NOTE BOOK

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CALCULUS 2

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DIFFERENTIALS

If $y = f(x)$ is a function of one variable, then $dy = f'(x)dx$ is called the differential. We also have differentials for more than 1 variable.

Given the function $z = f(x,y)$, then the differential dz is given by:

$$dz = f_x dx + f_y dy$$

This can be extended to function of 3 or more variables. For e.g. given the function $w = g(x,y,z)$, the differential is given by:

$$dw = g_x dx + g_y dy + g_z dz$$

EX: Let $w = \frac{x^3 y^6}{z^2}$

$$\Rightarrow dw = \frac{3x^2y^6}{z^2}dx + \frac{x^3 6x^5}{z^2}dy + \frac{2x^3y^6}{z^3}dz$$

PARTIAL DERIVATIVES AND CONTINUITY

A function $f(x,y)$ can have partial derivatives w.r.t. both x and y at a point without the function being continuous there. This is different from fct. of single variable, where the existence of a derivative implies continuity.

EX: Let $f(x,y) = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$

Notice that $f(0,0) = 1$ ($xy = 0 \rightarrow \begin{matrix} x=0 \\ y=0 \end{matrix}$ $0 \cdot 0 = 0$)

$$\lim_{(x,y)\to(0,0)} f(x,y) \Big|_{\text{along } y=x} = \lim_{(x,y)\to(0,0)} 0 = 0$$

Along the line $y=x$, $f(x,y)$ is constantly zero (except at the origin)

- f is not continuous at $(0,0)$

But $\frac{\partial f}{\partial x}(0,0) = 0 = \frac{\partial f}{\partial y}(0,0)$ (exists)

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{\overbrace{f(h,0)}^{1} - \overbrace{f(0,0)}^{1}}{h} = 0$$

Also $\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0} \frac{1-1}{h} = 0$

THM: Differentiability implies Continuity

- If the partial derivatives of f_x and f_y of a fct. $f(x,y)$ are continuous throughout an open region R, then f is differentiable at every point of R.

- If a fct. $f(x,y)$ is differentiable at (x_0, y_0), then f is continuous at (x_0,y_0)

EX: The function $f(x,y) = x^2y^3$ is everywhere differentiable, since $\frac{\partial f}{\partial x} = 2xy^3$ and $\frac{\partial f}{\partial y} = 3x^2y^2$ are everywhere continuous

REMARKS

Let $z = f(x,y)$

1. f is differentiable at (a,b), then f is continuous at (a,b). However, the converse is not true. Continuity does not imply differentiability

2. If f is differentiable at (a,b), then the partial derivative $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at (a,b). However, existence of partial derivatives doesn't imply differentiability. In fact, existence of partial derivatives doesn't guarantee continuity.

→ In single variable calculus, a fct. $f: \mathbb{R} \to \mathbb{R}$ is differentiable at $x=a$ if the following lim exists:

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = f'(a)$$

lim exists iff $\lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a} - f'(a)\right) = 0$

$$\Leftrightarrow \lim_{x\to a}\left(\frac{f(x)-f(a)\cdot f'(a)(x-a)}{x-a}\right) = 0 \Leftrightarrow \lim_{x\to a} \frac{f(x)-L(x)}{a} = 0$$

where $L(x) = f(a) + f'(a)(x-a)$ → $L(x)$ is the linear approx. to f at $x=a$

So, roughly speaking, a single variable fct. is diff. iff the differential between $f(x)$ and its linear $\simeq$ goes zero as we approach the point.

Now: $f: \mathbb{R}^2 \to \mathbb{R}$
$(x,y) \to f(x,y)$ / Let $h(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$
linear approx. = eqn. of the tan. plane

FORMAL DEFINITION OF DIFFERENTIABILITY

Consider $f: \mathbb{R}^2 \to \mathbb{R}$ and suppose that the partial derivatives f_x and f_y are defined at the pt. $(x,y) = (a,b)$. Define the linear fct

$$h(x,y) = f(a,b) - f_x(a,b)(x-a) - f_y(a,b)(y-b)$$

We say that f is differentiable at (a,b) if:

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y) - h(x,y)}{\|(x,y)-(a,b)\|} = 0$$

If either of the partial derivatives $f_x(a,b)$ and $f_y(a,b)$ do not exist, or the above limit doesn't exist or not zero, then f is not diff. at (a,b)

EX: Study the differentiability of the following fct at $(0,0)$

$$f(x,y) = \begin{cases} \dfrac{x\sqrt{y}}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{h\sqrt{0}}{h^2+0} - 0}{h} = 0$$

$$\frac{\partial f(0,0)}{\partial y} = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

Now, let's check if $\lim_{(x,y)\to(0,0)} \dfrac{f(x,y) - [f(0,0) - f_x(0,0)(x-0) - f_y(0,0)(y-0)]}{\sqrt{(x-0)^2 + (y-0)^2}}$

$$= \lim_{(x,y)\to(0,0)} \frac{\frac{x\sqrt{y}}{x^2+y^2} - 0 - 0 - 0}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{x\sqrt{y}}{(x^2+y^2)\sqrt{x^2+y^2}}$$

Now, lets see along $y=0$ and $y=x$ ⤵

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Along $x=0$, $\lim_{x\to 0} \frac{x\sqrt{y}}{(x^2+y^2)\sqrt{x^2+y^2}} = \lim_{x\to 0} 0 = 0$

Along $y=x$, $\lim_{x\to 0} \frac{x\sqrt{x}}{(x^2+x^2)\sqrt{2x^2}} = \lim_{x\to 0} \frac{1}{2x\sqrt{2x}} = \infty$

The fct. is not diff at $(0,0)$

EX2: Show that the fct. $f(x,y) = xy + 2x + y$ is diff at $(0,0)$

$f_x(x,y) = y+2 \implies f_x(0,0) = 2$
$f_y(x,y) = x+1 \implies f_y(0,0) = 1$
exists

$f(0,0) = 0$

$\boxed{h(x,y)} = f(0,0) + f_x(0,0)(x) + f_y(0,0)y = 0 + 2x + y = \boxed{2x+y}$ and

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - h(x,y)}{\sqrt{(x-0)^2+(y-0)^2}} = \lim_{(x,y)\to(0,0)} \frac{xy + \cancel{2x} + \cancel{y} \cancel{-2x} \cancel{-y}}{\sqrt{x^2+y^2}} = \frac{xy}{\sqrt{x^2+y^2}}$$

polar coordinates $= \lim_{r\to 0} \frac{r\cos\theta \, r\sin\theta}{\sqrt{r^2}} = \lim_{r\to 0} \frac{r^2\cos\theta\sin\theta}{|r|}$

But $-1 \leq \cos\theta\sin\theta \leq 1 \implies -|r| \leq \frac{r^2\cos\theta\sin\theta}{|r|} \leq |r|$

Since $\lim -|r| = \lim_{r\to 0} |r| = 0$,

By sandwich thm, $\lim \frac{r^2\cos\theta\sin\theta}{r} = 0$

$$\boxed{\lim_{(x,y)\to(0,0)} \frac{f(x,y) - h(x,y)}{\|(x-y)-(0,0)\|} = 0 \text{ and hence } f \text{ is diff at } (0,0)}$$

GRAPHICAL INTERPRETATION OF PARTIAL DERIVATIVES

Given a fct $f: \mathbb{R} \to \mathbb{R}$, we know that $f(x_0)$ gives the slope of the tg. line to the graph of f at x_0.

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Now let $f: \mathbb{R}^2 \to \mathbb{R}$

Fix $y = b$ (plane)

Let $g(x) = f(x, b)$

Then $\dfrac{\partial f}{\partial x}(a,b) = \dfrac{dg}{dx}(a)$

it is the slope of the tg line to the curve that results from the intersection of the plane $y=b$ and the $\underline{\text{surface}}$ at the pt. (a,b)

graph of y

Fix $x = a$

Let $h(y) = f(a,y)$ (fct. of y)

$\dfrac{\partial f}{\partial y}(a,b) = \dfrac{dh}{dy}(b)$

$\Rightarrow$ The partial derivative of f w.r.t. y at (a,b) is the slope of the tg. line at the intersection of the graph of y with the plane $x=a$

EX: The plane $x=1$ intersects the paraboloid $z = x^2+y^2$ in a parabola.
Find the slope of the tg line to the parabola at $(1,2,5)$

Sol. The slope is given by $\dfrac{\partial z}{\partial y}(1,2)$

$$\left.\frac{\partial z}{\partial y}\right|_{(1,2)} = \left. 2y \right|_{(1,2)} = (2)(2) = 4$$

To check, we can treat the parabola as the graph of single-variable fct. $z = y^2+1$ in the plane $x=1$ and ask for the slope at $y=2$.

The slope is given by the ordinary derivative:

$$\left.\frac{dz}{dy}\right|_{y=2} = \left.\frac{d}{dy}(1+y^2)\right|_{y=2} = \left. 2y\right|_{y=2} = 4$$