Second Order ODE and Stiffness

Let say we have a differential equation in the form of:

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0 \quad [1]$$

Here p and q are constants, and we are considering a homogeneous system. Let consider the solution by assumption first.

Let,

$$y = e^{rx}$$

Applying it to [1]:

$$r^2e^{rx} + p(re^{rx}) + q(e^{rx}) = 0 \; [2]$$

$$e^{rx}(r^2 + pr + q) = 0$$

$$Or, r^2 + pr + q = 0 \quad [3]$$

$$Equating\ to\ standard, ax^2 + bx + c = 0$$

Then, root:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad [4]$$

[3] might end up with three different types of solutions:

1. two real roots

2. both real roots are equal

3. two complex roots

1. Two real roots:

From [4], Real roots would come if $p^2 - 4q$>0

From [3]

let the two real roots are r1 and r2

Then,

$$y = Ae^{r1x} + Be^{r2x}$$

2. Equal real roots:

We would end up in this situation if $\mathrm{p}^2 - 4q = 0$

Then the solution is, $\mathrm{y} = \mathrm{Ae}^{\mathrm{rx}} + \mathrm{Bxe}^{\mathrm{rx}}$

3. Two complex roots:

Now the situation is: $\mathrm{p}^2 - 4q < 0$

Solution to this can be written in the form of:

y=$\mathrm{Ae}^{(\mathrm{a+ib})\mathrm{x}} + \mathrm{Be}^{(\mathrm{a\text{-}ib})\mathrm{x}} = \mathrm{e}^{\mathrm{a}}(\mathrm{Ae}^{\mathrm{ib}} + \mathrm{Be}^{\text{-}\mathrm{ib}})$

Now we would consider a second order damped RLC circuit to relate to stiffness:

$$\mathrm{L}\frac{\mathrm{d}^2\mathrm{Q}}{\mathrm{dt}^2} + \mathrm{R}\frac{\mathrm{dQ}}{\mathrm{dt}} + \frac{1}{\mathrm{C}}\mathrm{Q} = 0$$

Proceeding to solve:

$$\frac{\mathrm{d}^2\mathrm{Q}}{\mathrm{dt}^2} + \frac{\mathrm{R}}{\mathrm{L}}\frac{\mathrm{dQ}}{\mathrm{dt}} + \frac{1}{\mathrm{LC}}Q = 0 \qquad [5]$$

From [3]:

$$\mathrm{r}^2 + (\mathrm{R/L})\mathrm{r} + (1/\mathrm{LC}) = 0$$

Condition for real roots:

$$(R/L)^2 - 4/LC > 0$$

$$Or, \frac{R^2}{L^2} > \frac{4}{LC}$$

$$Or, \frac{R^2}{L} > \frac{4}{C}$$

$$Or, R^2C > 4L$$

$$Or, 1/4 > \frac{L}{R^2C}$$

$$Or, \frac{L}{R^2C} < 1/4$$

$$\approx \frac{L}{R^2C} << 1 \quad [6]$$

Solving, $\mathrm{r}_{1,2} = \frac{-R/L \pm \sqrt{(R/L)^2 - 4/LC + (\frac{2}{\mathrm{RC}})^2 - (\frac{2}{\mathrm{RC}})^2}}{2} = \frac{-\mathrm{R/L} \pm \sqrt{(\mathrm{R/L}-2/\mathrm{RC})^2}}{2}$

Now:

$$\mathrm{r}_1 = -(\mathrm{R/L}) \text{ and } \mathrm{r}_2 = -(1/\mathrm{RC})$$

Then the solution becomes:

$$\mathrm{Q} = \mathrm{Q}_1\mathrm{e}^{(-\mathrm{R/L})\mathrm{t}} + \mathrm{Q}_2\mathrm{e}^{(-1/\mathrm{RC})\mathrm{t}}$$

Referring to [6]:

$$\frac{L}{R^2C} \ll 1$$

$Or, L/R \ll RC$ [7]

$Or, R/L \gg 1/RC$ [8]

Let R/L=1000(1/RC), in this case circuit is highly damped. So according to the condition, maximum impact is coming from 2nd part of [5]. To solve this equation numerically, step size must be less than L/R to track things happening before “it is too late” because the circuit is highly damped.

Now to summarize, if step size is not maintained on fixed “smaller” level as discussed, numerical solution to this equation would be unstable, graph plot will blow out from the acceptable range. This special criterion which bounds the numerical solution of an equation between stability and instability, is called “stiffness”. Types of these equations are called stiff equations. Interestingly, step size dependency varies from one ODE solver method to another. So, one equation being stiff with one ODE solver method can produce stable result with another method.