PHYM004 Assessment 2: N-body Problem

Jay Malhotra

November 30, 2021

1 Introduction

The goal of this project was to implement a program capable of simulating the interaction between n point masses, given some initial conditions of position and velocity for each mass. By default, the program is set up to model attraction due to Newton's law of gravitation, but it could easily be adapted to model a number of other physical phenomena, such as electromagnetic forces.

This report examines the validity of the solution I developed. Section 2 examines the results of the simulation against certain analytical results and principles, and Section 3 demonstrates the usage of the program to simulate four stable 3-body orbits.

2 Analysis and validation of results

The test case that I chose is a 3-body system analogous to the Sun-Earth system with an additional body whose orbital parameters are based on Mercury, but slightly adjusted to demonstrate a more eccentric orbit and thus Kepler's first law. The eccentricity of this orbit may have been responsible for some interesting results, as will be discussed. Table 1 includes the raw figures used to set up this calculation. Figure 1 shows a 3D plot of the result given by running the program with these conditions.

Body name	Mass / kg	Position / m	Velocity / $\mathrm{m\,s^{-1}}$
Sun	2.0×10^{30}	$0\hat{\mathbf{x}} + 0\hat{\mathbf{y}} + 0\hat{\mathbf{z}}$	$0\hat{\mathbf{x}} + 0\hat{\mathbf{y}} + 0\hat{\mathbf{z}}$
EccentricPlanet	3.0×10^{23}	$6.0 \times 10^{10}\hat{\mathbf{x}} + 0\hat{\mathbf{y}} + 0\hat{\mathbf{z}}$	$0\hat{\mathbf{x}} + 6.0 \times 10^{4}\hat{\mathbf{y}} + 0\hat{\mathbf{z}}$
Earth	6.0×10^{24}	$1.5 \times 10^{11}\hat{\mathbf{x}} + 0\hat{\mathbf{y}} + 0\hat{\mathbf{z}}$	$0\hat{\mathbf{x}} + 3.0 \times 10^{4}\hat{\mathbf{y}} + 0\hat{\mathbf{z}}$

Table 1: Details of initial conditions used to set up the simulation which produced the results shown in Figure 1. Position and velocity are given as 3D vectors.

2.1 Conservation of angular momentum and Kepler's second law

At each time-step, the program logs the angular momentum of each body other than the first one. The angular momentum for a body in orbit around another body is calculated using

$$L = mrv, \tag{1}$$

where m is the mass of the body, r is the distance between the two bodies, and v is the velocity of the body. It can be shown that the area swept out by an orbit in some infinitesimal time is given by

$$\frac{dA}{dt} = \frac{L}{2m}, \tag{2}$$

so any demonstration that angular momentum is conserved is also a demonstration of Kepler's second law.

Angular momentum is generally calculated with reference to another body. For calculations involving our Solar System, this is usually the Sun, as the planets orbit around it while having comparatively little influence on one another. However, the program must be able to support non-standard configurations where arbitrarily choosing a body may not make sense, such as the three-body systems discussed in Section 3 where all the bodies have equal mass. To account for this, the program instead calculates angular momentum relative to the position of the centre of mass of the whole system of bodies. The value for velocity is taken as the simple magnitude of the vector, since the centre of mass should in theory be static.

Figure 2 is a plot showing the calculated angular momentum of the two non-Sun bodies at each time-step. It shows that angular momentum is generally conserved, but with varying levels of success: the 'Earth' body exhibits

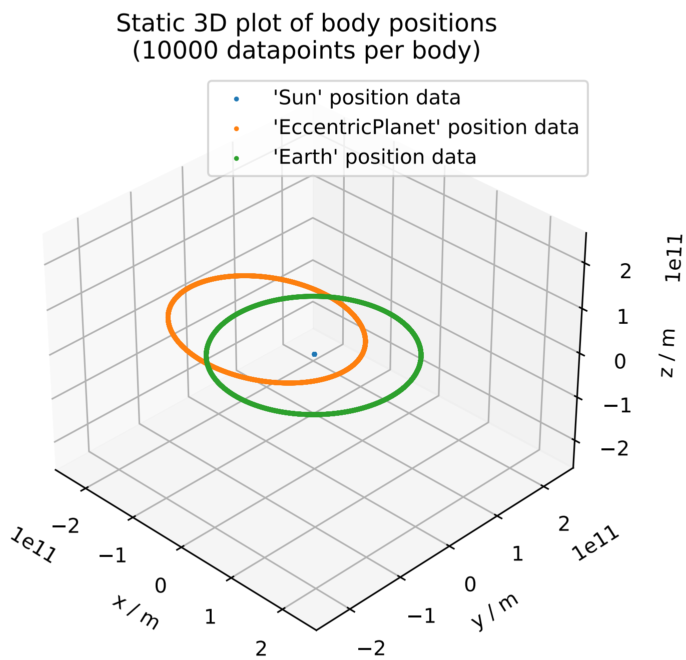


Figure 1: Static plot of 3-body system dynamics in x-y plane. The motion of the Sun around the barycentre(s) is included in the simulation, but it is too negligible to see in this plot.

fluctuations of around four or five orders of magnitude smaller than its angular momentum, whereas the 'EccentricPlanet' body has fluctuations that are only one order of magnitude smaller than its angular momentum.

Adjusting the time-step down does little to mitigate this. Figure 3 is a plot of the relative error of angular momentum as a function of time-step, for a simulation lasting 4×10^7 s. It shows that if the simulation time-step is not inappropriately large (i.e. allows for more than 100 or so steps) then the angular momentum's relative error is minimized, and that decreasing the time-step below the order of 10^3 seconds does little to address this issue.

2.2 Conservation of total energy

In theory, the total energy of each body (i.e. gravitational potential energy plus kinetic energy) should be conserved. Gravitational potential energy

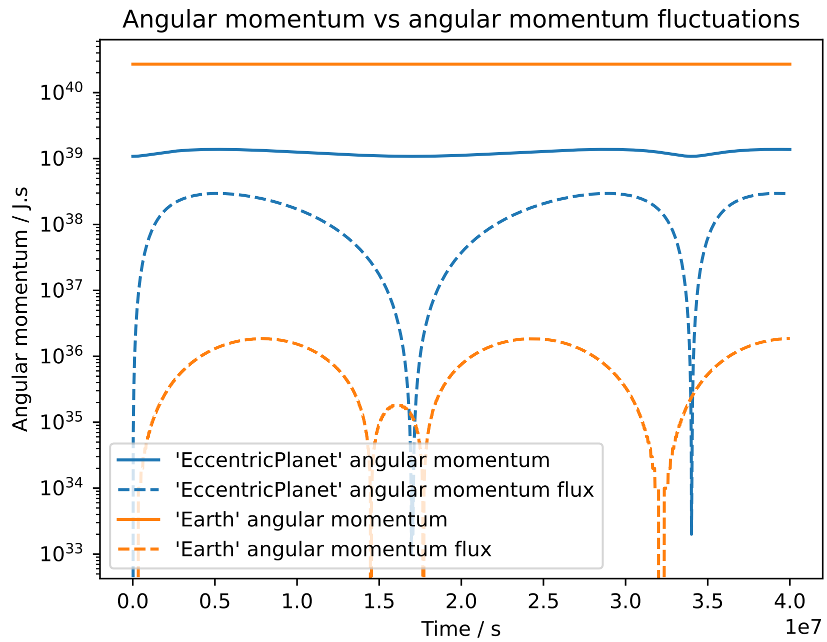


Figure 2: Plot showing total angular momentum, and angular momentum flux, as a function of simulation time for the two non-Sun bodies. 'Angular momentum flux' is defined as the absolute difference from the first calculated value of angular momentum at $t =$ simulation time-step $= 4 \times 10^3$s.

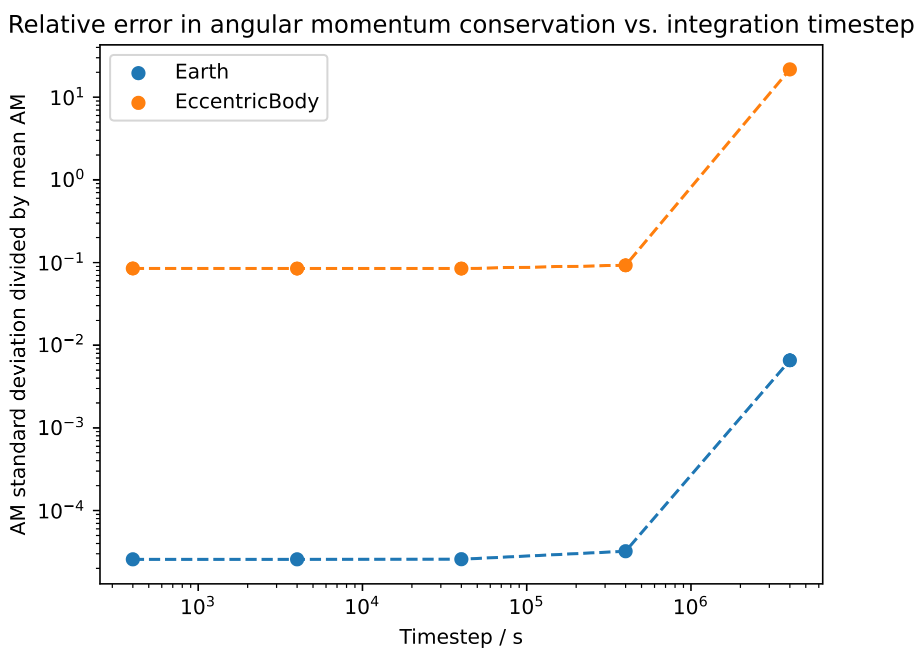


Figure 3: Plot showing the 'relative error in angular momentum conversation', i.e. the standard deviation of angular momentum divided by mean angular momentum, as a function of simulation time-step for the two non-Sun bodies. The total simulation time was 4×10^7 s.

is calculated as a sum over all other bodies in the simulation, but in this test case is mainly owed to the gravity of the Sun body. Kinetic energy is calculated using the square magnitude of the velocity vector, and not relative to the centre of mass or any other body.

The conservation of the total energy of each body was much stronger than the conservation of angular momentum. Figure 4 shows the total energy of each body and the fluctuations thereof as a function of time. In this case, both bodies have fluctuations that are approximately four or five orders of magnitude smaller than their actual total energy, which I consider to be an acceptable result. Figure 5 shows how changing the time-step affects the ability of the program to conserve total energy. In general, it is much more reliant on a low time-step than conservation of angular momentum, but the conservation is better overall.

2.3 Kepler's third law

Kepler's third law states that, for an elliptical orbit of a body around another body,

$$P^2 = \frac{4\pi^2}{G(M_1 + M_2)} a^3 \tag{3}$$

$$\implies \frac{P^2}{a^3} \cdot (M_1 + M_2) = \frac{4\pi^2}{G} \approxeq 5.9152 \times 10^{11}\,\mathrm{kg\,m^{-3}\,s^2} \tag{4}$$

where P is the period of the orbit, M_1 and M_2 are the masses of the bodies, and a is the semi-major axis of the orbit. Table 2 illustrates a comparison with the values from my code to the ones derived by this relation. It shows that Kepler's third law applies well to the Earth body, but not to the other, more eccentric orbit, though the relation is still correct to within one order of magnitude. This fact, along with a similar disparity between the two orbits in conservation of angular momentum, may indicate that eccentric orbits are more challenging to simulate accurately with this program.

3 Three-body problem

The program was able to successfully simulate a handful of three-body orbits, though it does require a very fine time-step to do so, possibly as a consequence

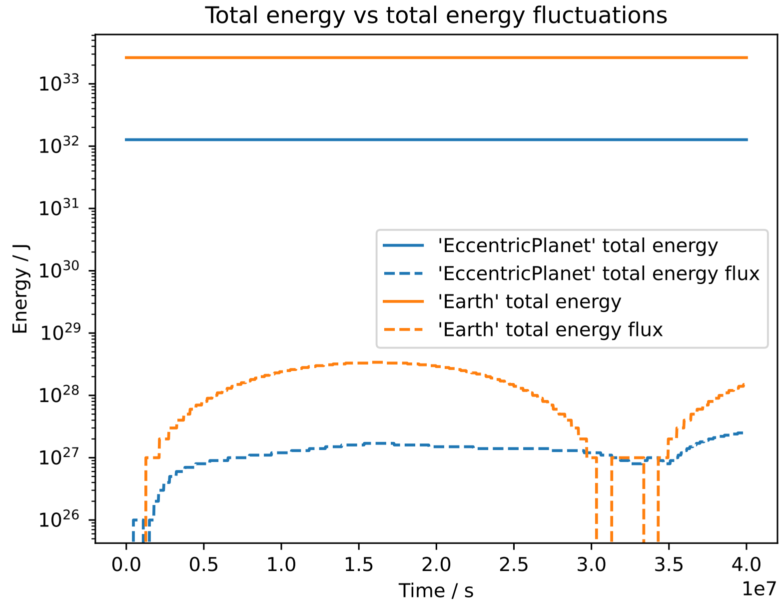


Figure 4: The absolute value of the total energy of each body, and the fluctuations in it (flux is defined in the same manner as in Figure 2) as a function of time elapsed. The time-step in this simulation was 4×10^3 s and the total time was 4×10^7 s.

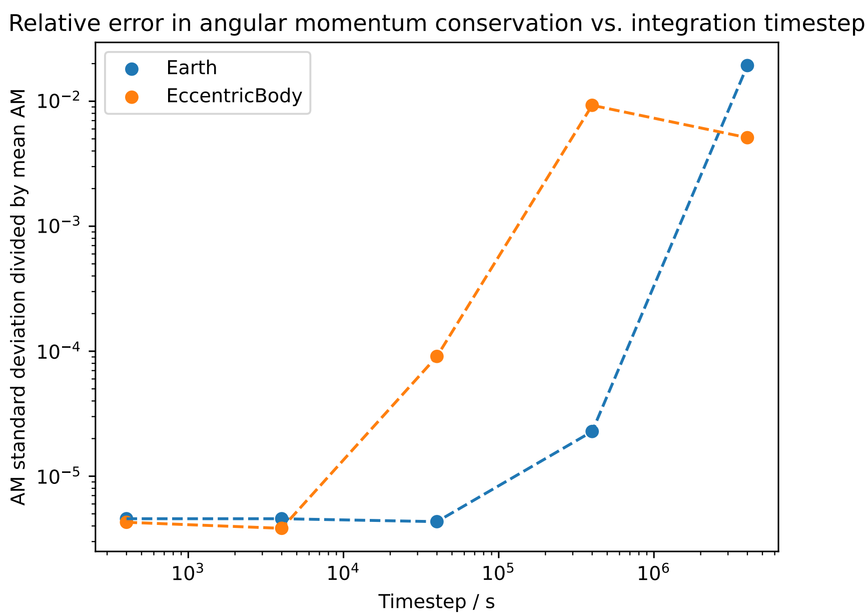


Figure 5: The relative error in the absolute value of total energy as a function of time-step. Relative error is calculated in the same manner as in Figure 3. Note that the dip towards the end of the relative error on the EccentricPlanet line is due to a sharp increase in the mean total energy.

Body name	P / s	a / m	LHS $\mathrm{kg\,m^{-3}\,s^2}$	LHS $\div \frac{4\pi^2}{G}$
Earth	3.21×10^7	1.53×10^{11}	5.75×10^{11}	0.973
EccentricPlanet	3.39×10^7	2.54×10^{11}	1.40×10^{11}	0.237

Table 2: Comparison of results from the simulation with Kepler's third law. Periods were derived from the periodicity seen in Figure 2, and semi-major axes are taken as half of the maximum value of x, y, or z in the data. LHS (left-hand-side) RHS (right-hand-side) both refer to Equation 4, with the last column being the result of dividing the empirical LHS with the theoretical RHS.

of the issues relating to conservation properties described above. Figure 6 shows the 3D plot results from simulating these orbits.

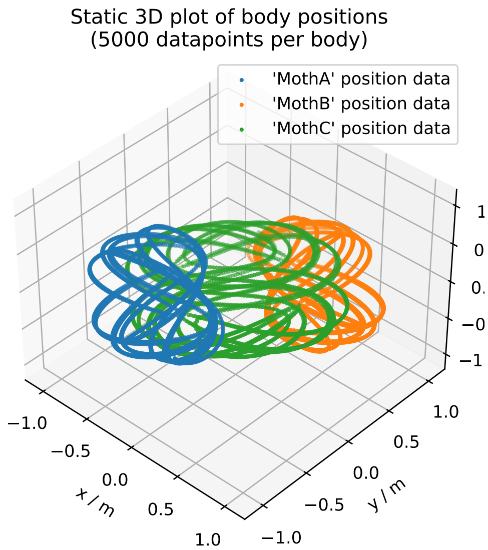
4 Conclusion

My program is able to adequately simulate the n-body problem, and reproduce known stable 3-body orbits, but it could use some work in ensuring that conservation properties are upheld. I did consider dynamically varying the time-step to keep angular momentum deviations within a given tolerance, but Figures 3 and 5 showed that past a certain point, lowering the time-step does not improve conservation properties, so it would first be necessary to decide what measures should be implemented to address these issues.

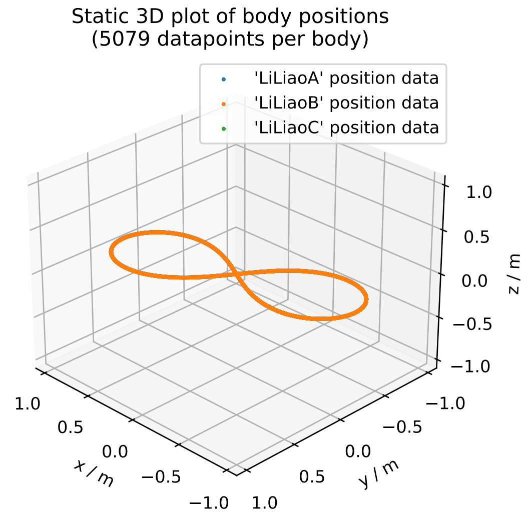
References

Dmitrašinović, V et al. (June 2018). "Linear stability of periodic three-body orbits with zero angular momentum and topological dependence of Kepler's third law: a numerical test". In: *Journal of Physics A: Mathematical and Theoretical* 51.31, p. 315101. ISSN: 1751-8121. DOI: `10.1088/1751-8121/aaca41`. URL: `http://dx.doi.org/10.1088/1751-8121/aaca41`.

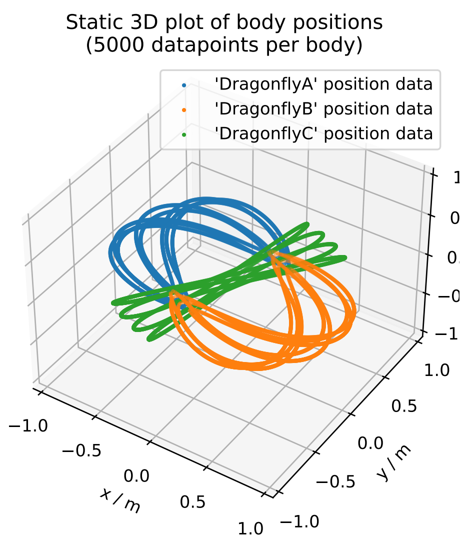
Li, XiaoMing and ShiJun Liao (Sept. 2017). "More than six hundred new families of Newtonian periodic planar collisionless three-body orbits". In: *Science China Physics, Mechanics & Astronomy* 60.12. ISSN: 1869-1927.



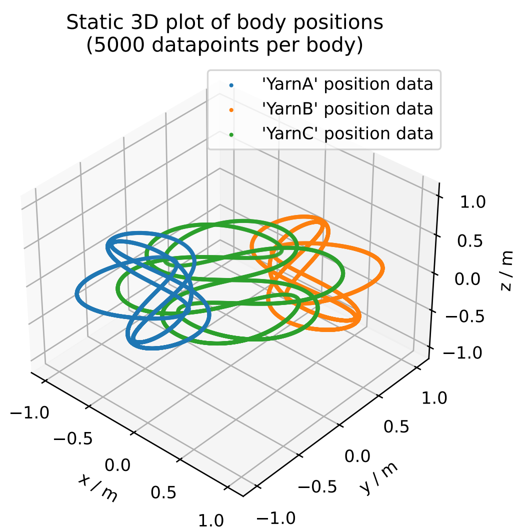
(a) VIIa.6.A. orbit from the 'VIIa - Moth' sequence of Dmitrašinović et al. 2018.



(b) First orbit from the 'I.A.1.i.c' sequence of Li and Liao 2017.



(c) II.4.A orbit from the 'II - Dragonfly' sequence of Šuvakov and V. Dmitrašinović 2013.



(d) VI.2.A orbit from the 'VI - Yarn' sequence of Šuvakov and V. Dmitrašinović 2013.

Figure 6: Various 3D plots of stable 3-body problems. The simulations were run for approximately twice their orbital period with a time-step of 4×10^{-5} s. The number of data-points stated in the title is after culling the data – i.e. selecting every n point to plot – for performance reasons.

DOI: 10.1007/s11433-017-9078-5. URL: http://dx.doi.org/10.1007/s11433-017-9078-5.

Šuvakov, Milovan and V. Dmitrašinović (Mar. 2013). "Three Classes of Newtonian Three-Body Planar Periodic Orbits". In: *Physical Review Letters* 110.11. ISSN: 1079-7114. DOI: 10.1103/physrevlett.110.114301. URL: http://dx.doi.org/10.1103/PhysRevLett.110.114301.

Acknowledgements

I would like to thank Ricky Reusser, whose helpful repository of stable 3-body orbits enabled me to find ones to simulate as well as properly cite them. His notebook containing this information can be found here: https://observablehq.com/@rreusser/periodic-planar-three-body-orbits