NOTE BOOK

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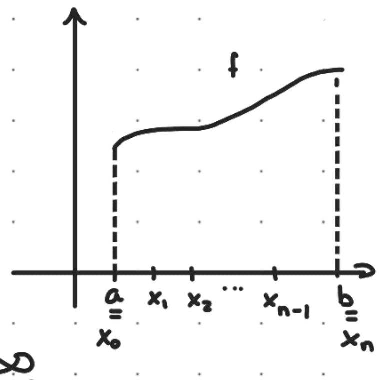
Multiple Integrals (Chap. 15)

RECAP:

Let f be a continuous fct. on $[a,b]$.
Divide the interval $[a,b]$ into equal n sub-intervals each of length $\Delta x = \frac{b-a}{n}$



x_k to be an arbitrary pt. $A \approx \sum_{k=1}^{n} f(x_k)\Delta x$
(called the approximation area)

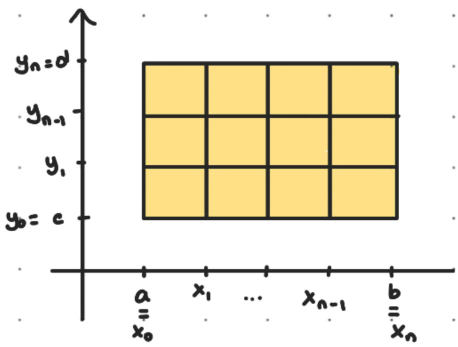
To get the exact area, we have to send $n \to \infty$
$(\Delta x \to 0)$

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x = \int_a^b f(x)\,dx$$

DOUBLE AND ITERATED INTEGRALS OVER RECTANGLES

DOUBLE INTEGRALS

Consider $R: \{(x,y) \mid a \le x \le y,\ c \le y \le d\}$



Divide $[a,b]$ into m subdivisions
Divide $[c,d]$ " n "

→ This will divide up our region into small rectangles

$$R = [a,b] \times [c,d]$$

These rectangles form a position of R $(\Delta A = \Delta x \Delta y)$
Δx = width of each rectangle along x-direction
Δy = height " " " " y-direction

(x_k, y_k) be arbitrary pt. in each one of the rect.

→ Each of these rectangles have a base area $\Delta A = \Delta x \Delta y$

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DOUBLE INTEGRALS AS VOLUMES

The volume under the surface is approximately

$$V \simeq \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j) \Delta A$$

$$V = \lim_{m,n \to +\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j) \Delta A = \iint_R f(x,y)\, dA$$

FUBINI'S THEOREM FOR CALCULATING DOUBLE INTEGRALS

$$\iint_R f(x,y)\, dA \to \int_{y=c}^{y=d} \left(\int_{x=a}^{x=b} f(x,y)\, dx \right) dy$$

$$\iint_R f(x,y)\, dA \to \int_{x=a}^{x=b} \int_{y=c}^{y=d} f(x,y)\, dy\, dx$$

THM: (Fubini's Thm)

If $f(x,y)$ is continuous fct. over a rectangular region
$R: a \leq x \leq b, \quad c \leq y \leq d$

$$\iint_R f(x,y)\, dA = \int_c^d \int_a^b f(x,y)\, dx\, dy = \int_a^b \int_c^d f(x,y)\, dy\, dx$$

these are called iterated integrals

EX1 $f(x,y) = 2xy$ (single integral w. constant) *

$$f(x,y) = \int f_x\, dx = \int 2x\underset{\text{constant}}{y}\, dx = 2y \frac{x^2}{2} + c = yx^2 + c(y)$$

EX2 $\displaystyle\int_1^{2y} 2xy\, dx = \left[x^2 y\right]_1^{2y} = (2y)^2 y - y = 4y^3 y - y$ *

EX3 Compute $\iint_R 6xy^2\, dx\, dy$ where $R = [2,4] \times [1,2]$

↓

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1st method (dx)

$$\int_1^2\left(\int_2^4 6xy^2\,dx\right)dy = \int_1^2 \left[3x^2y^2\right]_2^4 dy =$$

$$\int_1^2 3y^2(4^2-2^2)\,dy = 36\int_1^2 y^2\,dy =$$

$$= \frac{36}{3}\left[y^3\right]_1^2 = 84$$

2nd method (dy)

$$\int_2^4\left(\int_1^2 6xy^2\,dy\right)dx = \int_2^4 \left[2xy^3\right]_1^2 dy$$

$$= 14\int_2^4 x\,dx = \frac{14}{2}\left[x^2\right]_2^4 =$$

$$= 7(16-4) = 84$$

EX4 Compute $\iint_R xe^{xy}\,dA$ where $R = [-1,2]\times[0,1]$

1st method (dy)

$$\int_{-1}^2\left(\int_0^1 xe^{xy}\,dy\right)dx = \int_{-1}^2 \left[e^{xy}\right]_0^1 dx =$$

$\int xe^{xy}\,dy$

$u = xy \quad \frac{du}{dy} = x \rightarrow dy = \frac{du}{x}$

$= e^{xy} + c$

$$= \int_{-1}^2 (e^x - 1)\,dx = \left[e^x - x\right]_{-1}^2 = e^2 - 2 - e^{-1} - 1 =$$

$$= e^2 - e^{-1} - 3$$

2nd method (dx)

integr. by parts

$u = x \quad dv = e^{xy}dx$

$du = dx \quad v = \frac{1}{y}e^{xy}$

$$\int_0^1\int_{-1}^2 xe^{xy}\,dx\,dy = \int_0^1 \left[\frac{x}{y}e^{xy} - \int \frac{1}{y}e^{xy}dx\right]_{-1}^2 dy =$$

$$= \int_0^1 \left[\frac{x}{y}e^{xy} - \frac{1}{y^2}e^{xy}\right]_{-1}^2 dy = \int_0^1 \frac{2e^{2y}}{y} - \frac{1}{y^2}e^{2y} \ldots$$

too complicated

EX5: $\iint_R (x+y^2)\,dA$ where $R: \left\{(x,y) \,/\, \begin{matrix} 0\le x\le 1 \\ -1\le y\le 2\end{matrix}\right\}$

$[0,1]\times[-1,2]$

$$\int_{-1}^2\int_0^1 (x+y^2)\,dx\,dy = \int_{-1}^2 \left[\frac{x^2}{2} + y^2x\right]_0^1 dy =$$

$$= \int_{-1}^2 \left(\frac{1}{2}+y^2\right)dy = \left.\frac{1}{2}y + \frac{y^3}{3}\right]_{-1}^2 = 1 + \frac{8}{3} + \frac{1}{2} + \frac{1}{3} = \frac{9}{2}$$

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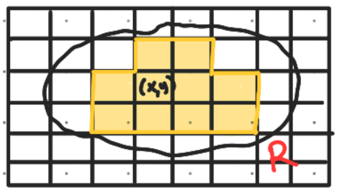
DOUBLE INTEGRAL OVER A GENERAL REGION

$$\lim_{m,n\to 0} \sum\sum f(x_i, y_i)\,\Delta x \Delta y$$

$$\Delta A = \Delta x \Delta y$$



We'll consider 2 types of Non-Rectangular Regions:

CASE 1: $R = \{(x,y) / a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$

$$\iint_R f(x,y)\,dA = \int_{x=a}^{x=b}\int_{y=g_1(x)}^{y=g_2(x)} f(x,y)\,dy\,dx$$

CASE 2: $R: \{(x,y) / h_1(y) \le x \le h_2(y);\ c \le y \le d\}$

Then: $\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$

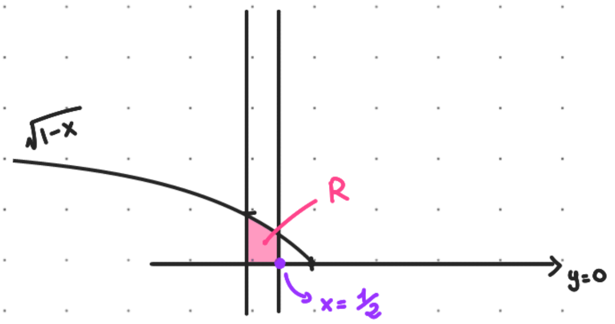
EX: $\iint_R 2xy\,dA$ $\quad R: (x,y) / 0 \le x \le \frac{1}{2};\ 0 \le y \le \sqrt{1-x}\}$

We are in the 1° case. Set up the integral

$$\iint_R 2xy\,dA = \int_{x=0}^{1/2}\int_{y=0}^{y=\sqrt{1-x}} 2xy\,dy\,dx = \int_0^{1/2} xy^2\Big]_0^{\sqrt{1-x}} dx$$

$$= \int_0^{1/2} x(1-x)\,dx = \frac{x^2}{2} - \frac{x^3}{3}\Big]_0^{1/2} = 1/12$$

Now we sketch the region ↓



FIBINUS THM FOR GENERAL REGION

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

y should be between fct. of x and x between constants

Bound x between fct. of y and y between constants

Ex Let R be the Region bounded by $y = 2x$ and $y = x^2$

Sketch R

$y = y$

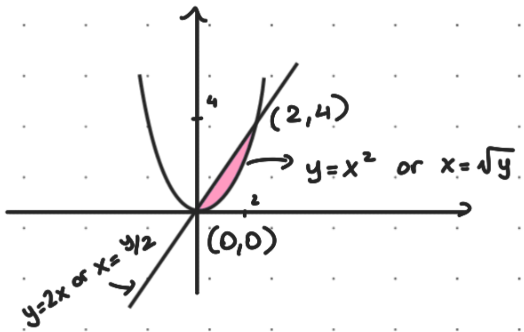
$x^2 = 2x$

$x^2 - 2x = 0$

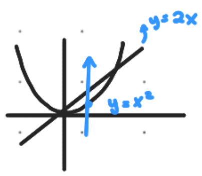
$x(x-2) = 0$

$x = 0$ or $x = 2$

$(0,0)$ $(2,4)$



$$\iint_R f(x,y)\,dA = \int_{x=0}^{x=2} \int_{y=x^2}^{y=2-x} f(x,y)\,dy\,dx = \int_0^4 \int_{x=y/2}^{x=\sqrt{y}} f(x)\,dx\,dy$$



you go from bottom to up along y vertical line ↑ (from bottom to up because that will be the positive one)

we enter the R horizontally now, from left to right