NOTE BOOK

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Algorithms

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```
HEAPIFY-DOWN(H, i):
    # i is a position in the heap, 1 ≤ i ≤ n
    LET n BE THE CURRENT SIZE OF THE HEAP
    IF 2i > n:        // think of the key, not value
        # i has no left, and no right, child
        RETURN
    ELIF 2i == n:
        # i has only the left child
        j = 2i
    ELSE:
        # i has both children
        IF KEY[H[right(i)]] < KEY[H[left(i)]]:
            j = right(i)
        ELSE:
            j = left(i)
    IF KEY[H[j]] < KEY[H[i]]:
        SWAP H[i] AND H[j]
        HEAPIFY-DOWN(H, j)
```

L: The function HEAPIFY-DOWN(H, i) fixes the heap property of H, provided that H has the ALMOST-HEAP-WITH-H[i]-TOO-LARGE property, in time O(log n)

P: We prove the claim by reverse induction (we start from i=n, and we move from i to i-1)

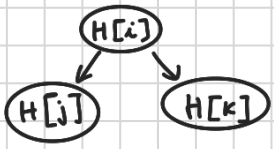
In general, if $i > 2n$, then i has no children, thus the A-H-W-H[i]-T-L property implies the heap property. Thus, the claim holds for $i > 2n$.

Otherwise, $i \leq 2n$. If the algorithm swaps $H[i]$ with $H[j]$, then - at the outset - $KEY[H[j]] < KEY[H[i]]$ ①

(If the algorithm does not swap, then the heap property holds.)

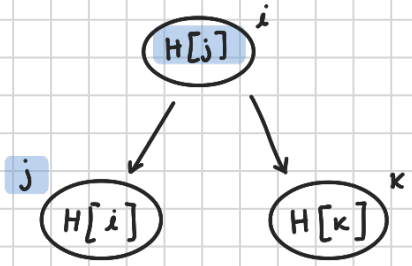
If the algorithm decided to swap $H[j]$ with $H[i]$, then the algorithm chose j as the child of i to consider

Thus, if k is the sibling of j, then $KEY[H[j]] \leq KEY[H[i]]$

IF we start from 

, then $KEY[H[j]] \leq \min(KEY[H[i]], KEY[H[k]])$

and the algorithm will swap $H[i]$ and $H[j]$, obtaining



Thus, the heap property now holds at i.

After the swap, the heap satisfies the ALMOST-HEAP-WITH-H[i]-TOO-LARGE property.

Since the number of levels is $O(\log n)$, and since each call takes $O(1)$ time, plus possibly the time for a new call at a lower level, the runtime is $O(\log n)$ ■

REMOVE (H, i):

```
LET n be the current number of items in H
SWAP H[i] WITH H[n]                                   O(1)
H[n] = NONE                                           O(1)
n -= 1                                                O(1)
IF KEY[H[i]] < KEY[H[parent(i)]]:
    // ALMOST-HEAP-WITH-H[i]-TOO-SMALL
    HEAPIFY-UP(H, i)                                  O(log n)
```

```
ELIF KEY[H[i]] > min(KEY[H[left(i)]], KEY[H[right(i)]]):   O(1)
    // ALMOST-HEAP-WITH-H[i]-TOO-LARGE:
    HEAPIFY-DOWN(H, i)                                     O(log n)
```

THM: REMOVE(H, i) removes the item in position i, while keeping the heap property of H, in O(log n) time.

HEAPS:

- ADD(H, v) takes O(log n) TIME;
- FINDMIN(H) = O(1) = ; // the minimum will be in position 1 because of the heap property.

```
DEF FINDMIN(H):
    RETURN H[1]
```

- REMOVE(H, i) = O(log n) = ;
- EXTRACTMIN(H) removes and returns the minimum of the heap

```
DEF EXTRACTMIN(H):
    v = FINDMIN(H)
    REMOVE(H, n)
    RETURN v
```

WITH THESE FUNCTIONS, we can already sort an array in O(n log n) time.

DEF HEAPSORT(V):

```
LET V BE AN ARRAY OF N ELEMENTS
INITIALISE A HEAP WITH N POSITIONS
FOR i = 0, 1, .., N-1
    LET x BE S.T. KEY[x] = V[i] AND VALUE[x] = V[i]
    ADD(H, V[i])
```

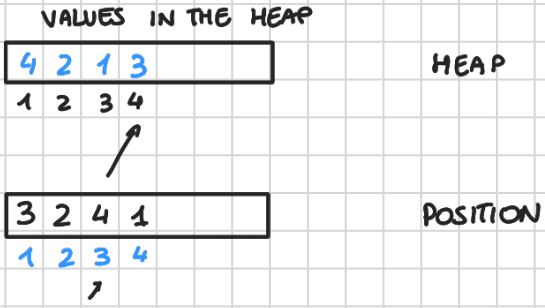
```
FOR i = 0, 1, .. , N-1
    V[i] = KEY[EXTRAMIN(H)]
RETURN V
```

THM: HEAPSORT sorts an array with N items in time $O(N \log N)$

One could want to be able to access items in the heap by value. In order to do so, values should be unique.

Suppose that the values form a subset of $\{1, 2, \dots, N\}$.

IF we use a vector/array POSITION that assigns to the generic value v the position of v in the heap, one gets a good solution.



THUS, POSITION[VALUE[v]] = i IF and only if VALUE[H[i]] = v.

- REMOVEV(H, value) REMOVES the (UNIQUE) ELEMENT of VALUE "value" in time $O(\log n)$

```
DEF REMOVEV(H, v):
    REMOVE(H, POSITION[v])
    POSITION[v] . NONE
```

Observe that, for everything to work out correctly, "Position" should be updated whenever we modify the heap.

We modify the heap in 3 ways:

- ADD AN ELEMENT ;
- REMOVE = = ;
- SWAP TWO = ;

Each of these operations requires no more than updating 2 entries of "Position". Thus, the extra work is only O(1) - TIME - The runtime of each of the functions remain same.