NOTE BOOK

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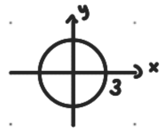
CALCULUS 2

Date: 14 / 10 / 2022

EX:

- $z = f(x,y) = \sqrt{9-x^2-y^2}$ $\quad 9-x^2-y^2 \geq 0$

DOMAIN: $9 \geq x^2+y^2$ Disc center at $(0,0)$ of radius 3



RANGE $\longrightarrow [0,3]$

START FROM DOMAIN

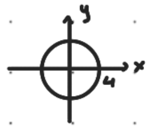
$$\boxed{\begin{aligned} &0 \leq x^2+y^2 \leq 9 \\ &-9 \leq -x^2-y^2 \leq 0 \\ &0 \leq 9-x^2-y^2 \leq 9 \\ &0 \leq \sqrt{9-x^2-y^2} \leq 3 \\ &0 \leq z \leq 3 \end{aligned}}$$

- $z = \dfrac{1}{\sqrt{16-x^2-y^2}}$

DOMAIN $16-x^2-y^2 > 0$ NOTE: not equal to zero because it's a fraction

$x^2+y^2 < 16$

interior of the circle centre $(0,0)$ at $r=4$



RANGE $[\frac{1}{4}; +\infty)$

START FROM DOMAIN

$$\boxed{\begin{aligned} &0 \leq x^2+y^2 < 16 \\ &-16 < x^2-y^2 \leq 0 \\ &0 < 16-x^2-y^2 \leq 16 \\ &0 < \sqrt{16-x^2-y^2} \leq 4 \\ &\frac{1}{0} < z \leq \frac{1}{4} \\ &\frac{1}{4} \leq z < +\infty \end{aligned}}$$

NOTE: $b < a < c$
$= \frac{1}{c} < \frac{1}{a} < \frac{1}{b}$
reciprocal

- $f(x,y) = \dfrac{1}{\ln(4-2x^2-y^2)}$

remember $\ln(1) = 0$

Df: $\ln(4-2x^2-y^2) \neq 0$ and $4-2x^2-y^2 > 0$

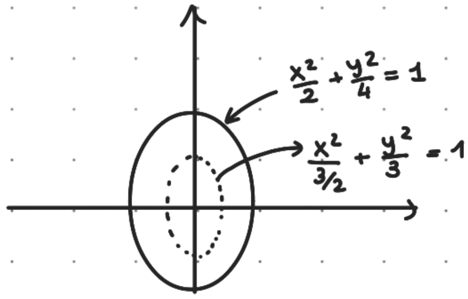
$4-2x^2-y^2 \neq 1$ $\qquad 2x^2+y^2 < 4$

$2x^2+y^2 \neq 3$ $\qquad \boxed{\frac{x^2}{2} + \frac{y^2}{4} < 1}$

$\boxed{\frac{x^2}{3/2} + \frac{y^2}{3} \neq 1}$ ellipse $\qquad$ interior of the ellipse

$x \to +\sqrt{\frac{3}{2}},\ -\sqrt{\frac{3}{2}}$ $\quad y \to \sqrt{3},\ -\sqrt{3}$ $\qquad x \to \sqrt{2},\ -\sqrt{2}$ $\quad y \to 2,\ -2$

DOMAIN: interior of the ellipse $\frac{x^2}{2}+\frac{y^2}{4}=1$ excluding the ellipse $2x^2+y^2=3$



LEVEL CURVES AND LEVEL SURFACES

There are two standard ways to picture the values of a function $f(x,y)$:

1. Draw and label curves in the domain in which f has a constant value.
2. Sketch the surface $z=f(x,y)$ in space.

Def: The set of points (x,y) in the plane where $f(x,y)$ is constant $(f(x,y)=c)$ is called level curve while the set of all points in space such that $f(x,y,z)=c$ is called level surface.

The set of all points $(x,y,f(x,y))$ in space, for (x,y) in the domain of f, is called the graph of f.

The graph of f is also called the surface $z=f(x,y)$

EX: Graph $f(x,y)=100-x^2-y^2$, plot the level curves $f(x,y)=0$ and $f(x,y)=51$

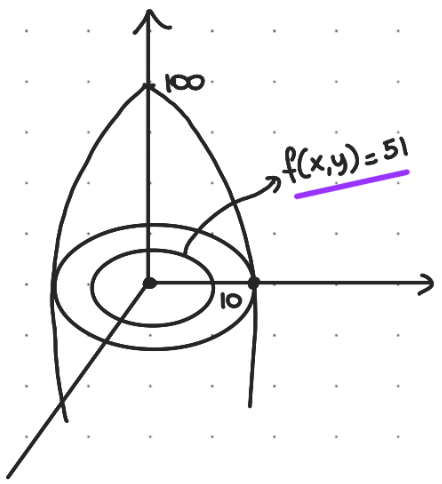
OBSERVATION → $z=100-x^2-y^2$ is a circular paraboloid

↓

DOMAIN → (x,y) plane

RANGE → $(-\infty, 100]$

$$\boxed{\begin{array}{c} x^2+y^2\geq 0 \\ -x^2-y^2\leq 0 \\ 100-x^2-y^2\leq 100 \\ z\leq 100 \end{array}}$$



- The level curve $f(x,y)=0$ i.e. $x^2+y^2=100$ is the set of all points in the circle of center $(0,0)$ and radius 10
- $f(x,y)=51 \Rightarrow x^2+y^2=49$
  $\Downarrow$
  circle of center $(0,0)$ at radius 7
- The level curve $f(x,y)=100$ consists of the origin only.

EX Describe the level surfaces of the function $f(x,y,z)=\sqrt{x^2+y^2+z^2}$

$\Rightarrow f(x,y,z)=c$

$\sqrt{x^2+y^2+z^2}=c$, $c>0$ is a sphere centered at the origin and radius c.

LIMITS AND CONTINUITY

The concept of limit for multi-variable functions is analogous to that for single-variable functions.

Def: We say that a function $f(x,y)$ approaches the limit L as (x,y) approaches (x_0,y_0) and write

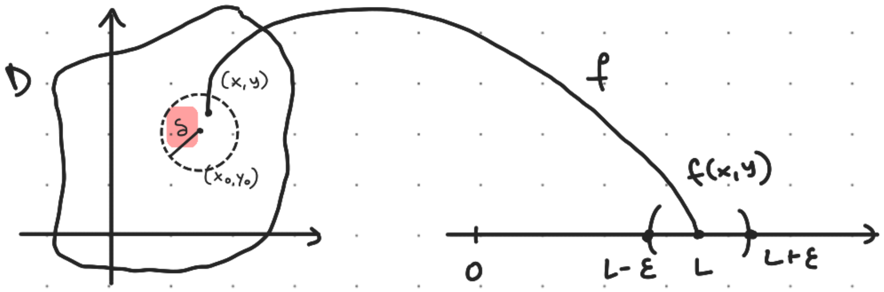
$$\boxed{\lim_{(x,y)\to(x_0,y_0)} f(x,y)=L}$$

i) $\forall \varepsilon>0$, $\exists$ corresponding $\delta>0$ such that for all (x,y) in the domain of f,

means $(x,y)\neq(x_0,y_0)$

$$|f(x,y)-L|<\varepsilon \text{ whenever } 0<\underbrace{\sqrt{(x-x_0)^2+(y-y_0)^2}}_{\text{open disk centered at }(x_0,y_0)}<\delta$$

This definition says that the distance between $f(x,y)$ and L becomes arbitrary small whenever the distance from (x,y) to (x_0, y_0) is made sufficiently small (but not 0)



δ is the radius of the disk centered at (x_0, y_0), for all the points (x,y) within this disk, the function values $f(x,y)$ lies inside the corresponding interval $(L-\varepsilon, L+\varepsilon)$

EXERCISE Show that $\lim\limits_{(x,y)\to(0,0)} \frac{4x^2y}{x^2+y^2} = 0$

Sol: Let $\varepsilon > 0$ and $\delta = \frac{\varepsilon}{4}$ and we get

$$|f(x,y) - 0| = \left|\frac{4x^2y}{x^2+y^2}\right| \quad \text{since } \frac{x^2}{x^2+y^2} \leq 1$$
$$< |4y|$$
$$< 4\delta = \varepsilon$$

EXERCISE 2 Prove that $\lim\limits_{(x,y)\to(0,0)} 2x = 0$

Sol: Given $\varepsilon > 0$, we need to find $\delta > 0$ if $(x,y) \neq (0,0)$ and $\sqrt{x^2+y^2} < \delta$, then

$$|2x| < \varepsilon$$

choose $\boxed{\delta = \frac{\varepsilon}{2}}$

Then, $|f(x,y) - 0| = |2x| = 2\sqrt{x^2}$
$$\leq 2\sqrt{x^2+y^2} < 2\delta = 2 \cdot \frac{\varepsilon}{2} = \varepsilon$$

EXERCISE 3 Show that $\lim\limits_{(x,y)\to(1,2)} (2x+y) = 4$

Sol: Given $\varepsilon > 0$, we need to find $\delta > 0$ s.t. if $0 < \sqrt{(x-1)^2 + (y-2)^2} < \delta$, then $|2x + y - 4| < \varepsilon$.

So Given $\varepsilon > 0$ and let $\delta = \frac{\varepsilon}{3}$
and $\sqrt{(x-1)^2 + (y-2)^2} < \delta$

Then $|2x + y - 4| = |2(x-1) + y - 2|$
$\leq 2|x-1| + |y-2| \longrightarrow$ since $|x-1| = \sqrt{(x-1)^2} \leq \sqrt{(x-1)^2 + (y-2)^2}$
$< 2\delta + \delta = 3\delta = \varepsilon$

PROPERTIES (of limits of functions of 2 var.)

Assume $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = L$ and $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = M$, then:

- $\lim\limits_{(x,y)\to(x_0,y_0)} (f(x,y) + g(x,y)) = L + M$ [Sum Rule]
- $\lim\limits_{(x,y)\to(x_0,y_0)} (f(x,y) - g(x,y)) = L - M$ [Difference Rule]
- $\lim Kf(x,y) = KL$ [K = any number]
- $\lim\limits_{(x,y)\to(x_0,y_0)} (f(x,y) \cdot g(x,y)) = L \cdot M$ [Product rule]
- $\lim\limits_{(x,y)\to(x_0,y_0)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}$ $(M \neq 0)$ [Quotient Rule]
- $\lim [f(x,y)]^n = L^n$ [Power Rule]

EXAMPLES

1. $\lim\limits_{(x,y)\to(0,1)} \frac{x - xy + 3}{x^2y + 5xy - y^3} = \frac{0 - 0\cdot 1 + 3}{0^2 1 + 5\cdot 0\cdot 1 - 1^3} = \frac{3}{-1} = -3$

2. $\lim\limits_{(x,y)\to(3,-4)} \sqrt{x^2 + y^2} = \sqrt{3^2 + (-4)^2} = 5$

3. $\lim\limits_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} = \frac{0}{0}$ (ind. form)

rationalize

$$= \lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} \cdot \frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} + \sqrt{y}} = \lim \frac{x\cancel{(x-y)}(\sqrt{x} + \sqrt{y})}{\cancel{x-y}}$$

$$= \lim_{(x,y)\to(0,0)} x(\sqrt{x} + \sqrt{y}) = 0$$

4. $\lim_{(x,y)\to(2,0)} \dfrac{\sqrt{2x-y}-2}{2x-y-4} = \dfrac{2-2}{4-0-4} = \dfrac{0}{0}$ (ind. form)

$= \lim_{(x,y)\to(2,0)} \dfrac{\cancel{\sqrt{2x-y}-2}}{(\cancel{\sqrt{2x-y}-2})(\sqrt{2x-y}+2)} = \lim_{(x,y)\to(2,0)} \dfrac{1}{\sqrt{2x-y}+2} =$

$= \lim_{(x,y)\to(2,0)} \dfrac{1}{\sqrt{4-0}+2} = \dfrac{1}{4}$

5. $\lim_{(x,y)\to(1,1)} \dfrac{xy-y-2x+2}{x-1} = \xrightarrow{\text{grouping}} \lim_{(x,y)\to(1,1)} \dfrac{y(x-1)-2(x-1)}{\cancel{x-1}} =$

$\lim_{(x,y)\to(1,1)} y-2 = -1$

NOTE

* When dealing with functions of a single variable, we also considered one-sided limits and stated:

$$\lim_{x\to c} f(x) = L \iff \lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x) = L$$

$\rightarrow c \leftarrow$

i.e. the limit L exists if and only if f(x) approaches L when x approaches c from either direction, the left or the right.

If $\lim_{x\to c^+} f(x) \neq \lim_{x\to c^-} f(x)$, then we say $\lim_{x\to c} f(x)$ doesn't exist.

* In the plane, there are infinite directions for which (x,y) might approach (x_0,y_0).
We don't have to restrict ourselves to approaching (x_0,y_0) from a particular direction, but rather we can approach that but along a path that is not a straight line.

Def: "Path" ⟶ A path is any curve passing through the point (x_0,y_0)

* For the limit to exist at a point (x_0,y_0), the limit must be the same along every approach path.
If we ever find paths with different limits, we say that the limit doesn't exist.

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TWO-PATHS TEST FOR NON-EXISTENCE OF LIMIT

If a function $f(x,y)$ has different limits along two different paths in the domain of f as (x,y) approaches (x_0,y_0); then $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ doesn't exist.

EX: Show that the function $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$ has no limit as (x,y) approaches $(0,0)$

$\Rightarrow$ SOLUTION (2 ways):

① Approach $(0,0)$ along the (x-axis)$(y=0)$, then

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2}{x^2} = 1 = L_1$$

Now, approach $(0,0)$ along the (y-axis)$(x=0)$,

$$\lim = \lim \frac{-y^2}{y^2} = -1 = L_2$$

$L_1 \neq L_2 \Rightarrow$ Limit doesn't exist!

② Consider the path $y = mx$

$$\Rightarrow f(x,y) = \frac{x^2 - m^2x^2}{x^2 + m^2x^2} = \frac{1-m^2}{1+m^2}$$

So, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \frac{1-m^2}{1+m^2}$ (depends on m)

for $m=1$ i.e. along the path $y=x$, we get

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0 = L_1$$

for $m=2$ i.e. along the path $y=2x$, we get

$$\lim f(x,y) = \frac{1-2^2}{1+2^2} = -\frac{3}{5} = L_2$$

$L_1 \neq L_2 \Rightarrow \lim\limits_{(x,y)\to(0,0)} f(x,y)$ doesn't exist

EX: Show that $\lim\limits_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2}$ doesn't exist

Consider the curve $y = mx^2$, $x \neq 0$
Along this curve, the function $f(x,y) = \frac{2x^2mx^2}{x^4+m^2x^4} = \frac{2m}{1+m^2}$

(so depends on m)

$\Rightarrow$ The limit varies with the path of approach.
For example:

if $m=1$, along the parabula $y=x^2$, we have
$\lim = \boxed{1 = L_1}$

if $m=0$, (x,y) approaches $(0,0)$ along the x-axis,
$\lim = \boxed{0 = L_2}$

So $\boxed{L_1 \neq L_2} \Longrightarrow \lim\limits_{(x,y)\to(0,0)} f(x,y)$ NO EXIST

EXERCISES (ASSIGNMENT)

1. Use the two-path test to show that $\lim\limits_{(x,y)\to(0,0)} \frac{x^6+2y^3}{x^4y+5x^6}$ doesn't exist.

Sol: (maybe !!) Consider $y = mx^2$, $x \neq 0$

then $f(x,y) = \frac{x^6 + 2m^3x^6}{mx^6+5x^6} = \frac{\cancel{x^6}(1+2m^3)}{\cancel{x^6}(m+5)} = $

$= \frac{1+2m^3}{m+5}$

If $m=1$, $\lim = \frac{1}{3} = L_1$
If $m=0$, $\lim = \frac{1}{5} = L_2$
$\Bigg\}$ $L_1 \neq L_2$

2. Find the following limit $\lim\limits_{(x,y)\to(1,1)} \frac{x^2y^2-1}{xy-1}$

$\lim\limits_{(x,y)\to(1,1)} \frac{x^2y^2-1}{xy-1} = \frac{(xy+1)\cancel{(xy-1)}}{\cancel{xy-1}} = 2$

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③ Let $f(x,y) = \frac{5x^2y^2}{x^2+y^2}$. Find $\lim_{(x,y)\to(0,0)} f(x,y)$

$$\lim_{(x,y)\to(0,0)} \frac{5x^2y^2}{x^2+y^2} = \frac{5\cancel{x^2y^2}}{\cancel{x^2y^2}\left(\underbrace{\frac{1}{y^2}}_{\infty}+\underbrace{\frac{1}{x^2}}_{\infty}\right)} = 0$$

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