Generating functions

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In this activity some exercises of the book Introduction to Probability [1] are solved.

Exercise 1, page 392

Let $Z_1, Z_2, \ldots, Z_n$ describe a branching process in which each parent has j offspring with probability p_j. Find the probability d that the process eventually dies out if

a) $p_0 = 1/2$, $p_1 = 1/4$, $p_2 = 1/4$.

b) $p_0 = 1/3$, $p_1 = 1/3$, $p_2 = 1/3$.

c) $p_0 = 1/3$, $p_1 = 0$, $p_2 = 2/3$.

d) $p_j = 1/2^{j+1}$, *for* $j = 0, 1, 2, \ldots$

e) $p_j = (1/3)(2/3)^j$, *for* $j = 0, 1, 2, \ldots$

f) $p_j = (e^{-2}2^j)/j!$, *for* $j = 0, 1, 2, \ldots$ *(estimate d nummerically).*

Let d the probability that the process will ultimately die out. Theorem 10.2 from page 380 [1] says that if the mean number m of offspring produced by a single parent is ≤ 1, then $d = 1$ and the process dies out with probability 1. But if $m > 1$ then $d < 1$ and the process dies out with probability d.

In the particular case of a), b) and c), the mean number m of offspring produced by a single parent is $m = p_1 + 2p_2 = 1 - p_0 + p_2$. If $m > 1$, d can be easily calculated by $d = p_0/p_2$.

a) $p_0 = 1/2$, $p_1 = 1/4$, $p_2 = 1/4$

The mean number m of offspring produced by a single parent is

$$m = \frac{1}{4} + 2\left(\frac{1}{4}\right) = \frac{3}{4} < 1.$$

Then, by theorem 10.2, follows that the process dies out with probability 1.

b) $p_0 = 1/3$, $p_1 = 1/3$, $p_2 = 1/3$

For this exercise $m = \frac{1}{3} + 2\left(\frac{1}{3}\right) = 1$. Therefore the process dies out with probability 1.

c) $p_0 = 1/3$, $p_1 = 0$, $p_2 = 2/3$

The mean number m of offspring produced by a single parent in this case is

$$m = 0 + 2\left(\frac{2}{3}\right) = \frac{4}{3} > 1.$$

The process dies out with probability $d = p_0/p_2 = \frac{1}{3}/\frac{2}{3} = 0.5$.

To solve d) and e) it is neccesary to remember that $h(z)$, the ordinary generating function for the p_i, is

$$h(z) = p_0 + p_1 z + p_2 z^2 + \cdots$$

and $m = h'(1)$. If $m \leq 1$, the process will surely die out and $d = 1$. To find the probability d when $m > 1$ one must find a root $d < 1$ of the equation

$$z = h(z).$$

d) $p_j = 1/2^{j+1}$, for $j = 0, 1, 2, \ldots$

The ordinary generating function of the problem is

$$\begin{aligned}
h(z) &= \frac{1}{2} + \frac{1}{2^2}z + \frac{1}{2^3}z^2 + \ldots \\
&= \frac{1}{2}\left(1 + \frac{1}{2}z + \frac{1}{2^2}z^2 + \ldots\right) \\
&= \frac{1}{2}\left[\left(\frac{1}{2}z\right)^0 + \left(\frac{1}{2}z\right)^1 + \left(\frac{1}{2}z\right)^2 + \ldots\right] \\
&= \frac{1}{2}\left(\frac{1}{1 - \frac{1}{2}z}\right) \\
&= \frac{1}{2 - z}.
\end{aligned}$$

To get this result it has been used that $1 + r + r^2 + \ldots = \frac{1}{1-r}$. Then

$$\begin{aligned}
h'(z) &= \frac{d}{dz}\left(\frac{1}{2 - z}\right) \\
&= \frac{d}{dz}(2 - z)^{-1} \quad \text{chain rule} \\
&= \frac{1}{(2 - z)^2},
\end{aligned}$$

and $m = h'(1) = \frac{1}{(2-1)^2} = 1 \leq 1$, therefore $d = 1$.

e) $p_j = (1/3)(2/3)^j$, for $j = 0, 1, 2, \ldots$

The ordinary generating function is

$$h(z) = \frac{1}{3}\left(\frac{2}{3}\right)^0 + \frac{1}{3}\left(\frac{2}{3}\right)^1 z + \frac{1}{3}\left(\frac{2}{3}\right)^2 z^2 + \frac{1}{3}\left(\frac{2}{3}\right)^3 z^3 + \ldots$$

$$
\begin{aligned}
&= \frac{1}{3}\left[1+\left(\frac{2}{3}z\right)^{1}+\left(\frac{2}{3}z\right)^{2}+\dots\right] \\
&= \frac{1}{3}\left(\frac{1}{1-\frac{2}{3}z}\right) \\
&= \frac{1}{3-2z}.
\end{aligned}
$$

Then, one can calculate $h'(z)$:

$$
\begin{aligned}
h'(z) &= \frac{d}{dz}\left(\frac{1}{3-2z}\right) \\
&= \frac{d}{dz}(3-2z)^{-1} \quad \text{chain rule} \\
&= \frac{2}{(3-2z)^2}.
\end{aligned}
$$

from which $m = h'(1) = \frac{2}{(3-2)^2} = 2$ and $d < 1$. To find the probability d we need to solve the equation $z = h(z)$. Using the previous result found for $h(z)$ we have

$$2z^2 - 3z + 1 = 0.$$

The roots of this equation are $z_1 = 1$ and $z_2 = 1/2$. Therefore, the probability d that the process eventually dies out is 0.5.

Excercise 3, page 392

In the chain letter problem (see Example 10.14) find your expected profit if

a) $p_0 = 1/2$, $p_1 = 0$, $p_2 = 1/2$.

b) $p_0 = 1/6$, $p_1 = 1/2$, $p_2 = 1/3$.

Show that if $p_0 > 1/2$, *you cannot expect to make a profit.*

The expected profit of the chain letter problem can be found by the expression $50m + 50m^{12}$, where $m = p_1 + 2p_2$.

a) $p_0 = 1/2$, $p_1 = 0$, $p_2 = 1/2$.

In this particular case $m = 0 + 2\left(\frac{1}{2}\right) = 1$. Then, the expected profit is: $50(1 + 1^{12}) - 100 = 0$.

b) $p_0 = 1/6$, $p_1 = 1/2$, $p_2 = 1/3$.

For this problem $m = \frac{1}{2} + 2\left(\frac{1}{3}\right) = \frac{7}{6}$ and the expected profit is

$$50\left[\frac{7}{6}+\left(\frac{7}{6}\right)^{12}\right] - 100 \approx 376.26 - 100 = 276.26.$$

Exercise 1, page 401

Let X be a continuous random variable with values in $[0,2]$ and density f_X. Find the moment generating function $g(t)$ for X if

a) $f_X(x) = \frac{1}{2}$.

b) $f_X(x) = \frac{1}{2}x$.

c) $f_X(x) = 1 - \frac{1}{2}x$.

d) $f_X(x) = |1 - x|$.

e) $f_X(x) = \frac{3}{8}x^2$.

The *moment generating function* $g(t)$ for X is define by Equation (1)

$$g(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x)\, dx. \tag{1}$$

The values of the variable X are in the interval $[0,2]$, therefore the moment generating funcion will be define by the integral in Equation (2)

$$g(t) = \int_{0}^{2} e^{tx} f_X(x)\, dx. \tag{2}$$

a) $f_X(x) = \frac{1}{2}$.

$$\begin{aligned} g(t) &= \int_0^2 e^{tx}\left(\frac{1}{2}\right) dx \\ &= \frac{1}{2}\int_0^2 e^{tx}\, dx \\ &= \frac{1}{2}\left[\frac{1}{t}\cdot e^{tx}\right]\Bigg|_0^2 \\ &= \frac{1}{2}\cdot\frac{e^{2t}-1}{t}. \end{aligned}$$

b) $f_X(x) = \frac{1}{2}x$.

$$\begin{aligned} g(t) &= \int_0^2 e^{tx}\left(\frac{1}{2}x\right) dx \\ &= \frac{1}{2}\int_0^2 xe^{tx}\, dx && \text{i.b.p} \\ &= \frac{1}{2}\left[\frac{x}{t}\cdot e^{tx} - \frac{1}{t^2}\cdot e^{tx}\right]\Bigg|_0^2 \end{aligned}$$

$$= \frac{1}{2} \cdot \frac{2te^{2t} - e^{2t} + 1}{t^2}.$$

c) $f_X(x) = 1 - \frac{1}{2}x$.

$$\begin{aligned} g(t) &= \int_0^2 e^{tx} \left(1 - \frac{1}{2}x\right) dx \\ &= \int_0^2 e^{tx}\, dx - \int_0^2 e^{tx} \left(\frac{1}{2}x\right) dx \\ &= \int_0^2 e^{tx}\, dx - \frac{1}{2} \int_0^2 xe^{tx}\, dx \end{aligned}$$

Note that these integrals were already calculated in a) and b), then

$$\begin{aligned} g(t) &= \int_0^2 e^{tx}\, dx - \frac{1}{2} \int_0^2 xe^{tx}\, dx \\ &= \frac{e^{2t} - 1}{t} - \frac{1}{2} \cdot \frac{2te^{2t} - e^{2t} + 1}{t^2} \\ &= \frac{3te^{2t} - 3t - 2te^{2t} + e^{2t} - 1}{2t^2} \\ &= \frac{e^{2t} - 2t + 1}{2t^2}. \end{aligned}$$

d) $f_X(x) = |1 - x|$.

Following the definition of absolute value, the density function f_X can be define by Equation (3)

$$f_X(x) = \begin{cases} 1 - x, & \text{if} \quad x \leq 1 \\ -1 + x, & \text{if} \quad x > 1. \end{cases} \tag{3}$$

Therefore the moment generating function will be define by

$$\begin{aligned} g(t) &= \int_0^2 e^{tx} |1 - x|\, dx \\ &= \int_0^1 e^{tx}(1 - x)\, dx + \int_1^2 e^{tx}(-1 + x)\, dx \\ &= \int_0^1 e^{tx}\, dx - \int_0^1 xe^{tx}\, dx - \int_1^2 e^{tx}\, dx + \int_1^2 xe^{tx}\, dx \\ &= \left[\frac{1}{t} e^{tx}\right]\Bigg|_0^1 - \left[\frac{x}{t} \cdot e^{tx} - \frac{1}{t^2} \cdot e^{tx}\right]\Bigg|_0^1 - \left[\frac{1}{t} e^{tx}\right]\Bigg|_1^2 + \left[\frac{x}{t} \cdot e^{tx} - \frac{1}{t^2} \cdot e^{tx}\right]\Bigg|_1^2 \\ &= \frac{1}{t} e^{2t} - \frac{1}{t^2} e^{2t} + \frac{2}{t^2} e^t - \frac{1}{t^2} - \frac{1}{t}. \end{aligned}$$

e) $f_X(x) = \frac{3}{8}x^2$.

$$g(t) = \int_0^2 e^{tx} \left(\frac{3}{8}x^2\right) dx$$

$$= \frac{3}{8}\int_0^2 x^2 e^{tx}\,dx.$$

Integrating by parts twice, the following result is obtained

$$\begin{aligned}
g(t) &= \frac{3}{8}\int_0^2 x^2 e^{tx}\,dx \\
&= \frac{3}{8}\left[e^{tx}\left(\frac{x^2}{t}-\frac{2x}{t^2}+\frac{2}{t^3}\right)\right]\Bigg|_0^2 \\
&= \frac{3}{8}\left[e^{2x}\left(\frac{4t^2-4t+2}{t^3}+\frac{2}{t^3}\right)\right].
\end{aligned}$$

Exercise 6, page 402

Let X be a continuous random variable whose characteristic function $k_X(\tau)$ is $k_X(\tau) = e^{-|\tau|}$, $-\infty < \tau < \infty$. Show directly that the density f_X of X is

$$f_X(x) = \frac{1}{\pi(1+x^2)}.$$

Having the characteristic function k_X, it is possible to determine the density function f_X by Equation (4)

$$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\tau x} k_X(\tau)\,d\tau. \tag{4}$$

The characteristic function of the problem is define by an absolute value, therefore

$$k_X(\tau) = \begin{cases} -\tau, & \text{if} \quad \tau \geq 0 \\ \tau, & \text{if} \quad \tau < 0. \end{cases} \tag{5}$$

Using this result, the density function f_X will be

$$\begin{aligned}
f_X(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ix\tau}e^{-|\tau|}\,d\tau \\
&= \frac{1}{2\pi}\left(\int_{-\infty}^{0} e^{-ix\tau}e^{\tau}\,d\tau\right) + \frac{1}{2\pi}\left(\int_{0}^{\infty} e^{-ix\tau}e^{-\tau}\,d\tau\right) \\
&= \frac{1}{2\pi}\left(\int_{-\infty}^{0} e^{\tau(1-ix)}\,d\tau\right) + \frac{1}{2\pi}\left(\int_{0}^{\infty} e^{-\tau(1+ix)}\,d\tau\right) \\
&= \frac{1}{2\pi}\lim_{R\to\infty}\left(\int_{-R}^{0} e^{\tau(1-ix)}\,d\tau\right) + \frac{1}{2\pi}\lim_{R\to\infty}\left(\int_{0}^{R} e^{-\tau(1+ix)}\,d\tau\right) \\
&= \frac{1}{2\pi}\lim_{R\to\infty}\left[\left(\frac{1}{1-ix}\right)e^{\tau(1-ix)}\right]\Bigg|_{-R}^{0} + \frac{1}{2\pi}\lim_{R\to\infty}\left[\left(-\frac{1}{1+ix}\right)e^{-\tau(1+ix)}\right]\Bigg|_{0}^{R}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2\pi}\left(\frac{1}{1-ix}+\frac{1}{ix+1}\right)\\
&= \frac{1}{2\pi}\cdot\frac{1+ix+1-ix}{(1-ix)(1+ix)}\\
&= \frac{1}{2\pi}\cdot\frac{2}{1-i^2x^2}\\
&= \frac{1}{\pi(1+x^2)}.
\end{aligned}$$

Exercise 10, page 403

Let $X_1, X_2, \ldots, X_n$ be an independent trials process with density

$$f(x) = \frac{1}{2}e^{-|x|}, \quad -\infty < x < \infty.$$

a) Find the mean and variance of $f(x)$.

b) Find the moment generating function for X_1, S_n, A_n, and S_n^*.

c) What can you say about the moment generating function of S_n^* as $n \to \infty$?

d) What can you say about the moment generating function of A_n as $n \to \infty$?

References

[1] Charles M. Grinstead and J. Laurie Snell. *Introduction to Probability.* AMS, 2003.