

Global attractor for a Cahn-Hilliard-chemotaxis model with logistic degradation

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Abstract

We consider a mathematical model coupling the Cahn-Hilliard system for phase separation with an additional equation describing the diffusion process of a chemical quantity whose concentration influences the physical process. The main application of the model refers to tumor progression, where the phase variable φ denotes the local proportion of active cancer cells and the chemical concentration σ may refer to a nutrient transported by the blood flow or to a drug administered to the patient. The resulting system is characterized by cross-diffusion effects similar to those appearing in the Keller-Segel model for chemotaxis; in particular, the nutrient tends to be attracted towards the regions where more active tumor cells are present (and consume it in a quicker way). Complementing various recent results on related models, we investigate here the long-time behavior of solutions under the perspective of infinite-dimensional dynamical systems. To this aim, we first identify a regularity setting in which the system is well posed and generates a closed semigroup according to the terminology introduced by Pata and Zelik. Then, partly based on the approach introduced by Rocca and the first author for the Cahn-Hilliard system with singular potential, we prove that the semigroup is strongly dissipative and asymptotically compact so guaranteeing the existence of the global attractor in a suitable phase space. Finally, we discuss the sign properties of σ and show that, if the initial datum σ_0 is a.e. strictly positive in the reference domain Ω with $\ln \sigma_0 \in L^1(\Omega)$, then, for every $0 < \tau < T < \infty$, there exists $\delta = \delta(\tau, T) > 0$ such that $\sigma(\cdot, \cdot) \geq \delta > 0$ a.e. in $\Omega \times (\tau, T)$. It is not clear, however, whether this property is uniform in T ; actually, we cannot exclude that δ could degenerate to 0 as T is let go to infinity.

Key words: Cahn-Hilliard, chemotaxis, Keller-Segel, singular potential, global attractor.

AMS (MOS) subject classification: 35B41, 35D30, 35K35, 35Q92, 92C17.

1 Introduction

Letting $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be a smooth bounded domain of boundary $\Gamma := \partial\Omega$, we consider the following PDE system:

$$\varphi_t - \Delta\mu = 0, \quad (1.1)$$

$$\mu = -\Delta\varphi + f(\varphi) - \chi\sigma, \quad (1.2)$$

$$\sigma_t - \Delta\sigma + \chi \operatorname{div}(\sigma\nabla\varphi) = -h(\sigma, \varphi)\sigma^2 + k(\sigma, \varphi)\sigma, \quad (1.3)$$

where the unknown variables φ (the normalized order parameter of a phase separation process), μ (the associated chemical potential) and σ (the concentration of a chemical substance) are defined for $x \in \Omega$ and $t \geq 0$. In the above model, relations (1.1)-(1.2) constitute a suitable form of the Cahn-Hilliard system, where f is the derivative of a configuration potential F (described below), whereas the second-order relation (1.3) is characterized by a cross-diffusion term with the same structure as in the Keller-Segel model for chemotaxis. As first observed in [19], this choice prescribes the chemical substance represented by σ to migrate towards the regions where one of the two phases is prevailing, in a way that is proportional to its local concentration. The magnitude of this effect is determined by the *chemotactic coefficient* $\chi > 0$ (and, of course, it is larger for larger χ). The logistic forcing term on the right-hand side of (1.3), where the functions h, k are assumed to be bounded, strictly positive, and Lipschitz continuous, from the one side has a sign-preserving effect guaranteeing (at least) the nonnegativity of σ , and, from the other side, provides a stabilizing effect as the quadratic term acts a source of a-priori estimates preventing the blow-up of σ . We note, however, that this effect is in competition with the quadratic growth of the cross-diffusion term, so that the actual behavior of solutions somehow depends on a comparison between the magnitude of the logistic source and that of the chemotactic coefficient χ , see Remark 2.5 below for more details.

The system is complemented by no-flux (i.e., homogeneous Neumann) boundary conditions for all unknowns. Concerning μ and σ , such conditions ensure, respectively, the conservation of mass and the absence of inflow (or outflow) of the chemical through Γ . Concerning φ , the no-flux condition in (1.2) corresponds to an orthogonality property of the (diffuse) interface with respect to Γ , a feature which is standardly assumed in Cahn-Hilliard-based models.

System (1.1)-(1.3) is mainly motivated by its applications to cancer growth processes. In such a setting, φ represents the local proportion of active tumor cells, whereas σ denotes the concentration or a nutrient (e.g., blood, or some protein, or possibly also some drug administered to the patient) affecting the evolution of the tumor mass. Actually, diffuse interface models for cancer evolution are becoming increasingly popular in the recent mathematical literature. In particular, many recent papers (see, e.g., [5, 6, 7, 8, 9, 25] and the references therein) have been devoted to the situation where (1.3) is replaced by a different relation, namely

$$\sigma_t - \Delta\sigma + \chi\Delta\varphi = b(\varphi, \sigma). \quad (1.4)$$

Compared to (1.3), (1.4) is mathematically simpler as it does not account for quadratic cross-diffusion effects. On the one hand, this prevents the blow-up of solutions even when the forcing term $b(\varphi, \sigma)$ is not of logistic type (but, for instance, has linear growth), but, on the other hand, relation (1.4) has no sign-preserving properties. Consequently, one cannot avoid that σ may attain negative values (which are somehow “nonphysical” as σ represents a concentration) even if the initial datum $\sigma_0 \geq 0$ a.e. in Ω . This fact, as first observed in [19], seems to be the main physical motivation for rather preferring relation (1.3).

As mentioned above, the function f in (1.2) denotes the derivative, or, more precisely, the subdifferential, of a configuration potential F of singular type, meaning that F is assumed to take finite values only in the interval $[-1, 1]$ (or, possibly, only in $(-1, 1)$), and to be identically $+\infty$ outside that interval, with the most relevant choice being represented by the so-called Flory-Huggins logarithmic potential defined by

$$F(r) = (1+r)\log(1+r) + (1-r)\log(1-r) - \frac{\lambda}{2}r^2, \quad r \in [-1, 1], \quad \lambda \geq 0. \quad (1.5)$$

Actually, as is customary for phase-separation models, the order parameter is normalized in such a way that the pure configurations are represented by $\varphi = \pm 1$; correspondingly, the minima of F are attained

in proximity of these states, and they are deeper for larger λ . Correspondingly, the “unphysical” states lying outside $[-1, 1]$ are penalized by letting $F(\varphi) \equiv +\infty$ for $|\varphi| > 1$ (see Assumption (A2) below for details). We point out that, for this class of models, the choice of a “singular” potential like (1.5) is furtherly motivated by the fact that it guarantees the a-priori uniform boundedness of φ and, in turn, the coercivity of the energy functional (cf. (3.3) below), which would be lost in the case when, e.g., F is a “smooth” double well polynomial potential (e.g. given by $F(r) = (r^2 - 1)^2$, which is a popular choice in the Cahn-Hilliard literature). To be precise, here we will consider a general class of singular potentials (including (1.5) as a particular case), so to apply to the present model (and in particular to its asymptotic analysis for large times) the abstract approach devised in [18] which does not rely on the specific expression (1.5).

As already observed, the first attempt in considering the coupling between the Cahn-Hilliard system (1.1)-(1.2) and the Keller-Segel-like relation (1.3) was carried out in [19], where the terminology “Cahn-Hilliard-Keller-Segel” model was proposed for this class of PDE systems. Since then, several papers have been devoted to analyzing this model or variants of it, see, e.g., [1, 4, 10, 12, 15, 19, 21] and the references therein. In these papers, existence and regularity properties of solutions have been addressed in several interesting situations (two- or three- space dimensions, occurrence or not of logistic forcing in (1.3) and of mass source in (1.1), presence of nonlinear sensitivity, different types of potentials, regularization of approximation schemes). Moreover, various different solution concepts have been proposed, including [15] “entropic” solutions, whose global existence can be proved even in three space dimensions with no need for stabilizing terms (a situation that is very different compared to the genuine Keller-Segel case for which blow-up occurs in that setting).

A closer look at this recent literature suggests in particular that, from the mathematical viewpoint, system (1.1)-(1.3) presents some specific features that make it rather different both from other Cahn-Hilliard-based models and from “standard” models of Keller-Segel type. Actually, on the one hand, the poor summability of the forcing term $\chi\sigma$ in (1.2) lowers the regularity properties of φ (in our case, however, the logistic source gives a relevant help) compared with what standardly happens for Cahn-Hilliard; on the other hand, φ has a fourth, rather than second order, dynamics with respect to space variables, which is something unusual in the Keller-Segel setting.

In this paper, we complement previous mathematical results on the Cahn-Hilliard-Keller-Segel model by analyzing the long-time behavior of solutions within the approach of infinite-dimensional dynamical systems and with the main purpose of proving existence of the global attractor in a suitable functional setting. This seems, indeed, to be the first attempt to address the long-time behavior of solutions to this model as previous works seem to consider only the case of finite time domains. As said, we deal with the case when a logistic forcing term is present in (1.3) and no mass source occurs in (1.1): in such a setting, it is not too difficult to identify a regularity class in which well-posedness can be shown. Actually, in order to have uniqueness, we need to prove a contractive estimate, and this requires rather good properties both for φ and σ in order to be able to deal with the quadratic cross-diffusion term in (1.3). In particular, for what concerns φ , the regularity we need (cf. (2.11) below) corresponds to the so-called “energy” class (according to the Cahn-Hilliard terminology), whereas for σ the minimal condition we are able to consider is a regularity setting of “fractional” type (cf. (2.14) below). We also notice that, in view of the occurrence of the singular potential, the phase-space is endowed with a nonstandard metric (cf. (2.12)), which is mutuated from the approach introduced in [18] for the Cahn-Hilliard system with singular potential. Then, assuming some compatibility condition on coefficients (cf. (2.15) and Remark 2.5 below), we can prove that the system is well-posed and generates a closed (according to the terminology in [17]) and asymptotically compact semigroup, which in turn guarantees the existence of the global attractor. As in [19], the proof of asymptotic compactness is based on the so-called “second energy estimate” for the Cahn-Hilliard system, a procedure providing (supposedly) optimal regularity properties for φ compatibly with the presence of a singular potential.

After proving existence of the global attractor, we investigate the sign properties of σ . Indeed, as relation (1.3) is evidently sign-preserving, the choice (2.14) of the phase space for σ needs to incorporate this information. On the other hand, it is also not difficult to see that, if one assumes stronger positivity conditions on the initial datum σ_0 (for instance, the assumption $\sigma_0 > 0$ almost everywhere with $\ln \sigma_0 \in L^1(\Omega)$ used in [15] for the construction of the “entropic solutions”), then this piece of information is maintained at least on finite time intervals. Hence, it may be worth investigating

whether, by applying parabolic regularization properties, one could prove that σ is separated from 0 in the uniform norm for every value of the time variable $t > 0$. Actually, we can show that the “entropic” reformulation of (1.3) proposed in [15] is suitable for applying a Moser iteration argument adapted from [22], which implies that $\sigma(x, t) \geq \delta(\tau, T) > 0$ for a.e. $(x, t) \in \Omega \times (\tau, T)$, where $0 < \tau < T$. However, as the estimates used in the proof apparently lack dissipativity, we cannot exclude that the value of δ might degenerate to 0 if one lets $T \nearrow \infty$ (see Remark 2.12 below for additional considerations on this point).

The plan of the paper is as follows: in the next section we introduce our assumptions in a rigorous framework and state all our mathematical results, whose proofs are detailed in the remainder of the paper. More precisely, Section 3 contains the a-priori estimates necessary for proving existence of weak solutions and dissipativity of the associated semigroup; there, the procedure to pass to the limit in a (hypothetical) approximation scheme is also sketched. The proof of the well-posedness of the system is completed in Section 4, which is devoted to uniqueness, whereas the subsequent Section 5 deals with the parabolic regularization estimates entailing asymptotic compactness of the solution semigroup (and consequently existence of the global attractor). Finally, in Section 6 a reinforced minimum principle property for equation (1.3) is shown.

2 Assumptions and main results

We start with introducing a set of notation which will be useful in order to rigorously formulate our mathematical results. Letting Ω be a smooth bounded domain of \mathbb{R}^d , $d \in \{2, 3\}$, of boundary Γ , we set $H := L^2(\Omega)$ and $V := H^1(\Omega)$. For the sake of simplicity, in all proofs we will directly consider the case when $d = 3$ (particularly, when dealing with embedding exponents or related inequalities). Of course, for $d = 2$ some of the results might be improved.

We will often write H in place of H^d (with similar notation for other spaces), in case vector-valued functions are considered. We denote by (\cdot, \cdot) the standard scalar product of H and by $\|\cdot\|$ the associated Hilbert norm. Moreover, we equip V with the standard norm $\|\cdot\|_V^2 = \|\cdot\|^2 + \|\nabla \cdot\|^2$. Identifying H with its dual space H' by means of the scalar product introduced above, we obtain the chain of continuous and dense embeddings $V \subset H \subset V'$. We indicate by $\langle \cdot, \cdot \rangle$ the duality pairing between V' and V , or, more generally, between X' and X where X is a generic Banach space continuously and densely embedded into H . Letting \mathbf{n} denote the outer unit normal vector to Ω , we also set

$$W := \{v \in H^2(\Omega) : \partial_{\mathbf{n}} v = 0 \text{ on } \partial\Omega\}, \quad (2.1)$$

which is a closed subspace of $H^2(\Omega)$ and in particular inherits its norm.

Next, for any functional $\xi \in V'$, we note as

$$\xi_{\Omega} := \frac{1}{|\Omega|} \langle \xi, 1 \rangle \quad (2.2)$$

the spatial average of ξ , where the duality can be replaced by an integral in case, say, $\xi \in H$. We note as H_0 , V_0 and V'_0 the closed subspaces of H , V , V' , respectively, consisting of the function(al)s having zero spatial average. Moreover, we observe that the Neumann Laplacian $B := (-\Delta)$, interpreted as a bounded linear operator

$$B : V \rightarrow V', \quad \langle Bv, w \rangle := \int_{\Omega} \nabla v \cdot \nabla w, \quad (2.3)$$

for $v, w \in V$, takes values in V'_0 and is invertible as it is restricted to the functions $v \in V_0$. We shall denote as $\mathcal{N} : V'_0 \rightarrow V_0$ the inverse of B acting over the functionals with zero spatial mean. Then, we observe that the norm

$$\|\xi\|_*^2 := \langle \xi, \mathcal{N}\xi \rangle, \quad \xi \in V'_0, \quad (2.4)$$

is a norm on V'_0 which is equivalent to the standard (dual) norm inherited from V' . We will use the above norm on occurrence. In particular, we may notice that, for $\xi \in V'_0$,

$$\int_{\Omega} |\nabla \mathcal{N}\xi|^2 = \langle B\mathcal{N}\xi, \mathcal{N}\xi \rangle = \langle \xi, \mathcal{N}\xi \rangle = \|\xi\|_*^2. \quad (2.5)$$

Assumption (A1). We assume the functions h and k on the right-hand side of (1.3) to satisfy the following structure hypotheses:

$$h, k \in C^1([0, +\infty) \times [-1, 1]), \quad \nabla h, \nabla k \in L^\infty([0, +\infty) \times [-1, 1]), \quad (2.6)$$

$$0 < \underline{h} \leq h(\sigma, \varphi) \leq \bar{h}, \quad 0 < \underline{k} \leq k(\sigma, \varphi) \leq \bar{k}, \quad \text{for every } (\sigma, \varphi) \in [0, +\infty) \times [-1, 1], \quad (2.7)$$

where $\underline{h}, \bar{h}, \underline{k}, \bar{k}$ are positive constants. Roughly speaking, the effect of h is to prevent the blowup of σ while the effect of k is that of avoiding it to degenerate to 0.

Assumption (A2). As we would like to consider a somehow abstract setting, we will assume general structure hypotheses on the singular potential F along the lines of the approach devised in [18]. To introduce the latter, we start with considering a (possibly multivalued) maximal monotone graph $\beta \subset \mathbb{R} \times \mathbb{R}$ (we refer to the monographs [2, 3] for the underlying convex analysis background), and, to avoid technicalities, we assume with no loss of generality the normalization

$$\overline{\text{dom } \beta} = [-1, 1], \quad 0 \in \beta(0), \quad (2.8)$$

where we recall that the *domain* $\text{dom } \beta \subset \mathbb{R}$ is the set of those $r \in \mathbb{R}$ such that $\beta(r)$ is not empty. The above choice is consistent with the ansatz that $\varphi = \pm 1$ corresponds to the pure states or configurations. Then, we observe (again, see [2] or [3]) that there exists a (unique) convex and lower semicontinuous function $\widehat{\beta} : \mathbb{R} \rightarrow [0, +\infty]$ with $\text{dom } \widehat{\beta} = [-1, 1]$ and $0 = \widehat{\beta}(0)$, such that $\beta = \partial \widehat{\beta}$, ∂ denoting here the subdifferential of convex analysis. We recall that, by definition, the domain $\text{dom } \widehat{\beta}$ is the set where $\widehat{\beta}$ takes finite values. Then, we may “reconstruct” F by setting

$$F(r) := \widehat{\beta}(r) - \frac{\lambda}{2} r^2, \quad f(r) = \beta(r) - \lambda r, \quad (2.9)$$

where $\lambda \geq 0$ is a given constant. In this way, F (f , respectively) is decomposed into its convex part $\widehat{\beta}$ (monotone part β , respectively) part and a quadratic (linear, respectively) perturbation. We may notice that, in the specific case of the Flory-Huggins potential (1.5), it turns out that $\text{dom } \widehat{\beta} = [-1, 1]$ as $\beta(r) = \log(1+r) - \log(1-r)$ is summable for $|r| \sim 1$.

Remark 2.1. As said, in agreement with the general theory of maximal monotone operators, the graph β may be multivalued; namely, for some $r \in [-1, 1]$ the (convex) set $\beta(r)$ may contain more than one element. For this reason, equation (1.2) should rather be formulated as

$$\mu = -\Delta\varphi + \eta - \lambda\varphi - \chi\sigma, \quad \text{where } \eta(\cdot, \cdot) \in \beta(\varphi(\cdot, \cdot)) \text{ almost everywhere,} \quad (2.10)$$

meaning that η is a suitable (and measurable) section of $\beta(\varphi)$. In particular, all the forthcoming estimates, as well as the regularity conditions (like, e.g., (2.18) below), should in fact be written in terms of the section η . However, to reduce technicalities, in the sequel we shall generally treat β as a single-valued function. Just in the definition of the “attractor space” (see (2.29) below), we preferred to maintain the notation of [18], from which this approach is inspired, where the multivalued structure of β explicitly occurs.

Phase space. In order to address the long-time analysis of system (1.1)-(1.3) by means of the theory of infinite-dimensional dynamical systems, we need to identify a suitable functional set where the system generates a dissipative process. In particular, according to the general theory (see the monographs [14, 24]), for initial data lying in such phase space, we need to be able to prove well-posedness of the system, continuity properties of trajectories with respect to the natural topology of the space, and existence of a bounded absorbing set (i.e., dissipativity).

To build the phase space, we consider the two variables separately: for what concerns φ , it is natural to assume the regularity corresponding to the finiteness of the physical energy and to the conservation of the total mass. Namely, we are naturally led to consider the set

$$\Phi_m := \{\varphi \in V : \widehat{\beta}(\varphi) \in L^1(\Omega), \varphi_\Omega = m\}, \quad (2.11)$$

where $m \in (-1, 1)$ is the fixed (and conserved in time) mean value of the initial datum.

Then, by [18, Lemma 3.8], it turns out that Φ_m is a complete metric space with respect to the metric

$$\text{dist}(\varphi_1, \varphi_2) := \|\varphi_1 - \varphi_2\|_V + \|\widehat{\beta}(\varphi_1) - \widehat{\beta}(\varphi_2)\|_{L^1(\Omega)}. \quad (2.12)$$

Remark 2.2. In the case of the Flory-Huggins potential (1.5) (as well as in all other cases when $\text{dom } \widehat{\beta} = [-1, 1]$), it is apparent that the condition $\widehat{\beta}(\varphi) \in L^1(\Omega)$ is equivalent to $-1 \leq \varphi \leq 1$ a.e. in Ω . Hence, in this case one may simply take Φ_m as the set of those functions $\varphi \in V$ such that $\varphi_\Omega = m$ and $-1 \leq \varphi \leq 1$ almost everywhere. Moreover, as this is a closed subset of V , one can also omit the second summand in (2.12) and use simply the V -norm.

For what concerns σ , the main issue is related to uniqueness. Actually, as noted in [19], due to the quadratic growth of the cross-diffusion term this is a nontrivial property when one considers (1.3). In our setting, the minimal regularity under which we can prove uniqueness (for the whole system) is of “fractional type” (and is a bit weaker than the assumptions taken in [19, Theorem 2.8] as here we do not have a mass source in (1.1)).

To introduce it, also inspired by the approach used in [13], we first define the linear unbounded abstract operator

$$A := (I - \Delta) : H \rightarrow H, \quad D(A) = W, \quad (2.13)$$

where the space W , i.e. the domain of A , is defined in (2.1). Then, noting that A is strictly positive, we can consider the fractional powers A^s of A for any $s \in \mathbb{R}$. We then set

$$\Sigma := \{\sigma \in D(A^{1/4}) = H^{1/2}(\Omega) : \sigma \geq 0 \text{ a.e. in } \Omega\}. \quad (2.14)$$

This is, of course a closed subset of $D(A^{1/4})$ (hence, a complete metric space). The phase space for our asymptotic analysis is then defined as the product space $\Phi_m \times \Sigma$.

Remark 2.3. Condition (2.14) accounts in particular for the nonnegativity of σ . Actually, the structure of the cross-diffusion term in (1.3) guarantees (the simplest proof of this fact is that based on Stampacchia’s truncation method) that, if $\sigma_0(\cdot) \geq 0$ almost everywhere in Ω , then it also holds that $\sigma(\cdot, \cdot) \geq 0$ almost everywhere in $\Omega \times (0, +\infty)$. This property will be improved in Theorem 2.11 below.

We are now ready to state our first theorem regarding well-posedness of the initial-boundary value problem for system (1.1)-(1.3) for initial data lying in the product space $\Phi_m \times \Sigma$, as well as dissipativity of the associated dynamical process in the same functional setting.

Theorem 2.4. *Let assumptions (A1)-(A2) hold. Let, moreover, the following compatibility condition hold:*

$$\chi^2 \leq 3\underline{h}. \quad (2.15)$$

For some $m \in (-1, 1)$, let also

$$(\varphi_0, \sigma_0) \in \Phi_m \times \Sigma. \quad (2.16)$$

Then, there exists one and only one triple (φ, μ, σ) defined over $\Omega \times (0, \infty)$ such that, for every $T > 0$,

$$\varphi \in H^1(0, T; V') \cap C^0([0, T]; V) \cap L^4(0, T; W) \cap L^2(0, T; W^{2,6}(\Omega)), \quad (2.17)$$

$$\widehat{\beta}(\varphi) \in C^0([0, T]; L^1(\Omega)), \quad \beta(\varphi) \in L^2(0, T; L^6(\Omega)), \quad (2.18)$$

$$\mu \in L^2(0, T; V), \quad (2.19)$$

$$\sigma \in H^1(0, T; D(A^{-1/4})) \cap C^0([0, T]; D(A^{1/4})) \cap L^2(0, T; D(A^{3/4})), \quad (2.20)$$

with $\sigma \geq 0$ almost everywhere in $\Omega \times (0, T)$. The triple (φ, μ, σ) satisfies equation (1.2) a.e. in $\Omega \times (0, T)$ with the boundary condition

$$\partial_n \varphi = 0, \quad \text{a.e. on } \Gamma \times (0, T), \quad (2.21)$$

together with the following weak formulations of (1.1) and (1.3):

$$\langle \varphi_t, \xi \rangle + \int_{\Omega} \nabla \mu \cdot \nabla \xi = 0, \quad \text{a.e. in } (0, T), \quad (2.22)$$

$$\langle \sigma_t, \xi \rangle + \int_{\Omega} \nabla \sigma \cdot \nabla \xi - \chi \int_{\Omega} \sigma \nabla \varphi \cdot \nabla \xi = \int_{\Omega} (-h(\sigma, \varphi)\sigma^2 + k(\sigma, \varphi)\sigma)\xi, \quad \text{a.e. in } (0, T), \quad (2.23)$$

where (in both relations) $\xi \in V$ is a generic test function, and the initial conditions

$$\varphi|_{t=0} = \varphi_0, \quad \sigma|_{t=0} = \sigma_0. \quad (2.24)$$

Moreover, if $S(t)$, $t \geq 0$, denotes the semigroup mapping associated to the above class of weak solutions, the dynamical process described by S is dissipative in the phase space $\Phi_m \times \Sigma$. Namely, there exists a number $R_1 \geq 0$ independent of the initial data such that the set

$$\mathcal{B}_0 := \{(\phi, s) \in \Phi_m \times \Sigma : \text{dist}(\phi, 0) + \|s\|_{D(A^{1/4})} \leq R_1\} \quad (2.25)$$

is absorbing, i.e., for any bounded set $B \in \Phi_m \times \Sigma$ of initial data there exists a time $T_1 > 0$ depending only on the “radius” R of B in the metric (2.12), such that

$$S(t)B \subset \mathcal{B}_0 \quad \text{for every } t \geq T_1. \quad (2.26)$$

Remark 2.5. It is worth making some comment on condition (2.15), which prescribes that the chemotactic effect is “small” compared to the stabilizing effect of the logistic term. Actually, existence of weak solutions without assuming this type of condition is still an open problem, at least in our case of a quadratic logistic source, whereas, as noted in [19], if the stabilizing part of the logistic term is superquadratic, then (2.15) can be avoided. Compared to [19, Theorem 2.2], here we have been able to improve (i.e. weaken) a bit condition (2.15), but we still need a form of it in order to close our a priori estimates.

As we have introduced the one-parameter family of mappings $t \mapsto S(t)$, it is worth noting that $S(\cdot)$ satisfies the semigroup properties in a proper way. In particular, it is clear that $S(0) = \text{Id}$ (the identity mapping) and that there holds the “concatenation property” $S(t_1) \circ S(t_2) = S(t_1 + t_2)$ for every $t_1, t_2 \geq 0$. Concerning the continuity properties of trajectories, one can first notice that, thanks to the continuity conditions in (2.17), (2.18) and (2.20), it turns out that the mapping $t \mapsto S(t)(\varphi_0, \sigma_0)$ is continuous from $[0, \infty)$ to $\Phi_m \times \Sigma$ (endowed with the distance (2.12)). Regarding continuity with respect to the initial data, this can be proved in a rather straightforward way; it is however even quicker to observe that the contractive estimate in the uniqueness proof (cf. (4.23) below) immediately implies that $S(\cdot)$ is a closed semigroup according to the terminology introduced by Pata and Zelik in [17], which is sufficient in order to apply all the machinery from the theory of infinite-dimensional dynamical systems. It is worth recalling here some basic definition from that theory: first of all, if X is the phase space of a dynamical process $S(\cdot)$, a set $K \subset X$ is said to be uniformly attracting whenever, for any bounded set $B \subset X$, one has

$$\lim_{t \nearrow \infty} \text{dist}(S(t)B, K) = 0, \quad (2.27)$$

where dist denotes the unilateral Hausdorff distance of the set $S(t)B$ from K , with respect to the metric of X , namely

$$\text{dist}(S(t)B, K) := \sup_{y \in S(t)B} \inf_{k \in K} d_X(y, k), \quad (2.28)$$

where d_X is the metric of X . Moreover, a set \mathcal{A} is the global (or “universal”) attractor of the semigroup $S(\cdot)$ iff \mathcal{A} is attracting and compact in X and is fully invariant with respect to $S(\cdot)$, i.e., $S(t)\mathcal{A} = \mathcal{A}$ for all $t \geq 0$.

For our system, the existence of the global attractor will be obtained by showing asymptotic compactness of the semigroup $S(\cdot)$, meaning, roughly speaking, that, for t sufficiently large, all solutions emanating from initial data satisfying (2.16) take values in a set that is bounded in some “better space”. To make precise this concept, also in view of the fact that the phase space $\Phi_m \times \Sigma$ and the distance (2.12) are somehow nonstandard, we introduce some more tools, again following the lines of the approach devised in [18]. As before, we shall consider the behavior of φ and σ “separately”, and, complementing respectively (2.11) and (2.14), we define

$$\Psi_m := \{\varphi \in W : \beta^0(\varphi) \in L^2(\Omega), \varphi_\Omega = m\}, \quad (2.29)$$

$$V_+ := \{\sigma \in V = D(A^{1/2}) : \sigma \geq 0 \text{ a.e. in } \Omega\}. \quad (2.30)$$

We recall that, for $r \in \text{dom}(\beta)$, $\beta^0(r)$ denotes the element of minimum modulus of the convex set $\beta(r)$ (recall that, according to the general theory detailed in [2, 3], we may consider multivalued maximal monotone operators β). Mimicking (2.12), the set Ψ_m can be naturally endowed with the distance

$$\text{dist}_1(\varphi_1, \varphi_2) := \|\varphi_1 - \varphi_2\|_{H^2(\Omega)} + \|\beta^0(\varphi_1) - \beta^0(\varphi_2)\|. \quad (2.31)$$

Remark 2.6. As observed in [19, Prop. 3.15] (and it is also easy to check directly), in the case when $\text{dom } \widehat{\beta} = [-1, 1]$ (as happens for (1.5)), the second summand in (2.31) can be avoided, and one can simply use the H^2 -norm. Recall that, in that case, the structure of the phase space is simpler as well (cf. Remark 2.2).

Then, an easy adaptation of the argument in [18, Prop. 3.15] gives

Proposition 2.7. *It holds that $\Psi_m \subset \Phi_m$ with compact immersion; namely, if $\{\varphi_n\}$ is a sequence in Ψ_m bounded with respect to the distance (2.31), then there exist $\varphi \in \Phi_m$ and a nonrelabelled subsequence of $\{\varphi_n\}$ such that $\text{dist}(\varphi_n, \varphi) \rightarrow 0$.*

We can then prove the following result providing existence of the global attractor in the regularity setting of Theorem 2.4:

Theorem 2.8. *Let assumptions (A1)-(A2) hold together with the compatibility condition (2.15). Then, the dynamical process $S(\cdot)$ associated to system (1.1)-(1.3) admits the global attractor $\mathcal{A} \subset \Psi_m \times V_+$. More precisely, \mathcal{A} is bounded in $\Psi_m \times V_+$, in the sense that there exists $M > 0$ such that*

$$\text{dist}_1(\phi, 0) + \|s\|_V \leq M, \quad \text{for every } (\phi, s) \in \mathcal{A}. \quad (2.32)$$

Moreover, for what concerns the phase variable, one has more precisely

$$\|\phi\|_{W^{2,6}(\Omega)} + \|\beta^0(\phi)\|_{L^6(\Omega)} \leq M, \quad \text{for every } (\phi, s) \in \mathcal{A}. \quad (2.33)$$

Exhibiting stronger a priori estimates, we can also prove some additional regularity of the nutrient states in the attractor. Namely, we have

Theorem 2.9. *Let the assumptions of Theorem 2.8 hold. Then, the elements of the attractor \mathcal{A} satisfy the following additional regularity estimate: there exists $M_1 > 0$ such that*

$$\|s\|_{H^2(\Omega)} \leq M_1, \quad \text{for every } (\phi, s) \in \mathcal{A}. \quad (2.34)$$

Remark 2.10. It is worth observing that the regularity properties of φ stated in (2.33) correspond to what is currently considered the “state of the art” for the Cahn-Hilliard equation with singular potential, at least in three dimensions of space (cf., e.g., [16]). On the other hand, regarding σ , in the “high-regularity regime” (1.3) behaves more or less like a semilinear parabolic equation; hence the only real obstacle that prevents us from further improving the asymptotic regularity of σ comes from the forcing terms depending on φ , whose smoothness, as said, is limited by (2.33).

The last question we would like to discuss is related to the nonnegativity of σ , which is incorporated in the definition (2.14) of the phase space. Actually, as noted in Remark 2.3, using the Stampacchia truncation method one can easily prove that such a property is maintained in time. On the other hand, it may be worth investigating whether this “minimum principle” could be improved, possibly in an uniform way with respect to time, by means of parabolic smoothing arguments. Actually, as observed in some previous papers dealing with related models (see, e.g., [15, 21]), it is very easy to prove that, if one additionally assumes

$$\sigma_0 > 0 \quad \text{a.e. in } \Omega, \quad \ln \sigma_0 \in L^1(\Omega), \quad (2.35)$$

then also this property is conserved in time. We also remark that this is far from being just an “academic” question, as the regularity (2.35) may serve as a source of additional a-priori estimates, on which is based, for instance, the concept of “entropic” solution developed in [15].

Here, assuming (2.35), we can show a time regularization property for $\ln \sigma$ implying that for strictly positive times $\sigma(t)$ is separated from 0 in the uniform norm. More precisely, we have

Theorem 2.11. Let the assumptions of Theorem 2.4 hold and let, in addition, (2.35) be satisfied. Then, for any $0 < \tau < T < \infty$ there exists $\delta = \delta(\tau, T) > 0$ such that

$$\sigma(x, t) \geq \delta \quad \text{for every } (x, t) \in \Omega \times [\tau, T]. \quad (2.36)$$

Remark 2.12. As noticed in the introduction, it is not obvious whether an estimate of the form (2.35) could be proved uniformly in time. It is actually clear that, if the initial datum satisfies $\sigma_0(x) \geq \delta_0$ a.e. in Ω for some constant $\delta_0 > 0$, then (2.35) holds with $\tau = 0$. On the other hand, the a-priori estimates leading to that relation apparently lack a dissipative character, whence one may expect that δ in (2.35) could degenerate to 0 as one lets $T \nearrow \infty$. As a consequence it is also not clear whether the states σ in the attractor are separated from 0 in the uniform norm.

Finally, for later convenience, it is useful to recall the so-called *uniform Grönwall lemma* (cf., e.g., [24, Lemma III.1.1]):

Lemma 2.13. Let $y, a, b \in L^1_{\text{loc}}(0, \infty)$ be three non-negative functions such that

$$y'(t) \leq a(t)y(t) + b(t) \quad \text{for a.e. } t > 0, \quad (2.37)$$

and let a_1, a_2, a_3 three non-negative constants such that

$$\sup_{t \geq 0} \int_t^{t+1} a(s) \leq a_1, \quad \sup_{t \geq 0} \int_t^{t+1} b(s) \leq a_2, \quad \sup_{t \geq 0} \int_t^{t+1} y(s) \leq a_3. \quad (2.38)$$

Then, it follows that $y(t+1) \leq (a_2 + a_3)e^{a_1}$ for every $t > 0$.

3 A priori estimates, existence and dissipativity

In this part we detail the basic a priori estimates leading to existence of weak solutions in the regularity setting of Theorem 2.4. Our argument partially follows the lines of the proofs given in previous contributions on related models, see, e.g., [12, 19, 21]. For this reason we will limit ourselves to detail the basic a priori estimates needed for proving existence, by formulating them in a way that could also yield the dissipativity of the associated dynamical process, which is a fundamental step of our long-time analysis. Moreover, we will work directly on the “original” system (1.1)-(1.3) without referring to any regularization scheme. Actually, the problem of the approximation (or discretization) of the system has been thoroughly discussed in the quoted references. However, we would like at least to recall a basic point: for this class of systems, the coercivity of the energy functional \mathcal{E} (cf. (3.3) below) is linked to the choice of a “singular” potential (as in Assumption (A2)), which guarantees a-priori uniform boundedness of φ and, consequently, a control of the coupling term $-\chi\sigma\varphi$. Any approximation of the system must take this fact into account: for instance, if β is replaced by a “smooth” regularization, as is customary for Cahn-Hilliard models, then the coercivity of \mathcal{E} is lost unless some further regularizing term is added in order to compensate the loss of boundedness of φ . Actually, in our view, this seems to be, the main difficulty arising when one approximates models in this class.

3.1 Dissipative energy estimate

We derive here a variant of the energy estimate (in itself, this can be seen as a direct consequence of the variational structure of the model), designed so to provide as a byproduct the existence of a bounded absorbing set with respect to the product metric in $\Phi_m \times \Sigma$. To obtain it, we proceed through several steps. First of all, we observe the mass conservation property

$$\frac{d}{dt} \int_{\Omega} \varphi = 0, \quad (3.1)$$

obtained integrating (1.1) over Ω and exploiting the no-flux conditions. Next, we test (1.1) by μ , (1.2) by φ_t , and (1.3) by $\ln \sigma - \chi\varphi$. Combining the resulting relations, it is not difficult to arrive at

$$\frac{d}{dt} \mathcal{E} + \|\nabla \mu\|^2 + \int_{\Omega} \sigma |\nabla(\ln \sigma - \chi\varphi)|^2 + \int_{\Omega} (h(\sigma, \varphi)\sigma^2 - k(\sigma, \varphi)\sigma)(\ln \sigma - \chi\varphi) = 0, \quad (3.2)$$

with the physical energy \mathcal{E} being given by

$$\mathcal{E}(\varphi, \sigma) = \frac{1}{2} \|\nabla \varphi\|^2 + \int_{\Omega} (F(\varphi) + \sigma(\ln \sigma - 1) - \chi \sigma \varphi). \quad (3.3)$$

Now, owing to condition (2.7) and to the fact

$$\|\varphi\|_{L^\infty(0,T;L^\infty(\Omega))} \leq 1, \quad (3.4)$$

which, as noted before, is a direct consequence of Assumption (A2) at least as far as we work on the “original” system and not on a regularization of it, it is not difficult to deduce

$$(h(\sigma, \varphi)\sigma^2 - k(\sigma, \varphi)\sigma)(\ln \sigma - \chi \varphi) \geq \frac{h}{2}\sigma^2 \ln(1 + \sigma) + \kappa \sigma \ln(1 + \sigma) - c. \quad (3.5)$$

Here and below, $c \geq 0$ and $\kappa > 0$ are computable constants depending on the constants $\bar{k}, \underline{k}, h$ and on the other assigned parameters of the system, with κ being used in estimates from below. The value of c, κ may vary on occurrence. Of course, it is clear that, if the a priori estimates were adapted to a (hypothetical) approximation, c, κ would not be allowed to depend on the approximation parameter(s).

That said, it is not difficult to check that the energy satisfies the coercivity property

$$\mathcal{E}(\varphi, \sigma) \geq \kappa (\|\sigma \ln(1 + \sigma)\|_{L^1(\Omega)} + \|\varphi\|_V^2 + \|\widehat{\beta}(\varphi)\|_{L^1(\Omega)}) - c, \quad (3.6)$$

as well as the boundedness condition

$$\mathcal{E}(\varphi, \sigma) \leq c (\|\sigma \ln(1 + \sigma)\|_{L^1(\Omega)} + \|\varphi\|_V^2 + \|\widehat{\beta}(\varphi)\|_{L^1(\Omega)} + 1). \quad (3.7)$$

In order to obtain a dissipative version of the energy estimate, we multiply (1.2) by $\varphi - \varphi_\Omega$, where we recall that φ_Ω stands for the (conserved, by (3.1)) spatial average of φ . Then, let us recall that (cf. [16])

$$\int_{\Omega} f(\varphi)(\varphi - \varphi_\Omega) \geq \kappa \int_{\Omega} \widehat{\beta}(\varphi) + \kappa \int_{\Omega} |\beta(\varphi)| - c, \quad (3.8)$$

where $\kappa > 0$ and $c \geq 0$ may now depend also on the “assigned” quantity $m = (\varphi_0)_\Omega$ and on the “nonconvexity” parameter λ . Hence, we deduce

$$\kappa \int_{\Omega} \widehat{\beta}(\varphi) + \kappa \int_{\Omega} |\beta(\varphi)| + \|\nabla \varphi\|^2 \leq c + \int_{\Omega} \mu(\varphi - \varphi_\Omega) + \chi \int_{\Omega} \sigma(\varphi - \varphi_\Omega). \quad (3.9)$$

Now, using the Poincaré-Wirtinger inequality there follows

$$\int_{\Omega} \mu(\varphi - \varphi_\Omega) = \int_{\Omega} (\mu - \mu_\Omega)(\varphi - \varphi_\Omega) \leq \frac{1}{2} \|\nabla \mu\|^2 + c_\Omega \|\nabla \mu\|^2, \quad (3.10)$$

where c_Ω is an embedding constant. Then, replacing (3.10) into (3.9), multiplying the result by $1/(2c_\Omega)$ and adding it to (3.2), using also (3.5) we obtain

$$\begin{aligned} \frac{d}{dt} \mathcal{E} + \frac{1}{2} \|\nabla \mu\|^2 + \int_{\Omega} \sigma |\nabla(\ln \sigma - \chi \varphi)|^2 + \kappa \left(\int_{\Omega} \widehat{\beta}(\varphi) + \int_{\Omega} |\beta(\varphi)| + \|\nabla \varphi\|^2 \right) \\ + \frac{h}{2} \int_{\Omega} \sigma^2 \ln(1 + \sigma) + \kappa \int_{\Omega} \sigma \ln(1 + \sigma) \leq c + \frac{\chi}{2c_\Omega} \int_{\Omega} \sigma(\varphi - \varphi_\Omega), \end{aligned} \quad (3.11)$$

for a possibly new value of the constant $\kappa > 0$. To control the last term in (3.11), we use (3.4) so to obtain

$$\frac{\chi}{2c_\Omega} \int_{\Omega} \sigma(\varphi - \varphi_\Omega) \leq c \int_{\Omega} \sigma \leq \frac{h}{4} \int_{\Omega} \sigma^2 \ln(1 + \sigma) - c. \quad (3.12)$$

We also observe that, as $|\varphi| \leq 1$ almost everywhere, there follows

$$\int_{\Omega} \widehat{\beta}(\varphi) - \frac{\lambda}{2} |\Omega| \leq \int_{\Omega} F(\varphi) \leq \int_{\Omega} \widehat{\beta}(\varphi). \quad (3.13)$$

Hence, using F or $\widehat{\beta}$ in the estimates is in fact equivalent (a similar equivalence holds for f and β). That said, neglecting the (positive) contribution of the cross-diffusion term and rearranging, (3.11) implies in particular

$$\begin{aligned} \frac{d}{dt}\mathcal{E} + \kappa &\left(\int_{\Omega} \widehat{\beta}(\varphi) + \|\varphi\|_V^2 + \|\sigma \ln(1 + \sigma)\|_{L^1(\Omega)} + 1 \right) \\ &+ \kappa \int_{\Omega} |\beta(\varphi)| + \frac{1}{2} \|\nabla \mu\|^2 + \frac{h}{4} \|\sigma^2 \ln(1 + \sigma)\|_{L^1(\Omega)} \leq c, \end{aligned} \quad (3.14)$$

or, in other words, owing to (3.7),

$$\frac{d}{dt}\mathcal{E} + \kappa\mathcal{E} + \kappa_0 \|\nabla \varphi\|^2 + \kappa \int_{\Omega} |\beta(\varphi)| + \frac{1}{2} \|\nabla \mu\|^2 + \frac{h}{4} \|\sigma^2 \ln(1 + \sigma)\|_{L^1(\Omega)} \leq c, \quad (3.15)$$

for possibly new values of the “small” positive constants κ and κ_0 (we preferred to keep the latter as a separate value as it will be managed in a while), which is a dissipative estimate for the physical energy \mathcal{E} and may be used for the construction of the absorbing set. For this purpose, however, it will be convenient to complement the above inequality with a further contribution. Namely, we test (1.2) by $-\Delta \varphi$ so to obtain

$$\|\Delta \varphi\|^2 \leq \lambda \|\nabla \varphi\|^2 + \|\nabla \mu\| \|\nabla \varphi\| + \chi \|\sigma\| \|\Delta \varphi\|, \quad (3.16)$$

whence, by simple manipulations,

$$\|\Delta \varphi\|^2 \leq (2\lambda + 1) \|\nabla \varphi\|^2 + \|\nabla \mu\|^2 + \chi^2 \|\sigma\|^2. \quad (3.17)$$

Then, multiplying (3.17) by $\delta > 0$ taken sufficiently small and summing the result to (3.15), it is not difficult to arrive at

$$\frac{d}{dt}\mathcal{E} + \kappa\mathcal{E} + \frac{\kappa_0}{2} \|\nabla \varphi\|^2 + \kappa \int_{\Omega} |\beta(\varphi)| + \kappa \|\Delta \varphi\|^2 + \frac{1}{4} \|\nabla \mu\|^2 + \frac{h}{8} \|\sigma^2 \ln(1 + \sigma)\|_{L^1(\Omega)} \leq c_0, \quad (3.18)$$

where $c_0 > 0$ is a computable constant depending only on the assigned parameters of the system and independent of the initial data.

Let us now assume that the initial data (φ_0, σ_0) are chosen, as in the statement of Theorem 2.4, in a bounded “ball” $B = \overline{B}(0, R)$ of the phase space of arbitrarily large, but otherwise assigned, radius $R > 0$, namely we have

$$\text{dist}(\varphi_0, 0) + \|\sigma_0\|_{D(A^{1/4})} \leq R. \quad (3.19)$$

By (3.7), the above implies in particular that there exists a computable monotone function $Q : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\mathcal{E}(0) \leq Q(R)$. Generally speaking, here and below Q will always denote a positive-valued function assumed to be monotone with respect of each of its arguments. The expression of Q , may only depend on the fixed parameters of the system and may vary on occurrence.

Applying the Grönwall lemma to (3.18), we then deduce

$$\mathcal{E}(t) \leq \mathcal{E}(0) e^{-\kappa t} + \frac{c_0}{\kappa} \leq Q(R) e^{-\kappa t} + \frac{c_0}{\kappa}, \quad (3.20)$$

for every $t \geq 0$. In addition to that, integrating (3.18) over $[t, t+1]$ for a generic $t \geq 0$, we deduce

$$\begin{aligned} \int_t^{t+1} \left(\frac{\kappa_0}{2} \|\nabla \varphi\|^2 + \kappa \int_{\Omega} |\beta(\varphi)| + \kappa \|\Delta \varphi\|^2 + \frac{1}{4} \|\nabla \mu\|^2 + \frac{h}{8} \|\sigma^2 \ln(1 + \sigma)\|_{L^1(\Omega)} \right) \\ \leq Q(R) e^{-\kappa t} + C_0, \end{aligned} \quad (3.21)$$

still for every $t \geq 0$. Here and below, $C_0 > 0$ is a computable constant, varying on occurrence, whose value is independent of the initial data. For instance, in the above formula one can take $C_0 = c_0(1 + \kappa^{-1})$. With this notation at hand, estimate (3.20), thanks also to (3.6), gives

$$\|\varphi(t)\|_V^2 + \|\widehat{\beta}(\varphi)\|_{L^1(\Omega)} + \|\sigma \ln(1 + \sigma)\|_{L^1(\Omega)} \leq Q(R) e^{-\kappa t} + C_0. \quad (3.22)$$

Note that the above is sufficient in order to control φ with respect to the “energy distance”. On the other hand, regarding σ , the information provided by (3.22) is very weak. Actually, as (2.14) suggests, the conditions on σ required for having well-posedness are much more restrictive compared to the information contained in the energy functional. Hence, in the sequel, we will need to derive some further dissipative estimates of better norms of σ . This procedure will be partially based on tools and techniques proposed in former papers dealing with related models.

3.2 Estimate of the singular potential

We go back to (3.9), where we can now neglect some nonnegative terms on the left-hand side, and modify a bit the control of the integral terms on the right-hand side. Using repeatedly (3.22), we first observe that

$$\chi \int_{\Omega} \sigma(\varphi - \varphi_{\Omega}) \leq c \int_{\Omega} \sigma \leq Q(R)e^{-\kappa t} + C_0. \quad (3.23)$$

Similarly,

$$\int_{\Omega} \mu(\varphi - \varphi_{\Omega}) \leq c \|\nabla \varphi\| \|\nabla \mu\| \leq (Q(R)e^{-\kappa t} + C_0) \|\nabla \mu\|. \quad (3.24)$$

Moreover, using (3.23)-(3.24) and integrating (1.2) over Ω , it is not difficult to deduce from (3.9) that

$$\|\beta(\varphi)\|_{L^1(\Omega)} + |\mu_{\Omega}| \leq c(1 + \|\nabla \mu\|)(Q(R)e^{-\kappa t} + C_0). \quad (3.25)$$

Squaring, integrating over a generic interval $(t, t+1)$ and using the control of $\|\nabla \mu\|^2$ resulting from (3.21), we also obtain

$$\int_t^{t+1} |\mu_{\Omega}|^2 \leq Q(R)e^{-\kappa t} + C_0, \quad (3.26)$$

whence, using (3.21) once more, we deduce

$$\int_t^{t+1} \|\mu\|_V^2 \leq Q(R)e^{-\kappa t} + C_0. \quad (3.27)$$

From the above, using (1.1) and standard arguments based on (2.4)-(2.5), we also infer

$$\int_t^{t+1} \|\varphi_t\|_*^2 \leq Q(R)e^{-\kappa t} + C_0. \quad (3.28)$$

3.3 “Unilateral” estimate of the singular term

We repeat here the argument devised in [19, Sec. 4]. Namely, observing that, by (2.8), $\beta(r)$ has the same sign of r , we test (1.2) by $-(\beta(\varphi)_-)^5$ (i.e., minus the fifth power of the negative part of $\beta(\varphi)$). As this is a monotone function of φ , the Laplacian, once integrated by parts, provides a nonnegative contribution. Hence we end up with

$$\|\beta(\varphi)_-\|_{L^6(\Omega)}^6 = - \int_{\Omega} \mu(\beta(\varphi)_-)^5 - \int_{\Omega} \lambda \varphi (\beta(\varphi)_-)^5 - \chi \int_{\Omega} \sigma (\beta(\varphi)_-)^5. \quad (3.29)$$

Now, the last term on the right-hand side is obviously nonpositive, while the other two integrals are controlled by observing that

$$- \int_{\Omega} \mu(\beta(\varphi)_-)^5 - \int_{\Omega} \lambda \varphi (\beta(\varphi)_-)^5 \leq c(\|\mu\|_{L^6(\Omega)} + \|\varphi\|_{L^6(\Omega)}) \|\beta(\varphi)_-\|_{L^6(\Omega)}^5. \quad (3.30)$$

Hence, by the uniform boundedness of φ and Sobolev’s embeddings, we end up with

$$\|\beta(\varphi)_-\|_{L^6(\Omega)} \leq c(\|\mu\|_V + 1), \quad (3.31)$$

whence, squaring and integrating over $(t, t+1)$ for a generic $t \geq 0$, using (3.27), we obtain

$$\int_t^{t+1} \|\beta(\varphi)_-\|_{L^6(\Omega)}^2 \leq Q(R)e^{-\kappa t} + C_0. \quad (3.32)$$

3.4 Improved summability of σ

In this part we adapt the argument devised in [19] to obtain a “decoupled” estimate for the cross-diffusion term in (1.3). This uses in an essential way the compatibility condition (2.15), which arises as the stabilizing contribution of the logistic source and the chemotactic effect provide terms that have the same growth rate, so that the estimate can be closed only by comparing the magnitude of the corresponding coefficients.

That said, we first observe that estimates (3.20)-(3.21) and (3.27) imply that, as the initial data satisfy condition (3.19), there exist a number R_0 independent of R and a time T_0 depending on R , such that, for every $t \geq T_0$, there holds

$$\mathcal{E}(t) + \int_t^{t+1} \left(\|\varphi\|_V^2 + \|\beta(\varphi)\|_{L^1(\Omega)} + \|\Delta\varphi\|^2 + \|\mu\|_V^2 + \|\sigma^2 \ln(1+\sigma)\|_{L^1(\Omega)} \right) \leq R_0. \quad (3.33)$$

Let us now test (1.3) by σ^p for “small” $p > 0$ to be chosen later on. Then, simple calculations give

$$\frac{1}{p+1} \frac{d}{dt} \|\sigma\|_{L^{p+1}(\Omega)}^{p+1} + \underline{h} \|\sigma\|_{L^{p+2}(\Omega)}^{p+2} + \frac{4p}{(p+1)^2} \int_{\Omega} |\nabla \sigma^{\frac{p+1}{2}}|^2 \leq \bar{k} \|\sigma\|_{L^{p+1}(\Omega)}^{p+1} + p\chi \int_{\Omega} \sigma^p \nabla \sigma \cdot \nabla \varphi. \quad (3.34)$$

Now, using (1.2), the last term is managed as follows:

$$\begin{aligned} p\chi \int_{\Omega} \sigma^p \nabla \sigma \cdot \nabla \varphi &= -\chi \frac{p}{p+1} \int_{\Omega} \sigma^{p+1} \Delta \varphi \\ &= \chi \frac{p}{p+1} \int_{\Omega} \mu \sigma^{p+1} - \chi \frac{p}{p+1} \int_{\Omega} f(\varphi) \sigma^{p+1} + \chi^2 \frac{p}{p+1} \int_{\Omega} \sigma^{p+2}. \end{aligned} \quad (3.35)$$

Following [19], the first two terms on the right-hand side can be controlled by observing, respectively, that

$$\chi \frac{p}{p+1} \int_{\Omega} \mu \sigma^{p+1} \leq c \|\mu\|_{L^4(\Omega)} \|\sigma^{\frac{p+1}{2}}\|_{L^4(\Omega)} \leq \frac{p}{(p+1)^2} \|\sigma^{\frac{p+1}{2}}\|_V^2 + c_p \|\mu\|_V^2 \|\sigma^{\frac{p+1}{2}}\|^2 \quad (3.36)$$

and that (here we also use that σ is nonnegative)

$$\begin{aligned} -\chi \frac{p}{p+1} \int_{\Omega} f(\varphi) \sigma^{p+1} &= -\chi \frac{p}{p+1} \int_{\Omega} \beta(\varphi) \sigma^{p+1} + \lambda \chi \frac{p}{p+1} \int_{\Omega} \varphi \sigma^{p+1} \\ &\leq \|\beta(\varphi)_-\|_{L^6(\Omega)} \|\sigma^{\frac{p+1}{2}}\|_{L^3(\Omega)} + c \|\sigma\|_{L^{p+1}(\Omega)}^{p+1} \\ &\leq \frac{p}{(p+1)^2} \|\sigma^{\frac{p+1}{2}}\|_V^2 + c_p \|\beta(\varphi)_-\|_{L^6(\Omega)}^2 \|\sigma^{\frac{p+1}{2}}\|^2 + c \|\sigma\|_{L^{p+1}(\Omega)}^{p+1}. \end{aligned} \quad (3.37)$$

Collecting (3.35)-(3.37), (3.34) gives

$$\begin{aligned} \frac{1}{p+1} \frac{d}{dt} \|\sigma\|_{L^{p+1}(\Omega)}^{p+1} + \left[\underline{h} - \chi^2 \frac{p}{p+1} \right] \|\sigma\|_{L^{p+2}(\Omega)}^{p+2} + \frac{2p}{(p+1)^2} \int_{\Omega} |\nabla \sigma^{\frac{p+1}{2}}|^2 \\ \leq c_p \left(1 + \|\mu\|_V^2 + \|\beta(\varphi)_-\|_{L^6(\Omega)}^2 \right) \|\sigma\|_{L^{p+1}(\Omega)}^{p+1}. \end{aligned} \quad (3.38)$$

The above procedure works for every $p \in (0, 1]$, but provides an additional estimate only if

$$\underline{h} - \chi^2 \frac{p}{p+1} \geq 0. \quad (3.39)$$

Actually, in the paper [19] it was directly assumed that the above condition was verified for $p = 1$, which corresponds to the fact that the chemotactic response coefficient χ is “small” compared to the logistic source. Here, we may observe that, as \underline{h} is assigned, it is clear that (3.39) holds at least for $p > 0$ sufficiently small. On the other hand, (3.38) seems to provide some regularity gain only for $p \geq 1/2$, which corresponds exactly to condition (2.15).

In the sequel, we will directly assume the worst scenario we are able to deal with, i.e. the case when (3.39) holds exactly for $p = 1/2$ (or, in other words, equality holds in (2.15)), noting that

the argument can be simplified if one directly assumes that (3.39) holds for $p = 1$ as done in [19]. For $p = 1/2$, using (3.38) and the uniform Grönwall lemma (Lemma 2.13), thanks also to (3.27) and (3.32), for every $t \geq T_0 + 1$ (where T_0 was defined at the beginning of this subsection) one deduces that

$$\|\sigma(t)\|_{L^{3/2}(\Omega)}^{3/2} + \int_t^{t+1} |\nabla \sigma|^{3/4} \leq Q(R_0), \quad (3.40)$$

which, by Sobolev's embeddings and elementary interpolation, implies

$$\|\sigma\|_{L^2(t,t+1;L^3(\Omega))} \leq Q(R_0). \quad (3.41)$$

3.5 “Bilateral” estimate of the singular term

We proceed similarly with Subsec. 3.3, namely we test (1.2) by $|\beta(\varphi)|\beta(\varphi)$, now with no need for taking the negative part. Actually, the choice of a quadratic function is motivated by the L^3 -regularity in (3.41). Using the monotonicity of the function $r \mapsto |\beta(r)|\beta(r)$, we have

$$\|\beta(\varphi)\|_{L^3(\Omega)}^3 \leq \int_{\Omega} \mu |\beta(\varphi)|\beta(\varphi) + \int_{\Omega} \lambda \varphi |\beta(\varphi)|\beta(\varphi) + \chi \int_{\Omega} \sigma |\beta(\varphi)|\beta(\varphi). \quad (3.42)$$

Now, proceeding similarly with Subsec. 3.3, it is not difficult to deduce

$$\|\beta(\varphi)\|_{L^3(\Omega)} \leq c(1 + \|\mu\|_{L^3(\Omega)} + \|\sigma\|_{L^3(\Omega)}). \quad (3.43)$$

Then, squaring the above relation and performing a further comparison of terms in (1.2), we get

$$\|\beta(\varphi)\|_{L^3(\Omega)}^2 + \|\Delta \varphi\|_{L^3(\Omega)}^2 \leq c(1 + \|\mu\|_{L^3(\Omega)}^2 + \|\sigma\|_{L^3(\Omega)}^2). \quad (3.44)$$

Hence, integrating over $(t, t+1)$ and using (3.27), (3.41), and elliptic regularity results, we deduce

$$\|\beta(\varphi)\|_{L^2(t,t+1;L^3(\Omega))} + \|\varphi\|_{L^2(t,t+1;W^{2,3}(\Omega))} \leq Q(R_0), \quad (3.45)$$

for every $t \geq T_0 + 1$.

Remark 3.1. If (3.39) holds for $p = 1$, it is easy to see that, modifying the above argument, one can directly get L^6 - and $W^{2,6}$ -space summability in (3.44). We will arrive at the same conclusion also for $p = 1/2$, but in this case the proof is a bit more involved.

3.6 “Parabolic” regularity of σ

We now repeat (3.34) with $p = 1$ and integrate by parts the last term as in (3.35). With (3.44) at hand, then one simply observes that

$$-\frac{\chi}{2} \int_{\Omega} \sigma^2 \Delta \varphi \leq \|\sigma\| \|\sigma\|_{L^6(\Omega)} \|\Delta \varphi\|_{L^3(\Omega)} \leq \frac{1}{4} \|\sigma\|_V^2 + c \|\Delta \varphi\|_{L^3(\Omega)}^2 \|\sigma\|^2. \quad (3.46)$$

Hence, (3.34), for $p = 1$, gives

$$\frac{1}{2} \frac{d}{dt} \|\sigma\|^2 + h \|\sigma\|_{L^3(\Omega)}^3 + \frac{3}{4} \int_{\Omega} |\nabla \sigma|^2 \leq c(1 + \|\Delta \varphi\|_{L^3(\Omega)}^2) \|\sigma\|^2. \quad (3.47)$$

Then, applying once more the uniform Grönwall lemma, using also (3.44), we deduce

$$\|\sigma(t)\| + \|\sigma\|_{L^3(t,t+1;L^3(\Omega))} + \|\sigma\|_{L^2(t,t+1;V)} \leq Q(R_0), \quad (3.48)$$

say, for every $t \geq T_0 + 2$.

The above condition, in turn, permits us to further improve the regularity of φ as anticipated in Remark 3.1. To this aim, we proceed similarly with Subsec. 3.5, now with an improved exponent. Namely, we test (1.2) by $|\beta(\varphi)|^4 \beta(\varphi)$ so to obtain the following analogue of (3.44):

$$\|\beta(\varphi)\|_{L^6(\Omega)}^2 + \|\Delta \varphi\|_{L^6(\Omega)}^2 \leq c(1 + \|\mu\|_{L^6(\Omega)}^2 + \|\sigma\|_{L^6(\Omega)}^2). \quad (3.49)$$

Hence, integrating once more over $(t, t+1)$ and using now (3.27), (3.48) and elliptic regularity results, we arrive at

$$\|\beta(\varphi)\|_{L^2(t,t+1;L^6(\Omega))} + \|\varphi\|_{L^2(t,t+1;W^{2,6}(\Omega))} \leq Q(R_0), \quad (3.50)$$

at least for $t \geq T_0 + 2$.

3.7 Fractional parabolic regularity for σ

Using the operator A introduced in (2.13), equation (1.3) can be rewritten as

$$\sigma_t + A\sigma + \chi \operatorname{div}(\sigma \nabla \varphi) = -h(\sigma, \varphi)\sigma^2 + (k(\sigma, \varphi) + 1)\sigma. \quad (3.51)$$

Then, setting $H := -h(\sigma, \varphi)\sigma^2 + (k(\sigma, \varphi) + 1)\sigma$, using Assumption (A1), (3.48) and interpolation, it is easy to check that

$$\|H\|_{L^2(t, t+1; D(A^{-1/4}))} \leq c\|H\|_{L^2(t, t+1; L^{3/2}(\Omega))} \leq Q(R_0), \quad \text{for a.e. } t \geq T_0 + 2, \quad (3.52)$$

where we used the dual embedding of

$$D(A^{1/4}) = H^{1/2}(\Omega) \subset L^3(\Omega). \quad (3.53)$$

Condition (3.52) allows us to multiply (3.51) by $A^{1/2}\sigma$ so to deduce

$$\frac{1}{2} \frac{d}{dt} \|A^{1/4}\sigma\|^2 + \|A^{3/4}\sigma\|^2 = (H, A^{1/2}\sigma) - \chi \int_{\Omega} \nabla \sigma \cdot \nabla \varphi A^{1/2}\sigma - \chi \int_{\Omega} \sigma \Delta \varphi A^{1/2}\sigma, \quad (3.54)$$

and we need to control the terms on the right-hand side. We first notice that

$$(H, A^{1/2}\sigma) \leq \frac{1}{4} \|A^{3/4}\sigma\|^2 + c\|A^{-1/4}H\|^2. \quad (3.55)$$

Next,

$$\begin{aligned} -\chi \int_{\Omega} \nabla \sigma \cdot \nabla \varphi A^{1/2}\sigma &\leq c\|\nabla \sigma\| \|\nabla \varphi\|_{L^\infty(\Omega)} \|A^{1/2}\sigma\| \\ &\leq c\|\nabla \varphi\|_{L^\infty(\Omega)} \|A^{1/2}\sigma\|^2 \\ &\leq c\|\nabla \varphi\|_{L^\infty(\Omega)} \|A^{1/4}\sigma\| \|A^{3/4}\sigma\| \\ &\leq c\|\varphi\|_{W^{2,6}(\Omega)}^2 \|A^{1/4}\sigma\|^2 + \frac{1}{4} \|A^{3/4}\sigma\|^2. \end{aligned} \quad (3.56)$$

Finally,

$$\begin{aligned} -\chi \int_{\Omega} \sigma \Delta \varphi A^{1/2}\sigma &\leq c\|\sigma\|_{L^3(\Omega)} \|\Delta \varphi\|_{L^3(\Omega)} \|A^{1/2}\sigma\|_{L^3(\Omega)} \\ &\leq c\|A^{1/4}\sigma\| \|\Delta \varphi\|_{L^3(\Omega)} \|A^{3/4}\sigma\| \\ &\leq c\|A^{1/4}\sigma\|^2 \|\Delta \varphi\|_{L^3(\Omega)}^2 + \frac{1}{4} \|A^{3/4}\sigma\|^2. \end{aligned} \quad (3.57)$$

Collecting (3.55)-(3.57), (3.54) gives

$$\frac{1}{2} \frac{d}{dt} \|A^{1/4}\sigma\|^2 + \frac{1}{4} \|A^{3/4}\sigma\|^2 \leq c\|A^{-1/4}H\|^2 + c\|\varphi\|_{W^{2,6}(\Omega)}^2 \|A^{1/4}\sigma\|^2. \quad (3.58)$$

Hence, recalling (3.48), (3.50) and (3.52), a further application of the uniform Grönwall lemma yields

$$\|\sigma(t)\|_{D(A^{1/4})} + \|\sigma\|_{L^2(t, t+1; D(A^{3/4}))} \leq Q(R_0), \quad (3.59)$$

at least for every $t \geq T_0 + 3$.

Estimate (3.59) is the last property we need in order to conclude the existence and dissipativity parts of the proof of Theorem 2.4. Indeed, from (3.22) and (3.59) it is clear that property (2.26) holds, where R_1 (i.e., the radius of the absorbing set w.r.t. the product distance of $\Phi_m \times D(A^{1/4})$) is a computable function of the quantities occurring on the right-hand sides of (3.22) and (3.59); moreover, one can take, for instance, $T_1 := T_0 + 3$ in accordance with the validity range of (3.59).

Remark 3.2. As our focus is mostly on the long-time behavior of solutions, we decided to present the a-priori bounds in a way that emphasizes their dissipative character. In particular, we may check that, in fact, the values R_1 and T_1 depend only on the norm $\|\sigma_0 \ln(1 + \sigma_0)\|_{L^1(\Omega)}$ and not on the (stronger) $D(A^{1/4})$ -norm of σ_0 . On the other hand, in order to apply the theory of infinite-dimensional dynamical systems, we also need to prevent (possible) nonuniqueness phenomena occurring “earlier” than the time T_1 . For this reason, we also need to achieve the regularity corresponding (for instance) to (3.59) (but the same applies to all the intermediate estimates that have been obtained through the uniform Grönwall lemma) over the time interval $(0, T_1)$. Of course this can be easily done by adapting the previous arguments and using the standard (rather than the uniform) Grönwall lemma for small values of the time variable. In this way, one can complement (for instance) (3.59) by the corresponding relation

$$\|\sigma\|_{L^\infty(0, T_1; D(A^{1/4}))} + \|\sigma\|_{L^2(0, T_1; D(A^{3/4}))} \leq Q(R, T_1), \quad (3.60)$$

where, now, the computable function Q depends also on the $D(A^{1/4})$ -norm of σ_0 , and not only on $\|\sigma_0 \ln(1 + \sigma_0)\|_{L^1(\Omega)}$.

3.8 End of proof of existence and dissipativity

In this part we detail how the estimates obtained so far imply existence of solutions and dissipativity of the associated semigroup in the regularity setting of Theorem 2.4.

First of all, we check (2.17)-(2.20), and, as we have decided to avoid detailing an approximation scheme, we will just see how these regularity properties follow from the a-priori estimates. In particular, as observed in Remark 3.2, we will refer to the “dissipative” estimates detailed before, but of course, working on a generic interval $(0, T)$, in fact we also need their “nondissipative” counterparts holding for small values of time.

That said, the first and the last regularity conditions in (2.17) follow, respectively, from (3.28), (3.50), and their non-dissipative counterparts. To get the third condition, we test (1.2) by $-\Delta\varphi$ so to obtain

$$\|\Delta\varphi\|^2 \leq \|\nabla\varphi\| (\|\nabla\mu\| + \chi\|\nabla\sigma\| + \lambda\|\nabla\varphi\|), \quad (3.61)$$

whence, observing that $\|\nabla\varphi\|$ is controlled uniformly in time by (3.20), the desired control follows by squaring (3.61) and integrating in time. Finally, we note that (3.20) implies at least that $\varphi \in L^\infty(0, T; V)$. In addition to that, (3.50) entails the second of (2.18).

The conditions proved so far are sufficient in order to apply the analogue of [18, Lemma 4.1] (which in turn is based on the abstract result [18, Lemma 3.9]), which implies the continuity properties appearing in the second (2.17) and in the first (2.18) (in fact, the lemma implies the absolute continuity of the function $t \mapsto \mathcal{E}(\varphi, \sigma)$).

Next, (2.19) follows directly from (3.27). Finally, concerning (2.20), one can first of all take advantage of (3.59) (and of the corresponding nondissipative relation (3.60)). Then, to obtain the regularity of the time derivative, this follows simply by testing (3.51) by $A^{-1/2}\sigma_t$ and proceeding similarly with Subsec. 3.7 in order to control the coupling and logistic terms (we leave the details to the reader). Note that, as a byproduct of this procedure, we deduce also the continuity property in (2.20) (which corresponds to the continuity of the σ -component of the trajectory with respect to the topology of Σ). Indeed, we have

$$\frac{1}{2} \frac{d}{dt} \|A^{1/4}\sigma\|^2 = (A^{1/4}\sigma_t, A^{1/4}\sigma) = (A^{-1/4}\sigma_t, A^{3/4}\sigma), \quad (3.62)$$

and we may observe that the right-hand side is summable as a consequence of the regularity properties proved so far.

As all the regularity properties in the statement have been achieved, it is easy to check that, at least formally, these suffice in order to interpret the system in the form (1.2) plus (2.21)-(2.24) (in particular, the initial data are recovered as both φ and σ are continuous in time with respect to suitable topologies). Finally, concerning dissipativity, for what concerns φ this is a direct consequence of (3.33), whereas for σ we can use (3.59). In particular, one may take $T_1 = T_0 + 3$ and choose $R_1 = Q(R_0)$ for some explicitly computable expression of Q also depending on the coercivity constant

$\kappa > 0$ in (3.6) and on the expression of the function Q appearing in (3.59). This completes the proof of Theorem 2.4 for what concerns existence and dissipativity. Of course, as already noted, this is a simplified proof in the sense that we worked directly on the “original” system; if an approximation scheme is used, then the procedure should be adapted in order to show that the estimates are sufficient to pass to the limit in the approximation. This seems, however, a very standard procedure, also in view of the fact that we are considering here a rather high regularity setting. It is worth observing that recovering existence of solutions by passing to the limit in an approximation might be a more delicate task in a lower regularity setting, see, e.g., [15].

4 Uniqueness

We prove here the uniqueness part of Theorem 2.4. The argument partially follows the lines of [19, Proof of Theorem 2.8]; however, as the current setting is different in several aspects (lack of a source term in (1.1), different hypotheses on the logistic terms in (1.3), fractional regularity setting for σ), we prefer to report the proof in full detail. That said, we consider a couple of solutions $(\varphi_1, \mu_1, \sigma_1)$ and $(\varphi_2, \mu_2, \sigma_2)$, both emanating from the same initial datum (φ_0, σ_0) satisfying (2.16), and we look for a contractive estimate. To obtain it, we first set $(\varphi, \mu, \sigma) := (\varphi_1, \mu_1, \sigma_1) - (\varphi_2, \mu_2, \sigma_2)$, and, taking the difference of system (1.1)-(1.3), we infer

$$\varphi_t - \Delta\mu = 0, \quad (4.1)$$

$$\mu = -\Delta\varphi + \beta(\varphi_1) - \beta(\varphi_2) - \lambda\varphi - \chi\sigma, \quad (4.2)$$

$$\sigma_t - \Delta\sigma + \chi \operatorname{div}(\sigma_1 \nabla \varphi) + \chi \operatorname{div}(\sigma \nabla \varphi_2) = (-h_1\sigma_1^2 + h_2\sigma_2^2 + k_1\sigma_1 - k_2\sigma_2), \quad (4.3)$$

where we have set $h_1 := h(\sigma_1, \varphi_1)$, $h_2 := h(\sigma_2, \varphi_2)$, and so on. We will work on a generic, but otherwise fixed, time interval $(0, T)$ and assume that both solutions turn out to satisfy the regularity conditions (2.17)-(2.20) over $(0, T)$.

Then, observing that $\varphi_\Omega = 0$ and recalling that \mathcal{N} denotes the inverse Neumann Laplacian over the functions with zero spatial mean, we may test (4.1) by $\mathcal{N}\varphi$ and (4.2) by φ . Taking the difference of the resulting relations, and using (4.4) together with the monotonicity of β , we deduce

$$\frac{1}{2} \frac{d}{dt} \|\varphi\|_*^2 + \|\nabla\varphi\|^2 \leq \lambda\|\varphi\|^2 + \chi \int_{\Omega} \sigma\varphi = \lambda\|\varphi\|^2 + \chi \int_{\Omega} (\sigma - \sigma_\Omega)\varphi, \quad (4.4)$$

the last equality holding as φ has zero spatial average. Then, adding $\|\varphi\|^2$ to both hand sides so to recover the full V -norm of φ on the left-hand side, we may observe that the right-hand side can then be managed this way:

$$(\lambda + 1)\|\varphi\|^2 + \chi \int_{\Omega} (\sigma - \sigma_\Omega)\varphi \leq \frac{1}{2} \|\varphi\|_V^2 + c\|\varphi\|_*^2 + c\|\sigma - \sigma_\Omega\|_*^2, \quad (4.5)$$

where we have also used Ehrling’s inequality. Hence, (4.4) gives

$$\frac{d}{dt} \|\varphi\|_*^2 + \|\varphi\|_V^2 \leq c\|\varphi\|_*^2 + c\|\sigma - \sigma_\Omega\|_*^2. \quad (4.6)$$

Dealing with equation (4.3) is a bit trickier as the spatial mean of σ is not conserved. To this aim, we first estimate the difference of the mean values by integrating (4.3) over Ω so to obtain

$$\frac{d}{dt} \sigma_\Omega = \frac{1}{|\Omega|} \int_{\Omega} (-h_1\sigma_1^2 + h_2\sigma_2^2 + k_1\sigma_1 - k_2\sigma_2). \quad (4.7)$$

Now, thanks to the global Lipschitz continuity of h (cf. (2.6)), it is not difficult to see that

$$\begin{aligned} \left| \int_{\Omega} (-h_1\sigma_1^2 + h_2\sigma_2^2) \right| &= \left| - \int_{\Omega} (h_1 - h_2)\sigma_1^2 - \int_{\Omega} h_2(\sigma_1 + \sigma_2)\sigma_1 \right| \\ &\leq c\|\sigma_1\|_{L^3(\Omega)}\|\sigma_1\|_{L^6(\Omega)}(\|\sigma\| + \|\varphi\|) + c(\|\sigma_1\| + \|\sigma_2\|)\|\sigma\|, \\ &\leq C(\|\sigma_1\|_V + 1)(\|\sigma\| + \|\varphi\|). \end{aligned} \quad (4.8)$$

Here and below C is a “large” constant depending on the value of some “known” norms of the two solutions. Here, for instance, C may depend on the $L^\infty(0, T; H)$ -norm of σ_2 and on the $L^\infty(0, T; L^3(\Omega))$ -norm of σ_1 ; the latter is controlled due to (2.20) and the continuous embedding $D(A^{1/4}) \subset L^3(\Omega)$.

The estimation of the term depending on k_1 and k_2 is analogous and in fact simpler. Mimicking the above procedure we actually obtain

$$\left| \int_{\Omega} (k_1 \sigma_1 - k_2 \sigma_2) \right| \leq c(1 + \|\sigma_1\|)(\|\sigma\| + \|\varphi\|) \leq C(\|\sigma\| + \|\varphi\|). \quad (4.9)$$

Hence, multiplying (4.7) by $2\sigma_\Omega$, and using (4.8)-(4.9), we infer

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |\sigma_\Omega|^2 &\leq C|\sigma_\Omega|(1 + \|\sigma_1\|_V)(\|\sigma\| + \|\varphi\|) \\ &\leq C_\epsilon(1 + \|\sigma_1\|_V^2)|\sigma_\Omega|^2 + \epsilon\|\varphi\|^2 + \epsilon\|\sigma - \sigma_\Omega\|^2, \end{aligned} \quad (4.10)$$

for “small” $\epsilon > 0$ to be chosen later on and correspondingly “large” $C_\epsilon > 0$.

Next, we subtract (4.7) from (4.3) so to obtain

$$\begin{aligned} (\sigma - \sigma_\Omega)_t - \Delta\sigma + \chi \operatorname{div}(\sigma_1 \nabla \varphi) + \chi \operatorname{div}(\sigma \nabla \varphi_2) \\ = (-h_1 \sigma_1^2 + h_2 \sigma_2^2 + k_1 \sigma_1 - k_2 \sigma_2) - (-h_1 \sigma_1^2 + h_2 \sigma_2^2 + k_1 \sigma_1 - k_2 \sigma_2)_\Omega \\ =: H - H_\Omega. \end{aligned} \quad (4.11)$$

Testing (4.11) by $\mathcal{N}(\sigma - \sigma_\Omega)$ we deduce

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\sigma - \sigma_\Omega\|_*^2 + \|\sigma - \sigma_\Omega\|^2 &= \chi \int_{\Omega} \sigma_1 \nabla \varphi \cdot \nabla \mathcal{N}(\sigma - \sigma_\Omega) \\ &\quad + \chi \int_{\Omega} \sigma \nabla \varphi_2 \cdot \nabla \mathcal{N}(\sigma - \sigma_\Omega) + \int_{\Omega} (H - H_\Omega) \mathcal{N}(\sigma - \sigma_\Omega). \end{aligned} \quad (4.12)$$

In order to control the terms on the right-hand side, we first observe that

$$\begin{aligned} \chi \int_{\Omega} \sigma_1 \nabla \varphi \cdot \nabla \mathcal{N}(\sigma - \sigma_\Omega) &\leq c \|\sigma_1\|_{L^3(\Omega)} \|\nabla \varphi\| \|\nabla \mathcal{N}(\sigma - \sigma_\Omega)\|_{L^6(\Omega)} \\ &\leq \frac{1}{6} \|\sigma - \sigma_\Omega\|^2 + \mathbf{C} \|\nabla \varphi\|^2, \end{aligned} \quad (4.13)$$

where the constant \mathbf{C} depends on the $L^\infty(0, T; L^3(\Omega))$ -norm of σ_1 , which is a known quantity in view of (2.20) and of the continuous embedding $D(A^{1/4}) \subset L^3(\Omega)$.

Similarly, using also (2.5), we have

$$\begin{aligned} \chi \int_{\Omega} \sigma \nabla \varphi_2 \cdot \nabla \mathcal{N}(\sigma - \sigma_\Omega) &\leq c \|\sigma\| \|\nabla \varphi_2\|_{L^\infty(\Omega)} \|\nabla \mathcal{N}(\sigma - \sigma_\Omega)\| \\ &\leq \frac{1}{12} \|\sigma\|^2 + c \|\varphi_2\|_{W^{2,6}(\Omega)}^2 \|\sigma - \sigma_\Omega\|_*^2 \\ &\leq \frac{1}{6} \|\sigma - \sigma_\Omega\|^2 + c |\sigma_\Omega|^2 + c \|\varphi_2\|_{W^{2,6}(\Omega)}^2 \|\sigma - \sigma_\Omega\|_*^2. \end{aligned} \quad (4.14)$$

It remains to control the last term on the right-hand side of (4.12). To this aim, we first point out that

$$\begin{aligned} \int_{\Omega} (H - H_\Omega) \mathcal{N}(\sigma - \sigma_\Omega) &\leq \|H - H_\Omega\|_{L^{6/5}(\Omega)} \|\mathcal{N}(\sigma - \sigma_\Omega)\|_{L^6(\Omega)} \\ &\leq c \|H\|_{L^{6/5}(\Omega)} \|\mathcal{N}(\sigma - \sigma_\Omega)\|_V \leq c \|H\|_{L^{6/5}(\Omega)} \|\sigma - \sigma_\Omega\|_*. \end{aligned} \quad (4.15)$$

To proceed, we observe that, by interpolation, one has

$$\|\sigma_1\|_V \leq c \|\sigma_1\|_{D(A^{1/4})}^{1/2} \|\sigma_1\|_{D(A^{3/4})}^{1/2}. \quad (4.16)$$

Hence, using (2.20), it is easy to get

$$\|\sigma_1\|_{L^4(0,T;V)} \leq C. \quad (4.17)$$

To control the first factor on the right-hand side of (4.15), we mimick the procedure in (4.8)-(4.9). Namely, (4.8) is modified as follows:

$$\begin{aligned} \| -h_1\sigma_1^2 + h_2\sigma_2^2 \|_{L^{6/5}(\Omega)} &\leq \|(h_1 - h_2)\sigma_1^2\|_{L^{6/5}(\Omega)} + \|h_2(\sigma_1 + \sigma_2)\sigma\|_{L^{6/5}(\Omega)} \\ &\leq c\|h_1 - h_2\|\|\sigma_1\|_{L^6(\Omega)}^2 + c(\|\sigma_1\|_{L^3(\Omega)} + \|\sigma_2\|_{L^3(\Omega)})\|\sigma\| \\ &\leq c(\|\sigma\| + \|\varphi\|)\|\sigma_1\|_{L^6(\Omega)}^2 + c(\|\sigma_1\|_{L^3(\Omega)} + \|\sigma_2\|_{L^3(\Omega)})\|\sigma\| \\ &\leq C(1 + \|\sigma_1\|_V^2)(\|\sigma\| + \|\varphi\|). \end{aligned} \quad (4.18)$$

Next, similarly with (4.9), we have

$$\|k_1\sigma_1 - k_2\sigma_2\|_{L^{6/5}(\Omega)} \leq C(\|\sigma\| + \|\varphi\|). \quad (4.19)$$

Actually, we point out that the constants C on the right-hand sides of (4.18) and (4.19) also depend on the $L^\infty(0, T; L^3(\Omega))$ -norms of σ_1 and σ_2 , which, again, are known quantities. Hence, (4.15) yields

$$\begin{aligned} \int_\Omega (H - H_\Omega) \mathcal{N}(\sigma - \sigma_\Omega) &\leq C(1 + \|\sigma_1\|_V^2)(\|\sigma - \sigma_\Omega\| + |\sigma_\Omega| + \|\varphi\|)\|\sigma - \sigma_\Omega\|_* \\ &\leq C(1 + \|\sigma_1\|_V^4)\|\sigma - \sigma_\Omega\|_*^2 + \frac{1}{6}\|\sigma - \sigma_\Omega\|^2 + c|\sigma_\Omega|^2 + c\|\varphi\|^2. \end{aligned} \quad (4.20)$$

Now, taking (4.13)-(4.20) into account, we deduce from (4.12) that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\sigma - \sigma_\Omega\|_*^2 + \frac{1}{2} \|\sigma - \sigma_\Omega\|^2 &\leq \mathbf{C} \|\varphi\|_V^2 + c|\sigma_\Omega|^2 \\ &\quad + C(1 + \|\sigma_1\|_V^4 + \|\varphi_2\|_{W^{2,6}(\Omega)}^2)\|\sigma - \sigma_\Omega\|_*^2, \end{aligned} \quad (4.21)$$

possibly for a new value of the constant \mathbf{C} (which, we recall, depends on the $L^\infty(0, T; L^3(\Omega))$ -norm of σ_1 and on the fixed parameters of the problem).

Let us now multiply (4.6) by $2\mathbf{C}$ and add the result to (4.21) so to get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\sigma - \sigma_\Omega\|_*^2 + 2\mathbf{C} \frac{d}{dt} \|\varphi\|_*^2 + \frac{1}{2} \|\sigma - \sigma_\Omega\|^2 + \mathbf{C} \|\varphi\|_V^2 \\ \leq c\mathbf{C} \|\varphi\|_*^2 + c|\sigma_\Omega|^2 + C(1 + \|\sigma_1\|_V^4 + \|\varphi_2\|_{W^{2,6}(\Omega)}^2 + \mathbf{C})\|\sigma - \sigma_\Omega\|_*^2. \end{aligned} \quad (4.22)$$

Then, we go back to (4.10) and choose $\epsilon = \min\{1/4, \mathbf{C}/2\}$ therein; we then sum it to (4.22) so to obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\sigma - \sigma_\Omega\|_*^2 + \frac{1}{2} \frac{d}{dt} |\sigma_\Omega|^2 + 2\mathbf{C} \frac{d}{dt} \|\varphi\|_*^2 + \frac{1}{4} \|\sigma - \sigma_\Omega\|^2 + \frac{\mathbf{C}}{2} \|\varphi\|_V^2 \\ \leq c\mathbf{C} \|\varphi\|_*^2 + C(1 + \|\sigma_1\|_V^2)|\sigma_\Omega|^2 + C(1 + \|\sigma_1\|_V^4 + \|\varphi_2\|_{W^{2,6}(\Omega)}^2 + \mathbf{C})\|\sigma - \sigma_\Omega\|_*^2. \end{aligned} \quad (4.23)$$

Then, using (4.17) (for σ_1) and the last of (2.17) (for φ_2) and applying Grönwall's lemma, we readily get the assert. The proof of Theorem 2.4 is complete.

5 Asymptotic compactness and construction of the attractor

In this part we detail the proof of Theorem 2.8. This will be achieved by showing additional regularity estimates holding uniformly for sufficiently large values of the time variable.

To this purpose, we start recalling that, for any set B of initial data bounded in $\Phi_m \times \Sigma$, any solution emanating from B takes values in the bounded absorbing set \mathcal{B}_0 for any $t \geq T_1$, where T_1 only depends on the “radius” of B in $\Phi_m \times \Sigma$. With a simple argument based on the uniform Grönwall lemma we will now see that any such solution takes values in a bounded set \mathcal{B}_1 of $\Psi_m \times V_+$ for every $t \geq T_1 + 2$. Using the fact that the immersion of $\Psi_m \times V_+$ into $\Phi_m \times \Sigma$ is compact (which

is clear for what concerns σ and follows from the result [19, Prop. 3.15] reported here as Prop. 2.7 for what concerns φ), the general theory of infinite-dimensional dynamical systems (applied to the closed semigroup $S(t)$, cf. [17]) then guarantees the existence of the global attractor in the sense of Theorem 2.8.

To deduce the additional regularity estimates, we first test (1.3) by $-\Delta\sigma$ so to obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla\sigma\|^2 + \|\Delta\sigma\|^2 &= \chi \int_{\Omega} \nabla\sigma \cdot \nabla\varphi \Delta\sigma \\ &\quad + \chi \int_{\Omega} \sigma \Delta\varphi \Delta\sigma + \int_{\Omega} (h(\sigma, \varphi)\sigma^2 - k(\sigma, \varphi)\sigma) \Delta\sigma. \end{aligned} \quad (5.1)$$

Then, the terms on the right-hand side are controlled as follows:

$$\chi \int_{\Omega} \nabla\sigma \cdot \nabla\varphi \Delta\sigma \leq c \|\nabla\sigma\| \|\nabla\varphi\|_{L^\infty(\Omega)} \|\Delta\sigma\| \leq \frac{1}{4} \|\Delta\sigma\|^2 + c \|\nabla\sigma\|^2 \|\varphi\|_{W^{2,6}(\Omega)}^2, \quad (5.2)$$

$$\chi \int_{\Omega} \sigma \Delta\varphi \Delta\sigma \leq c \|\sigma\|_{L^6(\Omega)} \|\Delta\varphi\|_{L^3(\Omega)} \|\Delta\sigma\| \leq \frac{1}{4} \|\Delta\sigma\|^2 + c \|\sigma\|_V^2 \|\varphi\|_{W^{2,3}(\Omega)}^2, \quad (5.3)$$

$$\int_{\Omega} (h(\varphi, \sigma)\sigma^2 - k(\varphi, \sigma)\sigma) \Delta\sigma \leq \frac{1}{4} \|\Delta\sigma\|^2 + c(1 + \|\sigma\|_{L^4(\Omega)}^4) \leq \frac{1}{4} \|\Delta\sigma\|^2 + c(1 + \|\sigma\|_V^4). \quad (5.4)$$

Collecting the above computations, (5.1) gives

$$\frac{1}{2} \frac{d}{dt} \|\nabla\sigma\|^2 + \frac{1}{4} \|\Delta\sigma\|^2 \leq c(1 + \|\sigma\|_V^2 + \|\varphi\|_{W^{2,6}(\Omega)}^2) \|\sigma\|_V^2. \quad (5.5)$$

Hence, recalling (3.50) and (3.59), and using the uniform Grönwall lemma, we deduce

$$\|\sigma(t)\|_V + \|\sigma\|_{L^2(t,t+1;W)} \leq Q(R_0), \quad (5.6)$$

say, for every $t \geq T_1 + 1$. At this point, a comparison of terms in (1.3) (equivalently, one could also use σ_t as a test function therein) also gives

$$\|\sigma_t\|_{L^2(t,t+1;H)} \leq Q(R_0), \quad (5.7)$$

still for $t \geq T_1 + 1$.

With (5.6)-(5.7) at disposal, we can go back to the Cahn-Hilliard system (1.1)-(1.2) and perform the so-called second-energy estimate. Namely, we test (1.1) by μ_t and the time derivative of (1.2) by φ_t (also this argument can be easily justified in an approximation scheme). Using the monotonicity of β , it is then easy to arrive at

$$\frac{1}{2} \frac{d}{dt} \|\nabla\mu\|^2 + \|\nabla\varphi_t\|^2 \leq \lambda \|\varphi_t\|^2 + \chi \int_{\Omega} \sigma_t \varphi_t. \quad (5.8)$$

Using Ehrling's lemma, the terms on the right-hand side are controlled as follows:

$$\lambda \|\varphi_t\|^2 + \chi \int_{\Omega} \sigma_t \varphi_t \leq \frac{1}{2} \|\nabla\varphi_t\|^2 + c \|\varphi_t\|_*^2 + c \|\sigma_t\|^2, \quad (5.9)$$

where we also used the fact that φ_t has zero spatial average due to conservation of mass. Replacing (5.9) into (5.8), we obtain

$$\frac{d}{dt} \|\nabla\mu\|^2 + \|\nabla\varphi_t\|^2 \leq c \|\varphi_t\|_*^2 + c \|\sigma_t\|^2. \quad (5.10)$$

Hence, recalling (3.27)-(3.28) and (5.7), we may apply once more the uniform Grönwall lemma so to deduce

$$\|\nabla\mu(t)\|^2 + \int_t^{t+1} \|\nabla\varphi_t\|^2 \leq Q(R_0), \quad (5.11)$$

for every $t \geq T_1 + 2$. More precisely, using (3.25) and (3.28), the above relation is improved to

$$\|\mu(t)\|_V^2 + \int_t^{t+1} \|\varphi_t\|_V^2 \leq Q(R_0), \quad (5.12)$$

still for $t \geq T_1 + 2$. Now, going back to inequality (3.49) and observing that its right-hand side is now controlled uniformly in time thanks to (5.6) and (5.12), we readily deduce

$$\|\beta(\varphi(t))\|_{L^6(\Omega)} + \|\varphi(t)\|_{W^{2,6}(\Omega)} \leq Q(R_0), \quad (5.13)$$

for $t \geq T_1 + 2$. Then, we observe that, for what concerns σ , the regularity prescribed in (2.32) corresponds exactly to the outcome of estimate (5.6), whereas, regarding φ , (5.13) yields directly (2.33) (which is of course stronger than (2.32)). The proof of Theorem 2.8 is then complete.

Remark 5.1. As already observed in Remark 2.1, the first condition in (5.13) should be intended to hold for the section η of the multi-function $\beta(\varphi)$ satisfying (2.10). As the definitions (2.29) of the space Ψ_m and (2.31) of the related metric use the “minimal section” β^0 , it may be worth observing that, as β^0 is, indeed, minimal, whenever the first (5.13) holds for some section η , it consequently holds also for $\beta^0(\varphi)$.

5.1 Further regularity of the attractor

We prove here the additional asymptotic regularity of σ stated in Theorem 2.9. To this aim, rewriting equation (1.3) in the form

$$\sigma_t + A\sigma = -\chi\nabla\sigma \cdot \nabla\varphi - \chi\sigma\Delta\varphi - h(\sigma, \varphi)\sigma^2 + (k(\sigma, \varphi) + 1)\sigma =: R(\sigma, \varphi), \quad (5.14)$$

using properties (5.6) and (5.13) with Sobolev’s embeddings, it is not difficult to infer

$$\|R(\sigma, \varphi)\|_{L^\infty(t, t+1; H)} \leq Q(R_0), \quad (5.15)$$

at least for $t \geq T_1 + 2$. In addition to that, we may observe that, as a further consequence of (5.6), for every $t \geq T_1 + 2$ there exists $\bar{t} \in [t, t+1]$ such that

$$\|\sigma(\bar{t})\|_{H^2(\Omega)} \leq Q(R_0). \quad (5.16)$$

Hence, applying over the time interval $[\bar{t}, \bar{t} + 2]$ parabolic regularity estimates of Agmon-Douglis-Nirenberg type to problem (5.14) with the “initial” condition $\sigma(\bar{t})$, we readily deduce

$$\|\sigma_t\|_{L^p(\bar{t}, \bar{t}+2; H)} + \|A\sigma\|_{L^p(\bar{t}, \bar{t}+2; H)} \leq Q(R_0), \quad (5.17)$$

which of course implies

$$\|\sigma_t\|_{L^p(t, t+1; H)} + \|A\sigma\|_{L^p(t, t+1; H)} \leq Q(R_0), \quad (5.18)$$

at least for $t \geq T_1 + 3$. Note that (5.17) and (5.18) hold for $p \in [1, \infty)$ with the expression of Q depending on p and possibly “exploding” as $p \nearrow \infty$.

Then, choosing p large enough (but finite), applying standard elliptic regularity and interpolation results, we also deduce

$$\|\sigma\|_{L^\infty(t, t+1; L^\infty(\Omega))} \leq Q(R_0). \quad (5.19)$$

The above uniform boundedness property is the key step for obtaining additional estimates. Actually, we can now take the time derivative of (1.3) and test it by σ_t so to deduce

$$\frac{1}{2} \frac{d}{dt} \|\sigma_t\|^2 + \|\nabla\sigma_t\|^2 = \chi \int_\Omega \sigma_t \nabla\varphi \cdot \nabla\sigma_t + \chi \int_\Omega \sigma \nabla\varphi_t \cdot \nabla\sigma_t + \int_\Omega (-h(\sigma, \varphi)\sigma^2 + k(\sigma, \varphi)\sigma)_t \sigma_t. \quad (5.20)$$

The terms on the right-hand side can be controlled as follows:

$$\chi \int_\Omega \sigma_t \nabla\varphi \cdot \nabla\sigma_t \leq \|\sigma_t\| \|\nabla\varphi\|_{L^\infty(\Omega)} \|\nabla\sigma_t\| \leq \frac{1}{4} \|\nabla\sigma_t\|^2 + Q(R_0) \|\sigma_t\|^2, \quad (5.21)$$

$$\chi \int_\Omega \sigma \nabla\varphi_t \cdot \nabla\sigma_t \leq \|\sigma\|_{L^\infty(\Omega)} \|\nabla\varphi_t\| \|\nabla\sigma_t\| \leq \frac{1}{4} \|\nabla\sigma_t\|^2 + Q(R_0) \|\nabla\varphi_t\|^2, \quad (5.22)$$

$$\int_\Omega (-h(\sigma, \varphi)\sigma^2 + k(\sigma, \varphi)\sigma)_t \sigma_t \leq Q(R_0) (\|\sigma_t\|^2 + \|\varphi_t\|^2), \quad (5.23)$$

where we have also used (5.13) with the continuous embedding $W^{2,6}(\Omega) \subset W^{1,\infty}(\Omega)$, (5.19), and Assumption (A1).

Using (5.21)-(5.23), (5.20) gives

$$\frac{d}{dt} \|\sigma_t\|^2 + \|\nabla \sigma_t\|^2 \leq Q(R_0) (\|\sigma_t\|^2 + \|\varphi_t\|_V^2), \quad (5.24)$$

whence, using (5.7), (5.12) and the uniform Grönwall lemma, we deduce

$$\|\sigma_t(t)\| + \|\sigma_t\|_{L^2(t,t+1;V)} \leq Q(R_0), \quad (5.25)$$

for every $t \geq T_1 + 4$. Finally, interpreting (1.3) as a time-dependent family of elliptic problems, namely

$$A\sigma = -\sigma_t - \chi \nabla \sigma \cdot \nabla \varphi - \chi \sigma \Delta \varphi - h(\sigma, \varphi) \sigma^2 + (k(\sigma, \varphi) + 1)\sigma, \quad (5.26)$$

we may observe that, as a consequence of (5.13) and (5.25), the right-hand side is bounded in H uniformly for $t \geq T_1 + 4$. Hence, applying once more standard elliptic regularity results, we deduce

$$\|\sigma(t)\|_{H^2(\Omega)} \leq Q(R_0), \quad (5.27)$$

for every $t \geq T_1 + 4$. As this corresponds exactly to (2.34), the proof of Theorem 2.9 is complete.

6 Proof of Theorem 2.11

In this section we analyze in more detail the sign properties of σ , so to prove Theorem 2.11. To this aim, partially following [15], we first provide a sort of “entropic” reformulation of (1.3), which is obtained by multiplying (1.3) by σ^{-1} and performing some integrations by parts. We point out that also this argument is formal; indeed, at this level we do not know whether σ^{-1} has any summability property and in principle we could not even exclude that there may exist regions of positive measure where σ is identically 0 (we will know that this in fact cannot happen as an outcome of the procedure). A simple way to make our procedure rigorous could be that of multiplying (1.3) by $T_n(\sigma^{-1})$, where T_n is the truncation operator at height n , and working on the relation obtained that way. Then, as the resulting estimates will turn out to be independent of n , at the end one may let $n \rightarrow \infty$. For the sake of brevity we omit the details of this argument and work directly on relation (1.3), whose “entropic” reformulation then reads

$$v_t - \Delta v - |\nabla v|^2 + h(\sigma, \varphi) \sigma - k(\sigma, \varphi) = -\chi \nabla v \cdot \nabla \varphi - \chi \Delta \varphi, \quad (6.1)$$

and where we have set $v := \ln \sigma$. Let us now consider a generic monotone function $\gamma : \mathbb{R} \rightarrow \mathbb{R}$, let $\widehat{\gamma}$ be some primitive of γ , and let us test (6.1) by $\gamma(v)$ so to deduce the general formula

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \widehat{\gamma}(v) + \int_{\Omega} (\gamma'(v) - \gamma(v)) |\nabla v|^2 - \int_{\Omega} k(\sigma, \varphi) \gamma(v) + \int_{\Omega} h(\sigma, \varphi) \sigma \gamma(v) \\ = -\chi \int_{\Omega} \gamma(v) \nabla v \cdot \nabla \varphi - \chi \int_{\Omega} \gamma(v) \Delta \varphi. \end{aligned} \quad (6.2)$$

Now, as we need to estimate the values of σ that are close to 0, we start with taking $\gamma(v) = -1$ for $v < 0$ and $\gamma(v) = 0$ for $v \geq 0$; namely, $\gamma(v) = -(\text{sign}(v))_- = \min\{\text{sign}(v), 0\}$ in such a way that $\widehat{\gamma}(v) = v_-$. Of course γ is a monotone function, but it is discontinuous at 0, hence the procedure is formal; to make it rigorous one could replace γ by a function γ_δ taking the value $\delta^{-1}r$ in the interval $[-\delta, 0]$ and coinciding with γ elsewhere, and then let $\delta \searrow 0$. We omit the details since also this argument is standard. Then, recalling that, if $v \in V$, then $\nabla v_- \in H$ with

$$\nabla v_- = \begin{cases} -\nabla v & \text{if } v \leq 0 \\ 0 & \text{otherwise} \end{cases} \quad (6.3)$$

almost everywhere in Ω (thanks to [23, Lemme 1.1 & Lemme 1.2] or [11, Lemma 7.6]) and neglecting some nonnegative terms from the left-hand side, (6.2) reduces to

$$\frac{d}{dt} \|v_-\|_{L^1(\Omega)} + \|\nabla v_-\|^2 \leq \int_{\Omega} h(\sigma, \varphi) \sigma + \chi \int_{\Omega} |\nabla v_-| |\nabla \varphi| + \chi \int_{\Omega} |\Delta \varphi|. \quad (6.4)$$

Now, integrating the above over $(0, T)$ and using (2.17) and (2.20), it is a standard matter to control the terms on the right-hand side. Hence, using also condition (2.35) on the initial datum, we deduce

$$\|v_-\|_{L^\infty(0,T;L^1(\Omega))} + \|v_-\|_{L^2(0,T;V)} \leq Q(T), \quad (6.5)$$

where the right-hand side depends on the reference time T as the above estimate does not have a dissipative character (we refer to Remark 2.12 for additional comments).

Notice also that (6.5), combined with Sobolev's embeddings, implies in particular that, for any “small” $\tau \in (0, T)$, there exists some $t_\tau \in (\tau/2, \tau)$ such that

$$\|v_-(t_\tau)\|_{L^6(\Omega)} \leq c\|v_-(t_\tau)\|_V \leq Q(T, \tau^{-1}). \quad (6.6)$$

Interpreting the above quantity as an “initial” datum for equation (6.1), we will now derive additional estimates holding for strictly positive times. To this aim, we repeat (6.2), taking now $\gamma(v) = -(v_-)^{p-1}$ for a generic exponent $p \geq 2$. Then, we readily obtain

$$\begin{aligned} & \frac{1}{p} \frac{d}{dt} \|v_-\|_{L^p(\Omega)}^p + \int_{\Omega} (v_-^{p-1} + (p-1)v_-^{p-2}) |\nabla v_-|^2 + \int_{\Omega} k(\sigma, \varphi) v_-^{p-1} - \int_{\Omega} h(\sigma, \varphi) \sigma v_-^{p-1} \\ & \leq \chi \int_{\Omega} v_-^{p-1} \nabla v_- \cdot \nabla \varphi + \chi \int_{\Omega} v_-^{p-1} \Delta \varphi = \chi \int_{\Omega} v_-^{p-1} \nabla v_- \cdot \nabla \varphi + \chi \int_{\Omega} v_-^{p-1} \Delta \varphi. \end{aligned} \quad (6.7)$$

In order to deal with the above relation, we go back to estimate (5.13) and make the following general consideration: modifying a bit the procedure given in the previous section and, in particular, applying Grönwall lemma on intervals of length τ (or, more precisely, τ/n for sufficiently large $n \in \mathbb{N}$, where n is the number of iterations), rather than on intervals of length 1, one could restate (5.13) as

$$\|\beta(\varphi(t))\|_{L^6(\Omega)} + \|\varphi(t)\|_{W^{2,6}(\Omega)} \leq Q(\tau^{-1}), \quad \text{for every } t \in [\tau, T] \text{ and every } \tau > 0, \quad (6.8)$$

where now the expression of Q may also depend on the specific choice of the initial datum as we are not looking for a dissipative estimate at this level. Hence, using (6.8) with Young's inequality, there follows

$$\chi \int_{\Omega} v_-^{p-1} \nabla v_- \cdot \nabla \varphi \leq \frac{1}{2} \int_{\Omega} v_-^{p-1} |\nabla v_-|^2 + Q(\tau^{-1}) \|v_-\|_{L^{p-1}(\Omega)}^{p-1}. \quad (6.9)$$

Let us now consider the last two terms on the left-hand side. To this aim, let us observe that, applying the above considerations to (5.27) and using Sobolev's embeddings, there follows

$$\|\sigma\|_{L^\infty(\tau, T; L^\infty(\Omega))} \leq c\|\sigma\|_{L^\infty(\tau, T; H^2(\Omega))} \leq Q(\tau^{-1}). \quad (6.10)$$

Using (6.10) and Assumption (A1), it is then clear that

$$\left| \int_{\Omega} (k(\sigma, \varphi) - h(\sigma, \varphi) \sigma) v_-^{p-1} \right| \leq Q(\tau^{-1}) \|v_-\|_{L^{p-1}(\Omega)}^{p-1}. \quad (6.11)$$

Next, we observe that the second term on the left-hand side of (6.7) (where we reduced the value of one constant as we have incorporated there the contribution of the first term on the right-hand side of (6.9)) can be equivalently rewritten as

$$\int_{\Omega} \left(\frac{1}{2} v_-^{p-1} + (p-1)v_-^{p-2} \right) |\nabla v_-|^2 = \frac{2}{(p+1)^2} \|\nabla v_-^{\frac{p+1}{2}}\|^2 + \frac{4(p-1)}{p^2} \|\nabla v_-^{\frac{p}{2}}\|^2. \quad (6.12)$$

Then, we control the last term in (6.7) as follows:

$$\begin{aligned} \chi \int_{\Omega} v_-^{p-1} \Delta \varphi & \leq c \|v_-^{\frac{p-2}{2}}\| \|v_-^{\frac{p}{2}}\|_V \|\Delta \varphi\|_{L^3(\Omega)} \\ & \leq Q(\tau^{-1}) \|v_-^{\frac{p-2}{2}}\| \|v_-^{\frac{p}{2}}\|_V \\ & \leq Q(\tau^{-1}) \|v_-^{\frac{p-2}{2}}\| (\|\nabla v_-^{\frac{p}{2}}\| + \|v_-^{\frac{p}{2}}\|_{L^1(\Omega)}) \\ & \leq \frac{2(p-1)}{p^2} \|\nabla v_-^{\frac{p}{2}}\|^2 + Q(\tau^{-1}) (1 + p \|v_-\|_{L^{p-1}(\Omega)}^{p-1}), \end{aligned} \quad (6.13)$$

where we have also used Young's inequality and the fact $p/2 \leq p - 1$ as we have assumed $p \geq 2$. Collecting (6.9)-(6.13) then (6.7) gives

$$\begin{aligned} & \frac{1}{p} \frac{d}{dt} \|v_-\|_{L^p(\Omega)}^p + \frac{2}{(p+1)^2} \|\nabla v_-^{\frac{p+1}{2}}\|^2 + \frac{2(p-1)}{p^2} \|\nabla v_-^{\frac{p}{2}}\|^2 \\ & \leq Q(\tau^{-1}) (1 + p\|v_-\|_{L^{p-1}(\Omega)}^{p-1}). \end{aligned} \quad (6.14)$$

The above differential inequality may serve as a starting point for a Moser iteration scheme with regularization, and, more precisely, we may follow the lines of the procedure given in [22, Proof of Lemma 3.3] (see in particular the differential inequality [22, (3.52)], which has exactly the same structure of or (6.14)), or the similar argument in [20, End of Proof of Thm 2.2] (actually the situation here is much simpler, in the sense that the regularity of the forcing terms in (6.4) is higher compared to both the quoted references).

We omit to report the details of the Moser scheme, as they are rather technical: the spirit of the procedure is to perform infinitely many iterations by working on time intervals of smaller and smaller length behaving like $\tau 2^{-k}$, $k \in \mathbb{N}$, and exploiting at each step the parabolic regularization effects. At the first iteration we can use (6.6) as a regularity condition on the “initial” datum; then (6.6) will be improved step by step exploiting the effects of the diffusion term.

It can then be shown that the resulting bounds are uniform for large p and that the behavior with respect to τ can also be quantitatively controlled (see the statement of [22, Lemma 3.3], where the argument was carried out for $\tau = 1$, and in particular compare formulas (3.50) and (3.51) therein). As a final outcome of the procedure we then get

$$\|v_-(t)\|_{L^\infty(\Omega)} \leq Q(\tau^{-1}, T), \quad \text{for every } t \geq \tau > 0, \quad (6.15)$$

which clearly implies (2.36) as the logarithm is unbounded near 0.

Remark 6.1. We observe that one could obtain a variant of the above result by an alternative procedure. Namely, if we multiply (1.3) by $-1/\sigma^2$ we deduce

$$\frac{d}{dt} \frac{1}{\sigma} + \frac{1}{\sigma^2} \operatorname{div}(\nabla \sigma) - \chi \frac{\nabla \sigma}{\sigma^2} \cdot \nabla \varphi - \chi \frac{1}{\sigma} \Delta \varphi = h(\sigma, \varphi) - \frac{k(\sigma, \varphi)}{\sigma}. \quad (6.16)$$

Then, noting that

$$\frac{1}{\sigma^2} \operatorname{div}(\nabla \sigma) = \operatorname{div}\left(\frac{\nabla \sigma}{\sigma^2}\right) - \nabla \sigma \cdot \nabla\left(\frac{1}{\sigma^2}\right) = -\Delta\left(\frac{1}{\sigma}\right) + 2\frac{|\nabla \sigma|^2}{\sigma^3} = -\Delta\left(\frac{1}{\sigma}\right) + 8|\nabla \sigma^{-1/2}|^2, \quad (6.17)$$

and setting $u := \sigma^{-1}$, we arrive at the relation

$$u_t - \Delta u + 8|\nabla u^{1/2}|^2 + \chi \nabla u \cdot \nabla \varphi - \chi u \Delta \varphi = h(\sigma, \varphi) - k(\sigma, \varphi)u, \quad (6.18)$$

which may also serve as a starting point for an alternative version of the Moser iteration argument. We preferred, however, to start from (6.1) as this requires the weaker assumption (2.35) on the initial datum, whereas using (6.18) we should likely assume some summability on $u_0 = \sigma_0^{-1}$, which is of course a more restrictive condition.

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