Book Note 1.6

Considering the eigenvalue problem for the eigenvalue λ and eigenvector $\vec{x}$:

$$A\vec{x} + \epsilon\vec{B}(\vec{x}) = \lambda\vec{x} \tag{1}$$

In order for this to qualify as an algebric equation, A has to be a linear operator matrix. Letting, $\epsilon\vec{B}(\vec{x})$ as small perturbation, $\vec{B}(\vec{x})$ does not have to be linear. Let the unperturbed eigensolution with eigenvalue a and eigenvector $\vec{e}$ has the form:

$$A\vec{e} = a\vec{e} \tag{2}$$

If matrix A is not symmetric, its transpose will have different eigenvector, let say $\vec{e^t}$ with the same eigenvalue. As left eigenvector of any matrix is equal to the right eigenvector of A^T, it can be written:

$$\vec{e^t}A = a\vec{e^t} \tag{3}$$

Restricting attention to the case where a is a single root with only one independent eigenvector $\vec{e}$. Then $\vec{e^t}$ is orthogonal to all other eigenvectors of A. Applying expansion in the power of ϵ starting from the unperturbed eigensolution:

$$\vec{x}(\epsilon) = \vec{e} + \epsilon\vec{x}_1 + \epsilon^2\vec{x}_2 + ... \tag{4}$$

$$\lambda(\epsilon) = a + \epsilon\lambda_1 + \epsilon^2\lambda_2 + ... \tag{5}$$

Considering expansion of coefficients of e^n from (4) and (5):
at $\epsilon^0 : \vec{x}(0) = \vec{e}$ **and** $\lambda(0) = a$
at $\epsilon^1 : \vec{x}(\epsilon) = \vec{e} + \epsilon\vec{x}_1$ **and** $\lambda(1) = a + \epsilon\lambda_1$ **Then applying into equation (1):**

$$A\vec{e} = a\vec{e} \tag{6}$$

and

$$A\vec{x}_1 + \vec{B}(\vec{e}) = a\vec{x}_1 + \lambda_1\vec{e} \tag{7}$$

rearranging equation (7):

$$(A - a)\vec{x}_1 = \lambda_1\vec{e} - \vec{B}(\vec{e}) \tag{8}$$

From the left side of equation (8):

$$\vec{e^t}.(A - a)\vec{x}_1 = [\vec{e^t}(A - a)].\vec{x}_1 = (a - a)\vec{e^t}.\vec{x}_1 = 0$$

Which means there is no component in the direction of $\vec{e}$, then (8) becomes:

$$\vec{e^t}.[\lambda_1\vec{e} - \vec{B}(\vec{e})] = 0 \tag{9}$$

Then, first perturbation of eigenvalue:

$$\lambda_1 = \frac{\vec{e^t}.\vec{B}(\vec{e})}{\vec{e^t}.\vec{e}} \tag{10}$$

If $\vec{B}(\vec{e})$ is nonlinear, then the eigenvalue is not independent of the magnitude of the eigenvector. Applying this back to equation (8):

$$(A - a)\vec{x}_1 = -\vec{B}\vec{e} + \frac{\vec{e^t}.\vec{B}(\vec{e})}{\vec{e^t}.\vec{e}}\vec{e} \tag{11}$$

In this equation, the second term refers to the part where $\vec{B}(\vec{e})$ is parallel to $\vec{e}$. Then equation (11) as a whole, refers to the part where $\vec{B}(\vec{e})$ is perpendicular to $\vec{e}$. So:

$$(A-a)\vec{x}_1 = -\vec{B}(\vec{e}) + \frac{\vec{e^t}.\vec{B}(\vec{e})}{\vec{e^t}.\vec{e}}\vec{e} = -\vec{B}(\vec{e})_{\perp} \qquad (12)$$

As the right side has no component in $\vec{e}$, (A-a) can be inverted to obtain solution for $\vec{x_1}$. But this will not be an unique solution as it is possible to add an arbitrary multiple of to without changing $(A-a)\vec{x_1}$, now the solution for $\vec{x_1}$ becomes:

$$\vec{x}_1 = -(A-a)^{-1}\vec{B}(\vec{e})_{\perp} + k_1\vec{e} \qquad (13)$$

Where k_1 is any arbitrary scalar. Now equation (13) becomes:

$$\vec{x}_1 = -(A-a)^{-1}\left(-\vec{B}(\vec{e}) + \frac{\vec{e^t}.\vec{B}(\vec{e})}{\vec{e^t}.\vec{e}}\vec{e}\right) + k_1\vec{e} \qquad (14)$$

Now, $-(A-a)^{-1}$ is restricted only in the orthogonal direction of $\vec{e}$. If the complete eigendecomposition of A is known, then writing x_1 as sum of all other eigenvectors $\vec{e}^{(j)}$, equation (14) becomes:

$$\vec{x}_1 = \sum_j \frac{\vec{e}^{(j)t}.\vec{B}(\vec{e})}{(a-a^j)(\vec{e}^{(j)t}.\vec{e}^{(j)})}\vec{e}^{(j)} + k_1\vec{e} \qquad (15)$$

References

[1] **Hinch, E. (1991). Perturbation Methods (Cambridge Texts in Applied Mathematics). Cambridge: Cambridge University Press. doi:10.1017/CBO9781139172189**