)^n$ samples. ¹

¹Please note that the authors do not grant permission for this paper to be used in training any algorithm or model.

1 Introduction

Few mathematical ideas are as pervasive or as powerful as convexity: recall that a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is *convex* if for every $x, y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$, we have $f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$. The mathematics of convex functions has been intensively studied for over a century, and continues to be central to a number of theoretical and applied disciplines ranging from mathematical analysis to machine learning [Roc97, BV04]. Within mathematical programming, for example, convexity of the objective function often governs computational tractability; to quote Rockafellar [Roc93], “the great watershed in optimization isn’t between linearity and nonlinearity, but convexity and non-convexity.”

This centrality of convexity motivates the following basic questions, which form the focus of this work:

How many samples are needed to *learn* a convex function?

How many samples are needed to *test* if a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is essentially convex?

We study each of these questions through the lens of theoretical computer science, viewing the former as a problem in *learning theory* [KV94] and the latter in *property testing* [Gol17, BY22]. For these problems to be well-defined, we must first specify a distance metric between functions on \mathbb{R}^n . Throughout this work, we employ the $L^2(\gamma)$ distance where $\gamma = \gamma_n$ denotes the measure corresponding to the n -dimensional standard Gaussian distribution $N(0, I_n)$.

Unsurprisingly, given their fundamental nature, both problems have been studied previously:

- There is work in statistics [SS11, GJ14, GS18, MCIS19, Düm24] that deals with optimal rates for convex regression where the perspective is that the dimension is fixed, and the number of samples is very large (could be more than exponential) and the question is to study the error rate as the number of samples goes to infinity. This is different than the standard perspective in learning theory where we are interested to more exactly quantify the number of samples needed as a function of the dimension.
- Turning to the problem of *testing* convexity, prior work has largely concerned discrete or highly constrained settings. Much of this literature [PRR06, BRY14a, PRV17, BE19, BBB20, LNV22] focuses on testing functions over finite domains or ranges under the Hamming distance, reflecting the field’s early connections to complexity theory [BLR90, RS96]. A few works such as [BRY14a] extend this line of work to real-valued functions under L^p metrics, but still assume bounded range and full query access.

We refer the reader to [Section 1.2](#) for a detailed discussion of related work on both testing and learning convex functions.

In this work, we study both *agnostic proper learning* and *(tolerant) testing* of convexity in high dimensions, in the statistically natural setting where one only has access to *samples* labeled by an unknown function f . The query-access model used extensively in property testing, while natural and powerful for applications to complexity theory [BLR90, RS96], is ill-suited to statistical problems. By contrast, sample-based access is the standard framework in settings such as empirical risk minimization, convex regression, and stochastic optimization, where functions are observed only through (possibly noisy) random samples rather than arbitrary queries [RM51, KSHdM02, HTF09, Bot10, SSBD14, GYDH15].

Before proceeding, we note that some smoothness assumption is essential for both problems: without any regularity, convexity cannot be meaningfully inferred from finitely many samples, even in one dimension. Accordingly, we restrict attention to *Lipschitz* functions—a natural and mild assumption also adopted in prior work on convex regression [MCIS19].

1.1 Our Results

We begin by introducing notation that will be used throughout the paper. Given a function $f \in L^2(\gamma)$, we will write $d_{\text{conv}}^L(f)$ for the $L^2(\gamma)$ distance of f to its nearest convex L -Lipschitz function. More formally,

$$d_{\text{conv}}^L(f) := \inf_{\substack{g: \mathbb{R}^n \rightarrow \mathbb{R} \\ g \text{ convex, } L\text{-Lipschitz}}} \|f - g\|_{L^2(\gamma)} \quad \text{where } \|f - g\|_{L^2(\gamma)} := \mathbf{E}_{\mathbf{x} \sim N(0, I_n)} \left[(f(\mathbf{x}) - g(\mathbf{x}))^2 \right]^{1/2}.$$

The $L^2(\gamma)$ metric is a standard choice in learning theory, property testing, and derandomization (see, for example, [KOS08, MORS10, Vem10, HKM13, KNOW14, CFSS17, DMN19, DS21, DMN21, KM22, DNS23, DNS24], among many others).

Although we focus on learning and testing convex functions over Gaussian space, all of our techniques and results extend naturally to functions over $[0, 1]^n$ under the uniform measure. We adopt the Gaussian setting as it is a more canonical model for studying real valued learning and testing problems.

1.1.1 Agnostic Proper Learning and Sample-Based Testing of Convex Functions

We construct an agnostic proper learning algorithm for Lipschitz convex functions with the following guarantees:

Theorem 1 (Agnostic proper learning of Lipschitz convex functions). Let $\varepsilon, L > 0$ and suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a L -Lipschitz function. There exists an algorithm which, given i.i.d. access to labeled samples $(\mathbf{x}, f(\mathbf{x}))$ where $\mathbf{x} \sim N(0, I_n)$, draws $n^{O(L^2/\varepsilon^2)}$ samples, runs in time $\exp(\tilde{O}(nL^2/\varepsilon^2))$, and outputs a function $g \in L^2(\gamma)$ such that

$$\|f - g\|_{L^2(\gamma)} \leq d_{\text{conv}}^L(f) + \varepsilon.$$

Additionally, the function g is convex and L -Lipschitz.

Using the proof of the Theorem above and the “testing-via-learning” paradigm [GGR98], we obtain a *tolerant* testing algorithm for Lipschitz convex functions:

Theorem 2 (Tolerant two-sided testing of Lipschitz convex functions). Let $\varepsilon, \varepsilon_0, L \geq 0$ and suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is L -Lipschitz. There is an algorithm which, given i.i.d. labeled samples $(\mathbf{x}, f(\mathbf{x}))$ where $\mathbf{x} \sim N(0, I_n)$, draws $n^{O(L^2/\varepsilon^2)}$ samples, runs in time $\exp(\tilde{O}(nL^2/\varepsilon^2))$, and has the following performance guarantee:

- If $d_{\text{conv}}^L(f) \leq \varepsilon_0$, then it outputs “accept” with probability 9/10;
- If $d_{\text{conv}}^L(f) \geq \varepsilon + \varepsilon_0$, then it outputs “reject” with probability 9/10.

In particular, setting $\varepsilon_0 = 0$ shows that for every fixed $\varepsilon > 0$, there exists a $\text{poly}(n)$ -time algorithm that distinguishes Lipschitz convex functions from those that are ε -far (in $L^2(\gamma)$) from all Lipschitz convex functions. The “testing-via-learning” approach underlies the best known sample-based testing algorithms for several other function classes including convex subsets of \mathbb{R}^n [KOS08, CFSS17], monotone Boolean functions [BT96, LRV22, LV25a], and k -monotone Boolean functions [Bla24].

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Our proof of [Theorem 1](#) proceeds in two steps:

1. **Low-Degree Approximation.** We begin with a standard observation (see, for example, Theorem 4.1 of [APVZ14] or alternatively [HSSV21, DMN21]) that every L -Lipschitz function over Gaussian space is well-approximated in L_2 by a $O(L^2)$ -degree polynomial. This allows us to learn such an approximation \tilde{f} from random samples via polynomial regression.
2. **Convex Regression.** We then *convexify* the learned polynomial \tilde{f} by projecting it onto the space of convex functions via convex regression [MCIS19]. This step is essential to obtain a *proper* learner—one whose output lies within the target class of convex L -Lipschitz functions. For our tolerant testing result, this is essential: the “testing via learning” reduction requires the learner to be proper.

Although the first step is conceptually straightforward, it has an independent implication:

Any $O(1)$ -Lipschitz function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ can be learned—and hence tested—from $n^{O(1/\varepsilon^2)}$ random samples in an information-theoretic sense.

Note, however, that this guarantee is purely information-theoretic and does not yield a computationally efficient learning algorithm for the function class. In general, however, fitting a function from the target class to the observed samples may be computationally intractable.

The second step introduces the main technical challenge: ensuring that the projection onto convex functions can be performed in finite (albeit exponential) time while preserving the $L^2(\gamma)$ error bound. These structural arguments form the core of the proof of [Theorem 2](#), drawing on Hermite analysis and geometric properties of convex functions to control how granular our (discretized) projection onto convex functions needs to be to yield sufficient approximation guarantees over all of Gaussian space.

1.1.2 Lower Bounds for Learning Convex Functions

We complement this upper bound with corresponding *lower bounds* for learning convex functions. In the information-theoretic setting, we show that learning an arbitrary linear function to constant accuracy requires $\Omega(n)$ queries, not just samples; see [Theorem 29](#). We further strengthen this by proving a quantitatively sharper bound for algorithms in the *correlational statistical query* (CSQ) model:

Theorem 3 (CSQ lower bound; informal, see [Theorem 38](#)). Suppose algorithm A , given access to L -Lipschitz function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ via correlational statistical queries, outputs a function $g \in L^2(\gamma)$ satisfying $\|f - g\|_{L^2(\gamma)} \leq d_{\text{conv}}^L(f) + \varepsilon$. Then A requires $n^{\text{poly}(L/\varepsilon)}$ correlational statistical queries with tolerance $n^{-\text{poly}(L/\varepsilon)}$.

CSQ algorithms only have access to the input f via approximate inner product queries of the form $\mathbf{E}_{z \sim N(0, I_n)}[f(z)h(z)]$, for query h chosen by the algorithm; we formally define the CSQ model in [Section 4](#). This model captures, for example, gradient descent with respect to squared loss, and the algorithm attaining our $n^{O(L^2/\varepsilon^2)}$ upper bound from [Theorem 1](#) via low-degree approximation is a CSQ algorithm. Thus, [Theorem 3](#) poses a barrier to improving upon the low-degree learning approach: any agnostic learning algorithm that fits the CSQ model, even if it uses properties of

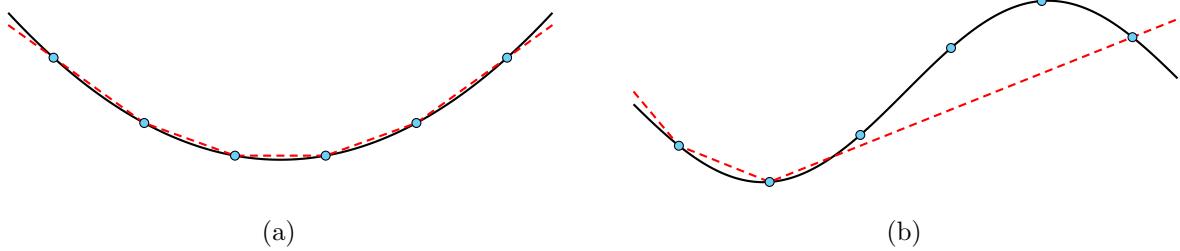


Figure 1: One-dimensional illustration of the empirical convex envelope (red, dashed) of (a) a convex function and (b) a far-from-convex function, with respect to the sampled points shown in cyan. (See Definition 42 for a precise definition.)

convex functions to do something more sophisticated than low-degree approximation, must use $n^{\text{poly}(L/\varepsilon)}$ statistical queries.

Our proof relies on the construction of a large family \mathcal{F} of functions with low pairwise-correlation, which is the standard approach for showing *weak learning* lower bounds in the CSQ model [BFJ⁺94, Szö09, Fel12]. Additionally, we require each function in \mathcal{F} to be non-trivially correlated with a convex Lipschitz function—namely a “projected ReLU”, which is a function of the form $x \mapsto \text{ReLU}(\langle x, u \rangle)$ for some $u \in \mathbb{S}^{n-1}$. This implies that an agnostic learning algorithm for convex functions gives a weak learner for \mathcal{F} , and yields our CSQ lower bound.

Our construction and proof strategy are inspired by and leverage the results of [DKZ20], who showed an $n^{\text{poly}(1/\varepsilon)}$ lower bound for agnostically learning ReLUs under the Gaussian distribution using statistical queries. While one might hope to immediately conclude the same lower bound for agnostically learning convex functions in our model (since ReLUs are convex), the key difference is that in our setting the input must be Lipschitz, whereas [DKZ20] build their hard family of inputs from Boolean functions. We adapt their construction to our purposes via careful application of the Ornstein-Uhlenbeck noise operator P_t , which satisfies a few crucial properties for our application:

- It takes bounded functions into Lipschitz ones, so that from a hard input $f : \mathbb{R}^n \rightarrow \{\pm 1\}$, we may obtain a hard Lipschitz input $P_t f$.
- It satisfies the self-adjointness property $\langle P_t f, g \rangle = \langle f, P_t g \rangle$, which allows us to map the approximation quality of a learned hypothesis g for the Lipschitz input $P_t f$ into the approximation quality of hypothesis $P_t g$ for the hard input f .
- Lipschitz functions are stable under noise in the sense that $\|P_t f - f\|_2^2 = O(t)$, which informally implies that most of the “signal” in the reduction survives our application of noise.

1.1.3 One-Sided Testing of Lipschitz Convex Functions

Our final result is a *one-sided* testing algorithm for Lipschitz convex functions—that is, an algorithm which never rejects a Lipschitz convex function.

Theorem 4 (One-sided testing of Lipschitz convex functions). Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be L -Lipschitz, and let $\varepsilon \geq 0$. There is an algorithm, ONE-SIDED-TESTER (Algorithm 1), which, given access to i.i.d. labeled samples $(\mathbf{x}, f(\mathbf{x}))$ where $\mathbf{x} \sim N(0, I_n)$, makes $O(\sqrt{n}L/\varepsilon)^n$ draws, runs in time $(nL/\varepsilon)^{O(n)}$, and has the following performance guarantee:

- If f is convex, then ONE-SIDED-TESTER outputs “accept” with probability 1;

- If $d_{\text{conv}}^L(f) \geq \varepsilon$, then ONE-SIDED-TESTER outputs “reject” with probability 9/10.

Our tester operates by constructing an *empirical convex envelope* (cf. [Definition 42](#)) of the sampled function values, which can be viewed as the function analogue of the convex hull of a collection of points. Given random samples $(\mathbf{x}_i, f(\mathbf{x}_i))$, the tester checks whether f coincides with its empirical convex envelope at these points. If f is convex, this always holds; conversely, we show that if f is L -Lipschitz and ε -far from convex, then with sufficiently many samples the test will detect a violation with high probability. See [Figure 1](#) for an illustration of the empirical convex envelope.

In particular, our analysis implies an associated *agnostic learning guarantee in $L^\infty(\gamma)$* : we show that when the input function f is an L -Lipschitz convex function, the empirical convex envelope learned by ONE-SIDED-TESTER uniformly approximates the convex envelope of f over all but an exponentially-small fraction of Gaussian space. Thus, the algorithm may also be viewed as a form of *uniform convex regression*.

1.2 Related Work

We briefly situate our results within the broader work on learning and testing convexity.

Learning Convex Functions. As discussed earlier, the statistical perspective for *convex regression* is interested in the error rate as the number of samples goes to infinity for fixed dimensions. This falls under the broader umbrella of *shape-constrained regression*. We refer the interested reader to [[SS11](#), [GJ14](#), [GS18](#), [MCIS19](#), [Düm24](#)] and the references therein for further background. By contrast, our results address the more general problem of *agnostic proper learning* from i.i.d. samples, with the goal of establishing PAC-style sample-complexity bounds rather than asymptotic error rates.

We note that low-degree approximation of Lipschitz functions over Gaussian space has been used a few times in the past; see, for example, [[APVZ14](#), [HSSV21](#), [DMN21](#)]. Such approximations also are the source of $n^{\text{poly}(1/\varepsilon)}$ sample-complexity bounds appearing in several agnostic learning problems beyond convexity, e.g., in the context of learning halfspaces [[DKK⁺21](#), [LV25b](#)]. However, with the exception of [[APVZ14](#)], most of these works focus on Boolean-valued function classes rather than real-valued ones.

Approximation Theory. As indicated earlier in [Section 1.1](#), our upper bounds for convexity testing follow from semi-agnostic learning algorithms for convex functions under the Gaussian L_2 distance. Related problems have been studied in approximation theory, both in the low-dimensional setting [[Son83](#), [Rot92](#)] as well as in high-dimensions [[HNW11](#)], with exponential-in- n sample lower bounds known for deterministic approximation under L^p distance [[HNW11](#)].

More broadly, classical results in convex approximation typically focus on deterministic settings, approximating a convex function by piecewise-linear or polynomial surrogates and bounding the error in terms of smoothness or curvature (see, e.g., [[DL93](#), [HNW11](#)]). Our setting differs in that we consider randomized, sample-based approximation under a fixed measure, leading to dimension-dependent but probabilistic guarantees.

Testing Convexity over Discrete Domains. The computational problem of recognizing convexity of quartic polynomials is known to be NP-hard [[AOPT13](#)]. (Deciding convexity of polynomials of odd-degree or quadratics is trivial, making degree 4 the first interesting case.) See [[AOPT13](#)] and the references therein for a history of the computational problem of recognizing

exact convexity of functions. The relaxed problem of recognizing *approximate* convexity of functions has primarily been considered in the discrete setting. The study of testing convexity of functions $f : [N] \rightarrow \mathbb{R}$ under the Hamming distance was initiated by Parnas, Ron, and Rubinfeld [PRR03], where convexity is defined as non-negativity of a discrete second-derivative. Their work was later extended by Ben-Eliezer [BE19], who obtained improved quantitative bounds; by Pallavoor, Raskhodnikova, and Verma [PRV17], who gave an improved bound parametrized by the size of the range of the function being tested; by Blais, Raskhodnikova, and Yaroslavtsev [BRY14b], who gave lower bounds on testing convexity of real-valued functions over the hypergrid $[N]^d$; and by Belovs, Blais, and Bommireddi [BBB20], who gave upper and lower bounds on the number of queries required to test convexity of real-valued functions over various discrete domains including the discrete line, the “stripe” $[3] \times N$, and the hypergrid $[N]^d$. These results, however, do not have any bearing on this work due to differences in domain and choice of distance metric.

L^p Testing of Convexity. The work most closely related to ours is that of Berman, Raskhodnikova, and Yaroslavtsev [BRY14a], who study L^p testing of convexity of an unknown $f : [0, 1]^n \rightarrow [0, 1]$ where $[0, 1]^n$ is endowed with the uniform measure. They provide tight bounds in the one-dimensional setting and exponential-in- n bounds for the high-dimensional case (see Corollary B.3 of [BRY14a]). Our results, in contrast, apply specifically to Lipschitz functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$, giving polynomial-in- n complexity algorithms for every fixed choice of the distance parameter ε . (We remark that our exponential-in- n algorithm from [Theorem 4](#) has the additional property of being a tester with one-sided error, which does not follow from a black-box application of testing-by-learning.)

Although one might hope to relate the two settings, this connection does not appear to be straightforward. The two frameworks are best viewed as complementary: ours imposes regularity assumptions that enable polynomial dependence on n , whereas [BRY14a] addresses unrestricted (i.e., possibly highly irregular) albeit bounded functions at exponential cost. Indeed, assuming Lipschitzness or bounded gradient is the standard regularity condition in convex analysis, optimization, and high-dimensional learning, and thus provides a more natural setting for analytic property testing.

Testing Set Convexity. A different line of research considers testing convexity of high-dimensional sets (i.e., Boolean-valued indicator functions of subsets of \mathbb{R}^n). Rademacher and Vempala [RV05] established exponential-in- n upper bounds for testing set convexity under the Lebesgue measure; they also gave a strong lower bound against a simple “line” tester for convexity. The [RV05] lower bound was subsequently strengthened by Blais and Bommireddi [BB20]. For testing set convexity under the Gaussian measure, the best-known upper bound follows from the agnostic learning algorithm for convex sets due to Klivans, O’Donnell, and Servedio [KOS08].

Strong lower bounds for sample-based testing under the Gaussian measure were established by Chen, Freilich, Servedio, and Sun [CFSS17], and the first lower bounds in the query model were obtained recently by Chen, De, Nadimpalli, Servedio, and Waingarten [CDN⁺25]. While our work shares conceptual and technical similarities with [KOS08, CFSS17, CDN⁺25], there does not appear to be a reduction or formal connection between the two settings. In particular, natural attempts to reduce testing convexity of functions to testing convexity of sets (such as mapping a function to its epigraph) fail to preserve the relevant distance measures, so neither problem appears to subsume the other.

Finally, beyond the Gaussian setting, the work of Black, Blais, and Harms [BBH24] also considers consider testing and learning convex subsets of the ternary hypercube $\{0, \pm 1\}^n$.

1.3 Discussion

We conclude with a number of open directions suggested by our work.

Analogy with Boolean Monotonicity. Convexity is formally connected to the monotonicity of the gradient: in one dimension, a function is convex if and only if its derivative is monotone, and more generally, convexity corresponds to the monotonicity of the gradient map. This connection suggests a natural analogy between convexity and monotonicity, and one might hope that insights from the rich body of work on learning and testing monotone Boolean functions can inform the study of convex function learning and testing.

For *learning*, our results may at first appear considerably stronger than existing bounds for learning monotone Boolean functions. (Recall that the sample complexity of agnostically learning monotone Boolean functions over $\{0, 1\}^n$ is $2^{\tilde{O}(\sqrt{n})}$ [BT96].) However, this stems from the $O(1)$ -Lipschitz assumption in our setting, which has no meaningful analog in the Boolean function setting. While all Boolean functions are technically $O(1)$ -Lipschitz, the underlying metric geometry differs between the Boolean setting and Gaussian space: in the Boolean space (see, for example, [LLRV25]), the typical L_1 (Hamming) distance between points in dimension n is of order n while in the Gaussian space the typical distance is of order \sqrt{n} . So, at least in some sense, our polynomial agnostic learning algorithm does correspond to the results for learning monotone functions with $2^{\Theta(\sqrt{n})}$ samples.

As for *testing*, in the Boolean setting the analogous quantity to d_{conv} is the *distance to monotonicity* d_{mon} , which has been extensively investigated in a rich line of work on monotonicity testing [GGLR98, CS13, KMS18]. Classical Boolean monotonicity testers rely on upper bounds on d_{mon} in terms of the *directed edge boundary*, and these structural relationships can be viewed as strengthenings of the classical edge-isoperimetric inequalities on the hypercube [Mar74, Tal93].

This analogy raises a natural question: can d_{conv} likewise give rise to structural inequalities that quantify the “non-convexity” of real-valued functions? Motivated by the role of the directed edge boundary in monotonicity testing, the following question suggests itself:

Question 5. Given an L -Lipschitz function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, is

$$d_{\text{conv}}(f) \leq \text{poly}(n) \cdot \mathbf{E}_{\mathbf{x} \sim N(0, I_n)} \left[\|\nabla_-^2 f(\mathbf{x})\|_{\text{op}} \right]$$

where $\nabla_-^2 f(\mathbf{x})$ is the negative-definite part of $\nabla^2 f(\mathbf{x})$?

If true, such an inequality would likely imply a convexity tester based on random samples of (or queries simulating) the Hessian of f , mirroring how monotonicity testers—both in discrete and continuous settings—leverage local gradient information [KMS18, Fer24]. Beyond its algorithmic implications, this inequality would also be of intrinsic geometric interest, as it would relate a *global* measure of non-convexity (namely, $d_{\text{conv}}(f)$) to *local* curvature data through the Hessian.

As a step toward understanding d_{conv} , we establish in [Section A](#) a quantitative relationship between $d_{\text{conv}}(f)$ and the negative degree-2 Hermite coefficients of f , which yields a lower bound on $d_{\text{conv}}(f)$ in terms of their magnitude.

We note that an independent line of work has identified an unexpected but powerful analogy between monotone Boolean functions and *convex subsets of Gaussian space* [DNS21, DS21, DNS22, DNS23, DNS24]. It is natural to ask which aspects of this correspondence extend further to the setting of convex functions over Gaussian space. As a first step in this direction, we note that [DNS21] establishes a quantitative sharpening of Hu’s correlation inequality for symmetric convex sets [Hu97], in the spirit of Talagrand’s quantitative correlation inequality [Tal96].

Testing Lower Bounds. Since we are concerned with functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$ whose outputs are real-valued, standard information-theoretic approaches to proving lower bounds become infeasible. A basic but significant obstacle—shared with many real-valued testing problems—is that each sample reveals a real number $f(x)$, which, at least naively, may contain an arbitrarily large amount of information. Presently, there is no general framework for establishing lower bounds for testing convexity of real-valued functions, even in the purely sample-based setting.

Therefore, we pose the following two questions on the true sample complexity of testing convexity of Lipschitz functions in the two-sided and one-sided error regimes. For two-sided testing, it is conceivable—again in analogy with monotonicity testing—that some polynomial dependence on n is unavoidable. We ask:

Question 6. Does testing convexity of 1-Lipschitz functions under the Gaussian distribution (with two-sided error) require $n^{\Omega(1)}$ samples? What if arbitrary queries are allowed?

Even for one-sided testing algorithms, we currently lack any nontrivial lower bounds on the sample complexity of testing convexity. It is instructive to recall how an exponential lower bound is obtained for the analogous problem in the setting of *set* convexity [CFSS17]. In particular, Lemma 29 of [CFSS17] establishes that for sufficiently small constant c , if 2^{cn} points are drawn independently from $N(0, I_n)$, then with high probability no point lies in the convex hull of the others. Consequently, such a draw of samples is consistent with the indicator function of a convex set, and a one-sided sample-based tester cannot detect any violation of convexity.

The difficulty in our setting is that a sample with no witnessed violation of convexity need not be *consistent* with any L -Lipschitz function that is far from convex. In particular, a naive interpolation through the sampled points may fail to be $O(L)$ -Lipschitz. Thus, it is not immediate that a one-sided tester must accept unless the sample explicitly contains a convexity violation. This leads to the following question:

Question 7. Is there a tester for convexity of 1-Lipschitz functions under the Gaussian distribution with one-sided error and sample complexity that is sub-exponential in n ?

Sample vs. Runtime Gaps. We note that while [Theorem 1](#) achieves a sample complexity of $n^{O(1/\varepsilon^2)}$, its runtime remains exponential in n . An analogous gap long persisted for properly learning monotone Boolean functions from samples: the best-known sample complexity was $2^{\tilde{O}(\sqrt{n})}$, whereas the best runtime was $2^{O(n)}$. Recent work by [LRV22, LV25a] closed this gap, giving a $2^{\tilde{O}(\sqrt{n})}$ -time agnostic proper learner. It is natural to ask whether a similarly efficient proper learning algorithm exists for convex functions, potentially closing the gap between information-theoretic and computational efficiency in our setting as well.

Lower Bounds in the Full SQ Model. Our $n^{\text{poly}(L/\varepsilon)}$ lower bound from [Theorem 3](#) applies to correlational statistical query (CSQ) algorithms, a setting that already captures the algorithm attaining our $n^{O(L^2/\varepsilon^2)}$ upper bound. Recall that CSQ algorithms only have access to the input f via approximate inner product queries of the form $\mathbf{E}_{z \sim N(0, I_n)}[f(z)h(z)]$. In contrast, the full SQ model allows approximate oracle access to $\mathbf{E}_{z \sim N(0, I_n)}[q(z, f(z))]$ for general query functions $q(x, y)$.

In the (distribution-specific) setting of *Boolean* functions, the CSQ model is equivalent to the full SQ model [BF02], because SQ queries to Boolean functions may be simulated using correlational queries. However, the same is not true in the real-valued setting, where there exist separations between the CSQ and SQ models, for example for the problem of ReLU regression [GV25]. Indeed, proving lower bounds against SQ algorithms in the real-valued setting is a notoriously challenging

problem; see [GGK20, CGKM22] and the discussion therein. We leave as an open question whether our $n^{\text{poly}(L/\varepsilon)}$ CSQ lower bound can be extended to the full SQ model, or whether our problem witnesses a separation between these models.

2 Preliminaries

We use boldfaced letters—such as $\mathbf{x}, \mathbf{f}, \mathbf{A}$ —to denote random variables (which may be real-valued, vector-valued, function-valued, or set-valued; the intended type will be clear from the context). We write $\mathbf{x} \sim \mathcal{D}$ to indicate that the random variable \mathbf{x} is distributed according to the probability distribution \mathcal{D} . Throughout, $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : \|x\|_2 = 1\}$ will denote the unit sphere in \mathbb{R}^n . We will write $B_p(r) \subseteq \mathbb{R}^n$ for the ℓ_p -ball of radius r , i.e.,

$$B_p(r) := \{x \in \mathbb{R}^n : \|x\|_p \leq r\}.$$

2.1 Hermite Analysis over Gaussian Space

Our notation and terminology follow Chapter 11 of [O'D14]. We say that an n -dimensional *multi-index* is a tuple $\alpha \in \mathbb{N}^n$, and we define

$$|\alpha| := \sum_{i=1}^n \alpha_i.$$

For $n \in \mathbb{N}_{>0}$, we write $L^2(\gamma_n)$ to denote the space of functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$ that have finite second moment under the Gaussian distribution, i.e., $f \in L^2(\gamma_n)$ if

$$\|f\|_{L^2(\gamma_n)}^2 := \mathbf{E}_{\mathbf{z} \sim N(0, I_n)} [f(\mathbf{z})^2] < \infty.$$

When the dimension n is clear from context, we will write $\gamma = \gamma_n$ instead. We view $L^2(\gamma)$ as an inner product space with

$$\langle f, g \rangle_{L^2(\gamma)} := \mathbf{E}_{\mathbf{z} \sim N(0, I_n)} [f(\mathbf{z})g(\mathbf{z})].$$

We will sometimes write $\langle f, g \rangle = \langle f, g \rangle_{L^2(\gamma)}$ for notational convenience. We recall the Hermite basis for $L^2(\gamma_1)$:

Definition 8 (Hermite basis). The *Hermite polynomials* $(h_j)_{j \in \mathbb{N}}$ are the univariate polynomials defined as

$$h_j(x) = \frac{(-1)^j}{\sqrt{j!}} \exp\left(\frac{x^2}{2}\right) \cdot \frac{d^j}{dx^j} \exp\left(-\frac{x^2}{2}\right).$$

In particular, we have

$$h_2(x) := \frac{x^2 - 1}{\sqrt{2}}. \tag{1}$$

The following fact is standard:

Fact 9 (Proposition 11.33 of [O'D14]). The Hermite polynomials $(h_j)_{j \in \mathbb{N}}$ form a complete, orthonormal basis for $L^2(\gamma_1)$. For $n > 1$ the collection of n -variate polynomials given by $(h_\alpha)_{\alpha \in \mathbb{N}^n}$ where

$$h_\alpha(x) := \prod_{i=1}^n h_{\alpha_i}(x)$$

forms a complete, orthonormal basis for $L^2(\gamma_n)$.

Given a function $f \in L^2(\gamma)$ and $\alpha \in \mathbb{N}^n$, we define its *Hermite coefficient on α* as $\widehat{f}(\alpha) = \langle f, h_\alpha \rangle$. It follows that $f : \mathbb{R}^n \rightarrow \mathbb{R}$ can be uniquely expressed as

$$f = \sum_{\alpha \in \mathbb{N}^n} \widehat{f}(\alpha) h_\alpha$$

with the equality holding in $L^2(\gamma)$; we will refer to this expansion as the *Hermite expansion* of f . One can check that Parseval's and Plancharel's identities hold in this setting:

$$\langle f, f \rangle = \sum_{\alpha \in \mathbb{N}^n} \widehat{f}(\alpha)^2 \quad \text{and} \quad \langle f, g \rangle = \sum_{\alpha \in \mathbb{N}^n} \widehat{f}(\alpha) \widehat{g}(\alpha).$$

The following lemma, which can be obtained via integration by parts and appears explicitly in [HSSV21], gives a term-by-term differentiation identity for a function in terms of its Hermite representation.

Lemma 10 ([HSSV21, Lemma 42]). Let $f \in L^2(\gamma)$, and let $i \in [n]$ be such that f is differentiable with respect to x_i and $\frac{\partial f}{\partial x_i} \in L^2(\mathbb{R}^n, \gamma)$. Then we have $\frac{\partial f}{\partial x_i} = \sum_{\alpha \in \mathbb{N}^n} \widehat{f}(\alpha) \frac{\partial h_\alpha}{\partial x_i}$.

Remark 11. Lemma 10 is stated for functions that are differentiable in x_i , but it holds just as well for Lipschitz functions, which in particular are differentiable almost everywhere and belong to the Gaussian Sobolev space $D^{1,2}$, see e.g., Section 2 of [AK18]; so we may proceed by approximating by polynomials, namely truncations of the Hermite expansion.

The (one-dimensional) Hermite polynomials enjoy the following differentiation formula.

Lemma 12 (Exercise 11.10(a) of [O'D14]). For each $j \geq 1$, we have $h'_j(z) = \sqrt{j} \cdot h_{j-1}(z)$.

We recall the Gaussian Poincaré inequality (see, e.g., [BGL14]):

Proposition 13. Suppose $f \in L^2(\gamma)$ is differentiable. Then

$$\mathbf{Var}_{\mathbf{x} \sim N(0, I_n)}[f(\mathbf{x})] \leq \mathbf{E}_{\mathbf{x} \sim N(0, I_n)} \left[\|f(\mathbf{x})\|_2^2 \right].$$

2.2 Noise Operator

We will make use of the *Ornstein–Uhlenbeck noise operator* and its regularizing properties; we refer the reader to Chapter 11 of [O'D14] as well as the monograph by Bakry, Gentil, and Ledoux [BGL14] for further background.

Definition 14 (Definition 11.12 and Proposition 11.37 of [O'D14]). For $t \geq 0$, the *Ornstein–Uhlenbeck noise operator* P_t is the linear operator defined on the space of functions $f \in L_1(\gamma_n)$ by

$$P_t f(x) := \mathbf{E}_{\mathbf{g} \sim N(0, I_n)} \left[f \left(e^{-t} x + \sqrt{1 - e^{-2t}} \mathbf{g} \right) \right].$$

Furthermore, if $f = \sum_{\alpha \in \mathbb{N}^n} \widehat{f}(\alpha) h_\alpha$, then

$$P_t f = \sum_{\alpha \in \mathbb{N}^n} e^{-t|\alpha|} \widehat{f}(\alpha) h_\alpha.$$

Fact 15 (Proposition 11.16 of [O'D14]). Let $f \in L_1(\gamma_n)$ and $t > 0$. Then $P_t f$ is a smooth function.

Fact 16 (Section 4.7.1 of [BGL14]). For all $f \in L_2(\gamma_n)$ and $t > 0$, we have the pointwise estimate

$$\|\nabla P_t f(x)\|_2^2 \leq \frac{P_t(f(x)^2) - (P_t f(x))^2}{e^{2t} - 1} \leq \frac{P_t(f(x)^2)}{2t}.$$

Taking expectations, we obtain:

Corollary 17. For all $f \in L_2(\gamma_n)$ and $t > 0$, we have that $|\nabla P_t f| \in L_2(\gamma_n)$.

The following commutation property will be useful:

Fact 18 (Section 2.7.1 of [BGL14]). For every smooth $f \in L_2(\gamma_n)$ and $t > 0$, we have

$$\nabla P_t f = e^{-t} P_t(\nabla f).$$

3 Agnostic Proper Learning and Two-Sided Sample-Based Testing

Write $\mathcal{C}(L)$ for the class of convex, L -Lipschitz functions:

$$\mathcal{C}(L) := \{f : \mathbb{R}^n \rightarrow \mathbb{R} : f \text{ is } L\text{-Lipschitz and convex}\}.$$

Recall our notation for the L_2 distance to the set of convex L -Lipschitz functions, which we may now write as

$$d_{\text{conv}}^L(f) = \inf_{g \in \mathcal{C}(L)} \|f - g\|_{L^2(\gamma)}.$$

By convention, given a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, by “sample from f ” we mean a sample of the form $(\mathbf{x}, f(\mathbf{x}))$ where $\mathbf{x} \sim N(0, I_n)$. The main result in this section is the following.

Theorem 1 (Agnostic proper learning of Lipschitz convex functions). Let $\varepsilon, L > 0$ and suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a L -Lipschitz function. There exists an algorithm which, given i.i.d. access to labeled samples $(\mathbf{x}, f(\mathbf{x}))$ where $\mathbf{x} \sim N(0, I_n)$, draws $n^{O(L^2/\varepsilon^2)}$ samples, runs in time $\exp(\tilde{O}(nL^2/\varepsilon^2))$, and outputs a function $g \in L^2(\gamma)$ such that

$$\|f - g\|_{L^2(\gamma)} \leq d_{\text{conv}}^L(f) + \varepsilon.$$

Additionally, the function g is convex and L -Lipschitz.

Combining the agnostic proper learning algorithm with a distance estimation step, we obtain a sample-based tolerant tester for Lipschitz convex functions:

Theorem 2 (Tolerant two-sided testing of Lipschitz convex functions). Let $\varepsilon, \varepsilon_0, L \geq 0$ and suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is L -Lipschitz. There is an algorithm which, given i.i.d. labeled samples $(\mathbf{x}, f(\mathbf{x}))$ where $\mathbf{x} \sim N(0, I_n)$, draws $n^{O(L^2/\varepsilon^2)}$ samples, runs in time $\exp(\tilde{O}(nL^2/\varepsilon^2))$, and has the following performance guarantee:

- If $d_{\text{conv}}^L(f) \leq \varepsilon_0$, then it outputs “accept” with probability 9/10;
- If $d_{\text{conv}}^L(f) \geq \varepsilon + \varepsilon_0$, then it outputs “reject” with probability 9/10.

Proof. Note that

$$d_{\text{conv}}^L(f)^2 \leq \mathbf{Var}[f] \leq \mathbf{E} \left[\|\nabla f\|^2 \right] \leq L^2$$

by the Poincaré inequality (Proposition 13), the fact that f is L -Lipschitz, and the fact that constant functions are convex. Hence we may assume that $\varepsilon \leq L$, since otherwise the tester can immediately accept.

On input $f : \mathbb{R}^n \rightarrow \mathbb{R}$, the tester uses the learning algorithm from Theorem 1 to learn a function $g \in \mathcal{C}(L)$ satisfying $\|f - g\|_2 \leq d_{\text{conv}}^L(f) + \varepsilon/4$ with probability at least $2/3$; assume henceforth that this event occurs. It then takes $m = O(L^4/\varepsilon^4)$ samples $(\mathbf{x}_i, f(\mathbf{x}_i))$, and accepts if and only if

$$\mathbf{Y} := \frac{1}{m} \sum_{i=1}^m (f(\mathbf{x}_i) - g(\mathbf{x}_i))^2 \leq \left(\varepsilon_0 + \frac{\varepsilon}{2} \right)^2.$$

The sample and time complexity claims are immediate, and we now prove correctness. Let $h := f - g$, so that $\mathbf{E}[\mathbf{Y}] = \mathbf{E}[h^2] = \|f - g\|_{L^2(\gamma)}^2$ and $\mathbf{Var}[\mathbf{Y}] = \frac{1}{m} \mathbf{Var}[h^2]$. Define the centered function $\bar{h} := h - \mathbf{E}[h]$. Then \bar{h} is centered and $2L$ -Lipschitz, and hence sub-Gaussian with sub-Gaussian norm $O(L)$. Then

$$\mathbf{Var}[h^2] = \mathbf{Var}[(\bar{h} + \mathbf{E}[h])^2] = \mathbf{Var}[\bar{h}^2 + 2\mathbf{E}[h]\bar{h}] \leq 2\mathbf{Var}[\bar{h}^2] + 2 \cdot \mathbf{E}[h]^2 \mathbf{Var}[\bar{h}].$$

By the sub-Gaussianity of \bar{h} , we have $\mathbf{Var}[\bar{h}^2] = O(L^4)$ and $\mathbf{Var}[\bar{h}] = O(L^2)$. Moreover, we have

$$\mathbf{E}[h]^2 \leq \mathbf{E}[h^2] = \|f - g\|_{L^2(\gamma)}^2 \leq \left(d_{\text{conv}}^L(f) + \frac{\varepsilon}{4} \right)^2 \leq 2d_{\text{conv}}^L(f)^2 + \varepsilon^2 \leq O(L^2).$$

We conclude that $\mathbf{Var}[h^2] \leq O(L^4)$. By Chebyshev's inequality,

$$\Pr \left[\left| \mathbf{Y} - \|f - g\|_{L^2(\gamma)}^2 \right| \geq \frac{\varepsilon^2}{16} \right] \leq \frac{\frac{1}{m} \cdot O(L^4)}{\varepsilon^4 / 256} \leq \frac{1}{10},$$

for suitable choice of $m = O(L^4/\varepsilon^4)$. Suppose $\left| \mathbf{Y} - \|f - g\|_{L^2(\gamma)}^2 \right| \leq \varepsilon^2/16$. Then if $d_{\text{conv}}^L(f) \leq \varepsilon_0$, we obtain

$$\mathbf{Y} \leq \|f - g\|_{L^2(\gamma)}^2 + \frac{\varepsilon^2}{16} \leq \left(\|f - g\|_{L^2(\gamma)} + \frac{\varepsilon}{4} \right)^2 \leq \left(d_{\text{conv}}^L(f) + \frac{\varepsilon}{2} \right)^2 \leq \left(\varepsilon_0 + \frac{\varepsilon}{2} \right)^2,$$

and the tester accepts. On the other hand, suppose $d_{\text{conv}}^L(f) > \varepsilon_0 + \varepsilon$. Then, recalling that g is a convex, L -Lipschitz function, we obtain

$$\mathbf{Y} \geq \|f - g\|_{L^2(\gamma)}^2 - \frac{\varepsilon^2}{16} > (\varepsilon_0 + \varepsilon)^2 - \frac{\varepsilon^2}{16} = \varepsilon_0^2 + 2\varepsilon_0\varepsilon + \frac{15\varepsilon^2}{16} > \varepsilon_0^2 + 2\varepsilon_0 \cdot \frac{3\varepsilon}{4} + \left(\frac{3\varepsilon}{4} \right)^2 = \left(\varepsilon_0 + \frac{3\varepsilon}{4} \right)^2,$$

and the tester rejects. By a union bound, the tester correctly accepts/rejects except with probability at most $\frac{2}{3} + \frac{1}{10} < \frac{1}{2}$, which may be made as small as desired by a standard majority vote. \square

Proof of the Main Result. We require the following lemmas. The first lemma says that a Lipschitz function is well-approximated by its low-degree Hermite expansion. This fact has been observed before, e.g. in [APVZ14, HSSV21] for domain $[0, 1]^n$; here we give a Gaussian version. Recall the notation $f^{\leq d} := \sum_{|\alpha| \leq d} \hat{f}(\alpha)h_\alpha$ for the degree- d Hermite expansion of f .

Lemma 19 (Lipschitz functions are approximately low-degree). Let $L, \varepsilon > 0$, let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be L -Lipschitz, and let $d := \lceil L^2/\varepsilon^2 \rceil$. Then $\|f - f^{\leq d}\|_{L^2(\gamma)} \leq \varepsilon$.

Additionally, we now give upper bounds on the number of samples required to approximately learn the low-degree Hermite coefficients of a Lipschitz function. We assume for simplicity that f has expectation bounded by some constant (this assumption is not essential, and the general case is recovered in the proof of [Theorem 1](#)).

Lemma 20 (Learning a single Hermite coefficient). Let $L, \xi > 0$, $\delta \in (0, 1)$, $\alpha \in \mathbb{N}^n$, and let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be L -Lipschitz and satisfy $|\mathbf{E}[f]| \leq O(1)$. Then

$$O\left(\frac{L^2 3^{|\alpha|}}{\xi^2} \log\left(\frac{1}{\delta}\right)\right) \text{ samples from } f$$

suffice to compute an estimate $\tilde{\mathbf{f}}(\alpha)$ satisfying $|\tilde{\mathbf{f}}(\alpha) - \hat{\mathbf{f}}(\alpha)| \leq \xi$ with probability at least $1 - \delta$. The running time is polynomial in the number of samples.

We have the following immediate consequence:

Corollary 21 (Learning a low-degree expansion). Let $L, \varepsilon > 0$, $\delta \in (0, 1)$, $d \in \mathbb{N}$, and let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be L -Lipschitz and satisfy $|\mathbf{E}[f]| \leq O(1)$. Then

$$O\left(\frac{L^2 n^{2.01d}}{\varepsilon^2} \log\left(\frac{1}{\delta}\right)\right) \text{ samples from } f$$

suffice to obtain coefficient estimates $\tilde{\mathbf{f}}(\alpha)$, for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq d$, such that the function $\mathbf{f} := \sum_{|\alpha| \leq d} \tilde{\mathbf{f}}(\alpha) h_\alpha$ satisfies $\|f^{\leq d} - \mathbf{f}\|_{L^2(\gamma)} \leq \varepsilon$ with probability at least $1 - \delta$. The running time is polynomial in the number of samples.

We will require a simple a priori upper bound on the Lipschitz constant of the low-degree approximation above in terms of distance to the origin:

Lemma 22 (Lipschitz constant of low-degree polynomials). Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a degree- d polynomial. Then for all $x \in \mathbb{R}^n$, we have

$$\|\nabla f(x)\|_2 \leq \|f\|_{L^2(\gamma)} d^{d+\frac{1}{2}} n^{d/2} (1 + |x|)^d.$$

The next lemma says that, given *query* access to a Lipschitz function g which is close to some convex Lipschitz function over a ball, we may learn (via convex regression) a good convex approximation to g using a finite number of queries. We will apply this result to an approximation \mathbf{f} of the low-degree restriction $f^{\leq d}$ over the restriction of the Gaussian measure to a sufficiently large ball. It is convenient to work with the centered radius- r ℓ^∞ -ball $B_\infty(r)$, i.e. an axis-aligned hypercube.

Remark 23. This number of queries used for the convex regression is exponential in n ; however, we make these queries to the learned approximation \mathbf{f} rather than the input function f , so we pay this cost in the running time but not in the sample complexity.

Lemma 24 (Learning a Lipschitz convex approximation). Let $r > 0$, and let μ be the probability measure γ conditioned to $B_\infty(r)$. There exists an algorithm which, given parameters $L, \varepsilon > 0$ and query access to some L -Lipschitz function $g : B_\infty(r) \rightarrow \mathbb{R}$, makes $(Lr\sqrt{n}/\varepsilon)^{O(n)}$ queries to g and outputs a function $\tilde{g} \in \mathcal{C}(L)$ that is a nearly optimal convex approximation of g over $B_\infty(r)$ in the following sense: for all convex, L -Lipschitz $h : B_\infty(r) \rightarrow \mathbb{R}$, we have

$$\|\tilde{g} - g\|_{L^2(B_\infty(r), \mu)} \leq \|h - g\|_{L^2(B_\infty(r), \mu)} + \varepsilon.$$

The running time is polynomial in the number of queries.

To translate back and forth between full Gaussian space and its restriction to an ℓ^∞ -ball, while approximately preserving the distance between Lipschitz functions, we use the following lemma.

Lemma 25 (Restricting Gaussian space to a box). There exists a universal constant $C > 0$ such that the following holds. Let $L, \tau \geq 1, \varepsilon \in (0, L)$, and let $r := C\sqrt{\log(nL\tau/\varepsilon)}$. Let μ be probability measure γ conditioned to $B_\infty(r)$. Then for all L -Lipschitz functions $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $|f(x^*) - g(x^*)| \leq \tau$ for some $x^* \in B_\infty(r)$, we have

$$\left| \|f - g\|_{L^2(\gamma)} - \|f - g\|_{L^2(B_\infty(r), \mu)} \right| \leq \varepsilon.$$

Remark 26. The condition that $|f(x) - g(x)| \leq \tau$ for some $x \in B_\infty(r)$ in Lemma 25 holds in particular if $\|f - g\|_2 \leq \tau/2$, since otherwise the choice of radius r is large enough to yield (say) $\|f - g\|_2 \geq 3\tau/4$, a contradiction.

Equipped with these results, we may prove [Theorem 1](#).

Proof of Theorem 1. We first claim that we may assume without loss of generality that $|\mathbf{E}[f]| \leq \varepsilon/100$. Indeed, the algorithm may first compute an approximation \bar{f} of $\mathbf{E}[f]$ up to $\varepsilon/100$ additive error using $O(L^2/\varepsilon^2)$ samples (by Chebyshev's inequality, since $\mathbf{Var}[f] \leq \mathbf{E}[\|\nabla f\|_2^2] \leq L^2$ by [Proposition 13](#), and the Lipschitz assumption), then proceed with the modified input function $f - \bar{f}$ and shift the final output again by \bar{f} .

We may also assume that $L \geq 1$ (by rescaling the input and ε if necessary) and that $\varepsilon < \frac{100}{99}L$ (otherwise, we get $\|f\|_{L^2(\gamma)} \leq \|f - \mathbf{E}[f]\|_{L^2(\gamma)} + |\mathbf{E}[f]| \leq L + \varepsilon/100 \leq \varepsilon$ by the triangle and Poincaré inequalities, and we may immediately output the constant zero function).

Let $\varepsilon' := \varepsilon/8$ and $d := \lceil (L/\varepsilon')^2 \rceil$. The algorithm first uses

$$O\left(\frac{L^2 n^{2.01d}}{\varepsilon^2}\right) = n^{O(L^2/\varepsilon^2)} \text{ samples from } f$$

to learn, via [Corollary 21](#), a degree- d approximation \mathbf{f} satisfying $\|f^{\leq d} - \mathbf{f}\|_{L^2(\gamma)} \leq \varepsilon'$ with probability at least $2/3$; assume henceforth that this event occurs. By [Lemma 19](#) and the triangle inequality, we have $\|f - \mathbf{f}\|_{L^2(\gamma)} \leq 2\varepsilon'$. Note that \mathbf{f} satisfies $\|\mathbf{f}\|_{L^2(\gamma)} \leq \|f\|_{L^2(\gamma)} + 2\varepsilon' \leq L + \varepsilon/100 + 2\varepsilon' \leq 2L$.

Let $r = \Theta\left(\frac{L}{\varepsilon}\sqrt{\log(nL/\varepsilon)}\right)$ be large enough to ensure that, when we set

$$L' := 2L(2ndr)^{2d} \geq \|\mathbf{f}\|_{L^2(\gamma)}(2ndr)^{2d} \geq \|\mathbf{f}\|_{L^2(\gamma)} d^{d+\frac{1}{2}} n^{d/2} (1 + r\sqrt{n})^d$$

as the upper bound on the Lipschitz constant of \mathbf{f} over $B_\infty(r)$ via [Lemma 22](#), we obtain

$$r \geq C' \sqrt{\log(nL'/\varepsilon)}$$

for sufficiently large choice of absolute constant $C' > 0$ so that, by [Lemma 25](#), ball $B_\infty(r)$ gives an ε' -approximation for the distance between \mathbf{f} and L -Lipschitz functions g for parameter $\tau = 6L$. (One may check that it is possible to choose r satisfying this.)

Let $\tilde{g} \in \mathcal{C}(L)$ be the function produced by [Lemma 24](#) with input $g = \mathbf{f}$, the choice of r above, and parameters L' and ε' . Note that the number of queries to \mathbf{f} , and thus the running time of this step, is

$$(L'r\sqrt{n}/\varepsilon)^{O(n)} = (Lndr/\varepsilon)^{O(dn)} = (nL/\varepsilon)^{O(nL^2/\varepsilon^2)} = \exp(\tilde{O}(nL^2/\varepsilon^2)).$$

We claim that \tilde{g} satisfies the guarantee of [Theorem 1](#). We already have that $\tilde{g} \in \mathcal{C}(L)$, so it remains to show that $\|\tilde{g} - f\|_{L^2(\gamma)} \leq d_{\text{conv}}^L(f) + \varepsilon$. Let $h \in \mathcal{C}(L)$ be any function satisfying $\|f - h\|_{L^2(\gamma)} \leq L$ (which exists because f is L -close to the constant $|\mathbf{E}[f]|$ function by the Poincaré inequality), so that $\|\mathbf{f} - h\|_{L^2(\gamma)} \leq L + 2\varepsilon' \leq 2L$. We have

$$\begin{aligned} \|\tilde{g} - f\|_{L^2(\gamma)} &\leq \|\tilde{g} - \mathbf{f}\|_{L^2(\gamma)} + \|\mathbf{f} - f\|_{L^2(\gamma)} \\ &\leq \|\tilde{g} - \mathbf{f}\|_{L^2(B_\infty(r), \mu)} + 3\varepsilon' && (\text{Lemma 25, see below}) \\ &\leq \|h - \mathbf{f}\|_{L^2(B_\infty(r), \mu)} + 4\varepsilon' && (\text{Lemma 24}) \\ &\leq \|h - \mathbf{f}\|_{L^2(\gamma)} + 5\varepsilon' && (\text{Lemma 25 via Remark 26}) \\ &\leq \|h - f\|_{L^2(\gamma)} + \|f - \mathbf{f}\|_{L^2(\gamma)} + 5\varepsilon' \\ &\leq \|h - f\|_{L^2(\gamma)} + 7\varepsilon'. \end{aligned}$$

To justify the first use of [Lemma 25](#), we claim that \tilde{g} satisfies $|\tilde{g}(x) - \mathbf{f}(x)| \leq 6L = \tau$ at some point $x \in B_\infty(r)$. Indeed, note that the function h satisfies (by [Lemma 25](#) via [Remark 26](#)) $\|h - \mathbf{f}\|_{L^2(B_\infty(r), \mu)} \leq \|h - f\|_{L^2(\gamma)} + \varepsilon' \leq 3L$; hence, if we had $|\tilde{g}(x) - \mathbf{f}(x)| > 6L$ for all $x \in B_\infty(r)$, we would obtain $\|\tilde{g} - \mathbf{f}\|_{L^2(B_\infty(r), \mu)} > 6L > \|h - \mathbf{f}\|_{L^2(B_\infty(r), \mu)} + \varepsilon'$, contradicting [Lemma 24](#).

Since the above holds for all $h \in \mathcal{C}(L)$ sufficiently close to f , we obtain $\|\tilde{g} - f\|_{L^2(\gamma)} \leq d_{\text{conv}}^L(f) + \varepsilon$ as desired. \square

We now prove the required lemmas.

3.1 Low-Degree Approximation

Lemma 19 (Lipschitz functions are approximately low-degree). Let $L, \varepsilon > 0$, let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be L -Lipschitz, and let $d := \lceil L^2/\varepsilon^2 \rceil$. Then $\|f - f^{\leq d}\|_{L^2(\gamma)} \leq \varepsilon$.

Proof. The condition that f is L -Lipschitz implies that $\|\nabla f(x)\|_{L^2(\gamma)} \leq L$ for all $x \in \mathbb{R}^n$. Consequently, we have

$$\begin{aligned} L^2 &\geq \mathbf{E}_{\mathbf{x} \sim N(0, I_n)} \left[\|\nabla f(\mathbf{x})\|_2^2 \right] = \sum_{i=1}^n \mathbf{E}_{\mathbf{x} \sim N(0, I_n)} \left[\left(\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}_i} \right)^2 \right] \\ &= \sum_{i=1}^n \mathbf{E}_{\mathbf{x} \sim N(0, I_n)} \left[\left(\sum_{\substack{\alpha \in \mathbb{N}^n \\ \alpha_i \neq 0}} \widehat{f}(\alpha) \sqrt{\alpha_i} \cdot h_{\alpha-e_i}(\mathbf{x}) \right)^2 \right] && (\text{Lemmas 10 and 12}) \\ &= \sum_{i=1}^n \sum_{\substack{\alpha, \beta \in \mathbb{N}^n \\ \alpha_i, \beta_i \neq 0}} \widehat{f}(\alpha) \widehat{f}(\beta) \sqrt{\alpha_i \beta_i} \cdot \mathbf{E}_{\mathbf{x} \sim N(0, I_n)} [h_{\alpha-e_i}(\mathbf{x}) \cdot h_{\beta-e_i}(\mathbf{x})] \end{aligned}$$

$$= \sum_{i=1}^n \sum_{\alpha_i \neq 0} \alpha_i \hat{f}(\alpha)^2 = \sum_{\alpha \in \mathbb{N}^n} |\alpha| \hat{f}(\alpha)^2, \quad (2)$$

where Equation (2) relied on the orthonormality of the Hermite polynomials. (Here $|\alpha| = \sum_{i=1}^n \alpha_i$.) As an immediate consequence of Equation (2), we have $|\hat{f}(\alpha)| \leq L$ for all $\alpha \neq 0^n$. Define the function

$$g := \sum_{|\alpha| \leq d} \hat{f}(\alpha) h_\alpha. \quad (3)$$

By Parseval's formula, we have

$$\mathbf{E}_{\mathbf{x} \sim N(0, I_n)} [(f(\mathbf{x}) - g(\mathbf{x}))^2] = \sum_{|\alpha| > d} \hat{f}(\alpha)^2 \leq \sum_{|\alpha| > d} \hat{f}(\alpha)^2 \cdot \frac{|\alpha|}{d} \leq \frac{L^2}{d} \leq \varepsilon^2,$$

which completes the proof. \square

Lemma 20 (Learning a single Hermite coefficient). Let $L, \xi > 0$, $\delta \in (0, 1)$, $\alpha \in \mathbb{N}^n$, and let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be L -Lipschitz and satisfy $|\mathbf{E}[f]| \leq O(1)$. Then

$$O\left(\frac{L^2 3^{|\alpha|}}{\xi^2} \log\left(\frac{1}{\delta}\right)\right) \text{ samples from } f$$

suffice to compute an estimate $\tilde{f}(\alpha)$ satisfying $|\tilde{f}(\alpha) - \hat{f}(\alpha)| \leq \xi$ with probability at least $1 - \delta$. The running time is polynomial in the number of samples.

Proof. We prove the claim for $\delta = 1/3$, and the general case follows by the standard median trick. The algorithm takes $s = O\left(\frac{L^2 3^{|\alpha|}}{\xi^2}\right)$ samples of the form $(\mathbf{x}_i, f(\mathbf{x}_i))$ and outputs $\tilde{f}(\alpha) := \frac{1}{s} \sum_{i=1}^s f(\mathbf{x}_i) h_\alpha(\mathbf{x}_i)$. By the definition of $\hat{f}(\alpha)$, we have

$$\mathbf{E}[\tilde{f}(\alpha)] = \mathbf{E}_{\mathbf{x} \sim N(0, I_n)} [f(\mathbf{x}) h_\alpha(\mathbf{x})] = \hat{f}(\alpha).$$

We now compute the variance. Let $\bar{f} := f - \mathbf{E}[f]$, so that for one sample $\mathbf{x} \sim N(0, I_n)$, we have

$$\begin{aligned} \mathbf{Var}[f(\mathbf{x}) h_\alpha(\mathbf{x})] &= \mathbf{Var}[\bar{f} h_\alpha + \mathbf{E}[f] h_\alpha] \\ &\leq 2 \mathbf{Var}[\bar{f} h_\alpha] + 2 \mathbf{E}[f]^2 \mathbf{Var}[h_\alpha] \\ &\leq 2 \mathbf{E}[(\bar{f} h_\alpha)^2] + 2 \mathbf{E}[f]^2 \mathbf{E}[h_\alpha^2] \\ &\leq 2 \mathbf{E}[\bar{f}^4]^{1/2} \mathbf{E}[h_\alpha^4]^{1/2} + O(1), \end{aligned}$$

the last step by the Cauchy-Schwarz inequality, the assumption that $|\mathbf{E}[f]| \leq O(1)$, and the fact that $\mathbf{E}[h_\alpha^2] = 1$. By hypercontractivity (e.g. [O'D14, Corollary 9.6]), we have $\mathbf{E}[h_\alpha^4]^{1/2} \leq 3^{|\alpha|} \mathbf{E}[h_\alpha^2] = 3^{|\alpha|}$. Since \bar{f} is L -Lipschitz and hence sub-Gaussian with sub-Gaussian norm $O(L)$ (see [Ver18, Theorem 5.2.3]), we have $\mathbf{E}[\bar{f}^4]^{1/2} = O(L^2)$. Hence $\mathbf{Var}[f(\mathbf{x}) h_\alpha(\mathbf{x})] = O(L^2 3^{|\alpha|})$, and the conclusion follows by Chebyshev's inequality. \square

Corollary 21 (Learning a low-degree expansion). Let $L, \varepsilon > 0$, $\delta \in (0, 1)$, $d \in \mathbb{N}$, and let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be L -Lipschitz and satisfy $|\mathbf{E}[f]| \leq O(1)$. Then

$$O\left(\frac{L^2 n^{2.01d}}{\varepsilon^2} \log\left(\frac{1}{\delta}\right)\right) \text{ samples from } f$$

suffice to obtain coefficient estimates $\tilde{\mathbf{f}}(\alpha)$, for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq d$, such that the function $\mathbf{f} := \sum_{|\alpha| \leq d} \tilde{\mathbf{f}}(\alpha) h_\alpha$ satisfies $\|f^{\leq d} - \mathbf{f}\|_{L^2(\gamma)} \leq \varepsilon$ with probability at least $1 - \delta$. The running time is polynomial in the number of samples.

Proof. Apply Lemma 20 for all coefficients $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq d$, of which there are at most

$$\binom{n+d-1}{d} \leq \left(\frac{(n+d) \cdot e}{d} \right)^d \leq (2en)^d,$$

with parameters $\xi = \varepsilon/(2en)^{d/2}$ and $\delta' = \delta/(2en)^d$. By Parseval's Theorem, we have

$$\left\| f^{\leq d} - \mathbf{f} \right\|_{L^2(\gamma)}^2 \leq \sum_{|\alpha| \leq d} (\hat{f}(\alpha) - \tilde{\mathbf{f}}(\alpha))^2 \leq (2en)^d \cdot \xi^2 = \varepsilon^2$$

except with failure probability at most $(2en)^d \cdot \delta' = \delta$ by the union bound. The number of samples required is at most

$$O\left((2en)^d \cdot \frac{L^2 3^d}{(\varepsilon/(2en)^{d/2})^2} \log\left(\frac{(2en)^d}{\delta}\right)\right) \leq O\left(\frac{L^2 n^{2.01d}}{\varepsilon^2} \log(1/\delta)\right). \quad \square$$

3.2 Lipschitz Convex Regression

We employ convex regression, via a quadratically constrained convex program, to find a Lipschitz convex approximation \tilde{g} of a function g over the ball $B_\infty(r)$. Our optimization problem, over a discrete grid-like point set X , follows the formulation of [MCIS19, Section 4]. We then use the promise that the input g is Lipschitz to argue that the learned approximation \tilde{g} is also a good approximation over the original space $B_\infty(r)$.

Lemma 24 (Learning a Lipschitz convex approximation). Let $r > 0$, and let μ be the probability measure γ conditioned to $B_\infty(r)$. There exists an algorithm which, given parameters $L, \varepsilon > 0$ and query access to some L -Lipschitz function $g : B_\infty(r) \rightarrow \mathbb{R}$, makes $(Lr\sqrt{n}/\varepsilon)^{O(n)}$ queries to g and outputs a function $\tilde{g} \in \mathcal{C}(L)$ that is a nearly optimal convex approximation of g over $B_\infty(r)$ in the following sense: for all convex, L -Lipschitz $h : B_\infty(r) \rightarrow \mathbb{R}$, we have

$$\|\tilde{g} - g\|_{L^2(B_\infty(r), \mu)} \leq \|h - g\|_{L^2(B_\infty(r), \mu)} + \varepsilon.$$

The running time is polynomial in the number of queries.

Proof. Let $\alpha := \frac{\varepsilon}{10L\sqrt{n}}$. Define the discrete set $X \subset B_\infty(r)$ by

$$X := \left\{ \sum_{i=1}^n \text{snap}_r(k_i \alpha e_i) : k \in \mathbb{Z}^n, |k_i| \leq \lceil r/\alpha \rceil \forall i \in [n] \right\}, \text{ where } \text{snap}_r(t) := \max(-r, \min(r, t)),$$

so that X forms an evenly-spaced, axis-aligned grid of side lengths at most α . Note that $|X| = O(r/\alpha)^n = O(Lr\sqrt{n}/\varepsilon)^n$. For each $x \in X$, define the cell P_x as

$$P_x := ([x_1, x_1 + \alpha e_1] \times \cdots \times [x_n, x_n + \alpha e_n]) \cap B_\infty(r)$$

Note that each P_x has diameter at most $\alpha\sqrt{n}$, and the collection

$$\mathcal{P} := \{P_x : x \in X\},$$

is a partition of $B_\infty(r)$ (up to a measure zero set). Also associate with each $x \in X$ the probability mass

$$\hat{\mu}(x) := \mu(P_x),$$

so that $\hat{\mu}$ is a probability distribution over X . The following claim establishes that $\hat{\mu}$ is a good approximation for μ regarding $L^2(\gamma)$ distances between L -Lipschitz functions:

Claim 27. Let $f_1, f_2 : B_\infty(r) \rightarrow \mathbb{R}$ be L -Lipschitz functions. Then

$$\left| \|f_1 - f_2\|_{L^2(B_\infty(r), \mu)} - \|f_1 - f_2\|_{L^2(X, \hat{\mu})} \right| \leq \frac{\varepsilon}{5}.$$

We first complete the proof of the lemma assuming the claim, and then prove the claim below.

The algorithm computes the probability masses $\hat{\mu}(x)$ (using the fact that each cell P_x is an axis-aligned box and the fact that the standard Gaussian is a product distribution), and then constructs the following quadratic program in the variables $\hat{g}(x), \hat{u}(x)$ for $x \in X$:

$$\text{minimize}_{x \in X} \sum_{x \in X} \hat{\mu}(x)(\hat{g}(x) - g(x))^2 \tag{4}$$

$$\text{subject to } \hat{g}(x) + \langle \hat{u}(x), y - x \rangle \leq \hat{g}(y) \quad \text{for all } x, y \in X \tag{5}$$

$$\|\hat{u}(x)\|_2^2 \leq L^2 \quad \text{for all } x \in X. \tag{6}$$

Note that this is a convex quadratically constrained quadratic program (QCQP), and therefore can be solved in polynomial time using an interior point method (see [BV04, §11.5]). The algorithm computes a solution \hat{g} whose objective value (4) is within $\varepsilon^2/100$ of the optimal value, and then outputs the following extension of \tilde{g} of \hat{g} to all of $B_\infty(r)$:

$$\tilde{g}(y) := \max_{x \in X} \hat{g}(x) + \langle \hat{u}(x), y - x \rangle \quad \text{for all } y \in B_\infty(r). \tag{7}$$

We first claim that $\tilde{g} \in \mathcal{C}(L)$. Indeed, \tilde{g} is convex since it is the pointwise maximum of a set of affine functions, and it is L -Lipschitz because each such affine function in (7) is L -Lipschitz by the constraint (6) on the vectors $\hat{u}(x)$.

It remains to show the near-optimality of \tilde{g} as an approximation of g . Note that, by the constraint (5), we have $\tilde{g}(x) = \hat{g}(x)$ for each $x \in X$. Now, let $h : B_\infty(r) \rightarrow \mathbb{R}$ be convex and L -Lipschitz. Let $\hat{h} : X \rightarrow \mathbb{R}$ be the restriction of h to X , and let $\hat{v}(x)$ be a subgradient of h at each $x \in X$. Then \hat{h}, \hat{v} form a feasible solution to the QCQP, and therefore $\|\hat{g} - g\|_{L^2(X, \hat{\mu})} \leq \|\hat{h} - g\|_{L^2(X, \hat{\mu})} + \varepsilon/10$ by the near-optimality of \hat{g} . We now have

$$\begin{aligned} \|\tilde{g} - g\|_{L^2(B_\infty(r), \mu)} &\leq \|\tilde{g} - g\|_{L^2(X, \hat{\mu})} + \frac{2\varepsilon}{10} && \text{(By Claim 27)} \\ &= \|\hat{g} - g\|_{L^2(X, \hat{\mu})} + \frac{2\varepsilon}{10} \\ &\leq \|\hat{h} - g\|_{L^2(X, \hat{\mu})} + \frac{3\varepsilon}{10} && \text{(Near-optimality of } \hat{g} \text{)} \\ &\leq \|h - g\|_{L^2(B_\infty(r), \mu)} + \frac{5\varepsilon}{10}, && \text{(By Claim 27)} \end{aligned}$$

as desired. \square

We now prove [Claim 27](#).

Proof of Claim 27. Recall that each cell $P_x \in \mathcal{P}$ has diameter at most $\alpha\sqrt{n}$, where

$$\alpha = \frac{\varepsilon}{10L\sqrt{n}}.$$

The idea is that f_1 and f_2 , being L -Lipschitz functions, cannot vary by more than $L \cdot \alpha\sqrt{n} = \varepsilon/10$ within each cell, and so space $L^2(X, \hat{\mu})$ is a good approximation of $L^2(B_\infty(r), \mu)$. Formally, we have the following. For each $x \in B_\infty(r)$, let $x^* \in X$ be the point defining the cell that x belongs to, that is, the point satisfying $x \in P_{x^*}$ (chosen arbitrarily in the measure-zero set where the choice is not unique). Let $f_1^*, f_2^* : B_\infty(r) \rightarrow \mathbb{R}$ be given by $f_1^*(x) := f_1(x^*)$ and $f_2^*(x) := f_2(x^*)$ for each $x \in B_\infty(r)$. Then, for each $x \in B_\infty(r)$, we have

$$|f_1(x) - f_1^*(x)| = |f_1(x) - f_1(x^*)| \leq L|x - x^*| \leq L \cdot \alpha\sqrt{n} = \frac{\varepsilon}{10},$$

and similarly $|f_2(x) - f_2^*(x)| \leq \varepsilon/10$. Moreover, we have $\|f_1^* - f_2^*\|_{L^2(B_\infty(r), \mu)} = \|f_1 - f_2\|_{L^2(X, \hat{\mu})}$ by the construction of measure $\hat{\mu}$. Hence, by the triangle inequality, we have

$$\begin{aligned} \|f_1 - f_2\|_{L^2(B_\infty(r), \mu)} &\leq \|f_1 - f_1^*\|_{L^2(B_\infty(r), \mu)} + \|f_1^* - f_2^*\|_{L^2(B_\infty(r), \mu)} + \|f_2^* - f_2\|_{L^2(B_\infty(r), \mu)} \\ &\leq \frac{2\varepsilon}{10} + \|f_1 - f_2\|_{L^2(X, \hat{\mu})}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \|f_1 - f_2\|_{L^2(X, \hat{\mu})} &= \|f_1^* - f_2^*\|_{L^2(B_\infty, \mu)} \\ &\leq \|f_1^* - f_1\|_{L^2(B_\infty, \mu)} + \|f_1 - f_2\|_{L^2(B_\infty, \mu)} + \|f_2 - f_2^*\|_{L^2(B_\infty, \mu)} \\ &\leq \frac{2\varepsilon}{10} + \|f_1 - f_2\|_{L^2(B_\infty, \mu)}. \end{aligned} \quad \square$$

3.3 Restriction to a Box

Lemma 22 (Lipschitz constant of low-degree polynomials). Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a degree- d polynomial. Then for all $x \in \mathbb{R}^n$, we have

$$\|\nabla f(x)\|_2 \leq \|f\|_{L^2(\gamma)} d^{d+\frac{1}{2}} n^{d/2} (1 + |x|)^d.$$

Proof. Write the Hermite decomposition

$$f(x) = \sum_{|\alpha| \leq d} \widehat{f}(\alpha) h_\alpha(x).$$

By the triangle and Cauchy-Schwarz inequalities, along with Parseval's theorem, we have

$$\|\nabla f(x)\|_2 = \left\| \sum_{|\alpha| \leq d} \widehat{f}(\alpha) \nabla h_\alpha(x) \right\|_2 \leq \sum_{|\alpha| \leq d} |\widehat{f}(\alpha)| \|\nabla h_\alpha(x)\|_2 \leq \|f\|_{L^2(\gamma)} \left(\sum_{|\alpha| \leq d} \|\nabla h_\alpha(x)\|_2^2 \right)^{1/2}. \quad (8)$$

For each α and $i \in [n]$, we have

$$\frac{\partial}{\partial x_i} h_\alpha(x) = \frac{\partial}{\partial x_i} \prod_{j=1}^n h_{\alpha_j}(x_j) = h'_{\alpha_i}(x_i) \prod_{j \neq i} h_{\alpha_j}(x_j).$$

Using the identity $h'_k(z) = \sqrt{k} \cdot h_{k-1}(z)$ (Lemma 12) along with the bounds $|h_k(z)| \leq ((1 + |z|)k)^k$ (which we prove in Claim 28 below), $|x_j| \leq |x|$, and $\alpha_j \leq |\alpha| \leq d$, we obtain

$$\left| \frac{\partial}{\partial x_i} h_\alpha(x) \right| \leq \sqrt{\alpha_i} \prod_{j=1}^n ((1 + |x_j|)\alpha_j)^{\alpha_j} \leq \sqrt{\alpha_i} ((1 + |x|)d)^{\sum_{j=1}^n \alpha_j} \leq \sqrt{\alpha_i} ((1 + |x|)d)^d.$$

Summing over $i \in [n]$,

$$\|\nabla h_\alpha(x)\|_2^2 \leq \sum_{i=1}^n \alpha_i \cdot ((1 + |x|)d)^{2d} \leq d((1 + |x|)d)^{2d}.$$

Plugging this bound into (8), we conclude that

$$\|\nabla f(x)\|_2 \leq \|f\|_{L^2(\gamma)} \left(n^d \cdot d((1 + |x|)d)^{2d} \right)^{1/2} \leq \|f\|_{L^2(\gamma)} n^{d/2} d^{d+1/2} (1 + |x|)^d. \quad \square$$

Claim 28. For every integer $k \geq 1$ and $z \in \mathbb{R}$, we have

$$|h_k(z)| \leq ((1 + |z|)k)^k.$$

Proof. We proceed by strong induction. The cases $k = 1$ and $k = 2$ are immediate, since $h_1(z) = z$ and $h_2(z) = \frac{z^2 - 1}{\sqrt{2}}$. Assume the claim holds for all $k' \leq k$. We now use the following recurrence relation, see e.g. [O'D14, Exercise 11.10(b)]:

$$\sqrt{(k+1)!} \cdot h_{k+1}(z) = z\sqrt{k!} \cdot h_k(z) - k\sqrt{(k-1)!} \cdot h_{k-1}(z).$$

Using the triangle inequality, upper bounding the terms $k!$ and $(k-1)!$ by $k!$, and applying the inductive hypothesis, we obtain

$$\begin{aligned} |h_{k+1}(z)| &\leq |z| |h_k(z)| + k |h_{k-1}(z)| \\ &\leq |z| \cdot ((1 + |z|)k)^k + k \cdot ((1 + |z|)(k-1))^{k-1} \\ &\leq |z| \cdot ((1 + |z|)(k+1))^k + ((1 + |z|)(k+1))^k \\ &= (1 + |z|) \cdot ((1 + |z|)(k+1))^k \\ &\leq ((1 + |z|)(k+1))^{k+1}. \end{aligned} \quad \square$$

Lemma 25 (Restricting Gaussian space to a box). There exists a universal constant $C > 0$ such that the following holds. Let $L, \tau \geq 1$, $\varepsilon \in (0, L)$, and let $r := C\sqrt{\log(nL\tau/\varepsilon)}$. Let μ be probability measure γ conditioned to $B_\infty(r)$. Then for all L -Lipschitz functions $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $|f(x^*) - g(x^*)| \leq \tau$ for some $x^* \in B_\infty(r)$, we have

$$\left| \|f - g\|_{L^2(\gamma)} - \|f - g\|_{L^2(B_\infty(r), \mu)} \right| \leq \varepsilon.$$

Proof. Let $h := f - g$ and $h_r : \mathbb{R}^n \rightarrow \mathbb{R}$ be given by $h_r(x) := h(x) \cdot \mathbf{1}_{x \in B_\infty(r)}$. Fix $x^* \in B_\infty(r)$ for which $|h(x^*)| \leq \tau$, which exists by hypothesis. Note that h is $2L$ -Lipschitz, and we conclude that

$$|h(0)| \leq |h(x^*)| + 2L|x^*| \leq \tau + 2Lr\sqrt{n}.$$

We then have, letting $A > 0$ be a sufficiently large absolute constant whose value in the rest of the proof may change from line to line,

$$\begin{aligned}
\left| \|h\|_{L^2(\gamma)}^2 - \|h_r\|_{L^2(\gamma)}^2 \right| &= \left| \int_{\mathbb{R}^n \setminus B_\infty(r)} h(x)^2 d\gamma(x) \right| \\
&\leq \left| \int_{\mathbb{R}^n \setminus B_\infty(r)} (|h(0)| + L|x|)^2 d\gamma(x) \right| \\
&\leq \left| \int_{\mathbb{R}^n \setminus B_\infty(r)} (\tau + 2Lr\sqrt{n} + L|x|)^2 d\gamma(x) \right| \\
&\leq A \left| \int_{\mathbb{R}^n \setminus B_\infty(r)} (\tau^2 + L^2r^2n + L^2|x|^2) d\gamma(x) \right| \\
&= A \left| (\tau^2 + L^2r^2n) \Pr_{\mathbf{x} \sim N(0, I_n)} [\|\mathbf{x}\|_\infty > r] + L^2 \int_{\mathbb{R}^n \setminus B_\infty(r)} |x|^2 d\gamma(x) \right|.
\end{aligned}$$

For the first term in the last line above, a union bound gives

$$\Pr_{\mathbf{x} \sim N(0, I_n)} [\|\mathbf{x}\|_\infty > r] \leq n \Pr_{\mathbf{z} \sim N(0, 1)} [|z| > r] \leq 2ne^{-r^2/2} \leq 2n \cdot \left(\frac{\varepsilon}{100AnL\tau} \right)^{10} < \frac{\varepsilon^2}{100A} \cdot \frac{1}{\tau^2 + nL^2r^2}$$

for sufficiently large choice of the constant C . For the second term, we upper bound each term $x_i^2 \leq r^2$ by another term $x_j^2 > r^2$ in the contribution of each $x \notin B_\infty(r)$, and then use the product formulation of the standard Gaussian distribution to obtain

$$\begin{aligned}
\int_{\mathbb{R}^n \setminus B_\infty(r)} |x|^2 d\gamma(x) &\leq n \int_{\mathbb{R}^n \setminus B_\infty(r)} \left(\sum_{i=1}^n x_i^2 \mathbf{1}_{|x_i| > r} \right) d\gamma(x) = n^2 \mathbf{E}_{\mathbf{z} \sim N(0, 1)} [z^2 \mathbf{1}_{|z| > r}] \\
&\leq An^2 \int_r^\infty z^2 e^{-z^2/2} dz \leq An^2 \int_r^\infty e^{-z^2/4} dz \leq An^2 \int_{r/2}^\infty e^{-t^2} dt \\
&\leq An^2 \operatorname{erfc}(r/2) \leq An^2 e^{-r^2/4} \leq \frac{\varepsilon^2}{100A} \cdot \frac{1}{L^2},
\end{aligned}$$

where the third inequality used the fact that $z^2 \leq e^{z^2/4}$ for $z \geq r$ as long as C is large enough, the penultimate inequality used the upper bound $\operatorname{erfc}(x) \leq e^{-x^2}$ for the complementary error function, and the last inequality holds again by the definition of r for large enough C . We conclude that

$$\left| \|h\|_{L^2(\gamma)}^2 - \|h_r\|_{L^2(\gamma)}^2 \right| \leq \frac{\varepsilon^2}{50},$$

and hence

$$\left| \|h\|_{L^2(\gamma)} - \|h_r\|_{L^2(\gamma)} \right| \leq \frac{\varepsilon}{7}.$$

We now claim that $\left| \|h_r\|_{L^2(\gamma)} - \|h_r\|_{L^2(B_\infty, \mu)} \right| \leq \varepsilon/10$, which will conclude the proof by the triangle inequality. By definition of the conditional probability measure μ and the fact that $h_r = 0$ outside $B_\infty(r)$, we have

$$\|h_r\|_{L^2(\gamma)}^2 = \Pr_{\mathbf{x} \sim N(0, I_n)} [\mathbf{x} \in B_\infty(r)] \cdot \|h_r\|_{L^2(B_\infty, \mu)}^2,$$

and hence, using a union bound and the fact that h_r is $2L$ -Lipschitz inside $B_\infty(r)$ with $|h_r(x^*)| \leq \tau$ for some $x^* \in B_\infty(r)$, we have

$$\begin{aligned} 0 &\leq \|h_r\|_{L^2(B_\infty, \mu)}^2 - \|h_r\|_{L^2(\gamma)}^2 = \|h_r\|_{L^2(B_\infty, \mu)}^2 \Pr_{\mathbf{x} \sim N(0, I_n)} [\mathbf{x} \notin B_\infty(r)] \\ &\leq \left(\sup_{B_\infty(r)} h_r^2 \right) \cdot n \Pr_{\mathbf{z} \sim N(0, 1)} [|\mathbf{z}| > r] \leq n \cdot (\tau + 4Lr\sqrt{n})^2 \cdot \Pr_{\mathbf{z} \sim N(0, 1)} [|\mathbf{z}| > r] \\ &\leq An(\tau^2 + L^2r^2n)e^{-r^2/2} \leq \frac{\varepsilon^2}{100}, \end{aligned}$$

for sufficiently large choice of C . We conclude that $\left| \|h_r\|_{L^2(B_\infty, \mu)} - \|h_r\|_{L^2(\gamma)} \right| \leq \varepsilon/10$. \square

4 Lower Bounds for Learning Convex Functions

In this section, we show two lower bound results for learning convex Lipschitz functions.

Our first result is an $\Omega(n)$ lower bound for *realizable* learning even against algorithms that make adaptive queries. Informally, we show this lower bound by observing that a random linear function $x \mapsto \langle x, \mathbf{u} \rangle$, where $\mathbf{u} \sim \mathbb{S}^{n-1}$, cannot be learned to good accuracy without learning $\Omega(n)$ orthogonal projections of \mathbf{u} . This result shows that the dependence on n of our $n^{O(L^2/\varepsilon^2)}$ agnostic learner from [Section 3](#) cannot be improved beyond polynomial factors for constant L and ε , even if the algorithm is allowed to make adaptive queries and is only required to succeed when the input function is convex.

Our second result is motivated by the following question. Our $n^{O(L^2/\varepsilon^2)}$ upper bound from [Section 3](#) is obtained by learning the low-degree coefficients of the Lipschitz input function. On the one hand, any algorithm that learns a low-degree approximation must use this many samples. On the other hand, it is conceivable that learning an approximation against *convex* functions only is easier than learning a general low-degree approximation, i.e. that our upper bound is sub-optimal because it does not use convexity to its full potential. Is it possible to bypass low-degree learning and do better than [Section 3](#)? We give evidence of a negative answer in the form of an $n^{\text{poly}(L/\varepsilon)}$ lower bound for agnostic learning in the weaker setting of correlational Statistical Query (CSQ) algorithms. As we will see, our agnostic learner from [Section 3](#) is itself a CSQ algorithm. Therefore, bypassing the $n^{\text{poly}(L/\varepsilon)}$ barrier would require a different algorithmic approach. This result builds upon an SQ lower bound of [\[DKZ20\]](#) for ReLU regression on piecewise-constant input functions (which are not Lipschitz); we show that, by appropriate application of noise, that result implies a lower bound in our setting, where the target class is the class of convex functions and the input must be Lipschitz.

4.1 An $\Omega(n)$ Lower Bound for Realizable Learning

In this subsection, we show the following result:

Theorem 29. There exists an absolute constant $\varepsilon > 0$ such that the following holds. Suppose A is an algorithm which, on input $f : \mathbb{R}^n \rightarrow \mathbb{R}$ that is promised to be convex and 1-Lipschitz, makes at most $q(n)$ adaptive queries to f and outputs a function $g \in L^2(\gamma)$ satisfying $\|f - g\|_{L^2(\gamma)} \leq \varepsilon$ with probability at least $2/3$. Then $q(n) = \Omega(n)$.

We start by defining a “hard distribution” over inputs $\mathbf{f} \in \mathcal{C}(1)$, namely the distribution over orthogonal projections onto uniformly random directions.

Definition 30 (Hard distribution). For each vector $u \in \mathbb{S}^{n-1}$, let $f_u \in \mathcal{C}(1)$ be the function given by $f_u(x) := \langle x, u \rangle$. Then, the distribution \mathcal{D}_{lin} over $\mathcal{C}(1)$ is given as follows. To sample a function $\mathbf{f} \sim \mathcal{D}_{\text{lin}}$, sample a unit vector $\mathbf{u} \sim \mathbb{S}^{n-1}$ (the *secret vector*), and then output the function $\mathbf{f} = f_{\mathbf{u}}$.

Note that \mathcal{D}_{lin} is supported on convex, 1-Lipschitz functions. We now show that no algorithm can learn a random function $\mathbf{f} \sim \mathcal{D}_{\text{lin}}$ with good probability using a sublinear number of queries. Concretely, we prove the following lemma, which implies [Theorem 29](#):

Lemma 31. There exists an absolute constant $\varepsilon > 0$ such that the following holds. Suppose A is an algorithm which, on input $\mathbf{f} \sim \mathcal{D}_{\text{lin}}$, makes at most $q(n)$ adaptive queries to \mathbf{f} and outputs a function $\mathbf{g} \in L^2(\gamma)$ such that

$$\Pr \left[\|\mathbf{f} - \mathbf{g}\|_{L^2(\gamma)} > \varepsilon \right] \leq \frac{1}{3},$$

where the probability is over the choice of \mathbf{f} and the internal randomness of A . Then $q(n) = \Omega(n)$.

We start with some simplifying observations about the structure of any candidate algorithm A . First, its queries may be assumed to be pairwise-orthogonal unit vectors. This means that each query may be seen as learning a new orthogonal projection of the secret vector \mathbf{u} .

Claim 32. Let A be an algorithm fulfilling the conditions of [Lemma 31](#) with some query complexity $q(n)$. Then there is an algorithm A' fulfilling the conditions of [Lemma 31](#) with the same query complexity $q(n)$, and such that the queries $x'_1, \dots, x'_{q(n)}$ made by A' always satisfy the following two properties:

1. For all $1 \leq i < j \leq q(n)$, we have $\langle x'_i, x'_j \rangle = 0$.
2. Each x'_i is a unit vector.

Proof. Given algorithm A , we construct A' which simulates A while satisfying the two properties above. We proceed inductively. In the i^{th} step, suppose algorithm A makes query x_i . Write

$$x_i = \sum_{j=1}^{i-1} c_j x'_j + x''_i,$$

where $c_j := \langle x_i, x'_j \rangle$ for each $j < i$ and $x''_i := x_i - \sum_{j=1}^{i-1} c_j x'_j$. If $x''_i = 0$, then A'' may skip this query (or e.g. repeat a previous query) and compute the required value $f(x_i)$ using the linearity of f and the outputs of previous queries:

$$f(x_i) = f \left(\sum_{j=1}^{i-1} c_j x'_j \right) = \sum_{j=1}^{i-1} c_j f(x'_j).$$

Otherwise, if $x''_i \neq 0$, then A' queries f at point $x'_i := \frac{x''_i}{\|x''_i\|}$, and again computes $f(x_i)$ by linearity:

$$f(x_i) = f \left(\sum_{j=1}^{i-1} c_j x'_j + \|x''_i\| \cdot x'_i \right) = \sum_{j=1}^{i-1} c_j f(x'_j) + \|x''_i\| f(x'_i).$$

As desired, x'_i is a unit vector which is orthogonal to x'_j for all $j < i$. \square

Second, the output of the algorithm may be assumed to be some linear function f_u :

Claim 33. Let A be an algorithm fulfilling the conditions of [Lemma 31](#) with distance parameter $\varepsilon > 0$ and query complexity $q(n)$. Then there is an algorithm A' fulfilling the conditions of [Lemma 31](#) with distance parameter 2ε and same query complexity $q(n)$, whose output is always some function f_u for $u \in \mathbb{S}^{n-1}$ in the notation of [Definition 30](#).

Proof. This is essentially the claim that, information-theoretically, any algorithm for realizable learning may be made proper by incurring a factor of 2 in the distance parameter. Formally, we have the following.

Algorithm A' simulates A until A produces output \mathbf{g} , and then considers two cases. If $\|\mathbf{g} - f_u\|_{L^2(\gamma)} \leq \varepsilon$ for some $u \in \mathbb{S}^{n-1}$, then A' outputs an arbitrary such f_u . Otherwise, by convention A' outputs f_{e_1} . To prove correctness, let E be the event that there exists $u \in \mathbb{S}^{n-1}$ for which $\|\mathbf{g} - f_u\| \leq \varepsilon$. Then, writing $A'(f)$ for the output of A' , for each possible input f in the support of \mathcal{D}_{lin} we have

$$\mathbf{Pr} \left[\|A'(f) - f\|_{L^2(\gamma)} > 2\varepsilon \right] = \mathbf{Pr} \left[\|A'(f) - f\|_{L^2(\gamma)} > \varepsilon \wedge E \right] + \mathbf{Pr} \left[\|A'(f) - f\|_{L^2(\gamma)} > \varepsilon \wedge \neg E \right],$$

where the probabilities above are over the internal randomness of the algorithm A' . We consider each term separately. When event E occurs, we have $A'(f) = f_u$ for some vector u satisfying $\|\mathbf{g} - f_u\| \leq \varepsilon$, so that, by the triangle inequality,

$$\|A'(f) - f\|_{L^2(\gamma)} = \|f_u - f\|_{L^2(\gamma)} \leq \|f_u - \mathbf{g}\|_{L^2(\gamma)} + \|\mathbf{g} - f\|_{L^2(\gamma)} \leq \varepsilon + \|\mathbf{g} - f\|_{L^2(\gamma)},$$

and hence we have the implication

$$\|A'(f) - f\|_{L^2(\gamma)} > 2\varepsilon \text{ and } E \implies \|\mathbf{g} - f\|_{L^2(\gamma)} > \varepsilon \text{ and } E.$$

When event E does not occur, then by definition of E and the fact that $f = f_v$ for some $v \in \mathbb{S}^{n-1}$, we have $\|\mathbf{g} - f\|_{L^2(\gamma)} > \varepsilon$. We conclude that

$$\begin{aligned} \mathbf{Pr} \left[\|A'(f) - f\|_{L^2(\gamma)} > 2\varepsilon \right] &\leq \mathbf{Pr} \left[\|\mathbf{g} - f\|_{L^2(\gamma)} > \varepsilon \text{ and } E \right] + \mathbf{Pr} \left[\|\mathbf{g} - f\|_{L^2(\gamma)} > \varepsilon \text{ and } \neg E \right] \\ &= \mathbf{Pr} \left[\|\mathbf{g} - f\|_{L^2(\gamma)} > \varepsilon \right], \end{aligned}$$

as desired. \square

We conclude the proof of [Lemma 31](#) with two claims: first, if the algorithm only learns $m < n$ components of the secret vector \mathbf{u} , then in expectation the learned components only contribute an m/n -fraction of the squared norm of \mathbf{u} ; and second, this expectation dictates how good an approximation of $f_{\mathbf{u}}$ the algorithm is able to learn.

Claim 34. Suppose algorithm A makes $m < n$ orthonormal queries $\mathbf{x}_1, \dots, \mathbf{x}_m$ to input $f_{\mathbf{u}}$. Write $\mathbf{c}_i := \langle \mathbf{u}, \mathbf{x}_i \rangle$ for the output of the i^{th} query. Then

$$\mathbf{E}_{\mathbf{c}_1, \dots, \mathbf{c}_m} \left[\sum_{i=1}^m \mathbf{c}_i^2 \right] = \frac{m}{n}.$$

Proof. By the linearity of expectation,

$$\mathbf{E}_{\mathbf{c}_1, \dots, \mathbf{c}_m} \left[\sum_{i=1}^m \mathbf{c}_i^2 \right] = \sum_{i=1}^m \mathbf{E}_{\mathbf{c}_i} \left[\mathbf{c}_i^2 \right] = \sum_{i=1}^m \mathbf{E}_{\mathbf{u}, \mathbf{x}_i} \left[\langle \mathbf{u}, \mathbf{x}_i \rangle^2 \right]. \quad (9)$$

By rotation invariance, we know that there exists a number $\alpha_n > 0$ such that $\mathbf{E}_{\mathbf{u}} [\langle \mathbf{u}, v \rangle^2] = \alpha_n$ for all $v \in \mathbb{S}^{n-1}$. But then, considering the orthonormal basis e_1, \dots, e_n , we have

$$n \cdot \alpha_n = \sum_{i=1}^n \mathbf{E}_{\mathbf{u}} [\langle \mathbf{u}, e_i \rangle^2] = \mathbf{E}_{\mathbf{u}} \left[\sum_{i=1}^n \langle \mathbf{u}, e_i \rangle^2 \right] = \mathbf{E}_{\mathbf{u}} [\|\mathbf{u}\|_2^2] = 1,$$

and hence $\alpha_n = 1/n$. Plugging this into (9) concludes the proof. \square

Claim 35. Suppose algorithm A makes $m \leq \frac{n}{100}$ orthonormal queries $\mathbf{x}_1, \dots, \mathbf{x}_m$ to input $\mathbf{f} = f_{\mathbf{u}}$. Write $\mathbf{c}_i := \langle \mathbf{u}, \mathbf{x}_i \rangle$ for the output of the i^{th} query. Then its output \mathbf{g} satisfies

$$\mathbf{E}_{\mathbf{f}, \mathbf{g}} [\langle \mathbf{f}, \mathbf{g} \rangle] \leq \frac{1}{10}.$$

Proof. Write $\mathbf{q}_i = (\mathbf{x}_i, \mathbf{c}_i)$ for the input-output pair corresponding to the i^{th} query. By the law of total expectation,

$$\mathbf{E}_{\mathbf{f}, \mathbf{g}} [\langle \mathbf{f}, \mathbf{g} \rangle] = \mathbf{E}_{\mathbf{q}_1, \dots, \mathbf{q}_m} \left[\mathbf{E}_{\mathbf{g}} \left[\mathbf{E}_{\mathbf{u}} [\langle f_{\mathbf{u}}, \mathbf{g} \rangle_{L^2(\gamma)} \mid \mathbf{g}] \mid \mathbf{q}_1, \dots, \mathbf{q}_m \right] \right].$$

Note that the secret vector \mathbf{u} and the algorithm's output \mathbf{g} are independent conditional on the queries $\mathbf{q}_1, \dots, \mathbf{q}_m$. Moreover, by Claim 33, we may assume that $\mathbf{g} = f_{\mathbf{v}}$, where $\mathbf{v} \in \mathbb{S}^{n-1}$ depends on the queries and the internal randomness of A . Let $\mathbf{x}_{m+1}, \dots, \mathbf{x}_n$ be any completion of $\mathbf{x}_1, \dots, \mathbf{x}_m$ into an orthonormal basis, and let $\mathbf{c}_i := \langle \mathbf{u}, \mathbf{x}_i \rangle$ for each $m+1 \leq i \leq n$ as well. Note that $f_{\mathbf{u}} = \sum_{i=1}^n \mathbf{c}_i f_{\mathbf{x}_i}$ and that, by symmetry, for each $m+1 \leq i \leq n$ we have $\mathbf{E} [\mathbf{c}_i \mid \mathbf{q}_1, \dots, \mathbf{q}_m] = 0$. We then have

$$\begin{aligned} \mathbf{E}_{\mathbf{f}, \mathbf{g}} [\langle \mathbf{f}, \mathbf{g} \rangle] &= \mathbf{E}_{\mathbf{q}_1, \dots, \mathbf{q}_m} \left[\mathbf{E}_{\mathbf{v}} \left[\mathbf{E}_{\mathbf{u}} [\langle f_{\mathbf{u}}, f_{\mathbf{v}} \rangle \mid \mathbf{q}_1, \dots, \mathbf{q}_m] \right] \right] \\ &= \mathbf{E}_{\mathbf{q}_1, \dots, \mathbf{q}_m} \left[\mathbf{E}_{\mathbf{v}} \left[\mathbf{E}_{\mathbf{u}} \left[\left\langle \sum_{i=1}^n \mathbf{c}_i f_{\mathbf{x}_i}, f_{\mathbf{v}} \right\rangle \right] \mid \mathbf{q}_1, \dots, \mathbf{q}_m \right] \right] \\ &= \mathbf{E}_{\mathbf{q}_1, \dots, \mathbf{q}_m} \left[\mathbf{E}_{\mathbf{v}} \left[\left\langle \sum_{i=1}^n \mathbf{E} [\mathbf{c}_i] f_{\mathbf{x}_i}, f_{\mathbf{v}} \right\rangle \mid \mathbf{q}_1, \dots, \mathbf{q}_m \right] \right] \\ &= \mathbf{E}_{\mathbf{q}_1, \dots, \mathbf{q}_m} \left[\mathbf{E}_{\mathbf{v}} \left[\left\langle \sum_{i=1}^m \mathbf{c}_i f_{\mathbf{x}_i}, f_{\mathbf{v}} \right\rangle \mid \mathbf{q}_1, \dots, \mathbf{q}_m \right] \right] \\ &= \mathbf{E}_{\mathbf{q}_1, \dots, \mathbf{q}_m} \left[\left\langle \sum_{i=1}^m \mathbf{c}_i f_{\mathbf{x}_i}, \mathbf{E}_{\mathbf{v}} [f_{\mathbf{v}} \mid \mathbf{q}_1, \dots, \mathbf{q}_m] \right\rangle \right]. \end{aligned}$$

Write $\mathbf{E} [\mathbf{v} \mid \mathbf{q}_1, \dots, \mathbf{q}_m] = \sum_{i=1}^n \mathbf{d}_i \mathbf{x}_i$, where vector of the expected coefficients $\mathbf{d} = (\mathbf{d}_1, \dots, \mathbf{d}_n)$ is a random variable determined by the queries and internal randomness of A . Then $\mathbf{E}_{\mathbf{v}} [f_{\mathbf{v}} \mid \mathbf{q}_1, \dots, \mathbf{q}_m] = \sum_{i=1}^n \mathbf{d}_i f_{\mathbf{x}_i}$. Moreover, we have $\|\mathbf{d}\|_2 \leq 1$ by Jensen's inequality. Observing as well that $f_{\mathbf{x}_1}, \dots, f_{\mathbf{x}_n}$ are orthonormal vectors in $L^2(\gamma)$, we have

$$\mathbf{E}_{\mathbf{f}, \mathbf{g}} [\langle \mathbf{f}, \mathbf{g} \rangle] = \mathbf{E}_{\mathbf{q}_1, \dots, \mathbf{q}_m} \left[\left\langle \sum_{i=1}^m \mathbf{c}_i f_{\mathbf{x}_i}, \sum_{i=1}^n \mathbf{d}_i f_{\mathbf{x}_i} \right\rangle \right] = \mathbf{E}_{\mathbf{q}_1, \dots, \mathbf{q}_m} \left[\sum_{i=1}^m \mathbf{c}_i \mathbf{d}_i \right]$$

$$\leq \mathbf{E}_{\mathbf{q}_1, \dots, \mathbf{q}_m} [\|\mathbf{c}\|_2 \|\mathbf{d}\|_2] \leq \sqrt{\mathbf{E}_{\mathbf{q}_1, \dots, \mathbf{q}_m} [\|\mathbf{c}\|_2^2]} \leq \sqrt{\frac{m}{n}} \leq \frac{1}{10},$$

where we used the Cauchy-Schwarz and Jensen's inequalities along with [Claim 34](#). \square

We may now conclude the proof of [Lemma 31](#), and therefore of [Theorem 29](#).

Proof of Lemma 31. Suppose the algorithm makes $m \leq \frac{n}{100}$ queries and outputs function \mathbf{g} . By the claims above, we obtain that $\mathbf{E}[\langle \mathbf{f}, \mathbf{g} \rangle] \leq 1/10$. On the other hand, since $\mathbf{g} = f_{\mathbf{v}}$ for some $\mathbf{v} \in \mathbb{S}^{n-1}$, we have, by the Cauchy-Schwarz inequality, that $\langle \mathbf{f}, \mathbf{g} \rangle \geq -1$. Therefore, by applying Markov's inequality to the random variable $1 + \langle \mathbf{f}, \mathbf{g} \rangle$, we obtain that, with probability at least $1 - 5/9 = 4/9 > 1/3$,

$$1 + \langle \mathbf{f}, \mathbf{g} \rangle \leq \frac{1 + \frac{1}{10}}{5/9} = 1.98,$$

and in this case we have $\langle \mathbf{f}, \mathbf{g} \rangle \leq 0.98$ and hence

$$\|\mathbf{f} - \mathbf{g}\|_{L^2(\gamma)}^2 = \|\mathbf{f}\|_{L^2(\gamma)}^2 + \|\mathbf{g}\|_{L^2(\gamma)}^2 - 2\langle \mathbf{f}, \mathbf{g} \rangle \geq 0.04.$$

\square

4.2 An $n^{\text{poly}(L/\varepsilon)}$ CSQ Lower Bound for Agnostic Learning

We start by defining the CSQ model and SQ dimension. While the CSQ model is often defined for query functions with bounded range only, in our setting it is natural to allow functions of bounded norm. The same convention appears e.g. in [\[DKKZ20, GGJ⁺20\]](#).

Definition 36 (CSQ algorithm). A *correlational statistical query* is a function $h \in L^2(\gamma)$ satisfying $\|h\|_2 \leq 1$. Given input function $f \in L^2(\gamma)$, correlational statistical query h , and *tolerance parameter* $\tau > 0$, the oracle $\text{CSTAT}(\tau)$ returns a value v such that $|v - \langle f, h \rangle| \leq \tau$. A *CSQ algorithm* is an algorithm which only accesses the input f via CSTAT oracle queries.

Definition 37 (SQ dimension). Given a family \mathcal{F} of $L^2(\gamma)$ functions with $\|f\|_{L^2(\gamma)} \leq 1$ for each $f \in \mathcal{F}$, the *SQ dimension* of \mathcal{F} , denoted $\text{SQDim}(\mathcal{F})$, is the largest integer d such that there exist distinct $f_1, \dots, f_d \in \mathcal{F}$ satisfying $|\langle f_i, f_j \rangle| \leq 1/d$ for all distinct $i, j \in [d]$.

The main result of this section is the following. Note that we write the result with Lipschitz constant $L = 1$ without loss of generality, since we may always reduce the general case to the 1-Lipschitz case by rescaling the input function f and the error parameter ε by a factor of $1/L$.

Theorem 38 (Formal statement of [Theorem 3](#)). There exist absolute constants $c, c' > 0$ such that the following holds. Suppose A is a CSQ algorithm which, on parameter $\varepsilon \geq n^{-c}$ and input $f : \mathbb{R}^n \rightarrow \mathbb{R}$ which is promised to be 1-Lipschitz, outputs a function $g \in L^2(\gamma)$ satisfying $\|f - g\|_{L^2(\gamma)} \leq d_{\text{conv}}^1(f) + \varepsilon$. Then A requires $n^{\Omega((1/\varepsilon)^{c'})}$ queries to $\text{CSTAT}(n^{-\Omega((1/\varepsilon)^{c'})})$.

At a high level, the proof proceeds as follows. First, we construct a family \mathcal{F} of Lipschitz functions which are hard to *weakly learn* (with respect to the L^2 distance) in the correlational SQ model. Then, we show that each function in \mathcal{F} is weakly approximated by a convex Lipschitz function, namely a projected ReLU function. Hence any agnostic learner for convex functions weakly learns \mathcal{F} , and therefore requires many CSQ queries. Our argument adapts a similar lower bound shown by [\[DKZ20\]](#) for piecewise-constant input functions, and indeed we use their results as building blocks for our argument.

A classic result in SQ learning [\[BFJ⁺94\]](#) is that lower bounds on $\text{SQDim}(\mathcal{F})$ imply lower bounds for weak learning using correlational statistical queries. While this result is typically stated for

Boolean or $[-1, 1]$ -valued functions, it holds just as well for functions of bounded norm. For completeness, we give here a complete proof by slightly adapting the elegant proof of the bounded range case by [Szö09].

Theorem 39 (SQ dimension lower bound for weak learning). Let \mathcal{F} be a family of $L^2(\gamma)$ functions with $\|f\|_{L^2(\gamma)} \leq 1$ for each $f \in \mathcal{F}$, and let $d := \text{SQDim}(\mathcal{F})$. Then every CSQ algorithm which, on each input function $f \in \mathcal{F}$, outputs a function $g \in L^2(\gamma)$ with $\|g\|_{L^2(\gamma)} \leq 1$ and satisfying $\langle f, g \rangle \geq d^{-1/3}$ requires at least $\frac{d^{1/3}}{2} - 2$ queries to CSTAT($d^{-1/3}$).

Proof. It is convenient to assume that the algorithm also issues a query with $h = g$ for its output hypothesis g , which increases the number of queries by at most 1. Let $f_1, \dots, f_d \in \mathcal{F}$ be functions satisfying $|\langle f_i, f_j \rangle| \leq 1/d$ for all distinct $i, j \in [d]$. Let $\tau := d^{-1/3}$, and let $h \in L^2(\gamma)$ be a query, so that $\|h\|_2 \leq 1$. We show that an adversarial oracle may answer the query with 0 without eliminating too many functions from \mathcal{F} . Specifically, let $A := \{i \in [d] : \langle f_i, h \rangle \geq \tau\}$. By the Cauchy-Schwarz inequality,

$$\left\langle h, \sum_{i \in A} f_i \right\rangle^2 \leq \left\| \sum_{i \in A} f_i \right\|_{L^2(\gamma)}^2 = \sum_{i, j \in A} \langle f_i, f_j \rangle \leq \sum_{i \in A} \left(1 + \frac{|A| - 1}{d} \right) \leq |A| + \frac{|A|^2}{d}.$$

On the other hand, by the definition of A we have $\langle h, \sum_{i \in A} f_i \rangle \geq |A|\tau$. We conclude that $|A| \leq \frac{d}{d\tau^2 - 1}$. Similarly, the set $B := \{i \in [d] : \langle f_i, h \rangle \leq -\tau\}$ satisfies $|B| \leq \frac{d}{d\tau^2 - 1}$. Therefore, at most $\frac{2d}{d\tau^2 - 1}$ functions in \mathcal{F} will be inconsistent with an oracle output of 0. Therefore, some f_i will be consistent with all the oracle outputs of 0 if the number of queries is smaller than

$$\frac{d(d\tau^2 - 1)}{2d} = \frac{d^{1/3} - 1}{2},$$

and in this case the output g will fail to satisfy $\langle f, g \rangle \geq d^{-1/3}$ for the input $f = f_i$. \square

We proceed with the proof of Theorem 38. We rely on the following building block from [DKZ20], which gives a family of Boolean-valued functions in $L^2(\gamma)$ with low pairwise-correlation, each of which is non-trivially correlated with the ReLU along some direction in \mathbb{R}^n .

Theorem 40 (Implicit in [DKZ20, Proposition 3.1 and Proposition 4.1]). There exists an absolute constant $c > 0$ such that the following holds. For all integers $1 \leq k \leq n^c$, there exists a family \mathcal{F}_k of $\mathbb{R}^n \rightarrow \{\pm 1\}$ functions, with $|\mathcal{F}_k| \geq n^{\Omega(k)}$, such that the following conditions hold:

1. For all distinct $f_i, f_j \in \mathcal{F}_k$, we have $\langle f_i, f_j \rangle \leq n^{-\Omega(k)}$.
2. For all $f \in \mathcal{F}_k$, there exists a unit vector $u \in \mathbb{S}^{n-1}$ such that $\mathbf{E}_{z \sim N(0, I_n)}[f(z)\text{ReLU}(\langle z, u \rangle)] \geq 1/\text{poly}(k)$.

Our proof relies on applying noise to the construction above via the Ornstein-Uhlenbeck operator P_t to obtain Lipschitz functions from the Boolean functions above. A key fact we require is that Lipschitz functions, including the projected ReLUs appearing above, enjoy the following stability property. (A similar, but more involved result for the L_1 metric appears in [DMN21].)

Proposition 41. Suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is 1-Lipschitz. Then

$$\|f - P_t f\|_{L^2(\gamma)}^2 \leq t.$$

Proof. Recall that $P_t f = \sum_{\alpha \in \mathbb{N}^n} e^{-t|\alpha|} \widehat{f}(\alpha) h_\alpha$, and also the formula

$$\mathbf{E}_{\mathbf{z} \sim N(0, I_n)} \left[\|\nabla f(\mathbf{z})\|_2^2 \right] = \sum_{\alpha \neq 0} |\alpha| \widehat{f}(\alpha)^2,$$

see [O'D14, Exercise 11.13(a)]. Therefore, by Parseval's formula,

$$\begin{aligned} \|f - P_t f\|_{L^2(\gamma)}^2 &= \sum_{\alpha \neq 0} (1 - e^{-t|\alpha|})^2 \widehat{f}(\alpha)^2 \\ &\leq \sum_{\alpha \neq 0} (1 - e^{-t|\alpha|}) \widehat{f}(\alpha)^2 \\ &\leq t \sum_{\alpha \neq 0} |\alpha| \widehat{f}(\alpha)^2 \\ &= t \cdot \mathbf{E}_{\mathbf{z} \sim N(0, I_n)} \left[\|\nabla f(\mathbf{z})\|_2^2 \right] \leq t. \end{aligned} \quad \square$$

We also recall [Fact 16](#), which in particular implies that for $f : \mathbb{R}^n \rightarrow [-1, 1]$, we have that $P_t f$ is $\frac{1}{\sqrt{2t}}$ -Lipschitz.

We are ready to prove our main result. Our proof follows a similar outline to the proof of Theorem 1.5 from [DKZ20], with additional steps to keep track of the Lipschitz condition and the effect of the noise operator P_t .

Proof of Theorem 38. First note that it suffices to prove the claim for L -Lipschitz inputs where $L = \text{poly}(1/\varepsilon)$, instead of 1-Lipschitz inputs. The former case reduces to the latter via the substitutions $f \mapsto f/L$ and $\varepsilon \mapsto \varepsilon/L$, which only affect the absolute constants in the statement of the theorem.

Let $k \leq n^c$, for the constant c from [Theorem 40](#). We use algorithm A to learn a function that is well-correlated with an arbitrary input from \mathcal{F}_k , which we will argue requires many queries via [Theorem 39](#).

Let $f \in \mathcal{F}_k$. Let $C(k) = \text{poly}(k)$ be a constant depending only on k such that the projected ReLU function $r(x) = \text{ReLU}(\langle x, u \rangle)$ from [Theorem 40](#) satisfies $\langle f, r \rangle \geq 1/C(k)$. Let $\varepsilon := \frac{1}{100C(k)^2}$. First, since the operator P_ε is self-adjoint, we have

$$\langle P_\varepsilon f, r \rangle = \langle f, P_\varepsilon r \rangle = \langle f, r \rangle + \langle f, P_\varepsilon r - r \rangle \geq \frac{1}{C(k)} - \|f\|_{L^2(\gamma)} \|P_\varepsilon r - r\|_{L^2(\gamma)} \geq \frac{1}{C(k)} - \varepsilon^{1/2},$$

where we used the bound on $\langle f, r \rangle$, the Cauchy-Schwarz inequality, the fact that $\|f\|_{L^2(\gamma)}^2 = 1$ because f is Boolean-valued, and [Proposition 41](#) (since r is 1-Lipschitz). Note also that $\|r\|_{L^2(\gamma)}^2 = 1/2$. We now have

$$\begin{aligned} \|C(k)P_\varepsilon f - r\|_{L^2(\gamma)}^2 &= \|C(k)P_\varepsilon f\|_{L^2(\gamma)}^2 + \|r\|_{L^2(\gamma)}^2 - 2C(k)\langle P_\varepsilon f, r \rangle \\ &\leq \|C(k)P_\varepsilon f\|_{L^2(\gamma)}^2 + \frac{1}{2} - 2 + 2C(k)\varepsilon^{1/2} \leq \|C(k)P_\varepsilon f\|_{L^2(\gamma)}^2 - 1. \end{aligned}$$

We execute algorithm A with input function $C(k)P_\varepsilon f$, which is an $O(1/\varepsilon)$ -Lipschitz function by [Fact 16](#) and the definition of ε . By hypothesis, since r is a convex function, algorithm A outputs a function $g \in L^2(\gamma)$ satisfying

$$\|C(k)P_\varepsilon f - g\|_{L^2(\gamma)} \leq \|C(k)P_\varepsilon f - r\|_{L^2(\gamma)} + \varepsilon,$$

and hence, by the choice of ε and recalling that $\|P_\varepsilon f\|_2 \leq \|f\|_{L^2(\gamma)} \leq 1$ because P_ε is a non-expansive operator, we have

$$\|C(k)P_\varepsilon f - g\|_{L^2(\gamma)}^2 \leq \|C(k)P_\varepsilon f\|_{L^2(\gamma)}^2 - 1 + \varepsilon^2 + 2\varepsilon C(k) \leq \|C(k)P_\varepsilon f\|_{L^2(\gamma)}^2 - \frac{1}{2}.$$

We conclude that

$$\begin{aligned} \langle P_\varepsilon g, f \rangle &= \langle g, P_\varepsilon f \rangle = \frac{1}{C(k)} \langle g, C(k)P_\varepsilon f \rangle \\ &= -\frac{1}{2C(k)} \left(\|C(k)P_\varepsilon f - g\|_{L^2(\gamma)}^2 - \|C(k)P_\varepsilon f\|_{L^2(\gamma)}^2 - \|g\|_{L^2(\gamma)}^2 \right) \\ &\geq -\frac{1}{2C(k)} \left(-\frac{1}{2} - \|g\|_{L^2(\gamma)}^2 \right) \geq \frac{1}{4C(k)} = \Omega(\varepsilon^{1/2}). \end{aligned}$$

Now, to satisfy [Theorem 39](#), we must produce a function that has at most unit norm. Let $g^* := \frac{P_\varepsilon g}{\|g\|_{L^2(\gamma)}}$, which has $\|g^*\|_2 \leq 1$. Note that we may assume that

$$\|g\|_{L^2(\gamma)} \leq \|C(k)P_\varepsilon f\|_{L^2(\gamma)} \leq C(k)$$

by the optimality of g (i.e. projecting onto a ball is a contractive operation), so we obtain

$$\langle g^*, f \rangle \geq \Omega\left(\frac{\varepsilon^{1/2}}{C(k)}\right) = \Omega(1/C(k)^2) = \Omega(1/\text{poly}(k)) \geq n^{-C}$$

for some absolute constant $C > 0$ depending on the constant c in $k \leq n^c$ and the degree of the polynomial. Since $\text{SQDim}(\mathcal{F}_k) = n^{\Omega(k)}$, [Theorem 39](#) implies that A requires at least $n^{\Omega(k)}$ queries to $\text{CSTAT}(n^{-\Omega(k)})$ to output such g^* . Since $\varepsilon = 1/\text{poly}(k)$, this concludes the proof. \square

5 One-Sided Sample-Based Testing

We now turn to convexity testing algorithms with *one-sided* error, i.e., algorithms that never incorrectly reject a convex function. Our main result here is the following:

Theorem 4 (One-sided testing of Lipschitz convex functions). Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be L -Lipschitz, and let $\varepsilon \geq 0$. There is an algorithm, **ONE-SIDED-TESTER** ([Algorithm 1](#)), which, given access to i.i.d. labeled samples $(\mathbf{x}, f(\mathbf{x}))$ where $\mathbf{x} \sim N(0, I_n)$, makes $O(\sqrt{n}L/\varepsilon)^n$ draws, runs in time $(nL/\varepsilon)^{O(n)}$, and has the following performance guarantee:

- If f is convex, then **ONE-SIDED-TESTER** outputs “accept” with probability 1;
- If $d_{\text{conv}}^L(f) \geq \varepsilon$, then **ONE-SIDED-TESTER** outputs “reject” with probability 9/10.

Note that in contrast to our upper bound for standard testing ([Theorem 2](#)), our sample complexity is exponentially worse. Our one-sided testing algorithm proceeds by constructing an empirical *convex envelope* of the function using labeled samples. We will ensure by design that when f is convex, its empirical convex envelope is close (in an $L^\infty(\gamma)$ sense) to f . On the other hand, when f is far from convex, it will be far from its empirical convex envelope; we can then detect this by drawing additional samples.

As with [Theorem 2](#), the testing algorithm of [Theorem 4](#) is in fact a learning algorithm but with a stronger, $L^\infty(\gamma)$ guarantee: if the input function f is L -Lipschitz and convex, then the empirical convex envelope constructed by **ONE-SIDED-TESTER** is itself a convex, L -Lipschitz function that is ε close in $L^\infty(\gamma)$ to f .

5.1 The Empirical Convex Envelope

We will require the following definition:

Definition 42 (Empirical convex envelope). For a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and m points $Y = \{y_1, y_2, \dots, y_m\} \subset \mathbb{R}^n$, we define the *empirical convex envelope* of f with respect to Y as the function $\bar{f}_Y : \mathbb{R}^n \rightarrow \mathbb{R}$ defined by

$$\bar{f}_Y(x) := \sup \left\{ \langle p, x \rangle + a : p \in \mathbb{R}^n, a \in \mathbb{R}, \|p\|_2 \leq L, \langle p, y_i \rangle + a \leq f(y_i) \text{ for all } y_i \in Y \right\}.$$

Informally, \bar{f}_Y is the “largest” convex function that is below f on all the sampled points $y \in Y$. We record two immediate properties of the empirical convex envelope for future use:

Lemma 43. Suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex, and let $Y = \{y_1, \dots, y_m\} \subseteq \mathbb{R}^n$. Let $\bar{f}_Y : \mathbb{R}^n \rightarrow \mathbb{R}$ be the empirical convex envelope of f with respect to Y .

1. The empirical convex envelope \bar{f}_Y is convex and L -Lipschitz.
2. Furthermore, if f is itself convex, then for all $y \in Y$, $\bar{f}_Y(y) = f(y)$.

Proof. Note that convexity of \bar{f}_Y is immediate from [Definition 42](#) since the supremum of affine functions is convex. Additionally, the constraint that $\|p\|_2 \leq L$ ensures that \bar{f}_Y is L -Lipschitz. Together, these imply the first item.

We now turn to the second item. Since $\bar{f}_Y(y_j) \leq f(y_j)$ for all $y_j \in Y$ by construction, it remains to show that $f(y_j) \leq \bar{f}_Y(y_j)$ for $y_j \in Y$. This is readily seen to hold by taking $p = \nabla f(y_j)$ and $a = f(y_j)$. Since f is L -Lipschitz, and by Rademacher’s theorem Lipschitz functions are differentiable almost everywhere, it follows that $\|\nabla f(y_j)\|_2 \leq L$. Since $\bar{f}_Y(y_j)$ is a supremum over all feasible p and a , it follows that $\bar{f}_Y(y_j) \geq f(y_j)$. \square

Finally, we note that the empirical convex envelope \bar{f}_Y can be computed on an input $x \in \mathbb{R}^n$ by the quadratically constrained convex program CECE (short for “compute empirical convex envelope”) given in [Figure 2](#). Before proceeding, we show that if f is convex, then CECE is always feasible; this lemma will later be used to argue correctness of our one-sided testing algorithm.

Lemma 44. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be L -Lipschitz, and let $Y = \{y_1, \dots, y_m\} \subseteq \mathbb{R}^n$. Let $\bar{f}_Y : \mathbb{R}^n \rightarrow \mathbb{R}$ be the empirical convex envelope of f with respect to Y .

If f is convex, then for $x \in \mathbb{R}^n$, $\text{CECE}(x) = \bar{f}_Y(x)$. Additionally, $\text{CECE}(x)$ runs in time $\text{poly}(m, n)$.

Here, we are assuming the real-valued model of computation, where infinite-length real-valued computations can be achieved with one unit operation. However, our results work equally well in the bounded-bit model, where we use and assume finite bits of precision, with some minor (and routine) modifications to the analysis.

Proof. Note that the quadratic program in [Figure 2](#) coincides with the definition of the empirical convex envelope from [Definition 42](#), and so correctness holds. Note that this program is indeed feasible when f is convex, since f itself is a solution to the quadratic program (see [Lemma 43](#)). The runtime is also immediate via standard guarantees for solving convex programs [BV04]. \square

$$\begin{aligned}
& \text{maximize} && \langle p, x \rangle + a \\
& \text{subject to} && p \in \mathbb{R}^n, a \in \mathbb{R} \\
& && \|p\|_2 \leq L \\
& && \langle p, y \rangle + a \leq f(y) \quad \forall y \in Y
\end{aligned}$$

Figure 2: The quadratic program CECE(x) which takes as input a collection of samples $Y = \{y_1, \dots, y_m\}$ labeled by an L -Lipschitz function f .

5.2 Algorithm 1 and Useful Preliminaries

We now formally describe our one-sided convexity testing algorithm, ONE-SIDED-TESTER (cf. [Algorithm 1](#)). We also record some useful lemmas before proving [Theorem 4](#). For the remainder of this section, we set

$$d := \frac{\varepsilon}{8L}$$

where ε, L are as in [Theorem 4](#).

Lemma 45. Let $Y = \{y_1, y_2, \dots, y_m\}$ be a set of points in \mathbb{R}^n such that for all $x \in B(1.1\sqrt{n})$, there exists $i \in [m]$ such that $\|x - y_i\|_2 \leq d$. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be an L -Lipschitz function and let \overline{f}_Y be its empirical convex envelope with respect to Y .

Then for all $x \in \mathbb{R}^n$, we have

$$|\overline{f}_Y(x) - f(x)| \leq 2L \cdot \min_{y \in Y} \|x - y\|_2.$$

In particular, for all $x \in B_2(1.1\sqrt{n})$, we have $|\overline{f}_Y(x) - f(x)| \leq 2Ld$.

Proof. Fix $x \in \mathbb{R}^n$, and let $y := \operatorname{argmin}_{y \in Y} \|x - y\|$. We then have

$$\begin{aligned}
|\overline{f}_Y(x) - f(x)| &\leq |\overline{f}_Y(x) - \overline{f}_Y(y)| + |\overline{f}_Y(y) - f(y)| + |f(x) - f(y)| && \text{(triangle inequality)} \\
&= |\overline{f}_Y(x) - \overline{f}_Y(y)| + |f(x) - f(y)| && (\overline{f}_Y(y) = f(y) \text{ for } y \in Y) \\
&\leq 2L \cdot \|x - y\|_2 && (f, \overline{f}_Y \text{ are } L\text{-Lipschitz})
\end{aligned}$$

as desired. \square

The following lemma shows that with high probability, a draw of roughly $\exp(n \log n)$ samples from $N(0, I_n)$ will be sufficiently “dense” in $B_2(1.1\sqrt{n})$ in the sense of (the hypothesis of) [Lemma 45](#).

Lemma 46. Let t_1 be as in [Algorithm 1](#), and suppose $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(t_1)} \sim N(0, I_n)$ are i.i.d. draws. Then with probability 0.99, for all $x \in B_2(1.1\sqrt{n})$ we have

$$\min_{i \in [t_1]} \|x - \mathbf{x}^{(i)}\| \leq d.$$

Proof. Note that $B_2(1.1\sqrt{n}) \subseteq B_\infty(1.1\sqrt{n})$. Partition $B_\infty(1.1\sqrt{n})$ into a hypergrid of at most $(1.1\sqrt{n}\ell^{-1})^n \leq (3nd^{-1})^n$ (hyper)cubes, each of side length $\ell := \lceil d/\sqrt{n} \rceil$. Suppose for every cube in the grid there exists a point $y \in Y$ in the cube. Then note that

$$\min_{y \in Y} \|x - y\|_2 \leq \ell \cdot \sqrt{n} \leq d$$

Input: Access to i.i.d. labeled samples $(\mathbf{x}, f(\mathbf{x}))$ where f is L -Lipschitz, $\varepsilon \geq 0$

Output: “Accept” or “reject”

ONE-SIDED-TESTER:

1. Set

$$t_1 := \left(\frac{cL\sqrt{n}}{\varepsilon} \right)^n$$

for a universal constant c , and draw t_1 samples $(\mathbf{x}^{(1)}, f(\mathbf{x}^{(1)})), \dots, (\mathbf{x}^{(t_1)}, f(\mathbf{x}^{(t_1)}))$. Let $\mathbf{Y} := \{\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(t_1)}\}$.

2. Let \mathcal{P} be a partition of $B_2(1.1\sqrt{n})$ into at most $(3nL/\varepsilon)^n$ (solid) hypercubes of side length $\ell = \lceil \varepsilon/(4L\sqrt{n}) \rceil$. For each hypercube $C \in \mathcal{P}$, check if there exists $\mathbf{x}^{(i)} \in C$; if not, halt and return “accept.”

3. Set $t_2 := \frac{5}{\varepsilon^4}$, and draw t_2 labeled samples $(\mathbf{y}^{(1)}, f(\mathbf{y}^{(1)})), \dots, (\mathbf{y}^{(t_2)}, f(\mathbf{y}^{(t_2)}))$. Let

$$\mathbf{Z} := \left\{ \mathbf{y}^{(j)} : \|\mathbf{y}^{(j)}\|_2 \leq 1.1\sqrt{n} \right\}.$$

4. Compute CECE($\mathbf{y}^{(j)}$) for all $j \in [t_2]$. If CECE($\mathbf{y}^{(j)}$) is infeasible for any j , then halt and return “reject.”
5. If $|f(\mathbf{y}^{(j)}) - \text{CECE}(\mathbf{y}^{(j)})| \geq \varepsilon/2$ for any $j \in [t_2]$, then halt and return “reject.” Otherwise, return “accept.”

Algorithm 1: A one-sided testing algorithm for Lipschitz convex functions.

thanks to our choice of ℓ .

Thus, in order to establish [Lemma 46](#), it suffices to show that after drawing t_1 samples, with probability at least $9/10$, every cube in the hypergrid contains at least one sample. We will show this by taking a union bound over the $(3nd^{-1})^n$ cubes in the hypergrid.

In this argument, it will be necessary to consider the cube in the hypergrid with minimal Gaussian measure. Note that the maximum ℓ_2 norm of any point in the hypergrid is at most $(1.1 + \ell)\sqrt{n}$. Therefore for all $x \in B_\infty(1.1\sqrt{n})$, we have

$$\frac{\exp(-\|x\|_2^2/2)}{(2\pi)^{n/2}} \geq \frac{\exp(-(1.1 + \ell)^2/(2n))}{(2\pi)^{n/2}} = \frac{1}{(2\pi)^{n/2}} \cdot \exp\left(-\left(1.1 + \frac{\varepsilon}{(4L\sqrt{n})}\right)^2 \cdot \frac{1}{2n}\right).$$

Since the Lebesgue measure of any cube is ℓ^n , the minimum Gaussian measure of any of the cubes in our hypergrid is at least

$$\alpha := \frac{1}{(2\pi)^{n/2}} \cdot \exp\left(-\left(1.1 + \frac{\varepsilon}{(4L\sqrt{n})}\right)^2 \cdot \frac{1}{2n}\right) \cdot \ell^n.$$

The probability that t_1 independent draws from $N(0, I_n)$ all fail to land in this cube is

$$(1 - \alpha)^{t_1} \leq \exp(-\alpha t_1).$$

Applying a union bound, it follows that the probability that there exists a cube in the hypergrid not containing any sample is at most

$$\exp(-\alpha t_1) \cdot 2^{O(n \log(1.1n/d))},$$

and it is readily verified that this is at most 0.01 for our choice of t_1 and ℓ , for some universal constant c . \square

Finally, the following lemma shows that random sampling allows us to distinguish between L -Lipschitz functions that are far in $L^2(\gamma)$:

Lemma 47. Suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is an L -Lipschitz function and $Y = \{y_1, y_2, \dots, y_m\}$ be a set of points in \mathbb{R}^n such that for all $x \in B(1.1\sqrt{n})$, there exists $i \in [m]$ such that $\|x - y_i\|_2 \leq d$, for $d = \varepsilon/(8L)$. Let \bar{f}_Y be the empirical convex envelope of f with respect to Y . If $\|f - \bar{f}_Y\|_{L^2(\gamma)} \geq \varepsilon$, then

$$\Pr_{\mathbf{x} \sim N(0, I_n)} \left[|f(\mathbf{x}) - \bar{f}_Y(\mathbf{x})| \geq \frac{\varepsilon}{2} \right] = \Omega\left(\frac{\varepsilon^4}{L^4}\right).$$

Proof. Note that

$$\begin{aligned} \Pr_{\mathbf{x} \sim N(0, I_n)} \left[|f(\mathbf{x}) - \bar{f}_Y(\mathbf{x})|^2 \geq \frac{\varepsilon^2}{4} \right] &\geq \Pr_{\mathbf{x} \sim N(0, I_n)} \left[|f(\mathbf{x}) - \bar{f}_Y(\mathbf{x})|^2 \geq \frac{\|f - \bar{f}_Y\|_{L^2(\gamma)}}{4} \right] \\ &\geq \left(\frac{9}{16}\right) \cdot \frac{\mathbf{E} \left[\|f - \bar{f}_Y\|^2 \right]^2}{\mathbf{E} \left[\|f - \bar{f}_Y\|^4 \right]}, \end{aligned}$$

where the second line follows from the Paley–Zygmund inequality and where all expectations are with respect to $N(0, I_n)$. Since $\|f - \bar{f}_Y\|_{L^2(\gamma)} \geq \varepsilon$, we have

$$\Pr_{\mathbf{x} \sim N(0, I_n)} \left[|f(\mathbf{x}) - \bar{f}_Y(\mathbf{x})|^2 \geq \frac{\varepsilon^2}{4} \right] \geq \frac{\Omega(\varepsilon^4)}{\mathbf{E} \left[\|f - \bar{f}_Y\|^4 \right]}. \quad (10)$$

We now apply Lemmas 45 and 46 to bound the denominator $\mathbf{E} \left[\|f - \bar{f}_Y\|^4 \right]$. Since all $x \in \mathbb{R}^n$ with $\|x\|_2 \leq 1.1\sqrt{n}$ satisfy $\min_{y \in Y} \|x - y\|_2 \leq d := \varepsilon/(4L)$,

$$\begin{aligned} \mathbf{E}_{\mathbf{x} \sim N(0, I_n)} \left[|f(\mathbf{x}) - \bar{f}_Y(\mathbf{x})|^4 \right] &\leq \Pr_{\mathbf{x} \sim N(0, I_n)} \left[|f(\mathbf{x}) - \bar{f}_Y(\mathbf{x})|^4 \mid \|x\|_2 \leq 1.1\sqrt{n} \right] \\ &\quad + \Pr_{\mathbf{x} \sim N(0, I_n)} \left[|f(\mathbf{x}) - \bar{f}_Y(\mathbf{x})|^4 \mid \|x\|_2 > 1.1\sqrt{n} \right] \cdot \Pr[\|x\|_2 > 1.1\sqrt{n}]. \end{aligned}$$

From Lemma 45, this is at most

$$\begin{aligned} &(2Ld)^4 + (2L)^4 \cdot \Pr_{\mathbf{x} \sim N(0, I_n)} \left[\min_{y \in Y} \|x - y\|_2^4 \mid \|x\|_2 > 1.1\sqrt{n} \right] \cdot \Pr_{\mathbf{x} \sim N(0, I_n)} [\|x\|_2 > 1.1\sqrt{n}] \\ &= (2Ld)^4 + (2L)^4 \cdot \text{poly}(n) \exp(-\Omega(n)) \\ &= \varepsilon^4/16 + o(1). \end{aligned}$$

Therefore, returning to Equation (10), we find that, for $\mathbf{x} \sim N(0, I_n)$, the probability that $|f(x) - \bar{f}_Y(x)| \geq \varepsilon/2$ is at least $\varepsilon^4/4$ for large enough n . \square

5.3 Proof of Theorem 4

First, note that the sample complexity of ONE-SIDED-TESTER is $t_1 + t_2 = O(\sqrt{n}L/\varepsilon)^n$ as desired. We establish correctness as well as the runtime bound below, starting with the former.

Correctness. We first show that if f is L -Lipschitz and convex, then ONE-SIDED-TESTER accepts with probability 1. First suppose that \bar{f}_Y is the empirical convex envelope of f with respect to a set $Y = \{y_1, y_2, \dots, y_{t_1}\}$ where there exists an $x \in B_2(1.1\sqrt{n})$ with $\min_{i \in [t_1]} \|x - y_i\| > d$. Then, Step 2 of the algorithm will return “accept.” Therefore, suppose instead that, for Y , all $x \in B_2(1.1\sqrt{n})$ satisfy $\min_{i \in [t_1]} \|x - y_i\| \leq d$. Thanks to Lemma 45 with our choice of $d = \varepsilon/(8L)$, we have that for all x with $\|x\|_2 \leq 1.1\sqrt{n}$, $|\bar{f}_Y(x) - f(x)| \leq \varepsilon/4$. Additionally, by Lemma 44, the subroutine COMPUTE-EMPIRICAL-CONVEX-ENVELOPE is always feasible on convex, L -Lipschitz f . Therefore, Algorithm ONE-SIDED-CONVEXITY-TESTER will always return ACCEPT on convex, L -Lipschitz input functions f .

For soundness, we argue that if the input is an L -Lipschitz function that is ε -far from the family of convex functions in ℓ_2 norm with respect to standard Gaussian measure, then the algorithm rejects with probability at least $2/3$. Suppose f is ε -far from convex. First, if COMPUTE-EMPIRICAL-CONVEX-ENVELOPE is infeasible at any x sampled, by Lemma 44, the algorithm will reject. Suppose we may not reject for this reason. Since its empirical Lipschitz convex estimate \bar{f}_Y is L -Lipschitz and convex by Lemma 43, this implies that

$$\|f - \bar{f}_Y\|_{L^2(\gamma)} \geq \varepsilon.$$

Suppose the algorithm samples a set Y of points in \mathbb{R}^n such that for all $x \in B(1.1\sqrt{n})$, there exists $y \in Y$ such that $\|x - y\|_2 \leq d$, for $d = \varepsilon/(8L)$; by Lemma 46 this occurs with probability at least 0.99. In this case, by Lemma 47, therefore, the probability that $\mathbf{x} \sim N(0, I_n)$ satisfies $|f(\mathbf{x}) - \bar{f}_Y(\mathbf{x})| \geq \varepsilon/2$ is at least $\varepsilon^4/4$. By a union bound together with a standard tail bound on the norm of a Gaussian vector, the probability that $\mathbf{x} \sim N(0, I_n)$ satisfies $|f(\mathbf{x}) - \bar{f}_Y(\mathbf{x})| \geq \varepsilon/2$ and $\|\mathbf{x}\|_2 \leq 1.1\sqrt{n}$ is at least $1 - \varepsilon^4/4 - e^{-n/200}$.

Since we check whether $|f(x) - \bar{f}_Y(x)| \geq \varepsilon/2$ on $5/\varepsilon^4$ points with $\|x\|_2 \leq 1.1\sqrt{n}$ in Algorithm 1, with very high probability at least one of the sampled points will satisfy $\|\mathbf{x}\|_2 \leq 1.1\sqrt{n}$ and $|f(\mathbf{x}) - \bar{f}_Y(\mathbf{x})| \geq \varepsilon/2$. Therefore, the algorithm will reject with probability at least $9/10$.

Runtime. Note that Step 1 takes time t_1 and Step 2 takes time

$$|\mathcal{P}| \cdot t_1 = (3nL/\varepsilon)^n \cdot \left(\frac{50L\sqrt{n}}{\varepsilon} \right)^n.$$

Finally, Step 3 requires time $\text{poly}(n, t_1) \cdot t_2$ by Lemma 44. It follows that Algorithm 1 uses time $(nL/\varepsilon)^{O(n)}$. \square

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A Convexity and the Hermite Spectrum

The main results of this section, [Lemma 50](#) and [Corollary 51](#), may be of independent interest and relate (distance to) convexity of a function to its degree-2 Hermite spectrum. We start with a simple observation:

Lemma 48. Let $f \in L^1(\gamma_n)$ be convex and let $t > 0$. Then $P_t f$ is convex.

Proof. This is an immediate consequence of the definition of convexity via the explicit formula for P_t given in [Definition 14](#). \square

We also record the following technical claim whose proof will be given shortly:

Claim 49. For all $f \in L^2(\gamma_n)$ and $t > 0$, it holds that

$$\frac{\partial^2 P_t f}{\partial x_i^2}(0) = \sum_{\alpha \in \mathbb{N}^n} \widehat{P_t f}(2\alpha + 2e_i) \sqrt{(2\alpha_i + 1)(2\alpha_i + 2)} \prod_{k \in [n]} (-1)^{\alpha_k} \sqrt{\frac{(2\alpha_k - 1)!!}{(2\alpha_k)!!}}.$$

We now give a simple necessary condition for convexity in terms of Hermite coefficients, using the fact that P_t preserves convexity.

Lemma 50. Let $f \in L^2(\gamma_n)$ be convex. Then $\widehat{f}(2e_i) \geq 0$ for all $i \in [n]$.

Proof. Suppose for a contradiction that $\widehat{f}(2e_i) < 0$ for some $i \in [n]$. For all $t > 0$, we have that $P_t f$ is convex (by Lemma 48) and smooth (by Fact 15). We will complete the proof of Lemma 50 by showing that, as $t \rightarrow \infty$, the characters $\widehat{P_t f}(2\alpha + 2e_i)$ with $\alpha \neq \vec{0}$ decay fast enough that their contribution in the formula above is dominated by $\widehat{P_t f}(2e_i) < 0$. This will contradict the convexity of $P_t f$. Formally, as $t \rightarrow \infty$ we have

$$\begin{aligned} \frac{\partial^2 P_t f}{\partial x_i^2}(0) &= \sum_{\alpha \in \mathbb{N}^n} \widehat{P_t f}(2\alpha + 2e_i) \sqrt{(2\alpha_i + 1)(2\alpha_i + 2)} \prod_{k \in [n]} (-1)^{\alpha_k} \sqrt{\frac{(2\alpha_k - 1)!!}{(2\alpha_k)!!}} \\ &= \sum_{\alpha \in \mathbb{N}^n} e^{-t(2|\alpha|+2)} \widehat{f}(2\alpha + 2e_i) \sqrt{(2\alpha_i + 1)(2\alpha_i + 2)} \prod_{k \in [n]} (-1)^{\alpha_k} \sqrt{\frac{(2\alpha_k - 1)!!}{(2\alpha_k)!!}} \\ &\leq e^{-2t} \left[\widehat{f}(2e_i) \sqrt{2} + \sum_{\alpha \neq \vec{0}} e^{-2t|\alpha|} \left| \widehat{f}(2\alpha + 2e_i) \right| \underbrace{\sqrt{(2\alpha_i + 1)(2\alpha_i + 2)}}_{\leq 2\alpha_i + 2 \leq 4|\alpha|} \prod_{k \in [n]} \underbrace{\sqrt{\frac{(2\alpha_k - 1)!!}{(2\alpha_k)!!}}}_{\leq 1} \right] \\ &\leq e^{-2t} \left[\widehat{f}(2e_i) \sqrt{2} + 4 \sum_{\substack{\alpha \neq \vec{0} \\ |\alpha| e^{-|\alpha|} \leq \frac{1}{e}}} \underbrace{|\alpha| e^{-t|\alpha|}}_{\leq |\alpha| e^{-|\alpha|} \leq \frac{1}{e}} e^{-t|\alpha|} \left| \widehat{f}(2\alpha + 2e_i) \right| \right] \\ &\leq e^{-2t} \left[\widehat{f}(2e_i) \sqrt{2} + \frac{4}{e} \sqrt{\sum_{\alpha \neq \vec{0}} e^{-2t|\alpha|}} \sqrt{\sum_{\alpha \neq \vec{0}} \widehat{f}(2\alpha + 2e_i)^2} \right]. \end{aligned} \tag{Cauchy–Schwarz}$$

It remains to bound the two series in the last line. We first observe that $\sum_{\alpha \neq \vec{0}} \widehat{f}(2\alpha + 2e_i)^2 < +\infty$, which is a consequence of the membership $f \in L^2(\gamma_n)$ via Plancherel's theorem. As for the first series, first note that, for each $\ell \in \mathbb{N}$, the number of multi-indices α satisfying $|\alpha| = \ell$ is at most n^ℓ . Hence as $t \rightarrow \infty$ we have

$$\sum_{\alpha \neq \vec{0}} e^{-2t|\alpha|} \leq \sum_{\ell \geq 1} n^\ell e^{-2t\ell} = \sum_{\ell \geq 1} e^{\ell \ln(n) - 2t\ell} \leq \sum_{\ell \geq 1} e^{-t\ell} = \frac{e^{-t}}{1 - e^{-t}} \rightarrow 0^+.$$

Thus $\frac{\partial^2 P_t f}{\partial x_i^2}(0) < 0$ for all sufficiently large t , contradicting the convexity of $P_t f$. \square

As an immediate consequence, we can relate the $L^2(\gamma)$ distance to convexity of a function to its *negative* degree-2 Hermite coefficients:

Corollary 51. Let $f \in L^2(\gamma_n)$. Then

$$d_{\text{conv}}(f) \geq \sum_{i \in [n]} \left(\widehat{f}(2e_i)^- \right)^2,$$

where $a^- := \max\{-a, 0\}$.

Proof. Let $g \in L^2(\gamma_n)$ be convex. Then $\widehat{g}(2e_i) \geq 0$ for each $i \in [n]$ by Lemma 50. Therefore, by Plancherel's theorem,

$$\|f - g\|_{L^2(\gamma)}^2 = \sum_{\alpha \in \mathbb{N}^n} \left(\widehat{f}(\alpha) - \widehat{g}(\alpha) \right)^2 \geq \sum_{i \in [n]} \left(\widehat{f}(2e_i)^- \right)^2. \quad \square$$

It remains to prove Claim 49. First, we recall an explicit formula for evaluating the univariate Hermite polynomials at 0.

Lemma 52. For all $j \in \mathbb{N}$, we have

$$h_{2j}(0) = (-1)^j \sqrt{\frac{(2j-1)!!}{(2j)!!}} \quad \text{and} \quad h_{2j+1}(0) = 0.$$

Proof. We proceed by induction. The claim holds for $j = 0$ since $h_0(x) = 1$ and $h_1(x) = x$. Assume the claim holds for some $j \in \mathbb{N}^n$. Recall that we have the following recurrence relation, see e.g. [O'D14, Exercise 11.10(b)].

$$\sqrt{(k+1)!} \cdot h_{k+1}(x) = x \sqrt{k!} \cdot h_k(x) - k \sqrt{(k-1)!} \cdot h_{k-1}(x).$$

Applying at $x = 0$ and rearranging, we obtain

$$h_{k+1}(0) = -\sqrt{\frac{k}{k+1}} \cdot h_{k-1}(0).$$

Taking $k = 2(j+1)$ yields $h_{2(j+1)+1}(0) = 0$ by the inductive hypothesis. Taking $k = 2j+1$ yields

$$h_{2(j+1)}(0) = -\sqrt{\frac{2j+1}{2j+2}} \cdot h_{2j}(0) = -\sqrt{\frac{2j+1}{2j+2}} \cdot (-1)^j \sqrt{\frac{(2j-1)!!}{(2j)!!}} = (-1)^{j+1} \sqrt{\frac{(2j+1)!!}{(2j+2)!!}},$$

as desired. \square

We are now ready to prove Claim 49, which we restate below for convenience.

Claim 49. For all $f \in L^2(\gamma_n)$ and $t > 0$, it holds that

$$\frac{\partial^2 P_t f}{\partial x_i^2}(0) = \sum_{\alpha \in \mathbb{N}^n} \widehat{P_t f}(2\alpha + 2e_i) \sqrt{(2\alpha_i + 1)(2\alpha_i + 2)} \prod_{k \in [n]} (-1)^{\alpha_k} \sqrt{\frac{(2\alpha_k - 1)!!}{(2\alpha_k)!!}}.$$

Proof. Applying Lemma 10 (via Corollary 17) and Lemma 12, we have

$$\frac{\partial P_t f}{\partial x_i} = \sum_{\alpha \in \mathbb{N}^n} \widehat{P_t f}(\alpha) \frac{\partial h_\alpha}{\partial x_i} = \sum_{\alpha_i \geq 1} \widehat{P_t f}(\alpha) \sqrt{\alpha_i} h_{\alpha-e_i}.$$

Hence $g := \frac{\partial P_t f}{\partial x_i}$ satisfies $\widehat{g}(\alpha) = \widehat{P_t f}(\alpha + e_i)\sqrt{\alpha_i + 1}$ for each $\alpha \in \mathbb{N}^n$. We wish to repeat this argument for the function g , which requires showing that $\frac{\partial g}{\partial x_i} = \frac{\partial^2 P_t f}{\partial x_i^2} \in L^2(\gamma_n)$ for the application of [Lemma 10](#). We may show this using the commutation property from [Fact 18](#) and the fact that P_t is a semigroup, as follows.

$$\frac{\partial g}{\partial x_i} = \frac{\partial}{\partial x_i} \left(\frac{\partial}{\partial x_i} P_{t/2} P_{t/2} f \right) = \frac{\partial}{\partial x_i} \left(e^{-t/2} P_{t/2} \left(\frac{\partial}{\partial x_i} P_{t/2} f \right) \right) = e^{-t/2} \frac{\partial}{\partial x_i} \left(P_{t/2} \left(\frac{\partial}{\partial x_i} P_{t/2} f \right) \right).$$

Now, one application of [Corollary 17](#) gives that $\frac{\partial}{\partial x_i} P_{t/2} f \in L^2(\gamma_n)$, and hence a second application gives that $\frac{\partial}{\partial x_i} P_{t/2} \left(\frac{\partial}{\partial x_i} P_{t/2} f \right) \in L^2(\gamma_n)$. Therefore, applying [Lemma 10](#) and [Lemma 12](#) to g yields

$$\begin{aligned} \frac{\partial^2 P_t f}{\partial x_i^2} &= \frac{\partial g}{\partial x_i} = \sum_{\alpha_i \geq 1} \widehat{g}(\alpha) \sqrt{\alpha_i} h_{\alpha - e_i} = \sum_{\alpha_i \geq 1} \widehat{P_t f}(\alpha + e_i) \sqrt{\alpha_i(\alpha_i + 1)} h_{\alpha - e_i} \\ &= \sum_{\alpha \in \mathbb{N}^n} \widehat{P_t f}(\alpha + 2e_i) \sqrt{(\alpha_i + 1)(\alpha_i + 2)} h_\alpha. \end{aligned}$$

Applying at 0 via [Lemma 52](#), the multi-indices α which are not composed of all even indices vanish, and we conclude the proof as follows.

$$\begin{aligned} \frac{\partial^2 P_t f}{\partial x_i^2}(0) &= \sum_{\alpha \in \mathbb{N}^n} \widehat{P_t f}(\alpha + 2e_i) \sqrt{(\alpha_i + 1)(\alpha_i + 2)} \prod_{k \in [n]} h_{\alpha_k}(0) \\ &= \sum_{\alpha \in \mathbb{N}^n} \widehat{P_t f}(2\alpha + 2e_i) \sqrt{(2\alpha_i + 1)(2\alpha_i + 2)} \prod_{k \in [n]} h_{2\alpha_k}(0) \\ &= \sum_{\alpha \in \mathbb{N}^n} \widehat{P_t f}(2\alpha + 2e_i) \sqrt{(2\alpha_i + 1)(2\alpha_i + 2)} \prod_{k \in [n]} (-1)^{\alpha_k} \sqrt{\frac{(2\alpha_k - 1)!!}{(2\alpha_k)!!}}. \quad \square \end{aligned}$$