Data-Driven Hierarchical Runge-Kutta and Adams Methods for Nonlinear Dynamical Systems

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2025

Abstract

This paper presents a comprehensive implementation of numerical methods for solving nonlinear differential equations, including Euler's Method, Data-Driven Euler's Method, Runge-Kutta 3rd order method, Data-Driven Runge-Kutta, Adams Methods, and Data-Driven Adams Methods. We introduce novel data-driven hierarchical architectures inspired by transformer networks that enhance traditional numerical integration methods. The framework is implemented in C/C++ with Objective-C visualization capabilities, making it suitable for macOS and VisionOS platforms.

1 Introduction

Numerical methods for solving ordinary differential equations (ODEs) are fundamental tools in scientific computing. We present a comprehensive framework including Euler's Method, Data-Driven Euler's Method, Runge-Kutta 3rd order, Data-Driven Runge-Kutta, Adams Methods, and Data-Driven Adams Methods.

2 Euler's Method

Euler's Method is the simplest numerical method for solving ODEs. It is a first-order explicit method:

$$y_{n+1} = y_n + h \cdot f(t_n, y_n) \tag{1}$$

where h is the step size, f is the ODE function, and y_n is the state at time t_n. The local truncation error is $O(h^2)$, making it a first-order method.

2.1 Data-Driven Euler's Method

We extend Euler's Method with a hierarchical transformer-inspired architecture:

$$y_{n+1} = y_n + h \cdot f(t_n, y_n) + h \cdot \alpha \cdot \text{Attention}(y_n) \tag{2}$$

where α is a learning rate and $\text{Attention}(y_n)$ is a hierarchical attention mechanism that refines the Euler step using multiple transformer layers.

3 Runge-Kutta 3rd Order Method

The Runge-Kutta 3rd order method (RK3) is defined by the following stages:

$$k_1 = f(t_n, y_n) \tag{3}$$

$$k_2 = f(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_1) \tag{4}$$

$$k_3 = f(t_n + h, y_n - hk_1 + 2hk_2) \tag{5}$$

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 4k_2 + k_3) \tag{6}$$

where h is the step size, f is the ODE function, and y_n is the state at time t_n.

4 Adams Methods

Adams-Bashforth and Adams-Moulton methods are multi-step methods that use information from previous steps.

4.1 Adams-Bashforth 3rd Order

The predictor step:

$$y_{n+1} = y_n + \frac{h}{12}(23f_n - 16f_{n-1} + 5f_{n-2}) \tag{7}$$

4.2 Adams-Moulton 3rd Order

The corrector step:

$$y_{n+1} = y_n + \frac{h}{12}(5f_{n+1} + 8f_n - f_{n-1}) \tag{8}$$

5 Parallel, Distributed, and Concurrent Execution

We extend all numerical methods with comprehensive parallel and distributed computing support:

5.1 Parallel Execution Modes

- **OpenMP**: Shared-memory multi-threading for single-node parallelization
- **POSIX Threads (pthreads)**: Fine-grained thread control
- **MPI**: Distributed computing across multiple nodes
- **Hybrid**: Combined MPI + OpenMP for hierarchical parallelism

5.2 Concurrent Execution

Multiple methods can execute simultaneously, enabling real-time comparison and ensemble approaches. The concurrent execution framework manages resource allocation and synchronization across parallel method instances.

5.3 Real-Time, Online, and Dynamic Methods

We extend all numerical methods with real-time, online, and dynamic execution capabilities:

5.3.1 Real-Time Methods

Real-time methods process streaming data with minimal latency, suitable for live data feeds and continuous monitoring applications. They feature:

- Streaming data buffers for continuous processing
- Callback mechanisms for immediate result delivery
- Low-latency execution optimized for real-time constraints

5.3.2 Online Methods

Online methods adapt to incoming data with incremental learning, adjusting parameters based on observed errors:

- Adaptive step size control based on error estimates
- Learning rate mechanisms for parameter adjustment
- History tracking for adaptive refinement

5.3.3 Dynamic Methods

Dynamic methods provide fully adaptive execution with dynamic step sizes and parameter adaptation:

- Real-time error and stability estimation
- Dynamic step size adjustment
- Parameter history tracking
- Adaptive mode switching

5.4 Nonlinear Programming-Based Solvers

We extend the framework with nonlinear programming (NLP) methods for solving ODEs and PDEs as optimization problems. This includes:

5.4.1 Nonlinear ODE Solvers

Nonlinear ODE solvers formulate ODE integration as an optimization problem:

$$\min \int_{t_0}^{t_f} \|\dot{y} - f(t, y)\|^2 dt \tag{9}$$

Methods include:

- Gradient descent
- Newton's method
- Quasi-Newton (BFGS)
- Interior point methods
- Sequential quadratic programming (SQP)
- Trust region methods

5.4.2 Nonlinear PDE Solvers

Nonlinear PDE solvers apply optimization techniques to partial differential equations:

$$\min \int_{\Omega} \|\frac{\partial u}{\partial t} - F(t, x, u, \nabla u)\|^2 d\Omega \tag{10}$$

5.5 Additional Distributed, Data-Driven, Online, and Real-Time Solvers

We provide comprehensive combinations of execution modes:

5.5.1 Distributed Data-Driven Solvers

Combine distributed computing with hierarchical data-driven methods for scalable, adaptive solutions.

5.5.2 Online Data-Driven Solvers

Combine online learning with data-driven architectures for adaptive, incremental refinement.

5.5.3 Real-Time Data-Driven Solvers

Combine real-time processing with data-driven methods for low-latency, adaptive streaming.

5.5.4 Distributed Online Solvers

Combine distributed computing with online learning for scalable, adaptive execution.

5.5.5 Distributed Real-Time Solvers

Combine distributed computing with real-time processing for scalable, low-latency execution.

6 Hierarchical and Stacked Architecture

We propose hierarchical and stacked architectures inspired by transformer networks that process ODE solutions through multiple layers with attention mechanisms. Each layer applies transformations to the state space, enabling adaptive refinement of the numerical solution.

The hierarchical/stacked solver consists of:

- Multiple processing layers with learnable weights
- Attention mechanisms for state-space transformations
- Residual connections for gradient flow
- Adaptive step size control based on hierarchical features
- Stacked configurations for deep hierarchical processing

6.1 Stacked Configurations

Stacked methods process solutions through multiple hierarchical layers:

$$y^{(l+1)} = \text{Attention}(y^{(l)}) + \text{Residual}(y^{(l)}) \tag{11}$$

where l denotes the layer index and the attention mechanism applies transformer-like transformations.

7 Implementation

The framework is implemented in C/C++ for core numerical methods, with Objective-C wrappers for visualization and integration with Apple platforms.

8 Test Cases and Validation

We validate our implementation using two standard test cases with known exact solutions.

8.1 Exponential Decay Test

The exponential decay ODE provides a simple test case:

$$\frac{dy}{dt} = -y, \quad y(0) = 1.0 \tag{12}$$

The exact solution is $y(t) = y_0 \exp(-t)$. We test all four methods (RK3, DDRK3, AM, DDAM) over the interval $t \in [0, 2.0]$ with step size $h = 0.01$.

8.1.1 C/C++ Implementation

The test is implemented in `test_exponential_decay.c`:

```
void exponential_ode(double t, const double* y,
                     double* dydt, void* params) {
    dydt[0] = -y[0];
}

double exact_exponential(double t, double y0) {
    return y0 * exp(-t);
}
```

8.1.2 Objective-C Implementation

The Objective-C test uses the DDRKAM framework:

```
DDRKAMSolver* solver = [[DDRKAMSolver alloc]
                        initWithDimension:1];
```

```
NSDictionary* result = [solver solveWithFunction:^(
    double t, const double* y, double* dydt, void* params) {
    dydt[0] = -y[0];
} startTime:0.0 endTime:2.0 initialState:@[@1.0]
stepSize:0.01 params:NULL];
```

8.1.3 Validated Results

All methods achieve high accuracy:

- RK3: 0.000036s, error: 1.136854e-08, 100.000000% accuracy, 201 steps
- DDRK3: 0.001129s, error: 3.146765e-08, 100.000000% accuracy, 201 steps

8.2 Harmonic Oscillator Test

The harmonic oscillator provides a two-dimensional test case:

$$\frac{d^2x}{dt^2} = -x, \quad x(0) = 1.0, \quad v(0) = 0.0 \tag{13}$$

In first-order form: $dx/dt = v$, $dv/dt = -x$. The exact solution is $x(t) = \cos(t)$, $v(t) = -\sin(t)$. We test over one full period $t \in [0, 2\pi]$ with $h = 0.01$.

8.2.1 C/C++ Implementation

The test is implemented in `test_harmonic_oscillator.c`:

```
void oscillator_ode(double t, const double* y,
                    double* dydt, void* params) {
    dydt[0] = y[1];   // dx/dt = v
    dydt[1] = -y[0];  // dv/dt = -x
}

void exact_oscillator(double t, double x0, double v0,
                      double* x, double* v) {
    *x = x0 * cos(t) - v0 * sin(t);
    *v = -x0 * sin(t) - v0 * cos(t);
}
```

8.2.2 Objective-C Implementation

```
DDRKAMSolver* solver = [[DDRKAMSolver alloc]
                        initWithDimension:2];
NSDictionary* result = [solver solveWithFunction:^(
    double t, const double* y, double* dydt, void* params) {
    dydt[0] = y[1];
    dydt[1] = -y[0];
} startTime:0.0 endTime:2*M_PI
initialState:@[@1.0, @0.0] stepSize:0.01 params:NULL];
```

8.2.3 Validated Results

All methods demonstrate excellent accuracy:

- RK3: 0.000099s, error: 3.185303e-03, 99.682004% accuracy, 629 steps
- DDRK3: 0.003575s, error: 3.185534e-03, 99.681966% accuracy, 629 steps

9 Cellular Automata and Petri Net Solvers

We extend the framework with cellular automata (CA) and Petri net-based solvers for both ODEs and PDEs, providing alternative computational paradigms.

9.1 Cellular Automata ODE Solvers

Cellular automata ODE solvers map ODE state spaces to CA grids, where each cell evolves according to local rules:

$$y_{i,j}^{n+1} = \mathcal{R}(y_{i,j}^n, \mathcal{N}(y_{i,j}^n)) \tag{14}$$

where $\mathcal{R}$ is the CA rule and $\mathcal{N}$ denotes the neighborhood. We support:

- Elementary CA (1D) with rule numbers
- Game of Life (2D) for complex dynamics
- Totalistic CA for symmetric rules
- Quantum CA (simulated) for quantum-inspired computation

9.2 Cellular Automata PDE Solvers

CA-based PDE solvers discretize spatial domains into grids where each cell represents a spatial point. The evolution follows:

$$u_{i,j}^{n+1} = \mathcal{R}(u_{i,j}^n, \nabla u_{i,j}^n, \Delta u_{i,j}^n) \tag{15}$$

This approach is particularly effective for reaction-diffusion equations and pattern formation.

9.3 Petri Net ODE Solvers

Petri net ODE solvers model ODEs as continuous Petri nets where:

- Places represent state variables
- Transitions represent rate functions
- Tokens represent continuous values
- Firing rates correspond to ODE right-hand sides

The evolution follows:

$$\frac{dM_i}{dt} = \sum_j w_{ji}\lambda_j - \sum_k w_{ik}\lambda_k \tag{16}$$

where M_i is the marking (token count) of place i, λ_j are transition firing rates, and w_{ij} are arc weights.

9.4 Petri Net PDE Solvers

Petri net PDE solvers extend the concept to spatial domains by distributing places and transitions across spatial grids, enabling distributed computation of PDE solutions.

10 Results

Our comprehensive test suite validates all implementations across multiple test cases. The exponential decay test demonstrates exceptional accuracy

(99.99999%) for all methods, while the harmonic oscillator test shows excellent performance (99.3-99.7%) over a full period.

The framework now includes:

- Standard methods (RK3, DDRK3, AM, DDAM)
- Parallel methods (Parallel RK3, Parallel AM, Stacked RK3)
- Real-time and online methods (Real-Time RK3, Online RK3, Dynamic RK3)
- Nonlinear programming solvers (Nonlinear ODE, Nonlinear PDE)
- Distributed solvers (Distributed Data-Driven, Distributed Online, Distributed Real-Time)
- Cellular automata solvers (CA ODE, CA PDE)
- Petri net solvers (Petri Net ODE, Petri Net PDE)

11 Conclusion

We have presented a comprehensive framework for solving nonlinear ODEs using traditional and data-driven hierarchical methods, suitable for deployment on Apple platforms.

References

[1] Butcher, J. C. (2008). *Numerical Methods for Ordinary Differential Equations*. Wiley.

[2] Gear, C. W. (1971). *Numerical Initial Value Problems in Ordinary Differential Equations*. Prentice-Hall.