# NOTE BOOK

Rokshana Ahmed

L = LEMMA
P = PROOF

## GALE - SHAPELEY ALGORITHM

- Initially, each $a_i \in A$, and each $b_j \in B$, is FREE
- While there exists some FREE $a_i$ that has not yet proposed to each $b_j \in B$
- Let $a_i$ be a FREE person that has not proposed to each $b_j \in B$
- Let $B' \subseteq B$ be the set of $b_j$ such that $a_i$ has not yet proposed to
- Let $b_j \in B'$ be the person from $B'$ that $a_i$ likes the most

$$a_1 : b_1 > b_2 > b_3$$
$$B' = \{ b_2, b_3 \}$$

$b_2$ is the most preferred in the $B'$ set

- IF $b_j$ is FREE :
  - MATCH UP $a_i$ and $b_j$ // $a_i$ & $b_j$ get engaged
  - $a_i$ and $b_j$ are not free anymore
- ELSE :
  - Suppose that $b_j$ is engaged to $a_K$
  - IF $b_j$ likes $a_K$ more than $a_i$
    - $a_i$ REMAINS FREE
  - ELSE :
    - the match between $b_j$ and $a_K$ is broken
    - $a_i$ and $b_j$ are matched up
    - $a_K$ becomes FREE
- RETURN THE FINAL LIST OF "MATCHES" AS THE MATCHING.

**NOTE**: This still doesn't define whether the final matching is perfect. Proof needed.

## TECHNICAL OBSERVATIONS

**L1**: Each $b \in B$ remains matched / engaged from the first time she gets a proposal until the end of the execution

↑

PROOF: When "b" gets the first proposal, she becomes engaged. (since it's her first time, NO REFUSE)

From then onwards she might get other proposals. She might either:

- accept
- reject

SHE WILL ALWAYS BE ENGAGED

If she accepts one, she'll switch partners (but she'll remain engaged);

If she rejects, she'll keep her previous partner.

**L2**: The engagements of the generic $b \in B$ get better (from her perspective) over the time

↑

P: b changes partner only if she gets a proposal from a better than her current one.

MONOTONE PROPERTY (keeps getting better)

## THE A SIDE HAS A DIFFERENT FATE

**L3**: For each sequence of proposals made by a decreases in quality over time.

↑

P: TRIVIAL (by the algorithm's def)

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NOTE: Theorem → more important than "lemma"

**Theorem(T)**: The algorithm terminates after at most $n^2$ iterations.

↑

**Proof (P)**: Each $a_i$ can propose to at most $|B| = n$ people from B. In each iteration of the algorithm, some $a_i$ proposes to some $b_j \in B$ that he had not yet proposed to earlier. ↓

Therefore, there can be at most $|A| \cdot |B| = n^2$ proposals, and iterations

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In general, people look for some "quantities" to bound the runtime of an algorithm

**L4**: IF $a \in A$ is FREE at some point in the execution, then there must exist some $b \in B$ to which $a$ has not yet made a proposal.

TECHNICAL OBS.

(THIS LEMMA LEADS TO THE FACT THAT WE WILL HAVE A PERFECT MATCHING IN THE END)

P: By contradiction, suppose that, at some point, $a^* \in A$ is FREE and he has proposed to everyone from B

By L1, each $b \in B$ remains engaged from the first proposal she gets, until the end. Thus, for $a^*$ to remain FREE after $n = |B|$ proposals, it must be that, at the time of his last proposals, each $b \in B$ was engaged.

But, recall that $|A| = |B| = n$.

For each $b \in B$ to be engaged, it must be that each $a \in A$ must be engaged. Thus, it is IMPOSSIMPLE that $a^*$ is FREE.

| THIS IS A PROOF BY CONTRADICTION |
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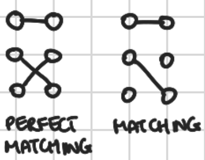
| • Basically, you start from an assumption and end up contradicting |

**L5**: The algorithm returns a perfect matching

P: When we match a to b, if b was already matched, then we break up the current engagement of b. Thus, the current matches from matching

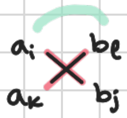



Suppose, by contradiction, that in the end $a \in A$ is FREE. Then, a has proposed to each $b \in B$. But, this contradicts L4. Thus, in the end, no $a \in A$ is FREE and, also, no $b \in B$ is FREE. ($|A| = |B|$ and the returned structure is a matching). Thus, the algorithm returns a perfect matching.

**THEOREM**: The algorithm returns a stable matching

**PROOF**: By L5, the algorithm returns a perfect matching (each person is matched to exactly one other person).

By contradiction, suppose that there exist two pairs in M, $\{a_i, b_j\}$, $\{a_k; b_\ell\}$ that are UNSTABLE




$a_i \cdots\!> b_\ell \cdots\!> b_j$
$b_\ell \cdots\!> a_i \cdots\!> a_k$

Then, $a_i$ prefers $b_\ell$ to $b_j$, and $b_\ell$ prefers $a_i$ to $a_k$. By the algorithm, $a_i$'s last proposals was to $b_j$

Now, let us consider two cases:

- $a_i$ did not propose to $b_\ell$ before $b_j$.
  Then, $a_i$ did not ever propose to $b_\ell$ ($b_j$ is $a_i$'s last proposal). But then, $a_i$ prefers $b_j$ to $b_\ell$ CONTRADICTION

- $a_i$ proposed to $b_\ell$ before $b_j$.
  Then, since $b_\ell$ ended up with $a_k$, and since L2 entails that $b_\ell$'s partners improve over time, it must be that $b_\ell$ prefers $a_k$ to $a_i$.
  CONTRADICTION

Therefore, no unstable pairs exist –
Thus, M is a stable matching.

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**DEF**: Let us say that $b_j \in B$ is a valid match for $a_i \in A$ if $\exists$ stable matching M such that $\{a_i, b_j\} \in M$

best partner from $a_i$'s perspective

**DEF**: Let $best(a_i)$, for $a_i \in A$, be the valid match $b^* \in B$ of $a_i$, that $a_i$ likes the best

**THEOREM**: The G-S algorithm returns $M = \{\{a_i, best(a_i)\} \mid a_i \in A\}$

**THEOREM**: " " " " $M = \{\{b_j, worst(b_j)\} \mid b_j \in B\}$

(This is a bit unfair since one gets the best and the others get the worst, all in the stable matching)