PHYM004 Project 2: Smoothed Particle Hydrodynamics

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1 Introduction

Smoothed-particle hydrodynamics (SPH) is an approach to the numerical simulation of fluid dynamics. Notable characteristics of this method include the fact that it is mesh-free, and that it is derived using Lagrangian mechanics, which gives it good conservation properties.

This report presents a one-dimensional SPH code which is capable of reproducing some basic analytical results. It features artificial viscosity calculation and a method for varying the smoothing length on a per-particle basis using a root-finding algorithm.

1.1 Equations of motion

Price (2012) provides a full first-principles derivation of SPH, but the key ideas are explained here.

SPH starts with the density estimate. This is a foundational quantity, and is used in almost every subsequent equation. It is calculated at any given point as the sum over all particles, which provide a contribution weighted by their distance:

$$\rho(\vec{r}_i) = \sum_j m_j W(|\vec{r}_i - \vec{r}_j|, h). \tag{1}$$

Here, $\vec{r}_i$ is a position (typically, but not necessarily, that of a particle) and $\vec{r}_j$ is the position of particle j, and m_j the mass of particle j. $W(\vec{r})$ is a function known as the weighting function, and is a function of a distance (here the particle separation) and a smoothing length h. The weighting function is related to a kernel $w(q)$ by $W(|\vec{r_i} - \vec{r_j}|) = \frac{1}{h} w(|\vec{r_i} - \vec{r_j}|/h)$.

The kernel is an important part of SPH. Most codes use an approximation to the Gaussian function which is truncated at a certain multiple of h. The value at which the spline is truncated is known as the 'compact support radius', and it increases computational efficiency by ignoring the negligible influence of extremely distant particles. This code uses the Schoenberg M_4 cubic spline kernel:

$$w(q) = \frac{2}{3} \begin{cases} \frac{1}{4}(2-q)^3 - (1-q)^3, & 0 \leq q < 1 \\ \frac{1}{4}(2-q)^3, & 1 \leq q < 2 \\ 0. & q \geq 2 \end{cases} \tag{2}$$

By applying Lagrangian mechanics and incorporating the above density estimate, it can be shown that the acceleration is given by:

$$\frac{d\vec{v}_i}{dt} = -\sum_j m_j \left(\frac{P_i}{\rho_i^2} + \frac{P_j}{\rho_j^2} \right) \nabla_a W_{ij}(h), \tag{3}$$

where $W_{ij}(h)$ is shorthand for $W(|\vec{r_i} - \vec{r_j}|, h)$. This equation is modified to add artificial viscosity and account for variable smoothing lengths as follows:

$$\frac{d\vec{v}_i}{dt} = -\sum_j m_j \left(\frac{P_i \Pi_{ij}}{\Omega_i \rho_i^2} \frac{\partial W_{ij}(h_i)}{\partial \vec{r_i}} + \frac{P_j}{\Omega_j \rho_j^2} \frac{\partial W_{ij}(h_j)}{\partial \vec{r_i}} \right). \tag{4}$$

Here, Ω_i is an expression defined by

$$\Omega_i \equiv 1 - \frac{\partial h_i}{\partial \rho_i} \sum_j m_j \frac{\partial W_{ij}(h_i)}{\partial h_i}, \tag{5}$$

and Π_{ij} is an artificial viscosity parameter, the definition of which is explained in Bate (1995).

1.2 Variable smoothing length

It is desirable to modify the smoothing length h on a per-particle basis, so that regions of higher density have a lower smoothing length (and thus higher resolution) and to keep the number of neighbours for each particle approximately constant. This is achieved by solving the following system of equations for h and ρ:

$$h_i(\rho_i) - \frac{\eta m_i}{\rho_i} = 0, \tag{6}$$

$$\rho_i(h_i) - \sum_j m_j W(|\vec{r}_i - \vec{r}_j|, h) = 0. \tag{7}$$

η in Equation 6 is a parameter that acts as a coefficient of the mean particle spacing $\frac{m}{\rho}$. It is related to the number of neighbours by $N_{\text{neigh}} \approx 4\eta$. The choice of η is arbitrary. For the tests described in this report, $\eta = 2$ is used to maintain approximately 8 neighbours for each particle.

The code solves the system of Equation 6 and Equation 7 by using a multidimensional root-finding subroutine from the GNU Scientific Library (GSL), with an analytically-derived Jacobian.

2 Results

2.1 Behaviour of density estimate along the axis

For this test, a random distribution of particles with $m = 1$ are used to test the effect of smoothing length on the density profile of the system. The variable smoothing length, as described in Section 1.2 is disabled for this test. Figure 1 is a plot showing different $\rho\vec{r}, h$ vs. x curves at different values of h. It shows that small values of h are very sensitive to the random nature of the distribution of particles, whilst large values of h effectively cancel out the random nature of the distribution and produce a parabola.

2.2 Relation between setup runtime and number of particles

This test also features a random distribution of particles and a constant smoothing length ($h = 0.3$). Figure **??** is a plot showing the results of this test.

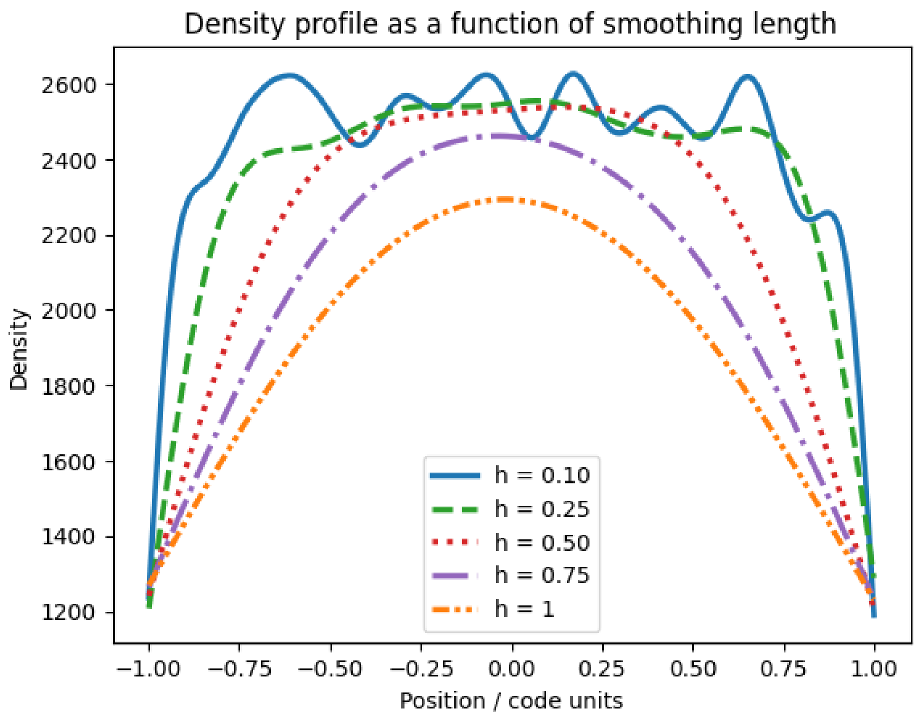

Figure 1: Plot showing density profiles at various values of h, for a distribution of 1000 particles. Note that for $h = 1$, the compact support radius of any given particle includes every single other particle in the distribution.

References

Price, Daniel J (2012). "Smoothed particle hydrodynamics and magnetohydrodynamics". In: *Journal of Computational Physics* 231.3, pp. 759–794.

Bate, Matthew Russell (1995). "The Role of Accretion in Binary Star Formation". PhD thesis. University of Cambridge.