

Dual Riemannian Newton Method on Statistical Manifolds

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Abstract

In probabilistic modeling, parameter estimation is commonly formulated as a minimization problem on a parameter manifold. Optimization in such spaces requires *geometry-aware methods* that respect the underlying information structure. While the *natural gradient* leverages the *Fisher information metric* as a form of Riemannian gradient descent, it remains a *first-order method* and often exhibits slow convergence near optimal solutions. Existing second-order manifold algorithms typically rely on the Levi-Civita connection, thus overlooking the dual-connection structure that is central to information geometry. We propose the *dual Riemannian Newton method*, a Newton-type optimization algorithm on manifolds endowed with a metric and a pair of dual affine connections. The dual Riemannian Newton method explicates how duality shapes second-order updates: when the retraction (a local surrogate of the exponential map) is defined by one connection, the associated Newton equation is posed with its dual. We establish local quadratic convergence and validate the theory with experiments on representative statistical models. Thus, the dual Riemannian Newton method thus delivers second-order efficiency while remaining compatible with the dual structures that underlie modern information-geometric learning and inference.

Index Terms

Differential geometry; information geometry; projection problem; Riemannian optimization; second-order methods; statistical inference

I. INTRODUCTION

In classical minimization over Euclidean spaces, the negative gradient provides the steepest descent direction, which means that first-order methods such as gradient descent are ubiquitous. To accelerate convergence, higher-order schemes with superlinear or even quadratic rates, most notably Newton’s method and trust-region methods, leverage second-order information to refine both directions and step sizes [1].

However, in many applications, the ambient space of the minimization is not Euclidean, but instead a manifold endowed with a nontrivial geometry. Typical examples include orthogonality/subspace constraints such as the Stiefel/Grassmann manifolds [2], and the cone of symmetric positive definite (SPD) matrices [3]. In these settings, the steepest descent direction is not the Euclidean negative gradient; it is the *Riemannian negative gradient*, defined by the underlying Riemannian metric. This viewpoint leads to Riemannian gradient descent [4]. In statistical manifolds, choosing the Fisher information matrix as the metric specializes Riemannian gradient descent to the well-known *natural gradient* method [5] also known as Fisher’s scoring [6]. While Riemannian gradient descent respects the geometry, it remains a first-order scheme and may converge slowly on ill-conditioned problems (c.f., [7]). To obtain superlinear convergence on manifolds, higher-order methods such as Riemannian Newton and Riemannian trust region methods have been developed, primarily under the Levi–Civita connection [4], and have seen success in minimization tasks like fixed-rank matrix completion [8] and independent component analysis [9].

Parameter estimation in probabilistic modeling is typically formulated as a minimization problem over a probabilistic model; for instance, minimizing the negative log-likelihood or a divergence between the model and data distributions. Higher-order manifold optimization methods have been widely used for parameter estimation of probabilistic models, although they have mainly been restricted to distributions with specific structural properties, such as Gaussian (mixture) models with special covariance structures, to reduce the number of interaction steps during optimization. These problems can be reformulated as Riemannian optimization tasks on the SPD [10], Stiefel, or Grassmann manifolds equipped with the Levi–Civita connection [11], [12]. Furthermore, second-order Newton methods have also been developed for the *exponential family* of distributions by equipping the statistical manifold with a pair of dual affine connections, the e-connection and the m-connection, under which the manifold becomes *dually flat* [13].

From the view of information geometry, in the minimization problem over a probabilistic model, the *natural gradient* method arises naturally as a first-order optimization method. However, when extending to second-order methods, it is necessary to compute the Hessian of the objective function. Unlike the Hessian defined via the Levi–Civita connection on standard manifolds such as the Stiefel or Grassmann manifold, the notion of the Hessian is not straightforward on a general statistical manifold. This difficulty motivates the use of dual affine connections, which provide a natural way of defining second-order structures in probabilistic modeling.

In this paper, we further generalize this line of research by introducing a Newton-type optimization framework on Riemannian manifolds endowed with *arbitrary dual affine connections*. Unlike the Levi–Civita or the e/m-connection setting (restricted to the exponential family), the proposed method, called the *dual Riemannian Newton method*, is applicable to minimization problems involving *general probability distributions*. Specifically, we consider the quadruple $(\mathcal{M}, g, \nabla, \nabla^*)$, where \mathcal{M} is the parameter manifold, g is the chosen Riemannian metric (for example, the Fisher information metric for statistical models), and ∇ and ∇^* are an arbitrary pair of dual affine connections (for example, α -connections). This formulation provides a unified geometric framework that allows Newton-type methods to be applied beyond the exponential family, enabling efficient second-order optimization directly on statistical manifolds.

We summarize our contribution in the following:

- 1) We propose a Newton-type optimization framework that can be applied to general minimization problems arising in parameter estimation for arbitrary probability distributions in Section III.
- 2) We establish the following theoretical properties of the proposed method: (i) Under a ∇ -based second-order retraction, the Newton equation must be solved using the dual connection ∇^* rather than within the ∇ connection in Algorithm 1; (ii) On dually flat manifolds and in affine coordinates, each step of the proposed dual Riemannian Newton method reduces to the Euclidean Newton step in Theorem IV.1; (iii) For classical ∇ -projection problems in information geometry, the dual Riemannian Newton method steps coincide with the natural-gradient updates in affine coordinates in Theorem IV.4; (iv) The dual Riemannian Newton method achieves local quadratic convergence in Theorem V.1.
- 3) We empirically demonstrate that the dual Riemannian Newton method achieves local quadratic convergence and show its superiority compared to the natural gradient and Adam methods [14] in Section VI.

The rest of the paper is summarized as follows. In Section II, we review the necessary mathematical background. Section III introduces the dual Riemannian Newton method and its practical implementation. Section IV develops its theoretical properties on dually flat manifolds in affine coordinates. Section V proves local quadratic convergence of the dual Riemannian Newton method. Section VI presents experiments verifying this convergence and demonstrating advantages over the natural gradient and the Adam methods.

II. PRELIMINARIES

We introduce the notations and provide a summary of the mathematical background relevant to this paper. We begin by introducing the dual affine connections, along with the Riemannian gradient and retraction. An affine connection is formally defined in Definition IX.1 in the Appendix.

Definition II.1 (Dual affine Connection and dualistic Structure). *Let (\mathcal{M}, g) be a Riemannian manifold, $\langle \cdot, \cdot \rangle$ the Riemannian inner product induced by a Riemannian metric g , and ∇ an affine connection on \mathcal{M} . The dual affine connection ∇^* with respect to g is the unique affine connection that satisfies*

$$X\langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X^* Z \rangle \quad (1)$$

for all vector fields $X, Y, Z \in \mathfrak{X}(\mathcal{M})$. If ∇ and ∇^* satisfy the above duality condition (1), then (g, ∇, ∇^*) is called a dualistic structure (or dual affine structure) on \mathcal{M} . Moreover, the dual connection determined by this condition satisfies

$$(\nabla^*)^* = \nabla. \quad (2)$$

Throughout this paper, we assume that both ∇ and ∇^* are *torsion-free*, an assumption that is usually satisfied in information geometry [15]: that is,

$$\nabla_X Y - \nabla_Y X = [X, Y], \quad \nabla_X^* Y - \nabla_Y^* X = [X, Y], \quad (3)$$

for any vector fields $X, Y \in \mathfrak{X}(\mathcal{M})$.

A concrete example illustrating the dual affine connection on a probabilistic model is α -connection presented in the following. Other examples include q -divergence [16] and the logarithmic divergence [17].

Example II.1 (α -geometry on a statistical manifold). Let $p(x; \xi)$ be the probability density function (PDF) of a random variable x , parameterized by $\xi = (\xi_1, \dots, \xi_n)^\top$. We denote the log-likelihood function by $\ell(\xi) = \ln p(x; \xi)$. The Riemannian metric on the parameter space, induced by the Fisher information, is given by

$$g_{ij}(\xi) = \mathbb{E}[\partial_i \ell(\xi) \partial_j \ell(\xi)], \quad 1 \leq i, j \leq n,$$

where $\partial_i = \frac{\partial}{\partial \xi_i}$.

Within the framework of α -geometry, the inner product of vector fields satisfies the following duality relation:

$$X \langle Y, Z \rangle = \langle \nabla_X^{(\alpha)} Y, Z \rangle + \langle Y, \nabla_X^{(-\alpha)} Z \rangle,$$

where $\nabla^{(\alpha)}$ and $\nabla^{(-\alpha)}$ form a dual pair of affine connections. The Christoffel symbols of the α -connection in local coordinates are given by [15]

$$\langle \nabla_{\partial_i}^{(\alpha)} \partial_j, \partial_k \rangle = \Gamma_{ij,k}^{(\alpha)}(\xi) := \mathbb{E}\left[\left(\partial_i \partial_j \ell + \frac{1-\alpha}{2} \partial_i \ell \partial_j \ell\right) \partial_k \ell\right].$$

These coefficients are symmetric in the lower indices, $\Gamma_{ij,k}^{(\alpha)}(\xi) = \Gamma_{ji,k}^{(\alpha)}(\xi)$, which implies that the α -connection is torsion-free. Moreover, raising an index using $g^{ks}(\xi)$ yields

$$\Gamma_{ij}^{(\alpha)s}(\xi) = \sum_{k=1}^n \Gamma_{ij,k}^{(\alpha)}(\xi) g^{ks}(\xi).$$

The pair $(\nabla^{(\alpha)}, \nabla^{(-\alpha)})$ is dual with respect to the Riemannian metric g . Notably, when $\alpha = 0$, the α -connection coincides with the Levi-Civita connection, the unique torsion-free and metric-compatible connection in Riemannian geometry. In information geometry, the special cases $\alpha = 1$ and $\alpha = -1$ correspond to the e-connection and m-connection, respectively, which play central roles in the geometry of exponential and mixture families.

Subsequently, we introduce the definition of the Riemannian gradient.

Definition II.2 (Riemannian Gradient). Let (\mathcal{M}, g) be a Riemannian manifold and let $f : \mathcal{M} \rightarrow \mathbb{R}$ be a smooth real-valued function. For any point $p \in \mathcal{M}$, the Riemannian gradient of f at p , denoted by $\text{grad } f(p)$, is the unique vector in the tangent space $T_p \mathcal{M}$ at p that satisfies

$$Df(p)[X_p] = \langle \text{grad } f(p), X_p \rangle \quad \text{for all } X_p \in T_p \mathcal{M},$$

where $Df(p)$ is the differential of f at p .

Here, the differential $Df(p)$ of a smooth function $f : \mathcal{M} \rightarrow \mathbb{R}$ at a point $p \in \mathcal{M}$ is defined as

$$Df(p)[X_p] = X_p(f)$$

for all $X_p \in T_p\mathcal{M}$, where $X_p(f)$ denotes the directional derivative of f at p in the direction X_p .

We next define the notion of a retraction on a manifold.

Definition II.3 (Retraction on a manifold). *Let \mathcal{M} be a smooth manifold. A smooth mapping $R : T\mathcal{M} \rightarrow \mathcal{M}$, where $T\mathcal{M} := \bigsqcup_{p \in \mathcal{M}} T_p\mathcal{M}$ denotes the tangent bundle of \mathcal{M} , is called a retraction if for any $p \in \mathcal{M}$, the restriction*

$$R_p := R|_{T_p\mathcal{M}} : T_p\mathcal{M} \rightarrow \mathcal{M}$$

satisfies the following two conditions:

- 1) $R_p(0_p) = p$ with the zero vector $0_p \in T_p\mathcal{M}$ at p , and
- 2) the differential of R_p at 0_p , denoted by $DR_p(0_p)$, satisfies

$$DR_p(0_p)[X_p] = X_p \quad \text{for all } X_p \in T_p\mathcal{M},$$

that is, $DR_p(0_p) = \text{id}_{T_p\mathcal{M}}$.

For the definition of the differential of a smooth map between manifolds, please refer to Definition IX.2 in the Appendix.

III. DUAL RIEMANNIAN NEWTON METHOD

In this section, we introduce our proposed method, called the *dual Riemannian Newton method*, which is a second-order optimization technique defined on a manifold equipped with dual affine connections.

A. Dual Riemannian Hessian

We begin by introducing an important ingredient of the proposed method, the *dual Riemannian Hessian* defined on a dualistic structure (g, ∇, ∇^*) as well as its symmetry.

Definition III.1 (Dual Riemannian Hessian). *Let $f : \mathcal{M} \rightarrow \mathbb{R}$ be a smooth function on a Riemannian manifold \mathcal{M} , and let ∇ and ∇^* be a pair of dual affine connections with respect*

to the metric g . The Hessian of f with respect to ∇ at a point $p \in \mathcal{M}$, denoted by $\text{Hess } f(p)$, is a linear map $\text{Hess } f(p) : T_p \mathcal{M} \rightarrow T_p \mathcal{M}$ given as

$$\text{Hess } f(p)[X_p] = \nabla_{X_p} \text{grad } f$$

for any $X_p \in T_p \mathcal{M}$, where $\text{grad } f$ is the Riemannian gradient of f with respect to the metric g . Similarly, the dual Hessian $\text{Hess}^* f(p) : T_p \mathcal{M} \rightarrow T_p \mathcal{M}$ of f with respect to ∇^* at the point p is given as

$$\text{Hess}^* f(p)[X_p] = \nabla_{X_p}^* \text{grad } f.$$

Proposition III.1 (Symmetry of the Hessian). *Let $f : \mathcal{M} \rightarrow \mathbb{R}$ be a smooth function defined on a Riemannian manifold $(\mathcal{M}, g, \nabla, \nabla^*)$, where ∇ and ∇^* are dual affine connections. Then both linear maps $\text{Hess } f(p)$ and $\text{Hess}^* f(p)$ are symmetric at any point $p \in \mathcal{M}$; that is, for any $X_p, Y_p \in T_p \mathcal{M}$, the following equations hold.*

$$\langle \text{Hess } f(p)[X_p], Y_p \rangle = \langle X_p, \text{Hess } f(p)[Y_p] \rangle, \quad \langle \text{Hess}^* f(p)[X_p], Y_p \rangle = \langle X_p, \text{Hess}^* f(p)[Y_p] \rangle.$$

Proof. For smooth vector fields $X, Y \in \mathfrak{X}(\mathcal{M})$, since both ∇ and ∇^* are torsion-free; that is,

$$\nabla_X Y - \nabla_Y X = [X, Y], \quad \nabla_X^* Y - \nabla_Y^* X = [X, Y], \quad (4)$$

these vector fields satisfy

$$X(p) = X_p, \quad Y(p) = Y_p$$

for all $p \in \mathcal{M}$. According to Definition II.1 in the preliminaries, we obtain

$$\begin{aligned} & \langle \nabla_X \text{grad } f, Y \rangle - \langle \nabla_Y \text{grad } f, X \rangle \\ &= X \langle \text{grad } f, Y \rangle - \langle \text{grad } f, \nabla_X^* Y \rangle - Y \langle \text{grad } f, X \rangle + \langle \text{grad } f, \nabla_Y^* X \rangle \\ &= X(Yf) - Y(Xf) - \langle \text{grad } f, \nabla_X^* Y - \nabla_Y^* X \rangle \\ &= [X, Y]f - [X, Y]f = 0. \end{aligned}$$

Similarly, for the dual connection ∇^* we have

$$\langle \nabla_X^* \text{grad } f, Y \rangle - \langle \nabla_Y^* \text{grad } f, X \rangle = 0,$$

which completes the proof. \square

B. Dual Riemannian Newton method

As mentioned in the introduction, parameter estimation in probabilistic modeling is often formulated as an optimization problem. In particular, the parameter space can be modeled as a Riemannian manifold, and the optimization problem takes the form:

$$\min_{p \in \mathcal{M}} f(p), \quad (5)$$

where $f : \mathcal{M} \rightarrow \mathbb{R}$ is a smooth function defined on a smooth Riemannian manifold \mathcal{M} . Typical examples include the negative log-likelihood function in maximum likelihood estimation (MLE) or divergence-based loss functions such as the Kullback–Leibler divergence, when estimating parameters of a probabilistic model. In the following, we introduce the dual Riemannian Newton method for solving this optimization problem.

First, we recall the following lemma concerning the covariant derivative; please refer to Definition IX.3 in the Appendix for its definition.

Lemma III.1 ([15]). *Consider a manifold \mathcal{M} with (g, ∇, ∇^*) . For any smooth curve $\gamma(t)$ and vector fields $Y(t), Z(t)$ along the curve, the following identity holds:*

$$\frac{d}{dt} \langle Y, Z \rangle = \left\langle \frac{D}{dt} Y, Z \right\rangle + \left\langle Y, \frac{D^*}{dt} Z \right\rangle.$$

Here, $\frac{D}{dt} Y$ (resp. $\frac{D^*}{dt} Y$) denotes the covariant derivative of $Y(t)$ induced by the affine connection ∇ (resp. ∇^*) along the curve $\gamma(t)$.

Based on the above lemma, we now derive our proposed dual Riemannian Newton method on a manifold $(\mathcal{M}, g, \nabla, \nabla^*)$. We recall a second-order retraction [4] associated with the affine connection ∇ , defined as $R_p(tX_p)$, where $p \in \mathcal{M}$ and $X_p \in T_p\mathcal{M}$. This retraction satisfies the standard conditions:

$$R_p(0_p) = p \quad \text{and} \quad DR_p(0_p)[X_p] = X_p.$$

By denoting $R_p(tX_p) := \gamma_{p,X_p}(t)$, these conditions are written as

$$\gamma_{p,X_p}(0) = p \quad \text{and} \quad \dot{\gamma}_{p,X_p}(t) \Big|_{t=0} = X_p. \quad (6)$$

Since $\gamma_{p,X_p}(t)$ is a second-order retraction, it also satisfies the condition:

$$\nabla_{\dot{\gamma}_{p,X_p}(t)} \dot{\gamma}_{p,X_p}(t) \Big|_{t=0} = 0. \quad (7)$$

As the first step, we derive the second-order approximation of $f(\gamma_p(tX_p))$. By applying Taylor's expansion to approximate the function $f(\gamma_p(tX_p))$, we have

$$\begin{aligned} & f(\gamma_p(tX_p)) \\ & \approx f(\gamma_p(0_p)) + t \left. \frac{d}{dt} f(\gamma_p(tX_p)) \right|_{t=0} + \frac{1}{2} t^2 \left. \frac{d^2}{dt^2} f(\gamma_p(tX_p)) \right|_{t=0} \\ & = f(p) + t \left. \langle \text{grad } f(\gamma_{p,X_p}(t)), \dot{\gamma}_{p,X_p}(t) \rangle \right|_{t=0} + \frac{1}{2} t^2 \left. \frac{d}{dt} \langle \dot{\gamma}_{p,X_p}(t), \text{grad } f(\gamma_{p,X_p}(t)) \rangle \right|_{t=0}, \end{aligned}$$

where the last equality follows from (6). Together with (6) and (7), we get

$$\begin{aligned} & f(p) + t \left. \langle \text{grad } f(\gamma_{p,X_p}(t)), \dot{\gamma}_{p,X_p}(t) \rangle \right|_{t=0} + \frac{1}{2} t^2 \left. \frac{d}{dt} \langle \dot{\gamma}_{p,X_p}(t), \text{grad } f(\gamma_{p,X_p}(t)) \rangle \right|_{t=0} \\ & = f(p) + t \langle \text{grad } f(p), X_p \rangle \\ & \quad + \frac{1}{2} t^2 \left(\left\langle \frac{D}{dt} \dot{\gamma}_{p,X_p}(t), \text{grad } f(\gamma_{p,X_p}(t)) \right\rangle + \left\langle \dot{\gamma}_{p,X_p}(t), \frac{D^*}{dt} (\text{grad } f(\gamma_{p,X_p}(t))) \right\rangle \right) \Big|_{t=0} \\ & = f(p) + t \langle \text{grad } f(p), X_p \rangle \\ & \quad + \frac{1}{2} t^2 \left(\left\langle \nabla_{\dot{\gamma}_{p,X_p}(t)} \dot{\gamma}_{p,X_p}(t), \text{grad } f(\gamma_{p,X_p}(t)) \right\rangle + \left\langle \dot{\gamma}_{p,X_p}(t), \frac{D^*}{dt} (\text{grad } f(\gamma_{p,X_p}(t))) \right\rangle \right) \Big|_{t=0} \\ & = f(p) + t \langle \text{grad } f(p), X_p \rangle + \frac{1}{2} t^2 \langle \text{Hess}^* f(p)[X_p], X_p \rangle. \end{aligned} \tag{8}$$

Therefore, the function $f(\gamma_p(X_p))$ can be locally approximated as

$$f(\gamma_p(X_p)) \approx f(p) + \langle \text{grad } f(p), X_p \rangle + \frac{1}{2} \langle \text{Hess}^* f(p)[X_p], X_p \rangle. \tag{9}$$

Subsequently, we have

$$f(\gamma_p(X_p + tY_p)) \approx f(p) + \langle \text{grad } f(p), X_p + tY_p \rangle + \frac{1}{2} \langle \text{Hess}^* f(p)[X_p + tY_p], X_p + tY_p \rangle.$$

As the second step, we derive Newton's equation with a second-order retraction related to ∇ from the above approximation. For simplicity, we define

$$\bar{f}(t) := f(p) + \langle \text{grad } f(p), X_p + tY_p \rangle + \frac{1}{2} \langle \text{Hess}^* f(p)[X_p + tY_p], X_p + tY_p \rangle. \tag{10}$$

By applying the self-adjoint property of the dual Hessian stated in Proposition III.1

$$\langle \text{Hess}^* f(p)[X_p], Y_p \rangle = \langle X_p, \text{Hess}^* f(p)[Y_p] \rangle,$$

we obtain

$$\begin{aligned} \left. \frac{d\bar{f}(t)}{dt} \right|_{t=0} & = \langle \text{grad } f(p), Y_p \rangle + \langle \text{Hess}^* f(p)[X_p], Y_p \rangle + t \left. \langle \text{Hess}^* f(p)[Y_p], Y_p \rangle \right|_{t=0} \\ & = \langle \text{grad } f(p), Y_p \rangle + \langle \text{Hess}^* f(p)[X_p], Y_p \rangle. \end{aligned} \tag{11}$$

By considering the stationary condition

$$\frac{d\bar{f}(t)}{dt} \Big|_{t=0} = 0 \quad \text{for all } Y_p \in T_p\mathcal{M},$$

we arrive at

$$\text{Hess}^* f(p)[X_p] = -\text{grad } f(p), \quad (12)$$

which corresponds to Newton's equation with a second-order retraction $R_p(X_p) = \gamma_{p,X_p}(1)$ related to ∇ . In other words, Equation (12) implies that, for each point $p \in \mathcal{M}$, the tangent vector $X_p \in T_p\mathcal{M}$ yielding the next iterate $p \leftarrow R_p(X_p)$ can be obtained by solving this equation.

Equation (12) motivates the Dual Riemannian Newton method presented in Algorithm 1. Surprisingly, this illustrates the *duality* of Newton's method formulated on $(\mathcal{M}, g, \nabla, \nabla^*)$. Specifically, if we choose the second-order retraction $\gamma_{p,X_p}(t)$ satisfying $\nabla_{\dot{\gamma}_{p,X_p}(t)} \dot{\gamma}_{p,X_p}(t) \Big|_{t=0} = 0$, the Newton step must be computed using its dual connection ∇^* through Equation (12).

Algorithm 1 Dual Riemannian Newton Method

Require: $(\mathcal{M}, g, \nabla, \nabla^*)$, initial point $p \in \mathcal{M}$ and retraction $R_p(\cdot)$.

Ensure: Approximate local minimizer of $f : \mathcal{M} \rightarrow \mathbb{R}$

- 1: **repeat**
- 2: Compute the Newton direction $X_p \in T_p\mathcal{M}$ by solving:

$$\text{Hess}^* f(p)[X_p] = -\text{grad } f(p)$$

- 3: Update: $p \leftarrow R_p(X_p)$

- 4: **until** convergence
-

For the *Riemannian gradient descent method* defined on $(\mathcal{M}, g, \nabla, \nabla^*)$, it is not necessary to solve the Newton equation (12). Instead, Step 2 in Algorithm 1 is replaced with

$$X_p = -s \text{ grad } f(p),$$

where s is a learning rate. Moreover, the *natural gradient method* can be viewed as a special case of the Riemannian gradient descent method when the Riemannian metric is chosen as the Fisher information matrix, while the *mirror descent method* [18] can be interpreted as the natural gradient method expressed in affine coordinates on a dually flat manifold. We will elaborate on this relationship in Section IV-B.

C. Computation of Dual Riemannian Newton method

We provide an explicit computation of the dual Riemannian Newton method under a local coordinate chart:

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{f} & \mathbb{R} \\ \varphi \downarrow & & f \circ \varphi^{-1} \\ \xi = (\xi_1, \dots, \xi_n) & & \end{array}$$

Here, \mathcal{M} is a smooth manifold of dimension n . Moreover, $\varphi : U \subset \mathcal{M} \rightarrow \mathbb{R}^n$ is a coordinate chart, and for any point $p \in U$, we denote the local coordinates by $\xi^p = (\xi_1^p, \dots, \xi_n^p)^\top := \varphi(p)$. Given a smooth function $f : \mathcal{M} \rightarrow \mathbb{R}$, its coordinate expression under φ is $f \circ \varphi^{-1} : \varphi(U) \subset \mathbb{R}^n \rightarrow \mathbb{R}$.

Firstly, we describe the Newton equation in (12) using the coordinate coefficients of a tangent vector. Observe that the Riemannian gradient $\text{grad } f(p)$ in a local coordinate chart φ is expressed as:

$$\text{grad } f(p) = \sum_{i,j=1}^n g^{ij}(\xi^p) \frac{\partial (f \circ \varphi^{-1})}{\partial \xi_j} \Big|_{\xi^p} \frac{\partial}{\partial \xi_i} \Big|_{\xi^p},$$

where $g^{ij}(\xi^p)$ is the (i, j) -component of the inverse Riemannian metric g^{-1} at the point $p \in \mathcal{M}$ under the local coordinates ξ , in detail,

$$[g^{ij}(\xi^p)] = [g_{ij}(\xi^p)]^{-1} \quad \text{with} \quad g_{ij}(\xi^p) := \left\langle \frac{\partial}{\partial \xi_i}, \frac{\partial}{\partial \xi_j} \right\rangle \Big|_{\xi^p}.$$

Here, $\left\{ \frac{\partial}{\partial \xi_1} \Big|_{\xi^p}, \dots, \frac{\partial}{\partial \xi_n} \Big|_{\xi^p} \right\}$ is the coordinate basis of the tangent space $T_p \mathcal{M}$. For simplicity, we write

$$\text{grad } f(p) = \sum_{k=1}^n a_k(\xi^p) \frac{\partial}{\partial \xi_k} \Big|_{\xi^p} \quad \text{with} \quad a_k(\xi^p) := \sum_{j=1}^n g^{kj}(\xi^p) \frac{\partial (f \circ \varphi^{-1})}{\partial \xi_j} \Big|_{\xi^p}.$$

Take an arbitrary tangent vector

$$X_p = \sum_{k=1}^n \beta_k(\xi^p) \frac{\partial}{\partial \xi_k} \Big|_{\xi^p}$$

under local coordinate ξ . Consequently, $\text{Hess}^* f(p)[X_p]$ can be calculated as:

$$\begin{aligned} & \text{Hess}^* f(p)[X_p] \\ &= \nabla_{X_p}^* \text{grad } f \\ &= (\beta_1 \ \dots \ \beta_n) \left[\begin{pmatrix} \frac{\partial a_1}{\partial \xi_1} & \dots & \frac{\partial a_n}{\partial \xi_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial a_1}{\partial \xi_n} & \dots & \frac{\partial a_n}{\partial \xi_n} \end{pmatrix} + \begin{pmatrix} \sum_{j=1}^n a_j \Gamma_{1j}^{*-1} & \dots & \sum_{j=1}^n a_j \Gamma_{1j}^{*-n} \\ \vdots & \ddots & \vdots \\ \sum_{j=1}^n a_j \Gamma_{nj}^{*-1} & \dots & \sum_{j=1}^n a_j \Gamma_{nj}^{*-n} \end{pmatrix} \right] \begin{pmatrix} \frac{\partial}{\partial \xi_1} \\ \vdots \\ \frac{\partial}{\partial \xi_n} \end{pmatrix} \Big|_{\xi^p}, \end{aligned} \tag{13}$$

where Γ_{ij}^{*k} denotes the Christoffel symbol of the dual affine connection ∇^* in the coordinate chart $\xi = (\xi_1, \dots, \xi_n)$, defined by

$$\nabla_{\frac{\partial}{\partial \xi_i}}^* \left(\frac{\partial}{\partial \xi_j} \right) \Big|_{\xi^p} = \sum_{k=1}^n \Gamma_{ij}^{*k}(\xi^p) \frac{\partial}{\partial \xi_k} \Big|_{\xi^p}.$$

For convenience, we define the matrix $\mathbf{H}^*(\xi) \in \mathbb{R}^{n \times n}$ whose (i, j) -element satisfies

$$\mathbf{H}_{ij}^*(\xi^p) = \frac{\partial a_j(\xi)}{\partial \xi_i} \Big|_{\xi^p} + \sum_{k=1}^n a_k(\xi^p) \Gamma_{ik}^{*j}(\xi^p). \quad (14)$$

The coordinate coefficients define a (chart-dependent) linear isomorphism

$$\Phi_p : T_p \mathcal{M} \longrightarrow \mathbb{R}^n, \quad \Phi_p(X_p) = (\beta_1(\xi^p), \dots, \beta_n(\xi^p))^\top,$$

which identify $T_p \mathcal{M}$ with \mathbb{R}^n and thus the Newton equation in (12) can be written as:

$$\mathbf{H}^{*\top}(\xi^p) \boldsymbol{\beta}(\xi^p) = -\mathbf{G}^{-1}(\xi^p) \nabla f(\xi^p), \quad (15)$$

where $\boldsymbol{\beta}(\xi^p) = (\beta_1(\xi^p), \dots, \beta_n(\xi^p))^\top \in \mathbb{R}^n$ is the Newton direction in local coordinates ξ , $\nabla f(\xi^p) = \left(\frac{\partial(f \circ \varphi^{-1})}{\partial \xi_1}, \dots, \frac{\partial(f \circ \varphi^{-1})}{\partial \xi_n} \right)^\top \Big|_{\xi^p} \in \mathbb{R}^n$ is the local coordinate expression of the gradient, $\mathbf{G}(\xi^p) = [g_{ij}(\xi^p)]$ is the Riemannian metric matrix, and $\mathbf{G}^{-1}(\xi^p) = [g^{ij}(\xi^p)]$ is its inverse.

Next, we will approximate the update based on a second-order retraction under a local coordinate chart. Let $\gamma_{p, X_p}(t) := R_p(tX_p)$ be a second-order retraction on the manifold. From its local chart (φ, ξ) , we have $\varphi(\gamma_{p, X_p}(t)) = (\xi_1(t), \dots, \xi_n(t))^\top$. Moreover, we obtain $\dot{\gamma}_{p, X_p}(0) = X_p$, and $\varphi(\gamma_{p, X_p}(0)) = (\xi_1(0), \dots, \xi_n(0))^\top = \varphi(p) = (\xi_1^p, \dots, \xi_n^p)^\top$. The tangent vector X_p can be expressed as $X_p = \sum_{k=1}^n \beta_k(\xi^p) \frac{\partial}{\partial \xi_k} \Big|_{\xi^p}$. Furthermore, the curve $\bar{\gamma}_{p, X_p}$ is defined as the geodesic [19] emanating from p in the direction X_p :

$$\bar{\gamma}_{p, X_p}(0) = p, \quad \dot{\bar{\gamma}}_{p, X_p}(0) = X_p, \quad \text{and} \quad \nabla_{\dot{\bar{\gamma}}_{p, X_p}(t)} \dot{\bar{\gamma}}_{p, X_p}(t) = 0, \quad \forall t \in [0, \epsilon].$$

Let $\varphi(\bar{\gamma}_{p, X_p}(t)) = (\bar{\xi}_1(t), \dots, \bar{\xi}_n(t))^\top$. The local coordinate form of the geodesic equation is

$$\ddot{\bar{\xi}}_i(t) + \sum_{j,k=1}^n \Gamma_{jk}^i(\bar{\xi}(t)) \dot{\bar{\xi}}_j(t) \dot{\bar{\xi}}_k(t) = 0, \quad \bar{\xi}_i(0) = \xi_i^p, \quad \dot{\bar{\xi}}_i(0) = \beta_i(\xi^p);$$

therefore at $t = 0$, we have

$$\ddot{\bar{\xi}}_i(0) = - \sum_{j,k=1}^n \Gamma_{jk}^i(\xi^p) \beta_j(\xi^p) \beta_k(\xi^p).$$

Since a second-order retraction agrees with the geodesic up to second order (, in other words, we require $\nabla_{\dot{\gamma}_{p,X_p}(t)} \dot{\gamma}_{p,X_p}(t) \Big|_{t=0} = 0$), we can take a specific second order retraction, a local-coordinate second-order retraction $\varphi(\gamma_{p,X_p}(t))$, of the form

$$\xi_i(t) = \xi_i^p + t \beta_i(\xi^p) - \frac{t^2}{2} \sum_{j,k=1}^n \Gamma_{jk}^i(\xi^p) \beta_j(\xi^p) \beta_k(\xi^p), \quad i = 1, \dots, n, \quad (16)$$

where the Christoffel symbols of the affine connection ∇ in the chart $\xi = (\xi_1, \dots, \xi_n)^\top$ are defined by

$$\nabla_{\frac{\partial}{\partial \xi_i}} \left(\frac{\partial}{\partial \xi_j} \right) \Big|_{\xi^p} = \sum_{k=1}^n \Gamma_{ij}^k(\xi^p) \frac{\partial}{\partial \xi_k} \Big|_{\xi^p}.$$

From (15) and (16), we arrive at the explicit computational procedure of Algorithm 1 described in Algorithm 2.

Algorithm 2 Coordinate-based Dual Riemannian Newton Method

Require: $(\mathcal{M}, g, \nabla, \nabla^*)$, local chart (φ, ξ) , initial coordinates $\xi^p = \varphi(p) \in \mathbb{R}^n$

Ensure: Approximate local minimizer of $f : \mathcal{M} \rightarrow \mathbb{R}$

1: **repeat**

2: Compute the Newton direction $\boldsymbol{\beta}(\xi^p) = (\beta_1(\xi^p), \dots, \beta_n(\xi^p))^\top$ by solving:

$$\mathbf{H}^{*\top}(\xi^p) \boldsymbol{\beta}(\xi^p) = -\mathbf{G}^{-1}(\xi^p) \nabla f(\xi^p)$$

3: Update coordinates using the second-order retraction approximation:

$$\xi_i^p \leftarrow \xi_i^p + \beta_i(\xi^p) - \frac{1}{2} \sum_{j,k=1}^n \Gamma_{jk}^i(\xi^p) \beta_j(\xi^p) \beta_k(\xi^p), \quad i = 1, \dots, n$$

4: **until** convergence

For comparison, we present the corresponding computation of the natural gradient descent method under a local coordinate chart. The natural gradient descent method corresponds to a Riemannian gradient descent using a first-order retraction, which is defined by

$$\xi_i(t) = \xi_i^p + t \beta_i(\xi^p),$$

where $\beta_i(\xi^p)$ denotes the i -th component of the Riemannian gradient at the point ξ^p . We write the computation procedure of the natural gradient descent method in Algorithm 3 for comparison with the dual Riemannian Newton method. Algorithm 3 further illustrates that, contrast to the dual Riemannian Newton Method, the natural gradient descent method does not require computing the Christoffel symbols Γ_{ij}^k and $\Gamma_{ij}^{*\,k}$ associated with the affine connections ∇ and ∇^* .

Algorithm 3 Coordinate-based Natural Gradient Descent Method

Require: Riemannian manifold $(\mathcal{M}, g, \nabla, \nabla^*)$, local chart (φ, ξ) , initial point $p \in \mathcal{M}$ with coordinates $\xi^p = \varphi(p) \in \mathbb{R}^n$; step size $s > 0$

Ensure: Approximate local minimizer of $f : \mathcal{M} \rightarrow \mathbb{R}$

1: **repeat**

2: Compute the natural gradient direction in coordinates:

$$\beta(\xi^p) = -s \mathbf{G}^{-1}(\xi^p) \nabla f(\xi^p)$$

3: Update local coordinates:

$$\xi_i^p \leftarrow \xi_i^p + \beta_i(\xi^p), \quad i = 1, \dots, n$$

4: **until** convergence

IV. THEORTICAL PROPERTIES

Although the introduction of the dual Riemannian Newton method does not rely on any specific structure of $(\mathcal{M}, g, \nabla, \nabla^*)$, for a dually flat manifold in the context of information geometry the dual Riemannian Newton method enjoys additional theoretical properties in specific coordinates, which will be introduced in this section.

A. Dual Riemannian Newton Method on dually flat manifold

First, we prove that each step of the dual Riemannian Newton method reduces to the Euclidean Newton step when applied to a dually flat manifold under affine coordinates.

Let ∇ be an affine connection on a manifold \mathcal{M} , and let $[\cdot, \cdot]$ denote the Lie bracket of vector fields defined by $[X, Y] := XY - YX$ for all $X, Y \in \mathfrak{X}(\mathcal{M})$. The *torsion tensor* T is the $\mathcal{C}^\infty(\mathcal{M})$ -bilinear map defined by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y], \tag{17}$$

and the *Riemannian curvature tensor* R is the $\mathcal{C}^\infty(\mathcal{M})$ -trilinear map defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z, \tag{18}$$

for all vector fields $X, Y, Z \in \mathfrak{X}(\mathcal{M})$. Using such tensors, we can introduce the dually flat manifold as follows:

Definition IV.1 (Dually Flat Manifold). *Let $(\mathcal{M}, g, \nabla, \nabla^*)$ be a Riemannian manifold endowed with dual affine connections ∇ and ∇^* . We say that \mathcal{M} is a dually flat manifold if both the torsion and curvature tensors of ∇ and ∇^* vanish:*

$$T = T^* = 0, \quad R = R^* = 0. \quad (19)$$

In the dually flat manifold, the following two affine coordinates naturally exist [15]: Let \mathcal{M} be a dually flat manifold equipped with (g, ∇, ∇^*) . A local chart (φ, ξ) of $U \subset \mathcal{M}$ along which all the connection coefficients $\{\Gamma_{ij}^k\}$ vanish identically is called a ∇ -affine coordinate neighborhood, and the associated local coordinate system (ξ_i) is called a ∇ -affine coordinate system. Similarly, we can also define ∇^* -affine coordinate system. For example, a probability distribution belongs to the exponential family if its density can be written as

$$p(x; \theta) = h(x) \exp \left(\sum_{i=1}^n \theta_i F_i(x) - \psi(\theta) \right),$$

where $\theta = (\theta_1, \dots, \theta_n)^\top \in \Theta \subset \mathbb{R}^n$ is the natural parameter, $F(x) = (F_1(x), \dots, F_n(x))^\top$ is the vector of sufficient statistics, $h(x)$ is the base measure, and $\psi(\theta)$ is the log-partition function that normalizes the distribution:

$$\psi(\theta) = \log \int h(x) \exp \left(\sum_{i=1}^n \theta_i F_i(x) \right) dx. \quad (20)$$

The expectation parameter is defined by the gradient $\eta = (\eta_1, \dots, \eta_n)^\top := \nabla_\theta \psi(\theta)$. In the framework of information geometry, θ defines an e-affine coordinate system under the exponential connection $\nabla^{(e)}$, while η defines an m-affine coordinate system under the mixture connection $\nabla^{(m)}$ [15]. The manifold $(\mathcal{M}, g, \nabla^{(e)}, \nabla^{(m)})$ is dually flat. Moreover, the explicit computation of the exponential and mixture connections $\nabla^{(e)}$ and $\nabla^{(m)}$ on a statistical manifold is provided in Example II.1.

In the subsequent discussion, for notational convenience, we define θ as the affine coordinate system with respect to the connection ∇ , and η as that with respect to the dual connection ∇^* on a dually flat manifold $(\mathcal{M}, g, \nabla, \nabla^*)$.

Recall the Newton equation (15) in a local coordinate ξ :

$$\mathbf{G}(\xi^p) \mathbf{H}^{*\top}(\xi^p) \boldsymbol{\beta}(\xi^p) = -\nabla f(\xi^p). \quad (21)$$

Under affine coordinate systems, $\mathbf{G}(\xi^p) \mathbf{H}^{*\top}(\xi^p)$ admits the following simplified form.

Theorem IV.1. Let $(\mathcal{M}, g, \nabla, \nabla^*)$ be a dually flat manifold. Suppose that $\theta = (\theta_1, \dots, \theta_n)^\top$ and $\eta = (\eta_1, \dots, \eta_n)^\top$ are affine coordinate systems associated with the affine connections ∇ and ∇^* , respectively. Let the corresponding local charts be (φ_θ, θ) and (φ_η, η) . For all $p \in \mathcal{M}$, the matrix $\mathbf{G}(\theta^p) \mathbf{H}^{*\top}(\theta^p)$ satisfies

$$(\mathbf{G}(\theta^p) \mathbf{H}^{*\top}(\theta^p))_{ij} = \frac{\partial^2(f \circ \varphi_\theta^{-1})}{\partial \theta_i \partial \theta_j} \Big|_{\theta^p}.$$

Proof. Observe that from [15], in a dually flat manifold, we have

$$(\mathbf{G}(\theta^p))_{ij} = \frac{\partial \eta_i}{\partial \theta_j} \Big|_{\theta^p} = \frac{\partial \eta_j}{\partial \theta_i} \Big|_{\theta^p}, \quad (\mathbf{G}(\eta^p))_{ij} = \frac{\partial \theta_i}{\partial \eta_j} \Big|_{\eta^p} = \frac{\partial \theta_j}{\partial \eta_i} \Big|_{\eta^p},$$

thereby implying

$$\mathbf{G}(\theta^p) = (\mathbf{G}(\eta^p))^{-1}.$$

Now consider Newton's equation under the η -coordinates:

$$\begin{aligned} \mathbf{H}^{*\top}(\eta^p) \boldsymbol{\beta}(\eta^p) &= -\mathbf{G}^{-1}(\eta^p) \nabla f(\eta^p) \\ &= -\nabla f(\theta^p). \end{aligned} \tag{22}$$

Together with the relationship between directional vectors in two coordinate systems given by

$$\boldsymbol{\beta}(\eta^p) = \mathbf{G}(\theta^p) \boldsymbol{\beta}(\theta^p), \tag{23}$$

Equation (22) gives

$$\mathbf{H}^{*\top}(\eta^p) \mathbf{G}(\theta^p) \boldsymbol{\beta}(\theta^p) = -\nabla f(\theta^p). \tag{24}$$

On the other hand, Newton's equation under the θ -coordinates is given as

$$\mathbf{G}(\theta^p) \mathbf{H}^{*\top}(\theta^p) \boldsymbol{\beta}(\theta^p) = -\nabla f(\theta^p). \tag{25}$$

Combining (24) and (25) gives

$$\mathbf{G}(\theta^p) \mathbf{H}^{*\top}(\theta^p) = \mathbf{H}^{*\top}(\eta^p) \mathbf{G}(\theta^p). \tag{26}$$

Since we have $\Gamma_{ij}^{*k}(\eta) = 0$ under η -coordinates, the dual Riemannian Hessian simplifies as

$$(\mathbf{H}^{*\top}(\eta^p))_{ij} = \frac{\partial a_i(\eta)}{\partial \eta_j} \Big|_{\eta^p},$$

where

$$\begin{aligned} a_k(\eta(\theta)^p) &:= \sum_{j=1}^n g^{kj}(\eta^p) \frac{\partial (f \circ \varphi_\eta^{-1})}{\partial \eta_j} \Big|_{\eta^p} \\ &= \sum_{j=1}^n \frac{\partial \eta_j}{\partial \theta_k} \Big|_{\theta^p} \frac{\partial (f \circ \varphi_\eta^{-1})}{\partial \eta_j} \Big|_{\eta^p} \\ &= \frac{\partial (f \circ \varphi_\theta^{-1})}{\partial \theta_k} \Big|_{\theta^p}. \end{aligned} \tag{27}$$

Finally, the product becomes

$$\begin{aligned}
(\mathbf{G}(\theta^p) \mathbf{H}^{*\top}(\theta^p))_{ij} &= (\mathbf{H}^{*\top}(\eta^p) \mathbf{G}(\theta^p))_{ij} \\
&= \sum_{k=1}^n \frac{\partial a_i(\eta)}{\partial \eta_k} \Big|_{\eta^p} \frac{\partial \eta_k}{\partial \theta_j} \Big|_{\theta^p} \\
&= \frac{\partial a_i(\eta(\theta))}{\partial \theta_j} \Big|_{\theta^p} \\
&= \frac{\partial^2(f \circ \varphi_\theta^{-1})}{\partial \theta_i \partial \theta_j} \Big|_{\theta^p},
\end{aligned} \tag{28}$$

which completes the proof. \square

From this simplified form, we reduce the dual Riemannian Newton method in affine coordinates to the classical Newton method in Euclidean space as follows: Since $\theta = (\theta_1, \dots, \theta_n)^\top$ are affine coordinates associated with the affine connection ∇ , we have $\Gamma_{ij}^k(\theta) = 0$. Consequently,

Line 2 of Algorithm 2 simplifies to

$$\beta(\theta^p) = -[\nabla^2 f(\theta^p)]^{-1} \nabla f(\theta^p),$$

where $(\nabla^2 f(\theta^p))_{ij} = \frac{\partial^2(f \circ \varphi_\theta^{-1})}{\partial \theta_i \partial \theta_j} \Big|_{\theta^p}$. Furthermore, **Line 3** becomes

$$\theta_i^p \leftarrow \theta_i^p + \beta_i(\theta^p), \quad i = 1, \dots, n,$$

which implies that the dual Riemannian Newton method in θ -coordinates matches the classical Newton method in Euclidean space. Similarly, if we choose retraction $R_p^*(tX_p) := \gamma_{p,X_p}^*(t)$ satisfying $\nabla_{\dot{\gamma}_{p,X_p}^*(t)}^* \dot{\gamma}_{p,X_p}^*(t) \Big|_{t=0} = 0$, we have $(\mathbf{G}(\eta^p) \mathbf{H}^\top(\eta^p))_{ij} = (\nabla^2 f(\eta^p))_{ij} = \frac{\partial^2(f \circ \varphi_\eta^{-1})}{\partial \eta_i \partial \eta_j} \Big|_{\eta^p}$, where $\mathbf{H}_{ij}(\eta^p) = \frac{\partial a_j(\eta)}{\partial \eta_i} \Big|_{\eta^p} + \sum_{k=1}^n a_k(\eta^p) \Gamma_{ik}^j(\eta^p)$. Analogously, in the η -coordinate system where the dual connection coefficients vanish – that is, $\Gamma_{ij}^k(\eta) = 0$ – **Line 2** of Algorithm 2 simplifies to

$$\beta(\eta^p) = -[\nabla^2 f(\eta^p)]^{-1} \nabla f(\eta^p),$$

and **Line 3** becomes the standard coordinate update:

$$\eta_i^p \leftarrow \eta_i^p + \beta_i(\eta^p), \quad i = 1, \dots, n.$$

which implies that the dual Riemannian Newton method in η -coordinates also matches the classical Newton method in Euclidean space.

B. Relationship with Natural Gradient and Mirror Descent under Projection Problems

In projection problems, we show that the dual Riemannian Newton method reduces to both the natural gradient and the mirror descent methods when restricted to auto-parallel submanifolds equipped with affine coordinates. However, we also show that when the objective function of the optimization has additional regularization terms, the method no longer coincides with either the natural gradient or the mirror descent methods.

In information geometry, the *projection problem* refers to finding the closest point on a submanifold to a given point in the ambient space, with respect to a specific divergence function, typically, the ∇ -divergence (also known as the Bregman divergence [20]). Under the framework of dually flat manifolds, such projections can be performed using either the ∇ -connection or the ∇^* -connection, leading to the concepts of ∇ -projection and ∇^* -projection, respectively.

First, we introduce the potential functions that are smooth and strictly convex, whose existence is theoretically supported by the following lemma.

Lemma IV.2 ([15]). *Let $(\mathcal{M}, g, \nabla, \nabla^*)$ be a dually flat manifold. Suppose $\theta = (\theta_1, \dots, \theta_n)^\top$ and $\eta = (\eta_1, \dots, \eta_n)^\top$ are affine coordinate systems with respect to the affine connections ∇ and ∇^* , respectively. Then there exist smooth and strictly convex functions $\psi(\theta) \in C^\infty$ and $\varphi(\eta) \in C^\infty$, called the potential functions, such that the following relations hold:*

$$\begin{aligned} \eta_i &= \frac{\partial \psi}{\partial \theta^i}, & \theta^i &= \frac{\partial \varphi}{\partial \eta^i}, \\ g_{ij}(\theta) &= g^{ij}(\eta) = \frac{\partial^2 \psi}{\partial \theta^i \partial \theta^j}, & g^{ij}(\theta) &= g_{ij}(\eta) = \frac{\partial^2 \varphi}{\partial \eta^i \partial \eta^j}, \end{aligned} \tag{29}$$

and we have

$$\psi(\theta) + \varphi(\eta) = \sum_{i=1}^n \theta_i \eta_i. \tag{30}$$

Using the potential functions ψ and φ , we define the ∇ -divergence.

Definition IV.2. *Let \mathcal{M} be a dually flat manifold associated with the dualistic structure (g, ∇, ∇^*) . For two points $p, q \in \mathcal{M}$, we define the ∇ -divergence $D(p\|q)$ as*

$$\begin{aligned} D(p\|q) &:= \psi(\theta^p) + \varphi(\eta^q) - \sum_{i=1}^n \theta_i^p \eta_i^q \\ &= \psi(\theta^p) - \psi(\theta^q) - \langle \nabla_\theta \psi(\theta^q), \theta^p - \theta^q \rangle_E, \end{aligned}$$

where $\{(\theta_i), (\eta_i)\}$ is dual affine coordinates on \mathcal{M} , and $\langle u, v \rangle_E$ denotes the standard Euclidean inner product, i.e., $\langle u, v \rangle_E = \sum_{i=1}^n u_i v_i$.

For example, in the exponential family, $\psi(\theta)$ is defined as Equation (20), and the dual potential function $\varphi(\eta)$ can be defined via the Legendre transform [20]:

$$\varphi(\eta) := \sup_{\theta \in \Theta} \left(\sum_{i=1}^n \theta_i \eta_i - \psi(\theta) \right). \quad (31)$$

The associated ∇ -divergence is the well-known Kullback–Leibler (KL) divergence.

When the projection problem is restricted to an *auto-parallel* submanifold, the resulting projection enjoys desirable properties such as *uniqueness* and *orthogonality* [20], which ensures a well-defined geometric correspondence between the point and the submanifold. In the subsequent analysis, we restrict the projection problem to auto-parallel submanifolds.

A formal definition of the auto-parallel submanifold is provided below: Let $\mathcal{S} \subset \mathcal{M}$ be a submanifold equipped with a local coordinate system $(u_a)_{1 \leq a \leq m}$. For any vector fields $\frac{\partial}{\partial u_a}, \frac{\partial}{\partial u_b} \in \mathcal{X}(\mathcal{S})$, where $\mathcal{X}(\mathcal{S})$ denotes the set of all smooth vector fields on \mathcal{S} , and for any point $p \in \mathcal{S}$, the covariant derivative satisfies

$$\nabla_{\frac{\partial}{\partial u_a}} \frac{\partial}{\partial u_b} \Big|_{u^p} \in T_p \mathcal{S},$$

then \mathcal{S} is called an *auto-parallel submanifold* of \mathcal{M} with respect to the connection ∇ defined on \mathcal{M} . Intuitively, the affine connection-induced covariant derivative of any tangent vector on the submanifold remains in its tangent space.

In the context of information geometry, an auto-parallel submanifold with respect to the e-connection is called an *e-flat submanifold* (typical for exponential families), whereas one with respect to the m-connection is called an *m-flat submanifold* (typical for mixture families).

We now provide a lemma of auto-parallel submanifolds on dually flat manifolds.

Lemma IV.3 ([15]). *Let $(\mathcal{M}, g, \nabla, \nabla^*)$ be a dually flat manifold, and let $(\theta_i)_{1 \leq i \leq n}$ be an affine coordinate system associated with the connection ∇ . A necessary and sufficient condition for an m -dimensional submanifold $\mathcal{S} \subset \mathcal{M}$ to be auto-parallel with respect to ∇ is the existence of a constant matrix $\mathbf{A} \in \mathbb{R}^{n \times m}$ of rank m , a constant vector $\mathbf{b} \in \mathbb{R}^n$, and a local coordinate*

system $(u_a)_{1 \leq a \leq m}$ on \mathcal{S} that is affine with respect to ∇ , such that the embedding of \mathcal{S} into \mathcal{M} is locally given by

$$\begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_n \end{bmatrix} = \mathbf{A} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} + \mathbf{b}.$$

The same property holds for η -coordinates under ∇^* . Please note that $(\mathcal{S}, g, \nabla, \nabla^*)$ also forms a dually flat submanifold of \mathcal{M} .

We show that the dual Riemannian Newton method is the same as the natural gradient descent method in the ∇ - and ∇^* -projection problems under the affine coordinates. Let \mathcal{S} be an auto-parallel submanifold. The ∇^* -projection is defined as

$$p^* = \arg \min_{p \in \mathcal{S}} D(p \| q),$$

where q is a fixed point outside \mathcal{S} . Similarly, the ∇ -projection is defined as

$$q^* = \arg \min_{q \in \mathcal{S}} D(p \| q).$$

Theorem IV.4. *Let $\mathcal{S} \subset \mathcal{M}$ be an auto-parallel submanifold (also a dually flat manifold with affine connections (∇, ∇^*)) and let $q \in \mathcal{M} \setminus \mathcal{S}$. In ∇ -affine coordinates $(u_a)_{1 \leq a \leq m}$ on \mathcal{S} , the procedure for solving the ∇^* -projection problem given as*

$$p^* = \arg \min_{p \in \mathcal{S}} D(p \| q)$$

by the dual Riemannian Newton method is identical to that by the natural gradient method.

Proof. Let us first define the function

$$f(u^p) := \psi(\theta(u^p)) + \varphi(\eta^q) - \sum_{i=1}^n \theta_i(u^p) \eta_i^q.$$

According to Lemma IV.3, the coordinate transformation from u to θ is given by

$$\begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_n \end{bmatrix} = \mathbf{A} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} + \mathbf{b},$$

where $\mathbf{A} \in \mathbb{R}^{n \times m}$ and $\mathbf{b} \in \mathbb{R}^n$ are constant matrix and vector, respectively. Therefore

$$\frac{\partial^2}{\partial u_a \partial u_b} \theta_i(u) = 0 \quad 1 \leq a, b \leq m, \quad 1 \leq i \leq n.$$

Since $(u_a)_{1 \leq a \leq m}$ are affine coordinates with respect to the connection ∇ on the submanifold \mathcal{S} , we have $\Gamma_{ab}^c(u) = 0$ for all $1 \leq a, b, c \leq m$, and it follows from Theorem IV.1 and Lemma IV.2 that

$$(\mathbf{G}(u^p) \mathbf{H}^{*\top}(u^p))_{ab} = \frac{\partial^2 f(u)}{\partial u_a \partial u_b} \Big|_{u^p} = \frac{\partial^2}{\partial u_a \partial u_b} \psi(\theta(u)) \Big|_{u^p} = (\mathbf{G}(u^p))_{ab}. \quad (32)$$

Moreover, according to $\Gamma_{ab}^c(u) = 0$ for any $1 \leq a, b, c \leq m$, **Line 3** of Algorithm 2 becomes

$$u_a^{p_k} \leftarrow u_a^p + \beta_a(u^p), \quad a = 1, \dots, m,$$

which completes the proof. \square

Similarly, for the ∇ -projection problem

$$q^* = \arg \min_{q \in \mathcal{S}} D(p\|q),$$

the dual Riemannian Newton method reduces to the natural Gradient method under affine coordinates by applying the same proof technique.

Interestingly, although the proposed dual Riemannian Newton method reduces to the natural gradient method for simple projection problems under affine coordinates, we show that it differs from both the natural gradient method and the mirror descent if we add a regularization term to the objective function. Consider the modified function:

$$\bar{f}(u^p) := \psi(\theta(u^p)) + \varphi(\eta^q) - \sum_{i=1}^n \theta_i(u^p) \eta_i^q + \sum_{a=1}^m (u_a^p)^2,$$

where we add a regularization term $\sum_{a=1}^m (u_a^p)^2$ to the objective function in ∇^* -projection problem. Observe that

$$(\mathbf{G}(u^p) \mathbf{H}^{*\top}(u^p))_{ab} = \frac{\partial^2 \bar{f}(u)}{\partial u_a \partial u_b} \Big|_{u^p} = (\mathbf{G}(u^p))_{ab} + 2 \mathbf{I}, \quad (33)$$

where \mathbf{I} denotes the identity matrix. This implies that the dual Riemannian Newton method differs from the natural gradient method. Specifically, in the natural gradient method, **Line 2** of Algorithm 3 (with learning rate $s = 1$) is $\beta(u^p) = -\mathbf{G}(u^p)^{-1} \nabla f(u^p)$. In contrast, in the dual Riemannian Newton method, **Line 2** of Algorithm 2 becomes $\beta(u^p) = -(\mathbf{G}(u^p) + 2\mathbf{I})^{-1} \nabla f(u^p)$; that is, $\mathbf{G}(u^p)^{-1}$ is replaced with $(\mathbf{G}(u^p) + 2\mathbf{I})^{-1}$ according to Equation (33).

Moreover, in affine coordinates, the natural gradient descent method is equivalent to mirror descent, so the dual Riemannian Newton method also differs from mirror descent. Mirror

descent carried out in the θ -coordinates coincides exactly with natural gradient descent in the η -coordinates. Concretely, the mirror-descent step in optimization problem $\min_{p \in \mathcal{M}} f(p)$ is

$$\theta^{p_{k+1}} = \arg \min_{\theta} \left\{ \langle \nabla f(\theta^{p_k}), \theta \rangle_E + \frac{1}{s_k} D(\theta \| \theta^{p_k}) \right\},$$

where

$$\nabla f(\theta) = \left(\frac{\partial(f \circ \varphi_\theta^{-1})}{\partial \theta_1}, \dots, \frac{\partial(f \circ \varphi_\theta^{-1})}{\partial \theta_n} \right)^\top$$

is the Euclidean gradient in an affine chart $\theta = \varphi_\theta(\cdot)$. Moreover, $D(\theta \| \theta^{p_k})$ is defined as

$$D(\theta \| \theta^{p_k}) := \psi(\theta) + \varphi(\eta(\theta^{p_k})) - \sum_{i=1}^n \theta_i \eta_i(\theta^{p_k}).$$

This is equivalent to the natural gradient update in the dual η -coordinates [18]:

$$\eta^{p_{k+1}} = \eta^{p_k} - s_k \mathbf{G}^{-1}(\eta^{p_k}) \nabla f(\eta^{p_k}),$$

where \mathbf{G} denotes the Fisher information metric (in η -coordinates).

V. CONVERGENCE ANALYSIS

In this section we show that in contrast to the natural gradient method, which typically exhibits only linear convergence [4], our proposal of the dual Riemannian Newton method achieves local quadratic convergence, which is the first second-order convergent algorithm within the framework of statistical manifold.

We begin by briefly introducing the properties of the *consistent matrix norm* $\|\cdot\|$ on $\mathbb{R}^{n \times n}$, which satisfies $\|\mathbf{I}\| = 1$ and the submultiplicative property $\|AB\| \leq \|A\| \|B\|$. A matrix norm $\|\cdot\|$ is said to be consistent with a vector norm $\|\cdot\|_v$ if

$$\|Ax\|_v \leq \|A\| \|x\|_v \quad \text{for all } A \in \mathbb{R}^{n \times n}, x \in \mathbb{R}^n.$$

For all three matrices \mathbf{E} , \mathbf{A} , and \mathbf{B} , we have the following two properties [7]:

- (i) If $\|\mathbf{E}\| < 1$, then $(\mathbf{I} - \mathbf{E})^{-1}$ exists and it satisfies

$$\|(\mathbf{I} - \mathbf{E})^{-1}\| \leq \frac{1}{1 - \|\mathbf{E}\|}.$$

- (ii) If \mathbf{A} is nonsingular and $\|\mathbf{A}^{-1}(\mathbf{B} - \mathbf{A})\| < 1$, then \mathbf{B} is also nonsingular and it satisfies

$$\|\mathbf{B}^{-1}\| \leq \frac{\|\mathbf{A}^{-1}\|}{1 - \|\mathbf{A}^{-1}(\mathbf{B} - \mathbf{A})\|}.$$

Using the above properties of the consistent matrix norm, we theoretically prove that the dual Riemannian Newton method achieves local quadratic convergence.

Theorem V.1. Let (φ, ξ) be a coordinate chart on a manifold \mathcal{M} of dimension n . Assume that there exists a point $p^* \in \mathcal{M}$ such that

$$\text{grad } f(p^*) = 0 \quad \text{and} \quad \text{Hess}^* f(p^*) \succ 0,$$

where $\text{Hess}^* f(p^*) \succ 0$ means

$$\langle \text{Hess}^* f(p^*)[X_{p^*}], X_{p^*} \rangle > 0 \quad \text{for any } X_{p^*} \in T_{p^*}\mathcal{M}.$$

Then there exists a neighborhood $\mathcal{U} \subset \mathcal{M}$ of p^* such that, for any initial point $p_0 \in \mathcal{U}$, Algorithm 2 generates an infinite sequence (p_k) that converges quadratically to p^* .

Proof. Fix an iteration number k . First, we introduce the notation

$$\xi^{p_k} := \varphi(p_k), \quad \varphi(R_{p_k}(X_{p_k})) := \hat{R}_{\xi^{p_k}}(\beta(\xi^{p_k})),$$

where the tangent vector X_{p_k} is represented in local coordinates as

$$X_{p_k} = \sum_{s=1}^n \beta_s(\xi^{p_k}) \frac{\partial}{\partial \xi_s} \Big|_{\xi^{p_k}},$$

and $\beta(\xi^{p_k}) = (\beta_1(\xi^{p_k}), \dots, \beta_n(\xi^{p_k}))^\top$.

Next, we denote the Riemannian gradient vector in local coordinates by

$$\mathbf{a}(\xi^{p_k}) := \mathbf{G}^{-1}(\xi^{p_k}) \nabla f(\xi^{p_k}),$$

which satisfies $\mathbf{a}(\xi^{p^*}) = 0$ at the optimal point p^* . According to Equation (14), the components of the dual-affine Hessian are given by

$$\mathbf{H}_{ij}^*(\xi^{p_k}) = \frac{\partial a_j(\xi)}{\partial \xi_i} \Big|_{\xi^{p_k}} + \sum_{s=1}^n a_s(\xi^{p_k}) \Gamma_{is}^*{}^j(\xi^{p_k}), \quad 1 \leq i, j \leq n.$$

Based on this expression, we define the Jacobian matrix

$$(\mathbf{D} \mathbf{a}(\xi^{p_k}))_{ij} := \frac{\partial a_i(\xi)}{\partial \xi_j} \Big|_{\xi^{p_k}},$$

and the connection-adjusted matrix

$$\hat{\Gamma}_{\xi^{p_k}, \mathbf{a}}^* := \mathbf{H}^{*\top}(\xi^{p_k}) - \mathbf{D} \mathbf{a}(\xi^{p_k}),$$

whose components are

$$(\hat{\Gamma}_{\xi^{p_k}, \mathbf{a}}^*)_{ij} = \sum_{s=1}^n a_s(\xi^{p_k}) \Gamma_{js}^*{}^i(\xi^{p_k}).$$

Therefore, the **Line 3** in Algorithm 2 is written as:

$$\xi^{p_{k+1}} = \hat{R}_{\xi^{p_k}} \left(-[\mathbf{H}^{*\top}(\xi^{p_k})]^{-1} \mathbf{a}(\xi^{p_k}) \right). \quad (34)$$

The proof proceeds through the following steps:

Step 1: Establish an upper bound for $\|(\mathbf{H}^{*\top}(\xi^{p_k}))^{-1}\|$.

Let $\kappa := \|(\mathbf{H}^{*\top}(\xi^{p^*}))^{-1}\|$. Since $\mathbf{a}(\xi)$ is a smooth vector field, it follows that $\mathbf{H}^{*\top}(\xi)$ is smooth, too; therefore, there exist $r_H^* > 0$ and $\gamma_H^* > 0$ such that

$$\|\mathbf{H}^{*\top}(\xi^p) - \mathbf{H}^{*\top}(\xi^q)\| \leq \gamma_H^* \|\xi^p - \xi^q\| \quad (35)$$

for all $\xi^p, \xi^q \in B_{r_H^*}(\xi^{p^*}) := \{\xi^p \in \mathbb{R}^d : \|\xi^p - \xi^{p^*}\| < r_H^*\}$. Let

$$\epsilon^* = \min \left\{ r_H^*, \frac{1}{2\kappa\gamma_H^*} \right\}.$$

Assuming that $\xi^{p_k} \in B_{\epsilon^*}(\xi^{p^*})$, it follows that

$$\begin{aligned} \|(\mathbf{H}^{*\top}(\xi^{p^*}))^{-1}(\mathbf{H}^{*\top}(\xi^{p_k}) - \mathbf{H}^{*\top}(\xi^{p^*}))\| &\leq \|(\mathbf{H}^{*\top}(\xi^{p^*}))^{-1}\| \cdot \|\mathbf{H}^{*\top}(\xi^{p_k}) - \mathbf{H}^{*\top}(\xi^{p^*})\| \\ &\leq \kappa \gamma_H^* \|\xi^{p_k} - \xi^{p^*}\| \leq \kappa \gamma_H^* \epsilon^* \leq \frac{1}{2}. \end{aligned}$$

It follows from the properties of the consistent matrix norm that $\mathbf{H}^{*\top}(\xi^{p_k})$ is nonsingular and

$$\|(\mathbf{H}^{*\top}(\xi^{p_k}))^{-1}\| \leq \frac{\|(\mathbf{H}^{*\top}(\xi^{p^*}))^{-1}\|}{1 - \|(\mathbf{H}^{*\top}(\xi^{p^*}))^{-1}(\mathbf{H}^{*\top}(\xi^{p_k}) - \mathbf{H}^{*\top}(\xi^{p^*}))\|} \leq 2\kappa.$$

It also follows that for all $\xi^{p_k} \in B_{\epsilon^*}(\xi^{p^*})$, the vector $\beta(\xi^{p_k}) := -[\mathbf{H}^{*\top}(\xi^{p_k})]^{-1} \mathbf{a}(\xi^{p_k})$ is well defined.

Step 2: Establish an upper bound for the difference of second-order retraction $\hat{R}_{\xi^{p_k}}(\beta(\xi^{p_k}))$ and first-order retraction $\xi^{p_k} + \beta(\xi^{p_k})$.

Since R is a retraction (thus a smooth mapping) and $\mathbf{a}(\xi^{p^*}) = 0$ at ξ^{p^*} , it follows that there exists r_R and $\gamma_R > 0$ such that

$$\|\hat{R}_{\xi^{p_k}}(\beta(\xi^{p_k})) - (\xi^{p_k} + \beta(\xi^{p_k}))\| \leq \gamma_R \|\xi^{p_k} - \xi^{p^*}\|^2$$

for all $\xi^{p_k} \in B_{\epsilon^*}(\xi^{p^*})$.

Step 3: Establish an upper bound for $\hat{\Gamma}_{\xi, \mathbf{a}}^* := \mathbf{H}^{*\top}(\xi) - D \mathbf{a}(\xi)$.

Note that $\hat{\Gamma}_{\xi, \mathbf{a}}^*$ is a linear operator. Again, by a smoothness argument, it follows that there exist constants r_Γ^* and γ_Γ^* such that

$$\|\hat{\Gamma}_{\xi^p, \mathbf{a}}^* - \hat{\Gamma}_{\xi^q, \mathbf{a}}^*\| \leq \gamma_\Gamma^* \|\xi^p - \xi^q\| \quad (36)$$

for all $\xi^p, \xi^q \in B_{r_\Gamma^*}(\xi^{p^*})$. In particular, since $\mathbf{a}(\xi^{p^*}) = 0$, we have $\hat{\Gamma}_{\xi^{p^*}, \mathbf{a}}^* = 0$, hence

$$\|\hat{\Gamma}_{\xi^p, \mathbf{a}}^*\| \leq \gamma_\Gamma^* \|\xi^p - \xi^{p^*}\|$$

holds for all $\xi^p \in B_{r_\Gamma^*}(\xi^{p^*})$.

Step 4: Establish an upper bound for $\|\mathbf{D}\mathbf{a}(\xi^p) - \mathbf{D}\mathbf{a}(\xi^q)\|$ for all $\xi^p, \xi^q \in B_{\min(\epsilon^*, r_\Gamma^*)}(\xi^{p^*})$.

From (35) and (36), we have, for all $\xi^p, \xi^q \in B_{\min(\epsilon^*, r_\Gamma^*)}(\xi^{p^*})$,

$$\begin{aligned} \|\mathbf{D}\mathbf{a}(\xi^p) - \mathbf{D}\mathbf{a}(\xi^q)\| &\leq \|\mathbf{H}^{*\top}(\xi^p) - \mathbf{H}^{*\top}(\xi^q)\| + \|\hat{\Gamma}_{\xi^p, \mathbf{a}}^* - \hat{\Gamma}_{\xi^q, \mathbf{a}}^*\| \leq \gamma_H^* \|\xi^p - \xi^q\| + \gamma_\Gamma^* \|\xi^p - \xi^q\| \\ &\leq (\gamma_H^* + \gamma_\Gamma^*) \|\xi^p - \xi^q\|. \end{aligned}$$

Final step. Observe that we have

$$\begin{aligned} \xi^{p_{k+1}} - \xi^{p^*} &= \hat{R}_{\xi^{p_k}} \left(-(\mathbf{H}^{*\top}(\xi^{p_k}))^{-1} \mathbf{a}(\xi^{p_k}) \right) - \xi^{p^*} \\ &= \xi^{p_k} - (\mathbf{H}^{*\top}(\xi^{p_k}))^{-1} \mathbf{a}(\xi^{p_k}) - \xi^{p^*} \\ &\quad + \hat{R}_{\xi^{p_k}} \left(-(\mathbf{H}^{*\top}(\xi^{p_k}))^{-1} \mathbf{a}(\xi^{p_k}) \right) - \xi^{p_k} + (\mathbf{H}^{*\top}(\xi^{p_k}))^{-1} \mathbf{a}(\xi^{p_k}). \end{aligned}$$

This, together with the bounds developed above, yields

$$\begin{aligned} \|\xi^{p_{k+1}} - \xi^{p^*}\| &\leq \left\| \xi^{p_k} - (\mathbf{H}^{*\top}(\xi^{p_k}))^{-1} \mathbf{a}(\xi^{p_k}) - \xi^{p^*} \right\| + \gamma_R \|\xi^{p_k} - \xi^{p^*}\|^2 \\ &\leq \left\| (\mathbf{H}^{*\top}(\xi^{p_k}))^{-1} (\mathbf{a}(\xi^{p_k}) - \mathbf{a}(\xi^{p^*}) - \mathbf{H}^{*\top}(\xi^{p_k})(\xi^{p_k} - \xi^{p^*})) \right\| + \gamma_R \|\xi^{p_k} - \xi^{p^*}\|^2 \\ &\leq \left\| (\mathbf{H}^{*\top}(\xi^{p_k}))^{-1} \right\| \cdot \left\| \mathbf{a}(\xi^{p_k}) - \mathbf{a}(\xi^{p^*}) - \mathbf{D}\mathbf{a}(\xi^{p_k})(\xi^{p_k} - \xi^{p^*}) \right\| \\ &\quad + \left\| (\mathbf{H}^{*\top}(\xi^{p_k}))^{-1} \right\| \cdot \left\| \hat{\Gamma}_{\xi^{p_k}, \mathbf{a}}^*(\xi^{p_k} - \xi^{p^*}) \right\| + \gamma_R \|\xi^{p_k} - \xi^{p^*}\|^2 \\ &\leq 2\kappa \cdot \frac{1}{2}(\gamma_H^* + \gamma_\Gamma^*) \|\xi^{p_k} - \xi^{p^*}\|^2 + 2\kappa\gamma_\Gamma^* \|\xi^{p_k} - \xi^{p^*}\|^2 + \gamma_R \|\xi^{p_k} - \xi^{p^*}\|^2 \\ &= (\kappa(\gamma_H^* + 3\gamma_\Gamma^*) + \gamma_R) \|\xi^{p_k} - \xi^{p^*}\|^2, \end{aligned}$$

which completes the proof. \square

Please note that, in the proof, γ_R is a constant associated with the retraction \hat{R}_ξ under the connection ∇ , while γ_H^* and γ_Γ^* are constants associated with $\mathbf{H}^{*\top}$ and $\hat{\Gamma}_{\xi, \mathbf{a}}^*$ under the dual connection ∇^* , respectively. This distinction also shows the underlying duality between ∇ and ∇^* in the context of information geometry.

VI. EXPERIMENTS

To verify the local quadratic convergence of the dual Riemannian Newton method, we conduct numerical evaluation under three concrete optimization problems: (1) Minimization of the KL divergence with ℓ_2 regularization for a log-linear model, (2) α -divergence minimization for Gaussian models, and (3) MLE for Beta mixtures.

Since the dual Riemannian Newton method requires a pair of dual affine connections, we adopt the α -connections defined on the probability distribution family $p(x; \xi)$, which are introduced in Example II.1 as the canonical affine connections in the framework of information geometry.

Moreover, we compare the dual Riemannian Newton method with the natural gradient method and Adam [14], which is widely used for optimization of neural networks, to examine how efficient the local quadratic convergence is compared with other approaches.

A. Experiment1: Log-linear Models with Higher-Order Interactions

We first examine the performance of optimization methods in the scenario of *interaction decomposition*, which is a central topic in multivariate discrete modeling: it hierarchically separates the main effects, pairwise effects, and higher-order interactions, clarifying which orders of interaction are necessary for reliable modeling and interpretation [21]. Specifically, *log-linear models* [22] provide a natural representation of this decomposition within the exponential family framework, endowed with the geometric structure of dual coordinates (θ, η) and dual flatness under $(\nabla^{(e)}, \nabla^{(m)})$. In practice, pairwise modeling, known as Boltzmann machines or Ising models, is widely used for learning and inference [23], and their tensor formulations are known to be connected to structured higher-order interactions as well as scaling/balancing perspectives [24]. In this context, we explicitly describe the log-linear modeling framework that realizes interaction decomposition for empirical distributions, and then formulate the optimization problems and algorithmic comparisons that follow.

More precisely, we use an n th-order tensor $\hat{\mathcal{P}} \in \mathbb{R}_{>0}^{2 \times 2 \times \dots \times 2}$ to represent an empirical distribution; that is, we assume that $\hat{\mathcal{P}}$ satisfies $\hat{\mathcal{P}} \in \mathcal{S}_\Omega$ for the set \mathcal{S}_Ω of all strictly positive distributions on $\{0, 1\}^n$. Its log-probability can be expressed as a log-linear model with interaction terms indexed by Ω , hence \mathcal{S}_Ω is explicitly described as

$$\mathcal{S}_\Omega := \left\{ \mathcal{P} \in \mathbb{R}_{>0}^{2 \times \dots \times 2} \mid \log \mathcal{P}(x) = \sum_{A \in \Omega} \theta^A \prod_{i \in A} x^i - \psi(\theta) \right\},$$

where $\Omega = \bigcup_{k=1}^n \{ i_1 i_2 \cdots i_k \mid 1 \leq i_1 < i_2 < \cdots < i_k \leq n \}$. Thus $\log \mathcal{P}(x)$ can be alternatively written as

$$\log \mathcal{P}(x) = \sum_i \theta^i x^i + \sum_{i < j} \theta^{ij} x^i x^j + \sum_{i < j < k} \theta^{ijk} x^i x^j x^k + \cdots + \theta^{1 \cdots n} x^1 x^2 \cdots x^n - \psi, \quad (37)$$

where $x = (x^1, \dots, x^n) \in \{0, 1\}^n$ is an n -dimensional binary vector, $\theta = (\theta^1, \dots, \theta^{1 \cdots n})$ is a natural parameter vector, and ψ is the log-partition function (normalizing constant). The corresponding expectation parameters $\eta = (\eta^1, \dots, \eta^{1 \cdots n})$ represent the expected values of variable combinations:

$$\begin{aligned} \eta^i &= \mathbb{E}[x^i] = \Pr(x^i = 1), \\ \eta^{ij} &= \mathbb{E}[x^i x^j] = \Pr(x^i = x^j = 1), \dots, \\ \eta^{1 \cdots n} &= \mathbb{E}[x^1 \cdots x^n] = \Pr(x^1 = \cdots = x^n = 1). \end{aligned}$$

The log-linear model belongs to the exponential family, which forms a dually flat manifold equipped with a pair of affine connections $(\nabla^{(e)}, \nabla^{(m)})$. The manifold is not flat if $\alpha \neq \pm 1$. The fully connected Boltzmann machine \mathcal{S}_B [23] is a special case of the log-linear model given in Equation 38, which considers up to second-order interactions of the model. Formally, Ω is replaced with $\mathcal{B} = \{i \mid 1 \leq i \leq n\} \cup \{ij \mid 1 \leq i < j \leq n\}$ and the resulting log probability is given as

$$\mathcal{S}_B := \left\{ \mathcal{P} \in \mathbb{R}_{>0}^{2 \times \cdots \times 2} \mid \log \mathcal{P}(x) = \sum_i \theta^i x^i + \sum_{i < j} \theta^{ij} x^i x^j - \psi(\theta) \right\}, \quad (38)$$

where θ^i are called biases, θ^{ij} interaction weights, and $\psi(\theta)$ the log-partition function. This also forms a dually flat submanifold equipped with the affine connections $(\nabla^{(e)}, \nabla^{(m)})$ [25]. The goal of optimization is to estimate the parameters θ such that the model distribution $\mathcal{P} \in \mathcal{S}_B$ approximates the empirical distribution $\hat{\mathcal{P}} \in \mathcal{S}_\Omega$, which is typically assumed to be $\hat{\mathcal{P}} \notin \mathcal{S}_B$.

It is often formulated as the minimization of the KL divergence; that is, the optimization problem is given as

$$\bar{\mathcal{P}} = \underset{\mathcal{P} \in \mathcal{S}_B}{\operatorname{argmin}} D_{KL}(\hat{\mathcal{P}}, \mathcal{P}). \quad (39)$$

By applying the *Legendre transformation* to $\psi(\theta)$, we have $\varphi(\eta)$ given as

$$\varphi(\eta) = \max_{\bar{\theta}} (\bar{\theta} \eta - \psi(\bar{\theta})), \quad \bar{\theta} \eta = \sum_{x \in \mathcal{B}} \bar{\theta}^x \eta^x.$$

Using both $\psi(\theta)$ and $\varphi(\eta)$, we can write the KL-divergence as the ∇ -divergence:

$$D_{KL}(\hat{\mathcal{P}}, \mathcal{P}) = D[\mathcal{P}, \hat{\mathcal{P}}] = \psi(\theta_{\mathcal{P}}) + \varphi(\eta_{\hat{\mathcal{P}}}) - \sum_i \theta_{\mathcal{P}}^i \eta_{\hat{\mathcal{P}}}^i - \sum_{i < j} \theta_{\mathcal{P}}^{ij} \eta_{\hat{\mathcal{P}}}^{ij}.$$

Please note that $\eta_{\hat{\mathcal{P}}} = (\eta_{\hat{\mathcal{P}}}^1, \dots, \eta_{\hat{\mathcal{P}}}^{1 \dots n})$ is a constant vector determined by $\hat{\mathcal{P}}$ and not treated as a variable, and $\theta_{\mathcal{P}} = (\theta^1, \dots, \theta^{(n-1)n})$ denotes the corresponding natural parameter vector in the Boltzmann machine.

As established in Theorem IV.4, the natural gradient method is equivalent to the dual Riemannian Newton method in the optimization problem (39). However, when a regularization term is added to the objective:

$$\bar{\mathcal{P}} = \underset{\mathcal{P} \in \mathcal{S}_{\mathcal{B}}}{\operatorname{argmin}} \left(D_{KL}(\hat{\mathcal{P}}, \mathcal{P}) + \lambda_1 \sum_i (\theta_{\mathcal{P}}^i)^2 + \lambda_2 \sum_{i < j} (\theta_{\mathcal{P}}^{ij})^2 \right),$$

with tunable hyperparameters λ_1 and λ_2 controlling the strength of regularization, the equivalence no longer holds. In this case, the natural gradient and dual Riemannian Newton methods follow different update dynamics due to the presence of the regularization term.

In this setting, the dual Riemannian Newton method with different α values (corresponding to different retractions $R^{(\alpha)}$) is expected to show faster convergence than the natural gradient method in both the θ - and η -coordinate systems (the latter being equivalent to the mirror descent) with line search satisfying the Wolfe conditions [4], as well as the Euclidean optimizer *Adam*, in the vicinity of a local optimum.

To construct the target distribution $\hat{\mathcal{P}}$, we first consider the exponential family \mathcal{S}_{Ω} and assign random interaction parameters θ^A for each interaction subset A with order $r = |A|$. Specifically, the parameter θ^A is sampled uniformly from the interval

$$\theta^A \sim \mathcal{U}\left[-\frac{\text{base_scale}}{r}, \frac{\text{base_scale}}{r}\right],$$

where the base scaling factor is set to `base_scale` = 1.0. This construction yields larger parameter magnitudes for lower-order interactions and smaller ones for higher-order interactions, ensuring numerical stability. The distribution $\hat{\mathcal{P}}$ is then obtained by evaluating and normalizing the unnormalized log-probabilities over all 2^n binary states, resulting in an exact ground-truth distribution without sampling noise. This synthetic distribution is used to verify the efficiency of the proposed dual Riemannian Newton method.

We conducted numerical experiments under two configurations: (i) $(\lambda_1, \lambda_2, n) = (0.3, 0.8, 8)$ with initial θ values sampled from Uniform($-0.1, 0.0$), and (ii) $(\lambda_1, \lambda_2, n) = (0.5, 0.5, 4)$ with initial θ values sampled from Uniform($-0.25, 0.2$).

The corresponding results are illustrated in Figure 1. Moreover, Table I reports the wall-clock runtime and the average per-iteration time for all methods under the identical initialization, measured on an Apple M3 Pro with 36GB of memory. In both cases, our proposal of the dual Riemannian Newton method is significantly faster than other optimization methods: natural gradient, mirror descent, and Adam. Moreover, the case of $\alpha = 0$ (that is, the Levi–Civita connection) does not achieve the fastest convergence in either experiment, further emphasizing the advantage of the proposed dual Riemannian Newton method through the flexibility of choosing an appropriate α -connection, which is not limited to $\alpha = 0$. As Theorem IV.1 shows, the case $\alpha = 1$ takes the least time, since it coincides with Newton’s method in Euclidean space. Moreover, based on the quadratic convergence result established in Theorem V.1, we have:

$$\|\xi^{p_{k+1}} - \xi^{p^*}\| \leq (\kappa(\gamma_H^* + 3\gamma_\Gamma^*) + \gamma_R) \|\xi^{p_k} - \xi^{p^*}\|^2.$$

When the optimization is performed under the α -connection, this inequality becomes:

$$\|\xi^{p_{k+1}} - \xi^{p^*}\| \leq \left(\kappa(\gamma_H^{(-\alpha)} + 3\gamma_\Gamma^{(-\alpha)}) + \gamma_R^{(\alpha)} \right) \|\xi^{p_k} - \xi^{p^*}\|^2.$$

This result implies that while the proposed dual Riemannian Newton method guarantees quadratic convergence for any choice of α -connection, the convergence rate depends on the affine connection and is governed by the coefficient

$$\left(\kappa(\gamma_H^{(-\alpha)} + 3\gamma_\Gamma^{(-\alpha)}) + \gamma_R^{(\alpha)} \right).$$

This theoretical observation aligns well with our experimental findings, highlighting the sensitivity of convergence speed to the choice of affine connection.

B. Experiment 2: α -divergence Minimization

We next consider the projection problem based on the α -divergence,

$$f = D_{\bar{\alpha}}(p\|q) = \frac{4}{1 - \bar{\alpha}^2} \left(1 - \int_{\mathcal{X}} p(x)^{\frac{1+\bar{\alpha}}{2}} q(x)^{\frac{1-\bar{\alpha}}{2}} dx \right).$$

The use of α -divergence other than the Kullback–Leibler divergence is advantageous because it allows flexible control over the projection behavior [26], [27], [28] and the robustness [29] against the model misspecification. In this experiment, $\bar{\alpha}$ is fixed at 3, and distributions p and q are given as

$$p(x) = \mathcal{N} \left(x \left| \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} \right. \right), \quad q(x) = \mathcal{N} \left(x \left| \begin{bmatrix} \mu \\ \mu \end{bmatrix}, \begin{bmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix} \right. \right).$$

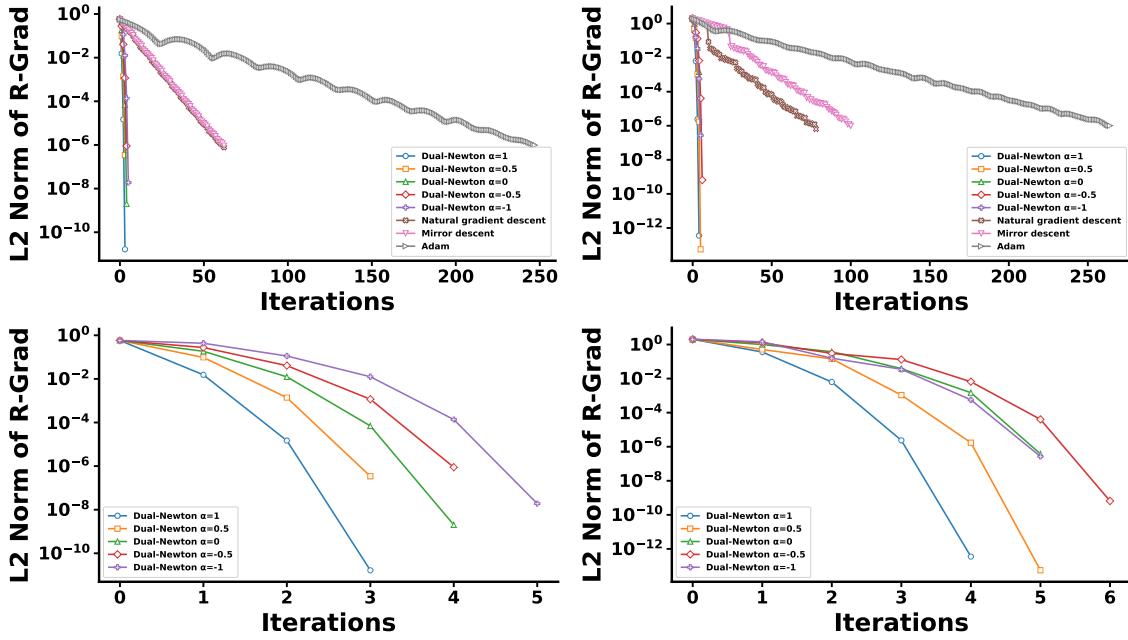


Fig. 1: Number of iterations for each method. (Left): $(\lambda_1, \lambda_2, N) = (0.5, 0.5, 4)$. (Right): $(\lambda_1, \lambda_2, N) = (0.3, 0.8, 8)$. The bottom row enlarges the first five or six steps of the dual Riemannian Newton method for clarity.

We set $\mu_1 = 2$, $\mu_2 = 1.5$, $\sigma_1 = 1.3$, and $\sigma_2 = 0.7$ and estimate the parameters μ, σ for q by minimizing the α -divergence. Here we introduce the α -connection in $q(x)$. The Fisher information matrix for this model is given by:

$$\mathbf{G}(\sigma) = \begin{pmatrix} \frac{2}{\sigma^2} & 0 \\ 0 & \frac{4}{\sigma^2} \end{pmatrix}.$$

The corresponding Christoffel symbols of the second kind associated with the α -connection are:

$$\Gamma_{ij}^{k(\alpha)}(\sigma) = \begin{cases} \begin{pmatrix} 0 & -\frac{1+\alpha}{\sigma} \\ -\frac{1+\alpha}{\sigma} & 0 \end{pmatrix}, & \text{if } k = 1, \\ \begin{pmatrix} \frac{1-\alpha}{2\sigma} & 0 \\ 0 & -\frac{1+2\alpha}{\sigma} \end{pmatrix}, & \text{if } k = 2. \end{cases}$$

It is straightforward to verify that the manifold is not flat when $\alpha \neq \pm 1$. Moreover, the α -divergence admits the following closed-form expression [30]:

$$\frac{4}{1-\bar{\alpha}^2} \left\{ 1 - (\sigma_1^2 \sigma_2^2)^{\frac{1-\bar{\alpha}}{4}} (\sigma^4)^{\frac{1+\bar{\alpha}}{4}} \left[\left(\frac{1+\bar{\alpha}}{2} \sigma^2 + \frac{1-\bar{\alpha}}{2} \sigma_1^2 \right) \left(\frac{1+\bar{\alpha}}{2} \sigma^2 + \frac{1-\bar{\alpha}}{2} \sigma_2^2 \right) \right]^{-\frac{1}{2}} \right\}$$

TABLE I: Total runtime, number of iterations, and average per-iteration time for each method (stopped when the Riemannian gradient is less than 10^{-6}) in Experiment 1. The lowest total runtime is underlined.

(a) $N = 4$				(b) $N = 8$			
Method	Iters	Total	Avg.	Method	Iters	Total	Avg.
Dual-Newton $\alpha = 1$	4	<u>0.001</u>	0.000333	Dual-Newton $\alpha = 1$	5	<u>0.006</u>	0.001177
Dual-Newton $\alpha = 0.5$	4	<u>0.001</u>	0.000178	Dual-Newton $\alpha = 0.5$	6	0.079	0.013139
Dual-Newton $\alpha = 0$	5	<u>0.001</u>	0.000172	Dual-Newton $\alpha = 0$	6	0.080	0.013410
Dual-Newton $\alpha = -0.5$	5	<u>0.001</u>	0.000174	Dual-Newton $\alpha = -0.5$	7	0.123	0.017535
Dual-Newton $\alpha = -1$	6	<u>0.001</u>	0.000174	Dual-Newton $\alpha = -1$	6	0.101	0.016900
Natural Gradient	63	0.014	0.000225	Natural Gradient	79	0.247	0.003132
Mirror Descent	63	0.039	0.000608	Mirror Descent	101	0.364	0.003590
Adam	248	0.008	0.000030	Adam	265	0.232	0.000876

$$\cdot \exp \left[-\frac{1-\bar{\alpha}^2}{8} \left(\frac{(\mu_1 - \mu)^2}{\frac{1+\bar{\alpha}}{2}\sigma^2 + \frac{1-\bar{\alpha}}{2}\sigma_1^2} + \frac{(\mu_2 - \mu)^2}{\frac{1+\bar{\alpha}}{2}\sigma^2 + \frac{1-\bar{\alpha}}{2}\sigma_2^2} \right) \right] \right\} .$$

In detail, in this experiment, we first set the divergence parameter to $\bar{\alpha} = 3$ in the α -divergence, and choose $\alpha = -0.4, -0.2, 0.0, 0.2$, or 0.4 to define the corresponding α -connections. The performance of the proposed dual Riemannian Newton method is then compared with that of the natural gradient and Adam methods. All methods are implemented in the (μ, σ) coordinate system, and the natural gradient method adopts a line search procedure satisfying the Wolfe conditions.

The corresponding results are illustrated in Figure 2. We also illustrate optimization paths from distinct retractions $R^{(\alpha)}$ in Figure 3. Moreover, Table II shows that the proposed dual Riemannian Newton method achieved convergence within 10–13 iterations, whereas the natural-gradient and Adam methods required 89 and 314 iterations, respectively. Despite similar per-iteration costs, the total runtime of the dual Riemannian Newton method is more than 10 times shorter than the other methods. As illustrated in Figure 3, its trajectories head almost straight toward the optimum, confirming that the proposed method efficiently captures the manifold curvature and converges more directly than first-order methods.

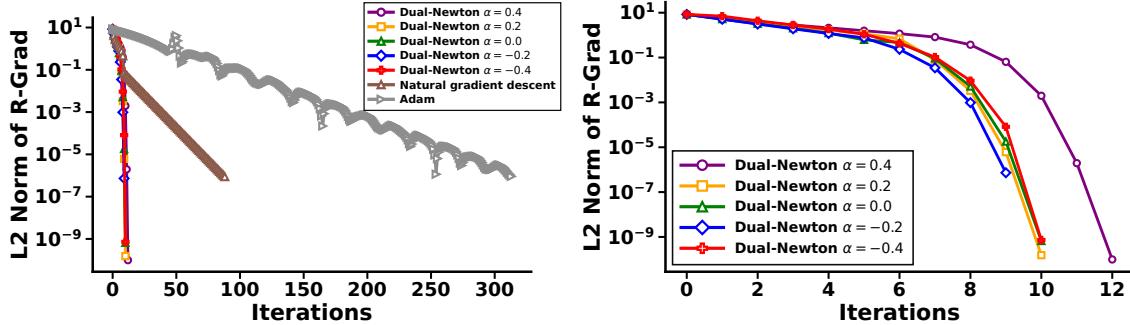


Fig. 2: (Left): The evolution of the Riemannian gradient norm versus iteration for each method. (Right): A magnified view of the first 12 steps of the dual Riemannian Newton method from the left panel.

TABLE II: Total runtime, iteration count, and average per-iteration time for each method (stopped when the Riemannian gradient is less than 10^{-6}) in Experiment 2. The lowest total runtime is underlined.

Method	Iterations	Total Time (s)	Avg. per Iteration (s)
Dual-Newton $\alpha = 0.4$	13	0.0007	0.000053
Dual-Newton $\alpha = 0.2$	11	0.0006	0.000053
Dual-Newton $\alpha = 0$	11	0.0006	0.000053
Dual-Newton $\alpha = -0.2$	10	<u>0.0005</u>	0.000054
Dual-Newton $\alpha = -0.4$	11	0.0008	0.000072
Natural Gradient	89	0.0079	0.000089
Adam	314	0.0184	0.000059

C. Experiment 3: Beta Mixture Model

Finally, to demonstrate the efficiency of the proposed dual Riemannian Newton method for *arbitrary* probability distributions beyond the exponential family, we apply it in this experiment to the maximum likelihood estimation of a fixed-weight *Beta mixture model*. The optimization problem is formulated as follows:

$$\min_{\{\alpha_k, \beta_k\}_{k=1}^K} \mathcal{L}(\alpha, \beta), \quad \mathcal{L}(\alpha, \beta) = - \sum_{i=1}^N \log \left[\sum_{k=1}^K w_k \text{Beta}(x_{i,1} | \alpha_k, \beta_k) \text{Beta}(x_{i,2} | \alpha_k, \beta_k) \right], \quad (40)$$

where K denotes the number of mixture components, α_k and β_k are the shape parameters of the k -th Beta component, and w_k are fixed mixture weights satisfying $\sum_{k=1}^K w_k = 1$. The objective

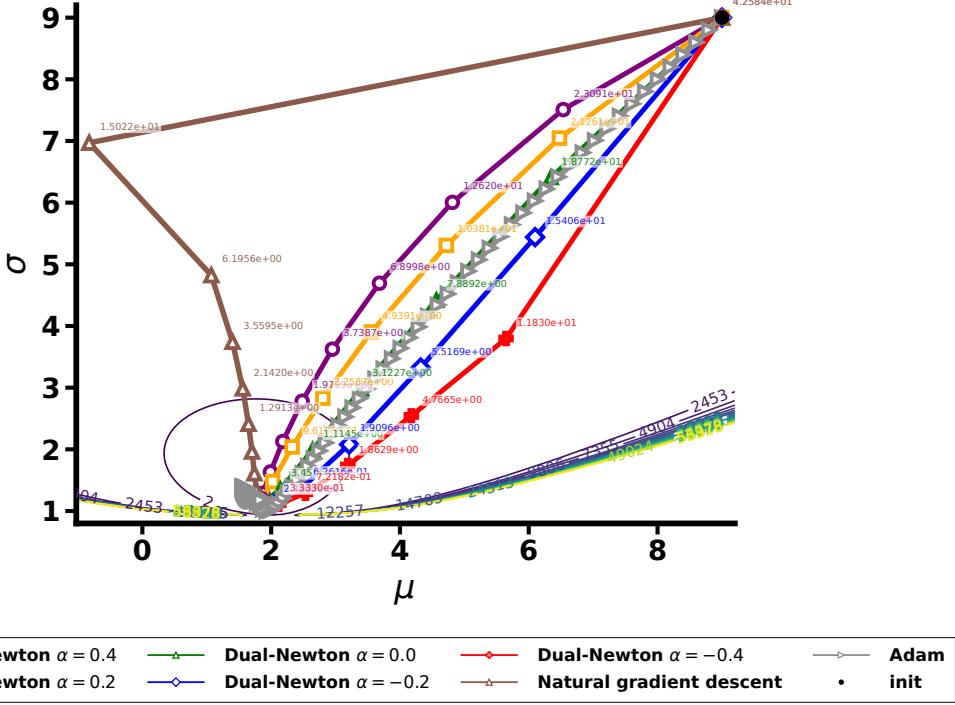


Fig. 3: Optimization paths for each method. The numbers over steps denote the α -divergence at that iteration (first five iterations shown). The bottom panel summarizes the legend for all methods.

$\mathcal{L}(\alpha, \beta)$ represents the negative log-likelihood of the observed data under the Beta mixture model. The dataset of $N = 5000$ samples is generated from a known mixture with parameters $w = [0.35, 0.4, 0.25]$, $\alpha_{\text{true}} = [2.0, 3.0, 5.0]$, and $\beta_{\text{true}} = [5.0, 2.0, 3.5]$. All methods are initialized from the same starting point. We compare the proposed dual Riemannian Newton method (with $\alpha \in \{0, 0.25, 0.5, 0.75, 1\}$) against the natural gradient method (with strong Wolfe line search) and a popularly used Euclidean optimizer, Adam.

The convergence speed is evaluated using the ℓ_2 norm of the Riemannian gradient at each iteration, as shown in Fig. 4. Moreover, Table III reports the wall-clock runtime and the average per-iteration time for all methods under a common setup (identical initialization and data). It can be observed that the dual Riemannian Newton method converges locally within eight iterations, achieving a significantly faster reduction of the Riemannian gradient norm than both the natural gradient method and Adam.

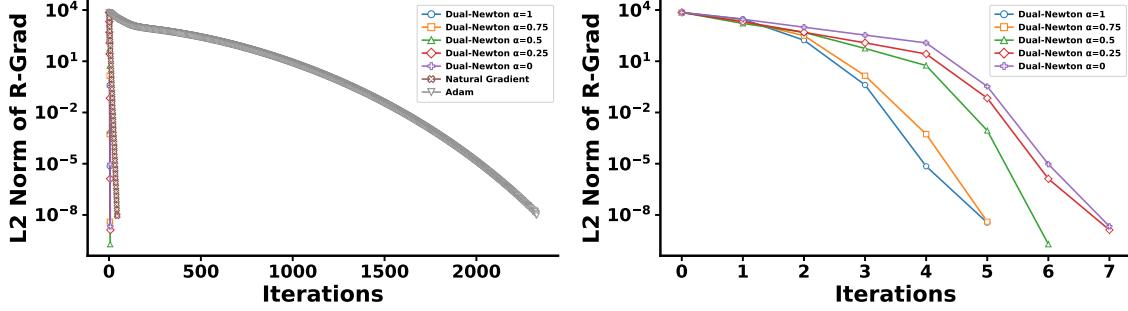


Fig. 4: (Left): The evolution of the Riemannian gradient norm versus iteration for each method. (Right): A magnified view for first seven steps of the dual Riemannian Newton method from the left panel.

TABLE III: Total runtime, iteration count, and average per-iteration time for each method (stopped when the Riemannian gradient is less than 10^{-8}) in Experiment 3. The lowest total runtime is underlined.

Method	Iterations	Total Time (s)	Avg. per Iteration (s)
Dual-Newton $\alpha = 1$	5	0.170	0.0284
Dual-Newton $\alpha = 0.75$	5	<u>0.146</u>	0.0243
Dual-Newton $\alpha = 0.5$	6	0.165	0.0236
Dual-Newton $\alpha = 0.25$	7	0.184	0.0230
Dual-Newton $\alpha = 0$	7	0.185	0.0231
Natural Gradient	45	0.410	0.0091
Adam	2288	6.891	0.0030

VII. DISCUSSION

In this section, we clarify how the choice of dual affine connections influences the descent property of the Newton direction through its connection to geodesic strict convexity. To this end, we take α -connections and discuss the choice of α . The key fact is that for a given objective function, there may exist a certain value of α with which the function exhibits *geodesic strict convexity*, which guarantees that each Newton step becomes a descent step and leads to stable convergence. In contrast, for other values of α , this convexity property may not hold, and the method may lose its descent guarantee. To make this relationship precise, we first introduce the definition of descent direction and geodesic strict convexity, and then analyze the positive

definiteness of the matrix $\mathbf{G}(\xi)\mathbf{H}^{(-\alpha)}(\xi)^\top$.

First, we introduce the definition of the descent direction.

Definition VII.1. Let (\mathcal{M}, g) be a Riemannian manifold and $f : \mathcal{M} \rightarrow \mathbb{R}$ be differentiable. A nonzero tangent vector $X_p \in T_p\mathcal{M}$ is a descent direction for f at $p \in \mathcal{M}$ if

$$\langle \text{grad } f(p), X_p \rangle < 0.$$

Equivalently, X_p is a non-ascent direction if $\langle \text{grad } f(p), X_p \rangle \leq 0$.

Given a suitably chosen α , the Newton direction can be guaranteed to serve as a descent direction for a particular objective function f at each iteration. Nevertheless, such a choice of α may not exist for any smooth functions f . We choose the retraction $R^{(\alpha)}$ associated with the connection $\nabla^{(\alpha)}$, and define the Newton direction as $X_p = -\text{Hess}^{(-\alpha)} f(p)^{-1}(\text{grad } f(p))$, which is the solution to the Hessian equation (12). Subsequently, under a local coordinate chart (φ, ξ) , the inner product of $\text{grad } f(p)$ and X_p can be written as

$$\langle \text{grad } f(p), X_p \rangle = -\nabla f^\top (\mathbf{G}(\xi)\mathbf{H}^{(-\alpha)}(\xi)^\top)^{-1} \nabla f,$$

where $\nabla f = \left(\frac{\partial(f \circ \varphi^{-1})}{\partial \xi_1}, \dots, \frac{\partial(f \circ \varphi^{-1})}{\partial \xi_n} \right)^\top$ is the gradient vector in local coordinates. If the matrix $\mathbf{G}(\xi)\mathbf{H}^{(-\alpha)}(\xi)^\top$ is symmetric positive definite (SPD) at every point ξ , then the Newton method is guaranteed to produce a descent direction. Note that $\mathbf{H}^{(-\alpha)}(\xi)\mathbf{G}(\xi) = \mathbf{G}(\xi)\mathbf{H}^{(-\alpha)}(\xi)^\top$, which implies that the product is symmetric. Furthermore, we will show that choosing the α -value for the matrix $\mathbf{G}(\xi)\mathbf{H}^{(-\alpha)}(\xi)^\top$ to be SPD is closely related to the notion of *geodesic strict convexity*, analogous to the classical Riemannian geometry setting equipped with the Levi-Civita connection. We clarify this relationship through the following definition.

Definition VII.2 (α -geodesically strictly). Let $(\mathcal{M}, g, \nabla^{(\alpha)}, \nabla^{(-\alpha)})$ be a smooth manifold and $f : \mathcal{M} \rightarrow \mathbb{R}$ be a twice-differentiable function. The α -geodesically strictly convex can be defined in the following two ways:

- 1) The function f is α -geodesically strictly convex in a neighborhood of $p \in \mathcal{M}$; that is, for every geodesic with $\nabla^{(\alpha)}$, $\gamma : [-\varepsilon, \varepsilon] \rightarrow \mathcal{M}$ with $\gamma^{(\alpha)}(0) = p$, $\dot{\gamma}^{(\alpha)}(0) = X_p$, we have

$$\left. \frac{d^2}{dt^2} f(\gamma^{(\alpha)}(t)) \right|_{t=0} > 0.$$

- 2) The Hessian with $\nabla^{(-\alpha)}$ of f at p is positive definite, i.e., for every nonzero tangent vector $X_p \in T_p\mathcal{M}$,

$$\langle \text{Hess}^{(-\alpha)} f(p)[X_p], X_p \rangle > 0.$$

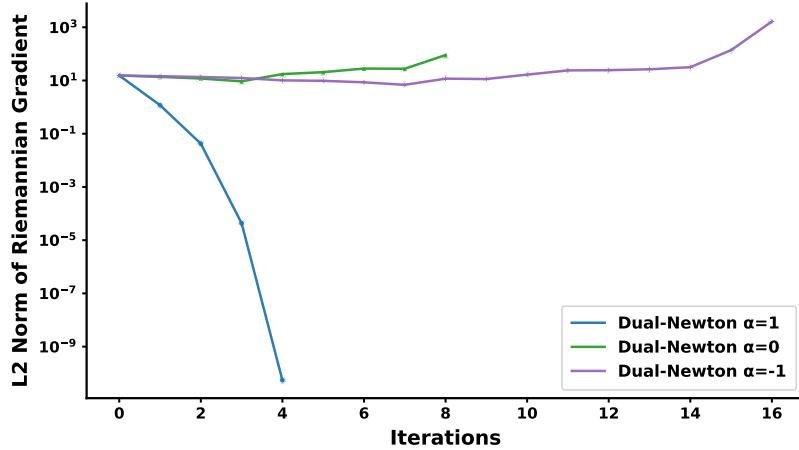


Fig. 5: The evolution of the Riemannian gradient norm versus iteration for each α connection. It demonstrates that every step is a descent step when $\alpha = 1$.

Equivalently, in a local coordinate chart (φ, ξ) with $\xi^p = \varphi(p)$, the tangent vector can be expressed as

$$X_p = \sum_{k=1}^n \beta_k(\xi^p) \frac{\partial}{\partial \xi_k} \Big|_{\xi^p},$$

and the above condition can be written in matrix form as

$$\boldsymbol{\beta}(\xi^p)^\top [\mathbf{G}(\xi^p) \mathbf{H}^{(-\alpha)}(\xi^p)^\top] \boldsymbol{\beta}(\xi^p) > 0,$$

where $\boldsymbol{\beta}(\xi^p) = (\beta_1(\xi^p), \dots, \beta_n(\xi^p))^\top$ is the coordinate representation of X_p .

Therefore, if we choose a value of α such that the function f is geodesically strictly convex, we can ensure that the Newton direction is a *descent direction* at every iteration.

For example, consider the objective function used in our first experiment:

$$f = D_{KL}(\hat{\mathcal{P}}, \mathcal{P}) + \lambda_1 \sum_i (\theta_{\mathcal{P}}^i)^2 + \lambda_2 \sum_{i < j} (\theta_{\mathcal{P}}^{ij})^2,$$

where $\lambda_1 > 0$ and $\lambda_2 > 0$. According to Theorem IV.1, if we choose $\alpha = 1$, and work with the geodesic $\gamma^{(1)}$ associated with the connection $\nabla^{(1)}$, then the following equality holds:

$$(\mathbf{G}(\theta) \mathbf{H}^{(-1)}(\theta)^\top)_{ij} = \frac{\partial^2 (f \circ \varphi_\theta^{-1})}{\partial \theta_i \partial \theta_j},$$

which implies that

$$\mathbf{G}(\theta) \mathbf{H}^{(-1)}(\theta)^\top \succ 0.$$

Thus, the function f is 1-geodesically strictly convex. However, for other values $\alpha \neq 1$, the function f may no longer be geodesically strictly convex, meaning that the matrix $\mathbf{G}(\theta) \mathbf{H}^{(-\alpha)}(\theta)^\top$ is not guaranteed to be SPD at all points in the manifold. For example, when the initial point is far from the optimum, a line-search strategy (e.g., enforcing the strong Wolfe conditions) can still ensure that Newton's method with $\alpha = 1$ produces a descent step at every iteration. In contrast, for $\alpha = 0$ and $\alpha = -1$, the Newton directions may not be descent directions, since $\mathbf{G}(\theta) \mathbf{H}^{(-\alpha)}(\theta)^\top$ may become non-positive definite or even non-invertible (e.g., for $(\lambda_1, \lambda_2, n) = (0.3, 0.8, 3)$) as shown in Figure 5.

VIII. CONCLUSION

We have introduced the *dual Riemannian Newton method* for optimization on statistical manifolds endowed with dual affine connections (∇, ∇^*) . The method explicitly leverages dual geometry: under a ∇ -based retraction, the associated Newton equation is solved with respect to ∇^* . On dually flat manifolds with affine coordinates, our proposed dual Riemannian Newton method reduces exactly to the Euclidean Newton step (a Hessian solve using the second derivatives of the objective function). For canonical ∇ -projection problems in information geometry, updates of the dual Riemannian Newton method coincide with those of the natural gradient method, clarifying when first-order and second-order geometry agree. We have established local quadratic convergence and observed consistent empirical improvements over first-order baselines in our numerical experiments.

IX. APPENDIX

In this appendix, we present the preliminaries of information geometry that were not included in the main text.

Definition IX.1 (Affine Connection). *Let \mathcal{M} be a smooth manifold and $\mathfrak{X}(\mathcal{M})$ be the set of all smooth vector fields on \mathcal{M} . An affine connection on \mathcal{M} is a map*

$$\nabla : \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \rightarrow \mathfrak{X}(\mathcal{M}), \quad (X, Y) \mapsto \nabla_X Y$$

that satisfies the following properties for all $X, Y, Z \in \mathfrak{X}(\mathcal{M})$, $f_1, f_2 \in C^\infty(\mathcal{M})$, and $a, b \in \mathbb{R}$:

- (1) $\nabla_{f_1 X + f_2 Y} Z = f_1 \nabla_X Z + f_2 \nabla_Y Z$,
- (2) $\nabla_X (aY + bZ) = a \nabla_X Y + b \nabla_X Z$,

$$(3) \quad \nabla_X(f_1 Y) = (X f_1) Y + f_1 \nabla_X Y.$$

Definition IX.2 (Differential of a Smooth Map Between Manifolds). *Let $F : \mathcal{M} \rightarrow \mathcal{N}$ be a smooth map between smooth manifolds. The differential of F at $p \in \mathcal{M}$, denoted by $DF(p)$, is the linear map*

$$DF(p) : T_p \mathcal{M} \rightarrow T_{F(p)} \mathcal{N},$$

defined by

$$DF(p)[X_p](f) = \xi(f \circ F) \quad \text{for all } X_p \in T_p \mathcal{M}, f \in C^\infty(\mathcal{N}),$$

where the right-hand side denotes the directional derivative of the composite function $f \circ F$ at p in the direction X_p , and $T_p \mathcal{M}$ is the tangent space of \mathcal{M} at p .

Definition IX.3 (Covariant Derivative). *Let \mathcal{M} be a smooth manifold with an affine connection ∇ , and let $\gamma : I \rightarrow \mathcal{M}$ be a smooth curve (with $I \subset \mathbb{R}$ an interval). An operator*

$$\frac{D}{dt} : \mathfrak{X}(\gamma) \rightarrow \mathfrak{X}(\gamma)$$

is called the covariant derivative along γ if, for any $Y, Z \in \mathfrak{X}(\gamma)$, $a, b \in \mathbb{R}$, and $f \in C^\infty(I)$, it satisfies all the following conditions:

- (1) $\frac{D}{dt}(aY + bZ) = a\frac{D}{dt}Y + b\frac{D}{dt}Z,$
- (2) $\frac{D}{dt}(fZ) = \frac{df}{dt}Z + f\frac{D}{dt}Z,$
- (3) $\frac{D}{dt}(V \circ \gamma)(t) = \nabla_{\dot{\gamma}(t)}V \quad \text{for all } V \in \mathfrak{X}(\mathcal{M}).$

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