Numerical Stability for Lugiato Lefever Equation (LLE)

Conceptual issues with numerical methods for solving DE:

1. Problem begins when $t \to \infty(propagation\ steps)$

2. Error adds up as order of O(h^p)

3. If the system is stable while approaching $t \to \infty$, $it\ is\ termed\ as\ \boldsymbol{absolute\ stability}$.

Stability analysis:

If we have any differential equation in the form:

$$y' = \lambda y(t)\ldots\ldots\ldots\ldots\ldots\ldots[1]$$

$$where\ y(0) = y0[initial\ condition]$$

We find the solution as:

$$y(t) = y0e^{\lambda t}\ldots\ldots\ldots\ldots\ldots[2]$$

Now let say we have an initial condition as y0+α, where α is a small change

Then, from equation 2:

$$y(t) = (y0 + \alpha)e^{\lambda t}\ldots\ldots\ldots\ldots\ldots[3]$$

We may say a small change in initial condition from [2] to [3], would not bring much change if λ is less than 0. So, while considering stability, our primary concern would be to keep $\lambda \leq 0$

Stability criterion for different solver methods:

1. Explicit Euler's Method:

We know for forward Euler's method:

$$y_{n+1} = y_n + hf(t_n, y_n)$$

$$= y_n + h\lambda y_n \,[from\,[1]]$$

$$= y_n(1 + \lambda h) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots [4]$$

So, if we start with y0 and intend to find the value at n step, equation [4] becomes:

$$y_n = (1 + \lambda h)^n y0$$

While we approach with our previously found stability criteria, we end up with:

$$|1 + \lambda h| < 1$$

Solving:

$$1 + \lambda h < 1$$

$$Or, \lambda h < 0$$

$$and,$$

$$-1 - \lambda h < 1$$

$$or, h < \frac{-2}{\lambda} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots [5]$$

So, the forward/explicit Euler method is conditionally stable, and we are bound with "sufficiently" small step size.

Stability in terms of LLE:

The normalized equation:

$$\frac{\partial\psi}{\partial t} = -(1+i\alpha)\psi + i|\psi|^2\psi - i\frac{\beta}{2}\frac{\partial^2\psi}{\partial\theta^2} + F$$

Here,

$$\alpha = frequency\ detunning$$

$$\beta = dispersion$$

$$F = pump\ power$$

Then neglecting nonlinearity and pumping, and applying β=-1, we have:

$$\frac{\partial\psi}{\partial t} = -(1+i\alpha)\psi + i\frac{1}{2}\frac{\partial^2\psi}{\partial\theta^2}\ldots\ldots\ldots\ldots\ldots[6]$$

Equation [6] is called the **Dispersive wave equation.**

Converting it into the mode number domain:

We may write, $\frac{\partial^2}{\partial\theta^2} \approx i^2m^2$

So, equation [6] becomes:

$$\frac{\partial\tilde{\psi}}{\partial t} = -(1+i\alpha + \mathrm{i}m^2/2)\tilde{\psi}[Fourier\ transform\ of\ spatial\ domain][7]$$

Performing integration on both sides of [7]:

$$\tilde{\psi}(t) = \widetilde{\psi_0}e^{-(1+i\alpha+\mathrm{i}m^2/2)t}$$

Now turning back to forward Euler method which we have just discussed:

$$\tilde{\psi}(t_n+1) = \check{\psi}(t_n) + h\check{\psi}(t_n)(-(1+i\alpha+\mathrm{i}m^2/2))$$

$$or,$$

$$\tilde{\psi}(t_n+1) = \tilde{\psi}(t_n)(1-(1+i\alpha+\mathrm{i}m^2/2)h)\ldots\ldots\ldots\ldots[8]$$

Now we would consider applying stability condition on equation [8]:

$$\lambda = -(1+i\alpha+\mathrm{i}m^2/2)$$

Then from [5],

$$h < \frac{-2}{\lambda}$$

Applying the value and taking the magnitude:

$$h < \frac{-2}{-(1+(\alpha+m^2/2)^2)^{1/2}}$$

$$Or, \boldsymbol{(1+(\alpha+m^2/2)^2)^{1/2}h > 2} \ldots\ldots\ldots.[\mathbf{9}]$$

Equation [9] is the stability criterion for LLE while implemented with forward Euler's method. But **with higher values of m**, it is critical to maintain the inequality of [9].