NOTE BOOK

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NETWORK DESIGN PROBLEM

$\boxed{1}$

$\boxed{0}$ $\boxed{1}$

$\underbrace{\boxed{1} \quad \boxed{1}}_{n_1}$ $\boxed{0}$ $\underbrace{\boxed{1} \quad \boxed{1}}_{n_2}$ $\boxed{0}$

- $n_1, n_2, \dots, n_k, \dots$

THM: IF $n \geq 2$ is an integer, then there exists a unique sequence of prime numbers $2 \leq p_1 < p_2 < p_3 < \dots < p_k$ and a unique sequence of natural numbers $n_1 \geq 1$, $n_2 \geq 1, \dots, n_k \geq 1$ S.T.

$$n = \prod_{i=1}^{k} p_i^{n_i}$$

- $15 = 3 \cdot 5 = 3^1 \cdot 5^1$
  $p_1 = 3 \quad p_2 = 5$
  $n_1 = 1 \quad n_2 = 1$
- $20 = 2^2 \cdot 5^1$
  $p_1 = 2 \quad p_2 = 5$
  $n_1 = 2 \quad n_2 = 1$

$2 = P_1, \quad 3 = P_2, \quad 5 = P_3 \dots$

P_i is the iTH prime number

$n_1, n_2, n_3, \dots, n_k \Rightarrow$ $\underbrace{\boxed{1} \dots \boxed{1}}$ $\quad p_1^{n_1} p_2^{n_2} \dots p_k^{n_k} = N_k$

ADDS $n_{k+1} \Rightarrow P_1^{n_1} P_2^{n_2} \dots P_k^{n_k} P_{k+1}^{n_{k+1}} = N_{k+1}$

$N_{k+1} \geq N_k$

HIT $\boxed{1}$ $N_{k+1} - N_k$ TIMES

$\boxed{0}$ $\boxed{1}$

$\underbrace{\boxed{1} \cdots \boxed{1}}_{n_1}$ $\boxed{0}$ $\underbrace{\boxed{1} \dots \boxed{1}}_{n_2}$ $\boxed{0}$

How many digits to represent $n_1, n_2, \dots, n_k$?

$$\sum_{i=1}^{k} (n_i + 1) = \sum_{i=1}^{k} n_k + k$$

"$n_i + 1$ bits to represent n_i"

"in binary we use $O(\log n_i)$ bits to represent n_i"

$\boxed{\begin{matrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{matrix}}$

$n_i \xrightarrow{\text{base } 2} b_1, b_2, \dots, b_k$

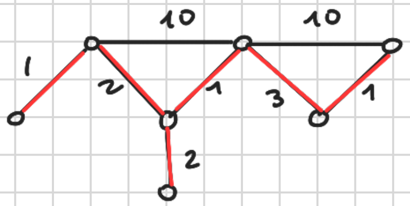
$0b_1 \; 0b_2 \; 0b_3 \; 0b_k$

$2\log_2 n_i = 2k$
$3\log_2 n_i = 3k$

$2\log_2 n_i$

$\log n_i + O(\log \log n_i)$

NETWORK DESIGN PROBLEM



$G(V,E)$ is a weighted, connected, graph.

We have $V = \{v_1, v_2, \dots, v_n\}$ locations.

Some pairs of locations, those in E, can be directly linked. Directly linking $\{v_i, v_j\} \in E$ costs $w(v_i, v_j) > 0$.

Assuming G is connected, what is the minimum price for indirectly connecting each pair of nodes of V?

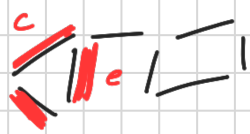
We aim to find subset $T \subseteq E$ of the edges so that:

- $G(V,T)$ is connected, and
- The cost of T, $cost(T) = \sum_{e \in T} c(e)$, is minimum.

L: Let T be an optimal solution to the network design problem. Then, $G(V,T)$ is a tree.

P: By definition, $G(V,T)$ has to be connected. We will show that $G(V,T)$ cannot contain cycles - thus, it has to be a tree (a connected graph with no cycles is a tree).

By contradiction, suppose that $G(V,T)$ contains a cycle C. Let "e" be any edge of the cycle.



$G(V, T-\{e\})$ is also connected,

since any path that went through the edge "e" can be rerouted through $C - \{e\}$.
Thus, $T-\{e\}$ is a valid (FEASIBLE) solution to the network design problem. (you can go from any node to any other node — that is, $G(V, T-\{e\})$ is connected).

The cost of this new solution is:

$$COST(T - \{e\}) = \sum_{e' \in T - \{e\}} c(e') = \left(\sum_{c' \in T} c(e') \right) - c(e) = COST(T) - c(e)$$

Recall that $c(e) > 0$, thus the $cost(T - \{e\}) = COST(T) - c(e) < COST(T)$. Thus, T is not an opt. solution.
CONTRADICTION ■

NOTE: This is also known as minimum spanning tree algo.

Thus, the network design problem is actually asking to find a subtree of $G(V, E)$ of minimum cost, and that connects each pair of vertices.

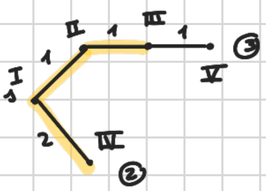
The latter problem is known as MINIMUM SPANNING TREE (or MST)

GREEDY APPROACHES?

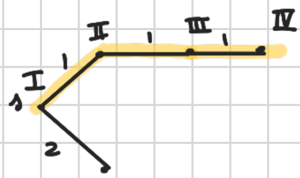
(1) PRIM'S ALGORITHM.

You start from an arbitrary node s. Let $S \leftarrow \{s\}$.

To select a new edge, we pick one having smallest cost, among those that take us from some node in S to some node in $V - S$.

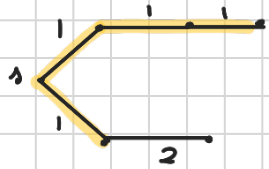


DIJKSTRA



PRIM

chooses edges that don't create cycles.



(2) KRUSKAL'S ALGORITHM

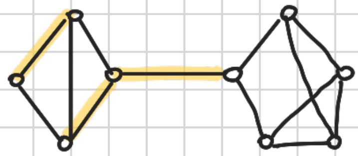
Sort the edges increasingly by cost. Let $T \leftarrow \emptyset$.
Scan the list of edges "e":

If "e" can be added to T w/o (without) creating cycles, add "e" to T.

③ REVERSE - KRUSKAL

Sort the edges decreasingly by cost. Scan the list of edges "e": IF "e" is part of a cycle (in the current graph), throw "e" away, o/w (otherwise) ADD "e" to T.

When is it "safe" to add an edge to a spanning tree T?

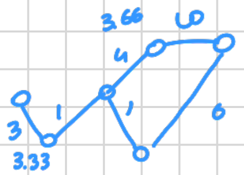


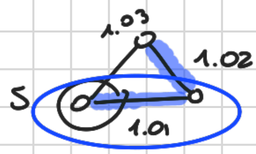
L: Assume that edge cost are pairwise distinct.

Let $\emptyset \subset S \subset V$ be a set of Nodes of $G(V, E)$.

Let $e \in E$ be an edge having smallest cost among the edges having one endpoint in S, and one in $V-S$.

Then, EACH MST of $G(V,E)$ contains "e".





δ is the minimum difference between distinct edges costs in $G(V,E)$