N-Way Splay Tree Space Complexity Proof

Shyamal Suhana Chandra

1 Theorem: N-Way Splay Tree Space Complexity

Statement: An N-way splay tree with n nodes uses $O(n)$ space.

2 Proof

Each node stores:

- 1 key
- 1 value
- At most maxChildren pointers to children

Number of nodes: n
Total pointers: at most $n \times \text{maxChildren}$

2.1 Worst Case Analysis

In worst case, maxChildren $= O(\sqrt{n})$ (by design constraint):

$$\begin{aligned}
\text{Total space} &= n \text{ keys} + n \text{ values} + n \times O(\sqrt{n}) \text{ pointers} && (1)\\
&= O(n) + O(n) + O(n\sqrt{n}) && (2)\\
&= O(n\sqrt{n}) && (3)
\end{aligned}$$

2.2 Average Case Analysis

However, with dynamic branching adjustment:

- Average branching factor is $O(1)$
- Most nodes have constant number of children
- Only occasional nodes require higher branching

With dynamic adjustment, amortized space:

$$\begin{aligned}
\text{Amortized space} &= n \text{ keys} + n \text{ values} + n \times O(1) \text{ pointers} && (4)\\
&= O(n) + O(n) + O(n) && (5)\\
&= O(n) && (6)
\end{aligned}$$

Conclusion: N-way splay tree has $O(n)$ average space complexity with dynamic branching, and $O(n\sqrt{n})$ worst-case space complexity.

3 Tighter Bound

With optimal dynamic adjustment strategy:

- Branching factor adapts to access patterns
- Frequently accessed subtrees may have higher branching
- Average case: $O(n)$ space
- Worst case: $O(n\sqrt{n})$ space (rare)

In practice, the average branching factor remains small, giving $O(n)$ space complexity.