NOTE BOOK

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RECAP

$x \in [a,b]$
$y \in [c,d]$

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

Remember:

$$\int_a^b \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy \qquad \int_c^d \int_{h_1(x)}^{h_2(x)} f(x,y)\,dy\,dx$$

$h_1(y) \le x \le h_2(y)$, $a \le y \le b$ $\qquad$ $h_1(x) \le y \le h_2(x)$, $a \le x \le b$

EX1: Evaluate the following integral:

$$\iint_R \frac{x}{1+xy}\,dA \qquad R: \left\{(x,y) \,\middle|\, \begin{matrix} 0 \le x \le 1 \\ 0 \le y \le 1 \end{matrix}\right\}$$

$$\int_0^1 \int_0^1 \frac{x}{1+xy}\,dy\,dx = \int_0^1 \frac{x}{x} \ln|1+xy| \Big]_0^1$$

$$= \int_0^1 \ln|1+x|\,dx = (1+x)\ln(1+x) - (1+x)\Big]_0^1$$

REMEMBER
$\int \ln x = x\ln x - x$

$$= 2\ln 2 - 2 + 1 =$$
$$= 2\ln 2 - 1$$

EX2:

$$\iint_R \frac{\ln y}{y^2}\,dA, \qquad R: \left\{(x,y) : \begin{matrix} 0 \le x \le \pi \\ e^{-2x} \le y \le e^{\cos x} \end{matrix}\right\}$$

$$\int_0^\pi \int_{e^{-2x}}^{e^{\cos x}} \frac{\ln(y)}{y}\,dy\,dx =$$

Let $u = \ln(y)$, $du = \frac{1}{y}dy$

For $y = e^{-2x}$, $u = \ln(e^{-2x}) = -2x$
For $y = e^{\cos x}$, $u = \ln(e^{\cos x}) = \cos x$

$$= \int_0^\pi \int_{-2x}^{\cos x} u\,du\,dx =$$

$$= \int_0^\pi \left[\frac{u^2}{2}\right]_{-2x}^{\cos x} dx = \frac{1}{2}\int_0^\pi (\cos^2 x - 4x^2)\,dx$$

$$= \frac{1}{2}\int_0^\pi \frac{4\cos 2x - 4x^2}{2}\,dx = \frac{1}{2}\left[\frac{x}{2} + \frac{\sin 2x}{4} - \frac{4}{3}x^3\right]_0^\pi = \boxed{\frac{\pi}{2} - \frac{2\pi^3}{3}}$$

$\cos(2x) = 2\cos^{2x} - 1$
$= 1 - 2\sin^2 x$

CALCULUS 2

Date: 21/ 11 / 2022

EX3: $\iint_R \frac{1}{xy}\,dA$; $R\left\{(x,y) \mid 1\le y\le e,\ y\le x\le y^2\right\}$

$$\int_1^e\int_y^{y^2}\frac{1}{xy}\,dx\,dy = \int_1^e \frac{1}{y}\ln|x|\Big]_y^{y^2} = \int_1^e \frac{1}{y}\left[\ln(y^2)-\ln(y)\right]dy$$

$$= \int_1^e \frac{\ln y}{y}\,dy = \frac{\ln y^2}{2}\Big]_1^e = \frac{1}{2}$$

EX4: $\iint_R (\sin(x)-y)\,dA$ R is the Region bounded by $y=\cos x$ and $y=0$, $x=\frac{\pi}{2}$ and $x=0$

$$\int_0^{\pi/2}\int_0^{\cos x}(\sin(x)-y)\,dy\,dx = \int_0^{\pi/2} y\sin x - \frac{y^2}{2}\Big]_0^{\cos x} dx =$$

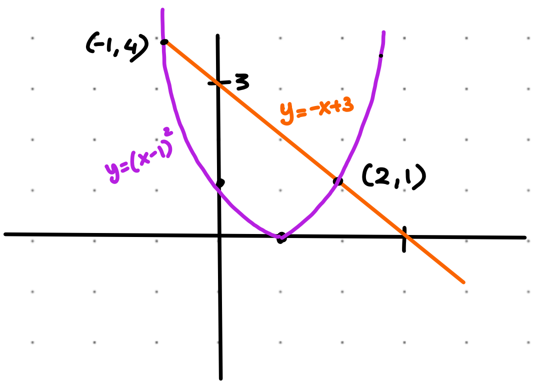
$$\int_0^{\pi/2} \sin x\cos x - \frac{\cos^2 x}{2} = \int_0^{\pi/2}\left[\frac{\sin 2x}{2} - \frac{1+\cos 2x}{4}\right]dx =$$

REMEMBER
$\sin^2 x = 2\sin x\cos x$

$\frac{\sin 2x}{2}$

$$= \frac{-\cos 2x}{4} - \frac{1}{4}x - \frac{\sin 2x}{4}\Big]_0^{\pi/2} = \frac{1}{2} - \frac{\pi}{4}$$

EX5: $\iint_R 4x^3\,dA$ / R is the region bounded by $y=(x-1)^2$ and $y=-x+3$



Steps

1. sketch the R
2. Find the intersection

$(x-1)^2 = -x+3$

$x^2-2x+1 = -x+3$

$x=2$, $x=1$

$y=1$, $y=4$

$$= \int_{x=1}^{x=2}\int_{y=(x-1)^2}^{y=-x+3} 4x^3\,dy\,dx = \int_{x=1}^{x=2} 4x^3y\Big]_{(x-1)^2}^{-x+3} dx = \frac{72}{5}$$

EX6: R is given by the points (0,0), (4,4), (6,0)

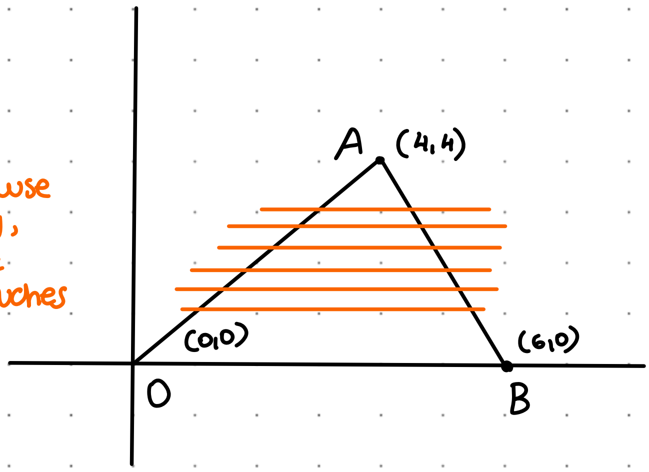
$$\iint_R f(x,y)\, dA$$

$-\frac{1}{2}x = y - 6$

AB → $y = -\frac{1}{2}x + 6$

$x = -2y + 12$

dx then dy because horizontally, every line always touches 2 fcts.



EQ OF STRAIGT LINE

$y - y_0 = m(x - x_0)$

$m = \frac{x_2 - x_1}{y_2 - y_1} = \frac{4-0}{4-6} = -\frac{4}{2} = -\frac{1}{2}$

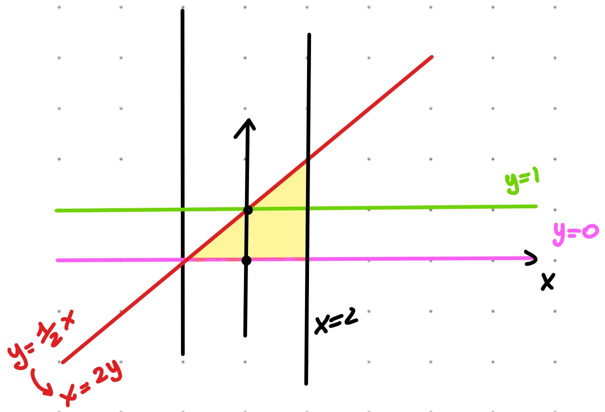
$y = -\frac{1}{2}x + 6$

EX7: $I = \int_0^1 \int_{2y}^2 e^{-x^2}\, dx\, dy$

we should switch from dxdy to dydx to make life easier.

$= \int_0^2 \int_0^{\frac{1}{2}x} e^{-x^2}\, dy\, dy$

we have to find the new bounds since we're switching so sketch R.



$= \int_0^2 e^{-x^2} [y]_0^{\frac{1}{2}x}\, dx$

$= \int_0^2 \frac{1}{2} x e^{-x^2} = -\frac{1}{4}[e^{-4} - 1]$

substitution

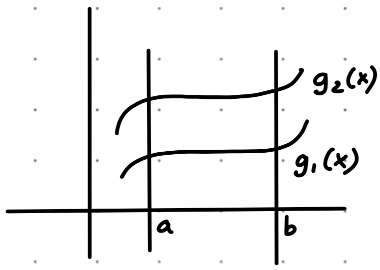
$u = -x^2$

$du = -2x du$

AREA BY DOUBLE INTEGRATION

So, we know Area of R = $\iint_R dA$ (dxdy / dydx)

$A = \int_a^b g_2(x) - g_1(x)\, dx$



PROOF: $\iint_R dA = \int_a^b \int_{g_1(x)}^{g_2(x)} dy\, dx = \int_a^b g_2(x) - g_1(x)$