# NOTE BOOK

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## RECAP:

In this section we are going to extend one of the most important ideas from calculus I into functions of 2 variables. We are going to start looking at identifying minimum and maximum of fcts.

# CALCULUS 2

Date: 11 / 11 / 22

- Partial derivative
  - extremes of a fct.
    - local max and min
    - absolute max and min
  - Chain rule
  - Directional Derivative
  - Tangent plane and Normal line

## SECTION 14.7. EXTREME VALUES AND SADDLE POINT

**Def:** Let $f(x,y)$ be defined on a Region $R$ containing the pt $(a,b)$ then :

1. $f(x,y)$ has relative min. at pt $(a,b)$ if $f(a,b) \le f(x,y)$ $\forall$ $(x,y)$ in an open disk centered at $(a,b)$
2. $f(a,b)$ is a local max. if $(a,b) \geq f(x,y)$ $\forall$ points in an open disk centered at $(a,b)$
3. $f$ has an absolute max at $(a,b)$ if $f(a,b) \geq f(x,y)$ for all $(x,y)$ in $R$
4. $f$ has absolute min at $(a,b)$ if $f(a,b) \le f(x,y)$ $\forall (x,y)$ in $R$

### DEF : Critical Points

A point $(a,b)$ of the domain of a function $f(x,y)$ is called a critical point (or stationary point) if

1. $f_x(a,b) = f_y(a,b) = 0$
2. Either $f_x(a,b)$ or $f_y(a,b)$ doesn't exist

NOTE THAT both partial derivatives must be zero at $(a,b)$.
If only one of the first order partial derivatives is zero at the point then the point will not be critical.
The value of the function of the critical point is called critical value.

# CALCULUS 2

Date: 11 / 11 / 2022

**EX:** Find the critical point for $f(x,y) = x^2 + 2xy - 4y^2 + 4x - 6y + 4$

$f_x = 2x + 2y + 4$ $\qquad$ $f_y = 2x - 8y - 6$

Now we set them to zero

$2x + 2y + 4 = 0$ $\qquad$ $2x - 8y - 6 = 0$

$2x = 8y + 6$

$8y + 6 + 2y + 4 = 0$

$10y = -10$

$\boxed{y = -1}$ and $\boxed{x = -1}$

$(-1,-1)$ is only critical point for $f(x,y)$

$\Rightarrow$ The main goal of determining critical points is to locate local max and local min.

## THM: First derivative test for Local Extreme Values

If the point $(a,b)$ is a (local) relative of the function $f(x,y)$ and the first order derivatives of $f(x,y)$ exist at $(a,b)$, then $f_x(a,b) = f_y(a,b) = 0$ and hence $(a,b]$ is a critical point.

**Proof** $\qquad g(x) = f(x,b) \qquad g(a) = f(a,b) \geq \underbrace{f(x,b)}_{g(x)} \quad \forall (x,y)$

Assume that $f$ has a relative extremum at $(a,b)$ then $g(x)$ also has a " " at $x=a$ (of the same kind as $f(x,y)$ at $x=0$)

$\Longrightarrow \quad g'(a) = 0 \quad \Rightarrow \quad f_x(a,b) = 0$

Similarly, we can define $h(y) = f(a,y)$ to show that $f_y(a,b) = 0$

The Thm says that critical points are the only possible points where $f(x,y)$ can assume extreme values.

→ If we substitute the values $f_x(a,b)=0$ and $f_y(a,b)=0$ into the equation

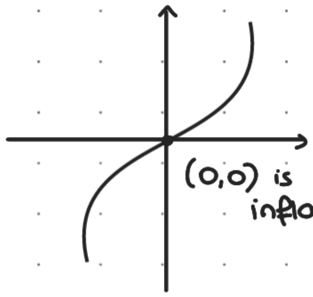
$$f_x(a,b)(x-a)+f_y(a,b)(y-b)-(z-f(a,b))=0$$

for the tangent plane to the surface $z=f(x,y)$ at $(a,b)$ the eqn reduces to :

$$0(x-a)+0(y-b)-z+f(a,b)=0 \Rightarrow z=f(a,b)$$

Thus, the surface have a horizontal tangent plane at local extremum.

* For a fct. of single variable, NOT every critical point gives rise to local extremum ! *




$f(x)=x^3$

$f'(x)=3x^2=0$

$\downarrow (0,0)$

**DEF : Saddle Point**

A function $f(x,y)$ has a saddle point at a critical point $(a,b)$ if in every open disk centered at $(a,b)$, there are domain points $(x,y)$ wher $f(x,y)>f(a,b)$ and domain points $(x,y)$ where $f(x,y)<f(a,b)$.

The corresponding point on the surface $z=f(x,y)$ is called saddle point on the surface.

EX: Find the local extreme value (if any) for $f(x,y)=y^2-x^2$

$f_x=-2x=0$
$f_y=2y=0$ > $(0,0)$ is the critical point and the only possible point where local extremas exist.

Along $y=0$, $f(0,x)=-x^2\leq 0=f(0,0)$

Along $x=0$, $f(y,0)=y^2\geq 0=f(0,0)$

$(0,0)$ is a saddle point as well

## THM: Second Derivative Test

Suppose that $f(x,y)$ and its first and second part derivatives are continuous throughout a disk centered at $(a,b)$ and that $f_x(a,b) = f_y(a,b) = 0$
Define the quantity:

$$\boxed{D = f_{x,x}(a,b)\, f_{y,y}(a,b) - \left[f_{x,y}(a,b)\right]^2}$$

Then :

1. If $D>0$ and $f_{x,x}(a,b) > 0$, $f$ has a local minimum at $(a,b)$
2. If $D>0$ and $f_{x,x}(a,b) < 0$, $f$ has a local max at $(a,b)$
3. If $D<0$, then $f$ has a saddle point at $(a,b)$
4. If $D=0$, then the test is inconclusive.

The expression D is called the descriminant or the Hession of f

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$

NOTE THAT if $D>0$, then both $f_{xx}(a,b)$ and $f_{yy}(a,b)$ will have the same sign, so in the first two cases, we could easily replace $f_{xx}(a,b)$ by $f_{yy}(a,b)$

### STEPS

1. Determine the critical points $(a,b)$ of the function where $f_{xx}(a,b) = f_{yy}(a,b) = 0$
2. Calculate the Descriminant for each critical point of f
3. Apply the 4 cases of the test to determine whether the point is a local max, local min or saddle point.

**EX1:** $f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$

<u>Step 1</u> $f_x = y - 2x - 2 = 0$ $\qquad f_y = x - 2y - 2 = 0$

$\downarrow$

$y = 2x+2 \longrightarrow x - 2(2x+2) - 2 = 0 \Rightarrow \boxed{x = -2}\ \boxed{y = -2}$

$-3x - 6 = 0$

so $(-2,2)$ is a critical point

Step 2 $\quad f_{xx} = -2 \quad f_{yy} = -2 \quad f_{xy} = 1$

$$D = (-2)(-2) - 1^2 = 3 > 0$$

and $f_{xx}(-2,-2) = -2 < 0$

$\Rightarrow$ $(-2,-2)$ is a local max pt. and the local max value is $f(-2,-2) = -2$

ex2 Let $f(x,y) = 3y^2 - 2y^3 - 3x^2 + 6xy$

Step 1. $\begin{cases} f_x = -6x + 6y = 0 \Rightarrow x = y \\ f_y = 6y - 6y^2 + 6x = 0 \end{cases}$

$$\left.\begin{aligned} &6x - 6x^2 + 6x = 0 \\ &-6x^2 + 12x = 0 \\ &6x(-x+2) = 0 \end{aligned}\right\} \begin{aligned} &\Rightarrow x = 0 \text{ or } x = 2 \\ &\Rightarrow y = 0 \text{ or } y = 2 \\ &\Rightarrow (0,0) \text{ or } (2,2) \end{aligned}$$

Step 2 $\quad f_{xx} = -6 \quad f_{yy} = 6 - 12y \quad f_{xy} = 6$

$$\begin{aligned} D &= (-6)(6-12y) - (-6)^2 = \\ &= -36 + 72y - 36 = \\ &= 72y - 72 = 72(y-1) \end{aligned}$$

$D(0,0) = -72 < 0$ so $(0,0)$ is a saddle point

$D(2,2) = 72(2-1) = 72 > 0$ and

$f_{xx}(2,2) = -6 < 0$ so $(2,2)$ is a local max for $f$

EX3 $f(x,y) = \frac{1}{3}x^3 + y^2 + 2xy - 6x - 3y + 4$

1 $\quad \begin{aligned} f_x &= x^2 + 2y - 6 = 0 \quad - \\ f_y &= 2y + 2x - 3 = 0 \\ \hline &x^2 - 2x - 3 = 0 \end{aligned}$

$(x-3)(x+1) = 0$

$$\boxed{\begin{aligned} &\Rightarrow x = 3 \text{ or } x = -1 \\ &\Rightarrow y = -\tfrac{3}{2} \text{ or } y = \tfrac{5}{2} \end{aligned}}$$

② $(3, -3/2)$ and $(-1, 5/2)$ critical points

$f_{xx} = 2x \quad f_{yy} = 2 \quad f_{xy} = 2$

$D = (f_{xx})(f_{yy}) - (f_{xy})^2 =$
$(2x)(2) - 2^2 = 4x - 4 = 4(x-1)$

$D(3, -3/2) = 4(3) - 4 = 8 > 0$ and $f_{yy} > 0$

then $(3, -3/2)$ is a local min pt.

$D(-1, 5/2) = 4(-1-1) = -8 < 0$

then $(-1, 5/2)$ is a saddle point

EX4 $f_x = e^x \sin y \quad f_y = e^x \cos y$

① $f_x = e^x \sin y = 0 \Rightarrow \sin y = 0 \Rightarrow y = 0 + k\pi$
$f_y = e^x \cos y = 0 \Rightarrow \cos y = 0 \Rightarrow y = \frac{\pi}{2}$

we know $e^x$ is always $> 0$

It has no critical point

EX ① $f(x,y) = x^2y + y^2 + xy$
② $f(x,y) = e^{-x^2-y^2}$
} TO DO AT HOME

REMEMBER TO REVISE INTEGRATION METHODS!