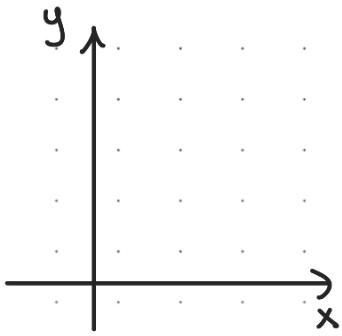
NOTE BOOK

Rokshana Ahmed

REVISION

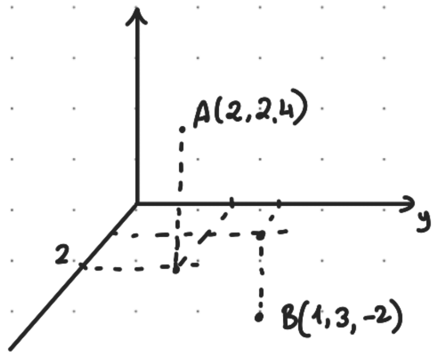
① A point A in space is represented as :

$$A(x,y,z)\begin{cases} x = \text{abscissa of } A \\ y = \text{ordinate of } A \\ z = \text{elevation} \end{cases}$$



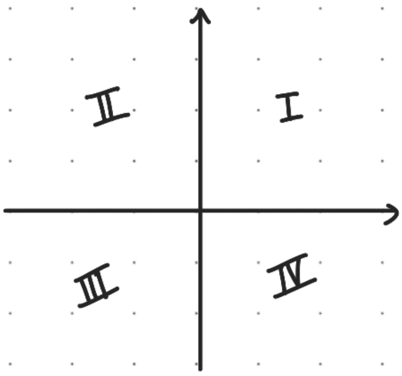
EXAMPLE :

Plot $A(2,2,4)$ $B(1,3,-2)$



1. REGIONS ON SPACE

In space $\begin{cases} z=0 \text{ is the } (x,y) \text{ plane} \\ y=0 \text{ is the } (x,z) \text{ plane} \\ x=0 \text{ is the } (y,z) \text{ plane} \end{cases}$ Coordinate planes



1st octant → $\left.\begin{matrix} xyz & x'yz \\ xyz' & x'yz' \\ xy'z & x'y'z \\ xy'z' & x'y'z \end{matrix}\right\}$ OCTANTS (8 regions)

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EX: Specify the regions determined by the following equations and inequalities:

① $z \geq 0$ → upper half-space including (x,y) plane

② $z = 1$ → plane parallel to xy plane passing $(0,0,1)$

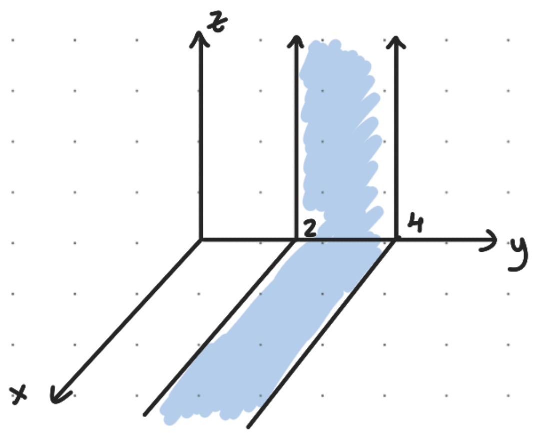
③ $z=0,\ x\geq 0,\ y\geq 0$

→ 1^{st} quadrant of the xy plane

④ $2 \leq y \leq 4$

$y=2$ → plane // to (x,z) plane thru $(0,2,0)$

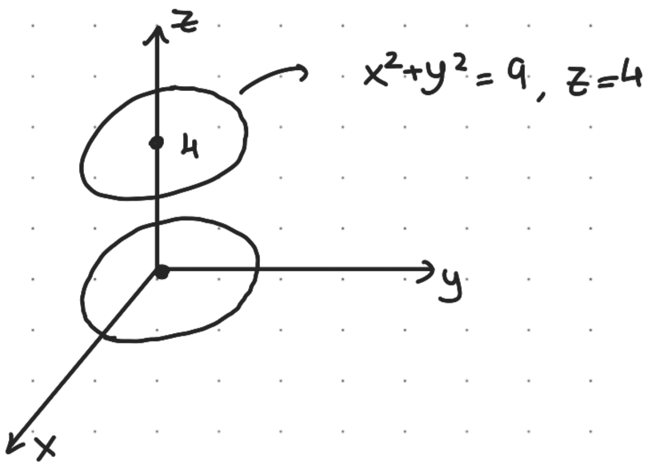
$y=4$ → plane // to (x,z) plane thru $(0,4,0)$



⑤ $x^2+y^2=9 \qquad yz=0$

↓

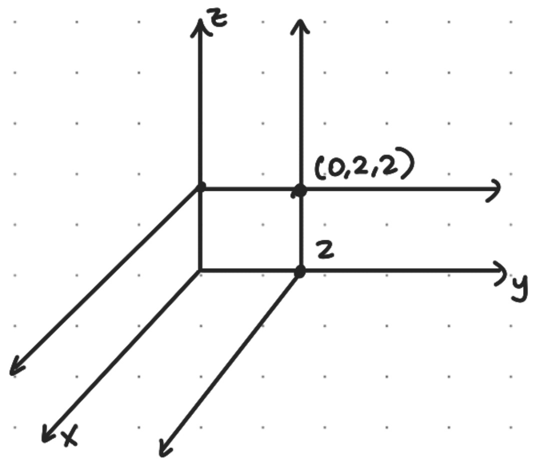
circle centered at $(0,0,0)$ in the (xy) plane of radius 3



⑥ $y=2$ and $z=2$

$y=2 \rightarrow$ plane // to (x,z) plane thru $(0,2,0)$

$z=2 \rightarrow$ plane // to (x,y) plane thru $(0,0,2)$



2. DATES AND SPHERE IN SPACE

The distance between $A(x_1,y_1,z_1)$ and $B(x_2,y_2,z_2)$ then $|AB| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

distance between A and B

ex : $A(2,1,5)$ and $B(-2,3,0)$

$|AB| = \sqrt{4+(-2)^2+5^2} = \sqrt{45} \simeq$

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2.1 EQUATION OF A SPHERE

A point $A(x,y,z)$ lies on a sphere of radius a and center $C(x_0,y_0,z_0)$ $\Rightarrow$ $|AC| = a$

$$\boxed{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2 = a^2}$$

ex: $(x+1)^2 + y^2 + (z-2)^2 = 9$

$\downarrow$

sphere centered at $(-1,0,2)$ and radius $r=3$)

Sketch

$x^2+y^2+z^2=1$

$x=0$ $\quad y^2+z^2=1 \longrightarrow$ circle center $(0,0,0)$ $r=1$ in the (yz) plane

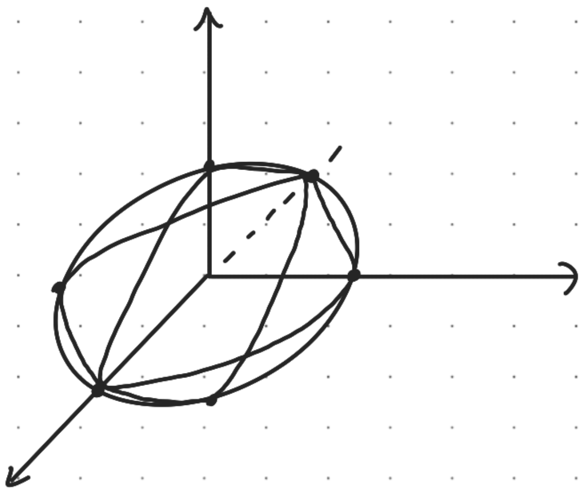
$y=0 \Rightarrow z=\pm1 \quad (0,0,1) \quad (0,0,-1)$

$z=0 \Rightarrow y^2=1 \Rightarrow y=\pm1 \quad (0,1,0) \quad (0,-1,0)$

$y=0$ $\quad x^2+z^2=1$

$x=0 \Rightarrow z=\pm1$

$z=0 \Rightarrow x=\pm1$



$x^2 + (y-2)^2 + z^2 = 1$

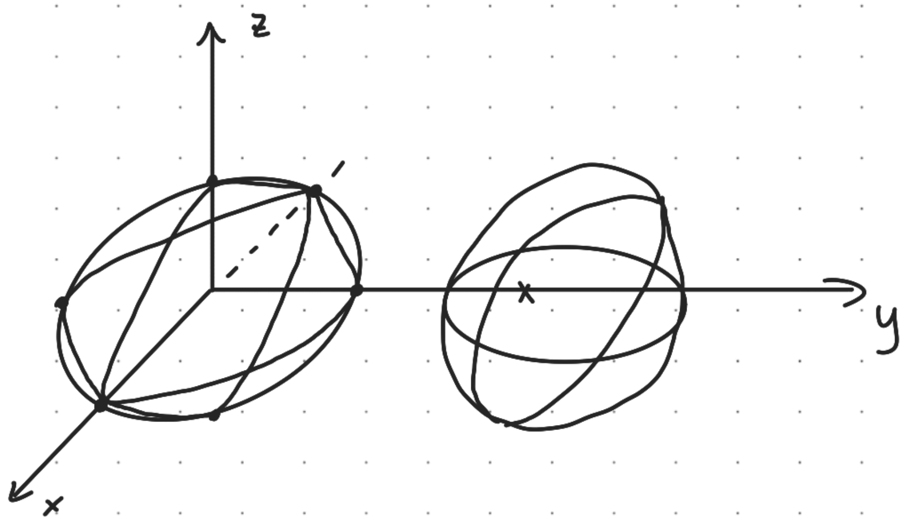
$(0,2,0)$ $+ r = 1$

$x = 0$ $(y-2)^2 + z^2 = 1$

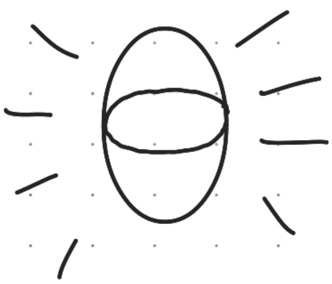
$(y-2)^2 = 1$

$y - 2 = 1$ or $y - 2 = -1$

$y = 3$ or $y = 1$



1. $x^2 + y^2 + z^2 > 1 \longrightarrow$ exterior of the unit sphere



$< 1 \rightarrow$ interior

$\leq 1 \rightarrow$ sphere together with interior

Wtf are you kidding me?

3. VECTORS IN R^3

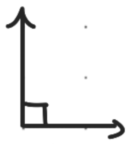
$\vec{v} = (v_1, v_2, v_3)$ $\quad |\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$

- if $\|\vec{u}\| = 1 \Rightarrow \vec{u}$ is a unit vector
  ↳ NORM

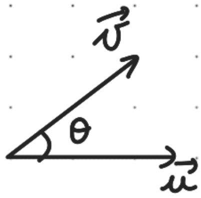
standard unit: $\vec{i} = (1,0,0)$ $\vec{j} = (0,1,0)$ $\vec{k} = (0,0,1)$

3.1 DOT PRODUCT

$\vec{u} = (u_1, u_2, u_3)$
$\vec{v} = (v_1, v_2, v_3)$



$\theta = 90°$



$u \cdot v = \|u\| \times \|v\| \cos\theta$

$\Rightarrow \vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$ $\quad \cos\theta = \dfrac{\vec{u} \cdot \vec{v}}{\|u\| \cdot \|v\|}$

$\vec{u}$ and $\vec{v}$ are orthogonal if $\vec{u} \cdot \vec{v} = 0$

PROPERTIES

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
2. $\vec{u} \cdot \vec{u} = \|u\|^2$
3. $0 \cdot \vec{u} = 0$
4. $\vec{u}(\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$

3.2 CROSS PRODUCT

1. $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$
2. $0 \times u = 0$
3. $u \times v = 0$ iff they are parallel

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$\vec{u} = (2,1,1)$ $\vec{v}\ (-4,3,1)$

$$\det\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix} = i(1-3) - j(2+4) + k(6+4)$$

$$= -2i - 6j + 10k$$

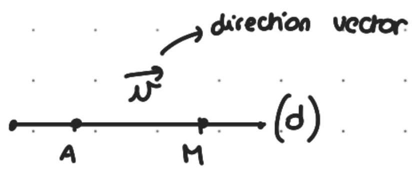
3.3 TRIPLE SCALAR PRODUCT

$$(\vec{u} \times \vec{v}) \cdot w$$

$$\det\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = u_1(v_2w_3 - v_3w_2) - u_2(v_1w_3 - v_3w_1) + u_3(v_1w_2 - v_2w_1)$$

3.4 LINES IN SPACE

$A = (x_A : y_A, z_A) \in (d)$



$\vec{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ direction vector

Let $M(x, y, z) \in d$

$$\underbrace{\overrightarrow{AM}}_{\begin{pmatrix} x-x_A \\ y-y_A \\ z-z_A \end{pmatrix}} = k\vec{v} \quad \begin{pmatrix} a \\ b \\ c \end{pmatrix} \longrightarrow \begin{cases} x - x_A = k \cdot a \\ y - y_A = kb \\ z - z_A = kc \end{cases}, \quad k \in \mathbb{R}$$

parametric

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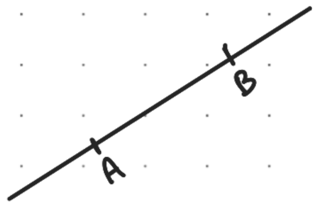
$$\frac{x-x_a}{a} = \frac{y-y_A}{b} = \frac{z-z_A}{c}$$ } cartesian equation

EX: $A(-3,2,-3)$

$B(1,-1,4)$

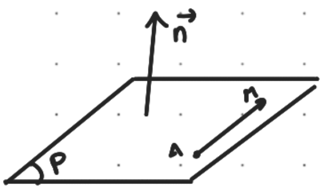
$$\vec{AB} = \begin{pmatrix} 4 \\ -3 \\ 7 \end{pmatrix} \begin{array}{l} \to a \\ \to b \\ \to c \end{array}$$



$$\begin{cases} x = -4t+3 \\ y = 3t+2 \\ z = -7t-3 \end{cases} \quad t \in \mathbb{R}$$ equation of a line

3.5 PLANE IN SPACE

To find an equation of a plane (P), we need $A \in (P)$ and a normal vector $\vec{n}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$



$$\begin{cases} v_1 - v_2 = -1 \\ v_1 + 2v_2 = 11 \end{cases}$$

EX:

$$\vec{AM} \cdot \vec{n} = 0$$

$$\begin{pmatrix} x-x_A \\ y-y_A \\ z-z_A \end{pmatrix} \quad \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\begin{cases} v_1 = -1+v_2 \\ (-1+v_2)+2v_2 = 11 \end{cases}$$

$$\begin{cases} v_1 = -1+v_2 \\ 3v_2 = 11+1 \end{cases}$$

$$\begin{cases} v_1 = -1+v_2 \\ 3v_2 = 12 \end{cases}$$

$$\begin{cases} v_1 = -1+v_2 \\ v_2 = 4 \end{cases} \quad \begin{cases} v_1 = 3 \\ v_2 = 4 \end{cases}$$

$$a(x-x_a)+b(y-y_A)+c(z-z_A)=0$$
$$Ax+By+Cz+ax+by+cz-ax_A-by_A-cz_A=0$$

(P) $A(0,0,1)$ $B(2,0,0)$ $C(0,3,0)$

$\vec{n}=\vec{AB}\times\vec{AC}$

$M(x,y,z)\in(P)$ $AM.(\vec{AB}\times\vec{AC})=0$

$\vec{AM}\cdot\vec{n}=0$

$$\vec{AM}=\begin{pmatrix}x\\y\\z-1\end{pmatrix}\quad \vec{AB}=\begin{pmatrix}2\\0\\-1\end{pmatrix}\quad \vec{AC}=\begin{pmatrix}0\\3\\-1\end{pmatrix}$$

note: $AM=\begin{pmatrix}x_M-x_A\\y_M-y_A\\z_M-z_A\end{pmatrix}=\begin{pmatrix}x\\y\\z-1\end{pmatrix}$

$$\begin{vmatrix}x & y & z-1\\2 & 0 & -1\\0 & 3 & -1\end{vmatrix}=0$$

$x(3)-y(-2)+(z-1)(6)=0$

$3x+2y+6z-6=0$ (P)

SHAPES

I. $\boxed{\text{Cylinders}}$ → circular, eliptical, parabolic

① Circular cylinder

$$1(\text{variable①})^2+1(\text{variable②})^2=a^2$$

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EX:

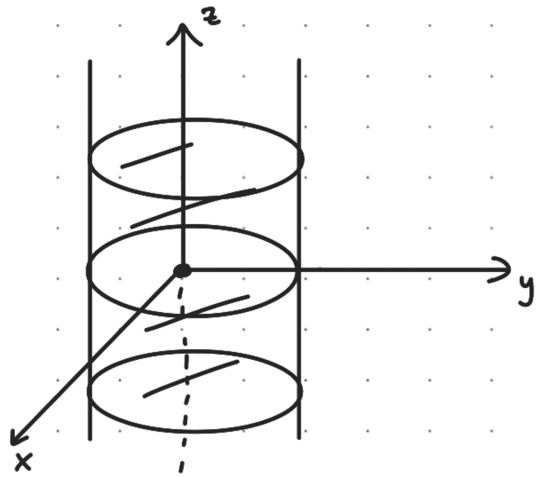
1. $x^2+y^2=4$ → is a circular cylinder along z-axis

$z=0$

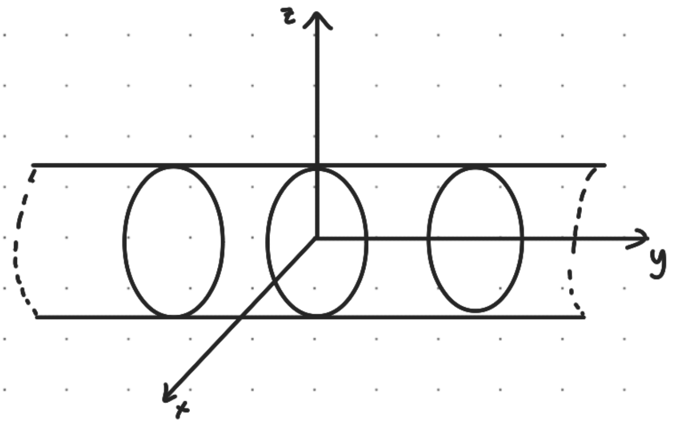
circle $(0,0,0)$
$r=2$

$z=3$

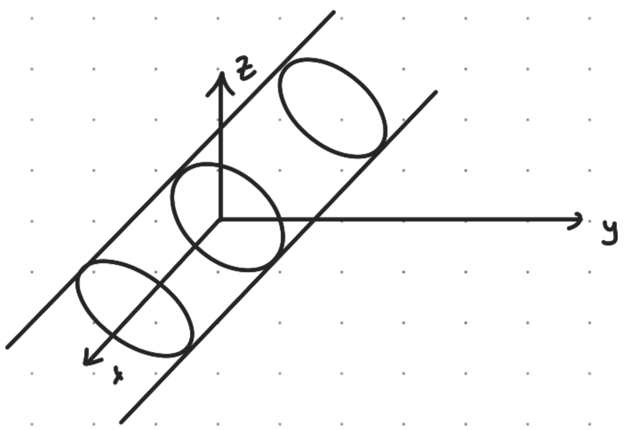
circle $(0,0,3)$
$r=3$



2. $x^2+z^2=4$ → circular cylinder around y-axis



3. $y^2+z^2=1$



② Elliptical Cylinder

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = c$$

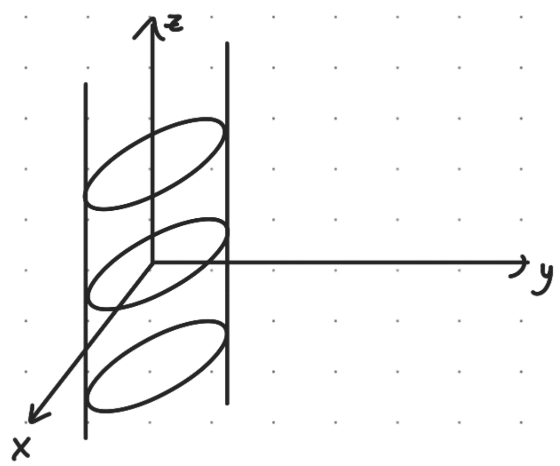
$a \neq b$, o.w it will be circular cylinder

Ex: $\frac{x^2}{4} + y^2 = 1$ → it is an elliptical cylinder along z-axis

$z=0$: $\frac{x^2}{4} + y^2 = 1$, ellipse of center $(0,0,0)$

$(\pm 2, 0, 0) \rightarrow y=0 \rightarrow \frac{x^2}{4} = 1 \rightarrow x^2 = 4 \rightarrow x = \pm 2$

$(0, \pm 1, 0) \rightarrow x=0 \rightarrow y^2 = 1 \rightarrow y = \pm 1$



$$\frac{(x-2)^2}{4} - y^2 = 1$$

↓

ellipse of center $(2,0,0)$ along z-axis

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① **CYLINDER** :
- circular
- elliptical
- parabolic

PARABOLIC CYLINDER

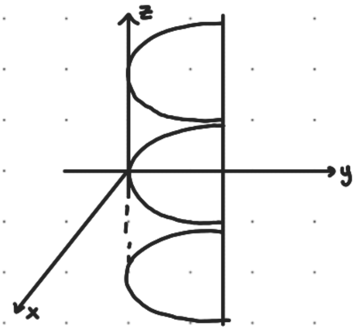
$\boxed{(\text{variable}) = (\text{variable})^2}$ $\xrightarrow{\text{it could be}}$ $\begin{cases} x = y^2 \\ y = x^2 \\ x = (y-1)^2 \\ z = x^2 \end{cases}$

REMINDER (parabola)
$y = a(x-h)^2 + k$
(h, k)

EX : $y = x^2 \longrightarrow$ parabolik cylinder around Z-axis

$z=0$ → z plane

$y = x^2 \rightarrow$ parabola in the (x,y) plane whose vertex is $(0,0)$ and opens in the y-axis



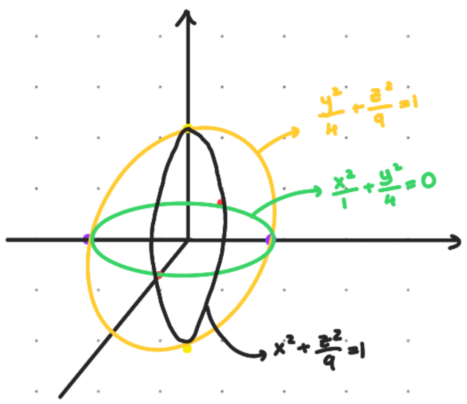
② **ELLIPSOID**

$\boxed{\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1}$ $\longrightarrow$ ellipsoid of center $(0,0,0)$

→ The intercepts are $(\pm a, 0, 0)$, $(0, \pm b, 0)$, $(0, 0, \pm c)$

EX1: $x^2 + \dfrac{y^2}{4} + \dfrac{z^2}{9} = 1$

$(\pm 1, 0, 0)$
$(0, \pm 2, 0)$
$(0, 0, \pm 3)$



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EX2 : $\frac{(x-2)}{9} + \frac{(y+1)^2}{4} + z^2 = 1 \longrightarrow$ ellipsoid centered at $(2,-1,0)$

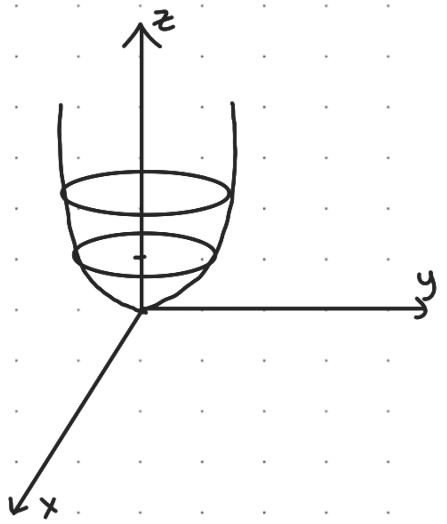
- $x=2$ and $y=-1 \Rightarrow z=\pm 1$ $(2,-1,1)$, $(2,-1,-1)$
- $x=2$ and $z=0 \Rightarrow (y+1)^2=4 \Rightarrow y+1=2$ or $y+1=-2$
  $\Rightarrow y=1$ or $y=-3$
  $(2,1,0)$ and $(2,-3,0)$
- $y=-1$ and $z=0 \Rightarrow (x-2)=\pm 3$ so $x=5$ or $x=-1$
  $(-1,-1,0)$, $(5,-1,0)$

3 PARABOLOID

$$\boxed{\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}}$$

ex. 1 $z = x^2+y^2$

- $z=0$
  $x^2+y^2=0$
  $(0,0,0)$
- $z=1$
  $x^2+y^2=1$
  $(0,0,1)$
  of radius 1
- $z=4$
  $x^2+y^2=4$
  $(0,0,4)$
  of radius 2



EX: 2 $x^2 + \frac{y^2}{9} = \frac{z}{2}$

$\downarrow$

$2x^2 + \frac{2}{9}y^2 = z$

$\downarrow$

$\frac{x^2}{1/2} + \frac{y^2}{9/2} = z$

so this is actually an elliptic parabula

STEP 1. $x=0 \Rightarrow z = \frac{2}{9}y^2$ parabula around z-axis with vertex $(0,0,0)$

STEP 2. $y=0 \Rightarrow z = 2x^2$ parabola around z-axis in the (xz) plane

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④ CONE

$$\boxed{(\text{variable})^2 = a(\text{variable})^2 + b(\text{variable})^2}$$

GENERAL FORMULA

this determines our axis

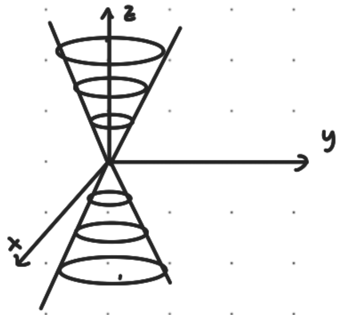
ex1: $z^2 = x^2 + y^2$ → a circular cone around z-axis

For $x=0$:

$z^2 = y^2$
$z = \pm y$
(eq. straight line)

For $y=0$:

$z^2 = x^2$
$z = \pm x$



ex2: $z = \sqrt{x^2 - y^2} \longrightarrow z > 0 \longrightarrow z^2 = x^2 + y^2$ circular cone with $z \geq 0$

$\Downarrow$

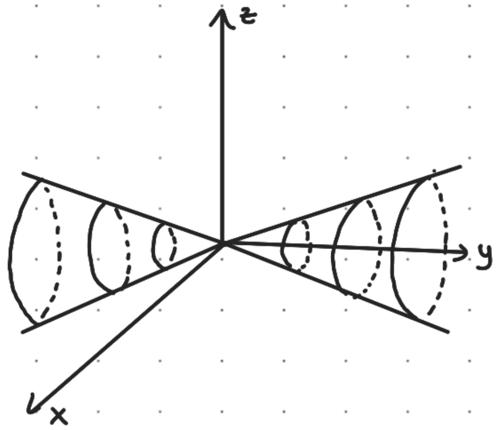
so circular cone upper plane

$z = -\sqrt{x^2 + y^2} \rightarrow z \leq 0$

EX3: $4x^2 + 9z^2 = 9y^2$

$y^2 = \frac{4}{9}x^2 + z^2$

$\Longrightarrow$ elliptic cone around y-axis



MULTIVARIABLE FUNCTION

D R

$(x, y) \longrightarrow z \quad f(x,y)$

$(x, y, z) \longrightarrow$

$\vdots$

$\mathbb{R}^n$

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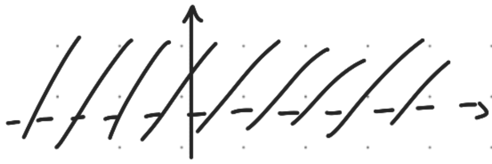
DOMAIN AND RANGE

1. $f(x,y) = x^2 + y^2$ Domain: (x,y) plane
Range: $[0, +\infty)$

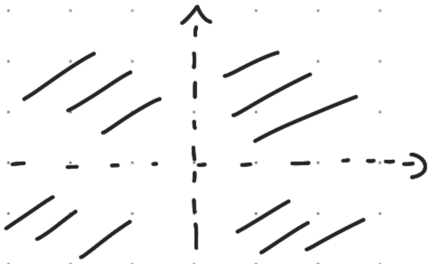
2. $f(x,y) = \frac{x}{y}$ Domain: (x,y) plane except the x-axis



she could ask to draw the domain at the exam.

Range: $(-\infty; +\infty)$

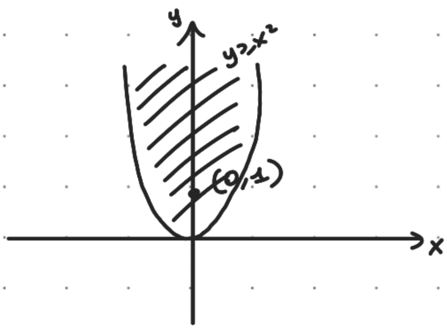
3. $f(x,y) = \frac{3}{xy}$ Domain: (xy) plane $-$ {x-axis and y-axis}



Range: $\mathbb{R}^*$ {z-axis without 0}

4. $f(x,y) = \sin(xy)$ Domain: (xy) plane
Range: $[-1, 1]$ along z-axis

5. $f(x,y) = \sqrt{y - x^2}$ Domain: $y \geq x^2$



Range: $[0; +\infty]$

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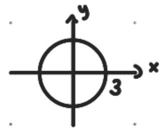
EX:

- $z = f(x,y) = \sqrt{9-x^2-y^2}$ $\quad 9-x^2-y^2 \geq 0$

DOMAIN: $9 \geq x^2+y^2$

Disc center at $(0,0)$ of radius 3



RANGE $\longrightarrow [0,3]$

START FROM DOMAIN

$$0 \leq x^2+y^2 \leq 9$$
$$-9 \leq -x^2-y^2 \leq 0$$
$$0 \leq 9-x^2-y^2 \leq 9$$
$$0 \leq \sqrt{9-x^2-y^2} \leq 3$$
$$0 \leq z \leq 3$$

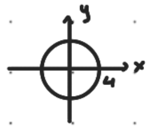
- $z = \dfrac{1}{\sqrt{16-x^2-y^2}}$

DOMAIN $16-x^2-y^2 > 0$

NOTE: not equal to zero because it's a fraction

$x^2+y^2 < 16$

interior of the circle centre $(0,0)$ at $r=4$



RANGE $[1/4 ; +\infty)$

START FROM DOMAIN

$$0 \leq x^2+y^2 < 16$$
$$-16 < x^2-y^2 \leq 0$$
$$0 < 16-x^2-y^2 \leq 16$$
$$0 < \sqrt{16-x^2-y^2} \leq 4$$
$$\frac{1}{0} < z \leq 1/4$$
$$\frac{1}{4} \leq z < +\infty$$

NOTE: $b < a < c$
$= \frac{1}{c} < \frac{1}{a} < \frac{1}{b}$
reciprocal

- $f(x,y) = \dfrac{1}{\ln(4-2x^2-y^2)}$

remember $\ln(1) = 0$

Df : $\ln(4-2x^2-y^2) \neq 0$ and $4-2x^2-y^2 > 0$

$4-2x^2-y^2 \neq 1$

$2x^2+y^2 \neq 3$

$$\boxed{\frac{x^2}{3/2} + \frac{y^2}{3} \neq 1}$$

ellipse

$x \to +\sqrt{\frac{3}{2}}, -\sqrt{\frac{3}{2}}$ $\quad y \to \sqrt{3}, -\sqrt{3}$

$2x^2+y^2 < 4$

$$\boxed{\frac{x^2}{2} + \frac{y^2}{4} < 1}$$

interior of the ellipse

$x \to \sqrt{2}, -\sqrt{2}$ $\quad y \to 2, -2$

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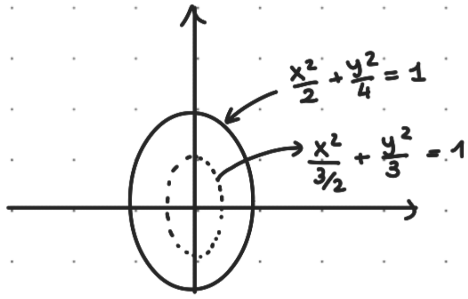
DOMAIN: interior of the ellipse $\frac{x^2}{2}+\frac{y^2}{4}=1$ excluding the ellipse $2x^2+y^2=3$



LEVEL CURVES AND LEVEL SURFACES

There are two standard ways to picture the values of a function $f(x,y)$:

1. Draw and label curves in the domain in which f has a constant value.
2. Sketch the surface $z=f(x,y)$ in space.

Def: The set of points (x,y) in the plane where $f(x,y)$ is constant $(f(x,y)=c)$ is called level curve while the set of all points in space such that $f(x,y,z)=c$ is called level surface.

The set of all points $(x,y,f(x,y))$ in space, for (x,y) in the domain of f, is called the graph of f.

The graph of f is also called the surface $z=f(x,y)$

EX: Graph $f(x,y)=100-x^2-y^2$, plot the level curves $f(x,y)=0$ and $f(x,y)=51$

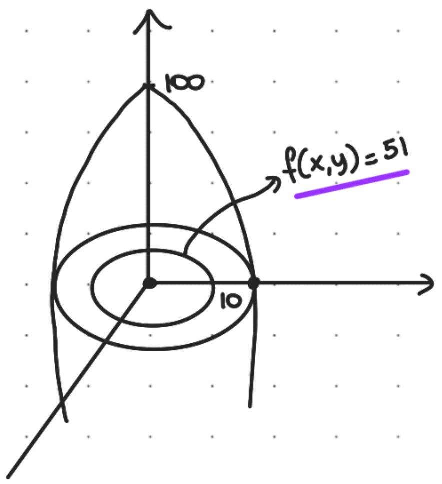
OBSERVATION → $z=100-x^2-y^2$ is a circular paraboloid

↓

DOMAIN → (x,y) plane

RANGE → $(-\infty,100]$

$$\boxed{\begin{array}{c} x^2+y^2\geq 0 \\ -x^2-y^2\leq 0 \\ 100-x^2-y^2\leq 100 \\ z\leq 100 \end{array}}$$



- The level curve $f(x,y)=0$ i.e. $x^2+y^2=100$ is the set of all points in the circle of center $(0,0)$ and radius 10
- $f(x,y)=51 \Rightarrow x^2+y^2=49$
  $\Downarrow$
  circle of center $(0,0)$ at radius 7
- The level curve $f(x,y)=100$ consists of the origin only.

EX Describe the level surfaces of the function $f(x,y,z)=\sqrt{x^2+y^2+z^2}$

$\Rightarrow f(x,y,z)=c$

$\sqrt{x^2+y^2+z^2}=c$, $c>0$ is a sphere centered at the origin and radius c.

LIMITS AND CONTINUITY

The concept of limit for multi-variable functions is analogous to that for single-variable functions.

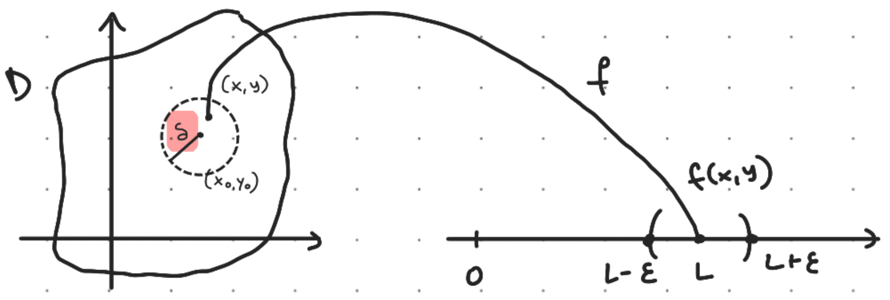
Def: We say that a function $f(x,y)$ approaches the limit L as (x,y) approaches (x_0,y_0) and write

$$\boxed{\lim_{(x,y)\to(x_0,y_0)} f(x,y)=L}$$

i) $\forall \varepsilon>0$, $\exists$ corresponding $\delta>0$ such that for all (x,y) in the domain of f,

$$|f(x,y)-L|<\varepsilon \text{ whenever } \overset{\text{means } (x,y)\neq(x_0,y_0)}{0<} \underbrace{\sqrt{(x-x_0)^2+(y-y_0)^2}}_{\text{open disk centered at }(x_0,y_0)}<\delta$$

This definition says that the distance between $f(x,y)$ and L becomes arbitrary small whenever the distance from (x,y) to (x_0,y_0) is made sufficiently small (but not 0)



δ is the radius of the disk centered at (x_0,y_0), for all the points (x,y) within this disk, the function values $f(x,y)$ lies inside the corresponding interval $(L-\varepsilon, L+\varepsilon)$

EXERCISE Show that $\lim\limits_{(x,y)\to(0,0)} \frac{4x^2y}{x^2+y^2} = 0$

Sol: Let $\varepsilon>0$ and $\delta = \frac{\varepsilon}{4}$ and we get

$$\left|f(x,y)-0\right| = \left|\frac{4x^2y}{x^2+y^2}\right| \quad \text{since } \frac{x^2}{x^2+y^2} \leq 1$$
$$< |4y|$$
$$< 4\delta = \varepsilon$$

EXERCISE 2 Prove that $\lim\limits_{(x,y)\to(0,0)} 2x = 0$

Sol: Given $\varepsilon>0$, we need to find $\delta>0$ if $(x,y)\neq(0,0)$ and $\sqrt{x^2+y^2} < \delta$, then

$$|2x| < \varepsilon$$

choose $\boxed{\delta = \frac{\varepsilon}{2}}$

Then, $|f(x,y)-0| = |2x| = 2\sqrt{x^2}$
$$\leq 2\sqrt{x^2+y^2} < 2\delta = 2\cdot\frac{\varepsilon}{2} = \varepsilon$$

EXERCISE 3 Show that $\lim\limits_{(x,y)\to(1,2)} (2x+y) = 4$

Sol: Given $\varepsilon>0$, we need to find $\delta>0$ s.t. if $0<\sqrt{(x-1)^2+(y-2)^2} < \delta$, then $|2x+y-4| < \varepsilon$.

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So Given $\varepsilon > 0$ and let $\delta = \frac{\varepsilon}{3}$
and $\sqrt{(x-1)^2 + (y-2)^2} < \delta$

Then $|2x + y - 4| = |2(x-1) + y - 2|$

$\leq 2|x-1| + |y-2|$ → since $|x-1| = \sqrt{(x-1)^2} \leq \sqrt{(x-1)^2 + (y-2)^2}$

$< 2\delta + \delta = 3\delta = \varepsilon$

PROPERTIES (of limits of functions of 2 var.)

Assume $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = L$ and $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = M$, then:

- $\lim\limits_{(x,y)\to(x_0,y_0)} \left(f(x,y) + g(x,y)\right) = L + M$ [Sum Rule]
- $\lim\limits_{(x,y)\to(x_0,y_0)} \left(f(x,y) - g(x,y)\right) = L - M$ [Difference Rule]
- $\lim Kf(x,y) = KL$ [K = any number]
- $\lim\limits_{(x,y)\to(x_0,y_0)} \left(f(x,y) \cdot g(x,y)\right) = L \cdot M$ [Product rule]
- $\lim\limits_{(x,y)\to(x_0,y_0)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}$ $(M \neq 0)$ [Quotient Rule]
- $\lim [f(x,y)]^n = L^n$ [Power Rule]

EXAMPLES

1. $\lim\limits_{(x,y)\to(0,1)} \frac{x - xy + 3}{x^2y + 5xy - y^3} = \frac{0 - 0\cdot 1 + 3}{0^2 1 + 5\cdot 0\cdot 1 - 1^3} = \frac{3}{-1} = -3$

2. $\lim\limits_{(x,y)\to(3,-4)} \sqrt{x^2 + y^2} = \sqrt{3^2 + (-4)^2} = 5$

3. $\lim\limits_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} = \frac{0}{0}$ (ind. form)

rationalize

$$= \lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} \cdot \frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} + \sqrt{y}} = \lim \frac{x\cancel{(x-y)}(\sqrt{x} + \sqrt{y})}{\cancel{x-y}}$$

$$= \lim_{(x,y)\to(0,0)} x(\sqrt{x} + \sqrt{y}) = 0$$

4. $$\lim_{(x,y)\to(2,0)} \frac{\sqrt{2x-y}-2}{2x-y-4} = \frac{2-2}{4-0-4} = \frac{0}{0} \ (\text{ind. form})$$

$$= \lim_{(x,y)\to(2,0)} \frac{\cancel{\sqrt{2x-y}-2}}{(\cancel{\sqrt{2x-y}-2})(\sqrt{2x-y}+2)} = \lim_{(x,y)\to(2,0)} \frac{1}{\sqrt{2x-y}+2} =$$

$$= \lim_{(x,y)\to(2,0)} \frac{1}{\sqrt{4-0}+2} = \frac{1}{4}$$

5. $$\lim_{(x,y)\to(1,1)} \frac{xy-y-2x+2}{x-1} = \xrightarrow{\text{grouping}} \lim_{(x,y)\to(1,1)} \frac{y(x-1)-2(x-1)}{\cancel{x-1}} =$$

$$\lim_{(x,y)\to(1,1)} y-2 = -1$$

NOTE

* When dealing with functions of a single variable, we also considered one-sided limits and stated:

$$\lim_{x\to c} f(x) = L \iff \lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x) = L$$

$\to c \leftarrow$

i.e. the limit L exists if and only if f(x) approaches L when x approaches c from either direction, the left or the right.

If $\lim_{x\to c^+} f(x) \neq \lim_{x\to c^-} f(x)$, then we say $\lim_{x\to c} f(x)$ doesn't exist.

* In the plane, there are infinite directions for which (x,y) might approach (x_0,y_0).
We don't have to restrict ourselves to approaching (x_0,y_0) from a particular direction, but rather we can approach that out along a path that is not a straight line.

Def: "Path" → A path is any curve passing through the point $(x_0;y_0)$

* For the limit to exist at a point (x_0,y_0), the limit must be the same along every approach path.
If we ever find paths with different limits, we say that the limit doesn't exist.

TWO-PATHS TEST FOR NON-EXISTENCE OF LIMIT

If a function $f(x,y)$ has different limits along two different paths in the domain of f as (x,y) approaches (x_0,y_0); then $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ doesn't exist.

EX: Show that the function $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$ has no limit as (x,y) approaches $(0,0)$

$\Rightarrow$ SOLUTION (2 ways):

① Approach $(0,0)$ along the (x-axis)($y=0$), then

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2}{x^2} = 1 = L_1$$

Now, approach $(0,0)$ along the (y-axis)($x=0$),

$$\lim = \lim \frac{-y^2}{y^2} = -1 = L_2$$

$L_1 \neq L_2 \implies$ Limit doesn't exist!

② Consider the path $y = mx$

$$\frac{2x^2-2x}{2x^2-2x+1}$$

$$=0$$

$$\Rightarrow f(x,y) = \frac{x^2-m^2x^2}{x^2+m^2x^2} = \frac{1-m^2}{1+m^2}$$

So, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \dfrac{1-m^2}{1+m^2}$ (depends on m)

for $m=1$ i.e. along the path $y=x$, we get

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0 = L_1$$

for $m=2$ i.e. along the path $y=2x$, we get

$$\lim f(x,y) = \frac{1-2^2}{1+2^2} = -\frac{3}{5} = L_2$$

$L_1 \neq L_2 \implies \lim\limits_{(x,y)\to(0,0)} f(x,y)$ doesn't exist

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EX: Show that $\lim\limits_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2}$ doesn't exist

Consider the curve $y = mx^2$, $x \neq 0$
Along this curve, the function $f(x,y) = \frac{2x^2mx^2}{x^4+m^2x^4} = \frac{2m}{1+m^2}$

(so depends on m)

$\Rightarrow$ The limit varies with the path of approach.
For example:

if $m=1$, along the parabula $y=x^2$, we have
$\lim = \boxed{1 = L_1}$

if $m=0$, (x,y) approaches $(0,0)$ along the x-axis,
$\lim = \boxed{0 = L_2}$

So $\boxed{L_1 \neq L_2} \Rightarrow \lim\limits_{(x,y)\to(0,0)} f(x,y)$ NO EXIST

EXERCISES (ASSIGNMENT)

1. Use the two-path test to show that $\lim\limits_{(x,y)\to(0,0)} \frac{x^6+2y^3}{x^4y+5x^6}$ doesn't exist.

Sol: (maybe !!)

Consider $y = mx^2$, $x \neq 0$

then $f(x,y) = \frac{x^6 + 2m^3x^6}{mx^6+5x^6} = \frac{\cancel{x^6}(1+2m^3)}{\cancel{x^6}(m+5)} = $

$= \frac{1+2m^3}{m+5}$

If $m=1$, $\lim = \frac{1}{3} = L_1$
If $m=0$, $\lim = \frac{1}{5} = L_2$
$\Big\}$ $L_1 \neq L_2$

2. Find the following limit $\lim\limits_{(x,y)\to(1,1)} \frac{x^2y^2-1}{xy-1}$

$\lim\limits_{(x,y)\to(1,1)} \frac{x^2y^2-1}{xy-1} = \frac{(xy+1)\cancel{(xy-1)}}{\cancel{xy-1}} = 2$

(3) Let $f(x,y) = \frac{5x^2y^2}{x^2+y^2}$. Find $\lim_{(x,y)\to(0,0)} f(x,y)$

$$\lim_{(x,y)\to(0,0)} \frac{5x^2y^2}{x^2+y^2} = \frac{5\cancel{x^2y^2}}{\cancel{x^2y^2}\left(\underbrace{\frac{1}{y^2}}_{\infty}+\underbrace{\frac{1}{x^2}}_{\infty}\right)} = 0$$

It's 00:44 and i probably wrote BULLSHIT :)

$$X^2+Z^2$$

NOTE BOOK

Rokshana Ahmed

EX 1

$$\lim_{(x,y)\to(0,0)} \frac{3xy}{3x^2+y^2}$$

① Along $x=0$; $\lim_{y\to 0} \frac{3xy}{3x^2+y^2} = 0$

Along $y=0$; $\lim_{x\to 0} \frac{0}{3x^2} = 0$ $\to L_1$

this doesn't imply that the limit exist

Along $y=x$; $\lim_{(x,y)\to(0,0)} \frac{3x^2}{3x^2+x^2} = \frac{3}{4} \to L_2$

$L_1 \neq L_2 \Rightarrow$ the limit doesn't exist

② $y = mx$

$$\lim_{(x,y)\to(0,0)} \frac{3xy}{3x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{3mx^2}{3x^2+m^2x^2} = \frac{3m}{m^2+3}$$

for $m=0$ i.e. $y=0 \Longrightarrow L_1 = 0$

for $m=1$ i.e. $y=x \Rightarrow L_2 = 3/4$

$L_1 \neq L_2$ so limit doesn't exist

EX. 2

$$\lim_{(x,y)\to(1,0)} \frac{2xy-2y}{x^2+y^2-2x+1}$$

① Along $y=0$, $\lim_{x\to 1} \frac{0}{x^2-2x+1} = 0 \to L_1$

Along $x=1$, $\lim_{y\to 0} \frac{2y-2y}{1^2+y^2-2+1} = \lim \frac{0}{y^2} = 0$

if these were different, we could say lim doesn't exist right away

so, $\frac{2xy-2y}{x^2+y^2-2x+1} = \frac{2y(x-1)}{(x-1)^2+y^2}$

Along $y = x-1$:

$$\lim \frac{2(x-1)^2}{(x-1)^2+(x-1)^2} = \boxed{1} \rightarrow L_2$$

So $L_1 \neq L_2 \Rightarrow$ Then limit doesn't exist

ex3 :

$$\lim_{(x,y)\to(0,0)} \frac{xy^3\cos x}{2x^2+y^6}$$

Along $x=0$, $\displaystyle\lim_{y\to 0} \frac{0}{y^6} = 0$

Along $y=0$, $\displaystyle\lim_{x\to 0} \frac{0}{2x^2} = 0$

L_1

Along $x=y^3$, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^6\cos y^3}{3y^6} =$

$$\lim \frac{\cos y^3}{3} = \frac{1}{3} \leftarrow L_2$$

Continuity

REMEMBER: $\displaystyle\lim_{x\to x_0} f(x) = f(x_0)$

- $\rightarrow$ $f(x_0)$ is defined
- $\rightarrow$ $\displaystyle\lim_{x\to x_0} f(x)$ exists
- $\rightarrow$ should be equal

A function $f(x,y)$ is continous at the point (x_0,y_0) if :

1. f is defined at (x_0,y_0)
2. $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists $\rightarrow$ so if you pick any path, along it, the limit should always be same.
3. $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$

So, a function is continuos if its continous at every point of it's domain.

EX1: Is the function $f(x,y) = \dfrac{x^2-y^2}{x^2-2xy-3y^2}$ contin. at $(1,-1)$?

Nope, it's not

EX2: Let $f(x,y) = \begin{cases} \dfrac{x^2-y^2}{x^2-2xy-3y^2} & x \neq -y \\ 5 & \text{if } x=-y \end{cases}$

① $f(1,-1) = 5$

② $\lim\limits_{(x,y)\to(1,-1)} f(x,y) = \lim\limits_{(x,y)\to(1,-1)} \dfrac{x^2-y^2}{x^2-2xy-3y^2}$

factorizing trial and error

$= \lim\limits_{(x,y)\to(1,-1)} \dfrac{(x-y)\cancel{(x+y)}}{\cancel{(x+y)}(x-3y)} = \dfrac{1+1}{1+3} = \dfrac{1}{2}$

$1 \quad -1 \quad x^2-2xy-3y^2 \quad -3 \quad -3,1 \quad 3,-1$

$(x-y)(x+3y) \to x^2+2xy-3y^2$ NOT RIGHT

so $(x+y)(x-3y)$

③ $f(1,-1) = 5 \neq \lim\limits_{(x,y)\to(1,-1)} = \dfrac{1}{2}$

f is not continous at $(1,-1)$

Partial Derivatives

$f: \mathbb{R} \longrightarrow \mathbb{R}$

$$f'(x_0) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$$

Def: Let $z = f(x,y)$
The partial derivative of $f(x,y)$ w.r. to x at the point (x_0, y_0), denoted by

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} \quad \text{or} \quad f_x(x_0,y_0) \quad \text{or} \quad \frac{\partial z}{\partial x}(x_0,y_0)$$

is given by:

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \lim_{h \to 0} \frac{f(x_0+h, y_0) + f(x_0,y_0)}{h}$$

The partial derivative of $f(x,y)$ w.r. to y at (x_0,y_0) is:

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)}, \quad f_y(x_0,y_0), \quad \left.\frac{\partial z}{\partial y}\right|_{(x_0,y_0)}$$

Ex: Let $f(x,y) = x^2y + 2x + y^2$

Find $f_x(x,y)$ using the limit definition

$$f_x(x,y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h} =$$

$$= \lim_{h \to 0} \frac{(x+h)^2y + 2(x+h) + y^3 - (x^2y + 2x + y^3)}{h} =$$

$$= \lim_{h \to 0} \frac{2xhy + h^2y + 2h}{h} = \lim_{h \to 0} 2xy + hy + 2 =$$

$$= 2xy + 2 \longrightarrow \text{So } f_x(x,y) = 2xy + 2$$

$$\lim_{h \to 0} \frac{\cancel{x^2y} + 2xhy + h^2y \cancel{+2x} + 2h \cancel{+y^3} \cancel{-x^2y} \cancel{-2x} \cancel{-y^3}}{h}$$

In general, if $f: U \subseteq \mathbb{R}^n \to \mathbb{R}$ is a function. Then the partial derivative of f at the point $a = (a_1, a_2 \ldots a_n)$ w.r.t the i^{th} variable x is defined as:

$$\frac{\partial f(a)}{\partial x_i} = \lim_{h \to 0} \frac{f(a_1, a_2, \ldots, a_{i-1}, a_i + h, \ldots, a_n) - f(a_1, a_2, \ldots, a_n)}{h}$$

NOTE BOOK

Rokshana Ahmed

CALCULUS 2

Date: 21 / 10 / 2022

EX1:

$f(x,y) = 3x - x^2y^2 + 2x^3y$

$f_x = \frac{\partial f}{\partial x} = 3 - 2xy^2 + 6x^2y \qquad f_y = \frac{\partial f}{\partial y} = -2x^2y + 2x^3$

EX2:

$f(x,y) = y\sin(xy)$ *

$f_x = y \cdot y\cos(xy) = y^2\cos(xy)$

$f_y = \frac{\partial y}{\partial y} \cdot \sin(xy) + y\frac{\partial \sin(xy)}{\partial y} = 1 \cdot \sin(xy) + yx\cos(xy)$

Here we have 2 y
so we can use product
rule

EX3:

$f(x,y) = \frac{2y}{y + \cos x}$

$f_x = \frac{\frac{\partial}{\partial x}\cancel{2y}^{0}(y+\cos x) - 2y\frac{\partial}{\partial x}(y+\cos x)}{(y+\cos x)^2} = \frac{2y\sin x}{(y+\cos x)^2}$

$f_y = \frac{\frac{\partial(2y)}{y} \cdot (y+\cos x) - 2y\frac{\partial}{\partial y}(y+\cos x)}{(y+\cos x)^2} =$

$= \frac{2(y+\cos x) - 2y}{(y+\cos x)^2} = \frac{2\cos x}{(y+\cos x^2)}$

EX4:

$f(x,y,z) = z^3 - x^2y$

$f_x = 2xy \qquad f_y = -x^2 \qquad f_z = 3z^2$

EX.6:

$f(x,y) = y^3\sin(x) + x^2\tan(y)$

$f_x = y^3\cos(x) + 2x\tan(y)$

$f_y = 3y^2\sin(x) + x^2\sec^2(y)$

so no need
of product
rule

ex. 7:

$$f(x,y) = x\ln(xy) + y\sin(xy)$$

partial $\cdot \frac{x}{xy}$

$$f_x = \frac{\partial}{\partial x}(x)\ln(xy) + x\frac{\partial(\ln(xy))}{\partial x} + y^2\cos(xy) =$$

$$= \ln(xy) + 1 + y^2\cos(xy)$$

$$f_y = \cancel{x}\cdot\frac{x}{\cancel{x}y} + \sin(xy) + yx\cos(xy)$$

ex. 8:

$$z = y^x$$

$$\frac{\partial z}{\partial x} = y^x \cdot \ln(y) \qquad \frac{\partial z}{\partial x} = xy^{x-1}$$

ex. 9:

$$f(x,y,z,w) = \frac{xw^2}{y+\sin(zw)} *$$

$$f_x = \frac{w^2}{y+\sin(zw)} \qquad f_y = \frac{2xw\,(y+\sin(zw)) - xw^2(z\cos(zw))}{y+\sin(zw)^2}$$

$$\hookrightarrow f_x = \frac{\frac{\partial(xw^2)}{\partial y}(*) - xw^2(1)}{*^2}$$

(BEHIND THE SCENES)

Implicit Differentiation

ex: $x^2y + xz + yz^2 = 8$ where $z = f(x,y)$

Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

remember, in this case, z isn't costant cuz $z = f(x,y)$

$$\frac{\partial}{\partial x}\left(x^2y + xz + yz^2\right) = 0$$

$$2xy + z + x\frac{\partial z}{\partial x} + 2yz\frac{\partial z}{\partial x} = 0$$

$$2xy + z\ (x+2yz)\frac{\partial z}{\partial y} = 0$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{-2xy - z}{x+2yz}$$

CALCULUS 2

Date: 21/10/2022

ex: $\frac{\partial}{\partial y}\left(x^2y + xz + yz^2\right)\left(\frac{\partial}{\partial y} 8\right)$ where $z=f(x,y)$

Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

$$x^2 + x\frac{\partial z}{\partial y} + z^2 + y2z\frac{\partial z}{\partial y} = 0 \rightarrow x^2 + z^2 + (x+2yz)\frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial y} = \frac{-x^2 - z^2}{x+2yz}$$

ex3: $2\cos(x+2y) + \sin(yz) - 1 = 0$

$-2\cdot 2\cos(x+2y)$ $\quad \cos(yz)\left(z + y\frac{\partial z}{\partial y}\right)$ [for $\frac{\partial z}{\partial y}$]

$$-2\sin(x+2y) + y\cos(yz)\cdot\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = \frac{2\sin(x+2y)}{y\cos(yz)}$$

$$-4\sin(x+2y) + z\cos(yz) + y(\cos(yz))\frac{\partial z}{\partial y} = 0$$

$$\Rightarrow \frac{\partial z}{\partial y} = \frac{4\sin(x+2y) - z\cos(yz)}{y\cos(yz)}$$

ex4: $\frac{\partial}{\partial x}(yz - \ln z) = \frac{\partial}{\partial x}(x+y)$ $\qquad z=f(x,y)$!

$$y\frac{\partial z}{\partial x} - \frac{1}{z}\frac{\partial z}{\partial x} = 1 \Rightarrow \frac{\partial z}{\partial x} = \frac{z}{yz-1}$$

Higher Order partial Derivatives

$$z = f(x,y) < \begin{matrix} f_x \\ f_y \end{matrix}$$

$$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} \qquad f_{yy} = (f_y)_y = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x} \qquad (f_y)_x = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\,\partial y}$$

mixed partial der
(the order is important)

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EX1: $f(x,y) = x\cos y + ye^x$

$f_x = \cos(y) + ye^x \longrightarrow f_{xx} = ye^x$
$f_y = -x\sin y + e^x \longrightarrow f_{yy} = -x\cos y$

$f_{xy} = -\sin y + e^x \qquad f_{yx} = -\sin y + e^x$

EX 2. $f(x,y,z) = 1 - 2xy^2z + x^2y$, Find $f_{yxyz} = \dfrac{\partial^4 f}{\partial z\,\partial y\,\partial x\,\partial y}$

(A)
$f_y = -4xyz + x^2$
$f_{yx} = -4yz + 2x$
$f_{yxy} = -4z$
$f_{yxyz} = -4$

Schwartz's Theorem

> It states that symmetry of 2^{nd} partial derivatives will always hold at a point if the second partial derivatives are continuous around that point.

EX :

$$\text{det } f(x,y) = \begin{cases} \overbrace{\dfrac{xy(x^2-y^2)}{x^2+y^2}}^{u} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

claim : $\dfrac{\partial^2 f}{\partial x\,\partial y}(0,0) \neq \dfrac{\partial^2 f}{\partial y\,\partial x}(0,0)$

For $(x,y) \neq (0,0)$

$$\frac{\partial f}{\partial x} = \frac{[y(x^2-y^2) + xy(2x)](x^2+y^2) - 2x(xy)(x^2-y^2)}{(x^2+y^2)^2}$$

$$= \frac{y(x^4-y^4) + 2x^2y(x^2+y^2) - 2x^2y(x^2-y^2)}{(x^2+y^2)}$$

so $\frac{\partial f}{\partial x} = \frac{y(x^4+4x^2y^2-y^4)}{(x^2+y^2)^2}$ $(x,y) \neq (0,0)$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h+0,0) - \cancel{f(0,0)}^{\,0}}{h} = 0$$

$$\boxed{\frac{\partial f}{\partial x} = \begin{cases} \frac{y(x^4+4x^2y^2-y^4)}{(x^2+y^2)^2} & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}}$$

1^{st} partial derivative in respect to x

For $(x,y) \neq 0,0$:

$$\frac{\partial f}{\partial y} = \frac{[x(x^2-y^2)+xy(-2y)](x^2+y^2) - 2y(xy)(x^2-y^2)}{(x^2+y^2)^2} =$$

$$= \frac{x(x^4-4x^2y^2-y^4)}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,0+h) - \cancel{f(0,0)}^{\,0}}{h} = 0$$

so

$$\boxed{\frac{\partial f}{\partial y} = \begin{cases} \frac{x(x^4-4x^2y^2-y^4)}{(x^2+y^2)^2} & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}}$$

1^{st} partial derivative in respect to y

$$\Rightarrow \frac{\partial^2 f}{\partial y \partial x}(0,0) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(0,0) = \lim_{h\to 0} \frac{\frac{\partial f}{\partial x}(0,h) - \cancel{\frac{\partial f}{\partial x}(0,0)}^{\,0}}{h} =$$

$$= \frac{-h}{h} = -1$$

$$\Rightarrow \frac{\partial^2 f}{\partial x \partial y}(0,0) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(0,0) = \lim_{h\to 0} \frac{\overbrace{\frac{\partial f}{\partial y}(h,0)}^{h} - \cancel{\frac{\partial f}{\partial y}(0,0)}^{\,0}}{h} = 1$$

initial claim: $\frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y \partial x}(0,0)$

$\rightarrow \quad 1 \neq -1$ because not continuous at $(0,0)$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) =$$

$$= \frac{[(x^4-4x^2y^2-y^4)+x(4x^3-8xy^2)][x^2+y^2]^2 - x(x^4-4x^2y^2-y^4)(2)(x^2+y^2)(2x)}{(x^2+y^2)^4}$$

$$= \frac{x^6+9x^4y^2-9x^2y^4-y^6}{(x^2+y^2)^3}$$

$$\boxed{\frac{\partial^2 f}{\partial x \partial y} \ast = \begin{cases} \frac{x^6+9x^4y^2-9x^2y^4-y^6}{(x^2+y^2)^3} & (x,y) \neq (0,0) \\ 1 & (x,y) = (0,0) \end{cases}}$$

Second partial derivative

Now we have to see if 2nd partial derivative is cont.:

Along $y = mx$

$$\ast \quad \frac{x^6+9x^4m^2x^2-9x^2m^4x^4-m^6x^6}{(x^2+m^2x^2)^3} = \frac{\cancel{x^6}(1+9m^2-9m^4-m^6)}{\cancel{x^6}(1+m^6)} =$$

$$= \frac{1+9m^2-9m^4-m^6}{1+m^6}$$

so $\lim\limits_{(x,y)\to(0,0)} \frac{\partial^2 f}{\partial x \partial y} = \frac{1+9m^2-9m^4-m^6}{1+m^6}$

limit doesn't exist since it depends on m

NOTE: Since we know $\frac{\partial^2 f}{\partial x \partial y}$ isn't contin., we don't need to calculate the other one by definition of theorem

NOTE BOOK

Rokshana Ahmed

DIFFERENTIALS

If $y=f(x)$ is a function of one variable, then $dy = f'(x)dx$ is called the differential. We also have differentials for more than 1 variable.

Given the function $z=f(x,y)$, then the differential dz is given by:

$$dz = f_x dx + f_y dy$$

This can be extended to function of 3 or more variables. For e.g. given the function $w = g(x,y,z)$, the differential is given by:

$$dw = g_x dx + g_y dy + g_z dz$$

EX: Let $w = \frac{x^3y^6}{z^2}$

$$\Rightarrow dw = \frac{3x^2y^6}{z^2}dx + \frac{x^3 6x^5}{z^2}dy + \frac{2x^3y^6}{z^3}dz$$

PARTIAL DERIVATIVES AND CONTINUITY

A function $f(x,y)$ can have partial derivatives w.r.t. both x and y at a point without the function being continuous there. This is different from fct. of single variable, where the existence of a derivative implies continuity.

EX: Let $f(x,y) = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$

Notice that $f(0,0) = 1$ $\left(xy=0 \rightarrow \begin{matrix} x=0 \\ y=0 \end{matrix} \; 0 \cdot 0 = 0 \right)$

$$\lim_{(x,y)\to(0,0)} f(x,y) \Big|_{\text{along } y=x} = \lim_{(x,y)\to(0,0)} 0 = 0$$

Along the line $y=x$, $f(x,y)$ is constantly zero (except at the origin)

- f is not continuous at $(0,0)$

But $\frac{\partial f}{\partial x}(0,0) = 0 = \frac{\partial f}{\partial y}(0,0)$ (exists)

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{\overbrace{f(h,0)}^{1} - \overbrace{f(0,0)}^{1}}{h} = 0$$

Also $\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0} \frac{1-1}{h} = 0$

THM: Differentiability implies Continuity

- If the partial derivatives of f_x and f_y of a fct. $f(x,y)$ are continuous throughout an open region R, then f is differentiable at every point of R.

- If a fct. $f(x,y)$ is differentiable at (x_0,y_0), then f is continuous at (x_0,y_0)

EX: The function $f(x,y) = x^2y^3$ is everywhere differentiable, since $\frac{\partial f}{\partial x} = 2xy^3$ and $\frac{\partial f}{\partial y} = 3x^2y^2$ are everywhere continuous

REMARKS

Let $z = f(x,y)$

1. f is differentiable at (a,b), then f is continuous at (a,b). However, the converse is not true Continuity does not imply differentiability

2. If f is differentiable at (a,b), then the partial derivative $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at (a,b). However, existence of partial derivatives doesn't imply differentiability. In fact, existence of partial derivatives doesn't guarantee continuity.

→ In single variable calculus, a fct. $f: \mathbb{R} \to \mathbb{R}$ is differentiable at $x=a$ if the following lim exists:

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = f'(a)$$

lim exists iff $\lim_{x\to a} \left(\frac{f(x)-f(a)}{x-a} - f'(a) \right) = 0$

$$\Leftrightarrow \lim_{x\to a} \left(\frac{f(x)-f(a) \cdot f'(a)(x-a)}{x-a} \right) = 0 \Leftrightarrow \lim_{x\to a} \frac{f(x)-L(x)}{a} = 0$$

where $L(x) = f(a) + f'(a)(x-a)$ → $L(x)$ is the linear approx. to f at $x=a$

So, roughly speaking, a single variable fct. is diff. iff the differential between $f(x)$ and its linear $\simeq$ goes zero as we approach the point.

Now: $f: \mathbb{R}^2 \to \mathbb{R}$
$(x,y) \to f(x,y)$ / Let $h(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$
linear approx. = eqn. of the tan. plane

FORMAL DEFINITION OF DIFFERENTIABILITY

Consider $f: \mathbb{R}^2 \to \mathbb{R}$ and suppose that the partial derivatives f_x and f_y are defined at the pt. $(x,y) = (a,b)$. Define the linear fct

$$h(x,y) = f(a,b) - f_x(a,b)(x-a) - f_y(a,b)(y-b)$$

We say that f is differentiable at (a,b) if:

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y) - h(x,y)}{\|(x,y)-(a,b)\|} = 0$$

If either of the partial derivatives $f_x(a,b)$ and $f_y(a,b)$ do not exist, or the above limit doesn't exist or not zero, then f is not diff. at (a,b)

EX: Study the differentiability of the following fct at $(0,0)$

$$f(x,y) = \begin{cases} \frac{x\sqrt{y}}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{h\sqrt{0}}{h^2+0} - 0}{h} = 0$$

$$\frac{\partial f(0,0)}{\partial y} = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

Now, let's check if $\displaystyle\lim_{(x,y)\to(0,0)} \frac{f(x,y) - [f(0,0) - f_x(0,0)(x-0) - f_y(0,0)(y-0)]}{\sqrt{(x-0)^2+(y-0)^2}}$

$$= \lim_{(x,y)\to(0,0)} \frac{\frac{x\sqrt{y}}{x^2+y^2} - 0 - 0 - 0}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{x\sqrt{y}}{(x^2+y^2)\sqrt{x^2+y^2}}$$

Now, lets see along $y=0$ and $y=x$ ↴

Along $x=0$, $\lim_{x \to 0} \frac{x\sqrt{y}}{(x^2+y^2)\sqrt{x^2+y^2}} = \lim_{x\to 0} 0 = 0$

Along $y=x$, $\lim_{x \to 0} \frac{x\sqrt{x}}{(x^2+x^2)\sqrt{2x^2}} = \lim_{x\to 0} \frac{1}{2x\sqrt{2x}} = \infty$

The fct. is not diff at $(0,0)$

EX2: Show that the fct. $f(x,y) = xy + 2x + y$ is diff at $(0,0)$

$$\left.\begin{aligned} f_x(x,y) &= y+2 \implies f_x(0,0) = 2 \\ f_y(x,y) &= x+1 \implies f_y(0,0) = 1 \end{aligned}\right\} \text{exists}$$

$$f(0,0) = 0$$

$\boxed{h(x,y)} = f(0,0) + f_x(0,0)(x) + f_y(0,0)y = 0 + 2x + y = \boxed{2x+y}$ and

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - h(x,y)}{\sqrt{(x^2-0)^2 + (y-0)^2}} = \lim_{(x,y)\to(0,0)} \frac{xy + \cancel{2x} + \cancel{y} - \cancel{2x} - \cancel{y}}{\sqrt{x^2+y^2}} = \frac{xy}{\sqrt{x^2+y^2}}$$

polar coordinates $= \lim_{r\to 0} \frac{r\cos\theta \, r\sin\theta}{\sqrt{r^2}} = \lim_{r\to 0} \frac{r^2\cos\theta\sin\theta}{|r|}$

But $-1 \leq \cos\theta\sin\theta \leq 1 \implies -|r| \leq \frac{r^2\cos\theta\sin\theta}{|r|} \leq |r|$

Since $\lim_{-|r|} = \lim_{r\to 0} |r| = 0$,

By sandwich thm, $\lim \frac{r^2\cos\theta\sin\theta}{r} = 0$

$$\boxed{\lim_{(x,y)\to(0,0)} \frac{f(x,y) - h(x,y)}{\|(x-y) - (0,0)\|} = 0 \text{ and hence } f \text{ is diff at } (0,0)}$$

GRAPHICAL INTERPRETATION OF PARTIAL DERIVATIVES

Given a fct $f: \mathbb{R} \to \mathbb{R}$, we know that $f(x_0)$ gives the slope of the tg. line to the graph of f at x_0.

Now let $f: \mathbb{R}^2 \to \mathbb{R}$

Fix $y=b$ (plane)

Let $g(x) = f(x,b)$

Then $\frac{\partial f}{\partial x}(a,b) = \frac{dg}{dx}(a)$

it is the slope of the tg line to the curve that results from the intersection of the plane $y=b$ and the surface at the pt. (a,b)

Graph of y

Fix $x=a$

Let $h(y) = f(a,y)$ (fct. of y)

$\frac{\partial f}{\partial y}(a,b) = \frac{dh}{dy}(b)$

$\Rightarrow$ The partial derivative of f w.r.t. y at (a,b) is the slope of the tg. line at the intersection of the graph of y with the plane $x=a$

EX: The plane $x=1$ intersects the paraboloid $z=x^2+y^2$ in a parabola.
Find the slope of the tg line to the parabola at $(1,2,5)$

Sol. The slope is given by $\frac{\partial z}{\partial y}(1,2)$

$$\left.\frac{\partial z}{\partial y}\right|_{(1,2)} = \left.2y\right|_{(1,2)} = (2)(2) = 4$$

To check, we can treat the parabola as the graph of single-variable fct. $z = y^2+1$ in the plane $x=1$ and ask for the slope at $y=2$.

The slope is given by the ordinary derivative:

$$\left.\frac{dz}{dy}\right|_{y=2} = \left.\frac{d}{dy}(1+y^2)\right|_{y=2} = \left.2y\right|_{y=2} = 4$$

NOTE BOOK

Rokshana Ahmed

CALCULUS 2

Date: 28 / 10 / 2022

CHAIN RULE FOR MULTI-VARIABLE FUNCTION

Aim: How to find derivatives of multivariable fct involving parametries or composition

The chain rule for functions of a single variable says that when $y = f(x)$ is a differentiable fct. of x and $x = g(t)$ is a differentiable fct. of t, then y is a differentiable fct. of t and

$$\boxed{\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}}$$

Another notation which is probably familiar to most people is:

$$\underline{F(x) = f(g(x)) \implies F'(x) = f'(g(x)) \cdot g'(x)}$$

For functions of 2 or more variables, the Chain rule has several forms. The form depends on how many are involved, but once this is taken into account, it works like Chain Rule in single variable.

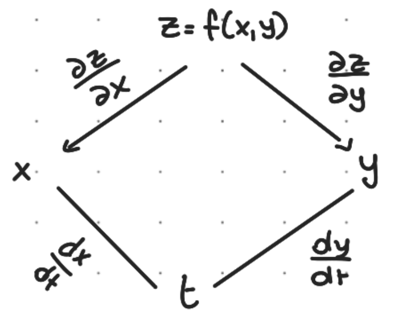
FUNCTION OF TWO VARIABLES: (case of 1 ind. var and 2 intermediate var.)

THM: If $z = f(x,y)$ is differentiable and if $x = x(t)$, $y \neq y(t)$ are differentiable functions of f, then the composite $z = f(x(t), y(x))$ is a differentiable fct of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}^{=\frac{\partial z}{\partial x}} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y}^{=\frac{\partial z}{\partial y}} \cdot \frac{dy}{dt} \quad \text{or}$$

$$\frac{dz}{dt} = f_x(x(t), y(t)) \cdot x'(t) + f_y(x(t), y(t)) \cdot y'(t)$$

The branch diagram provides a convenient way to remember the Chain Rule. To find $\frac{dz}{dt}$, start at z, and read down each route to t, multiplying derivatives along the way, then add the products



Dependent variable

Intermediate vars

Independent var

NOTE: The meaning of the dependant variable "z" is different on each side of the preceding eqn.
On the left-hand side, it refers to the composite function $z = f(x(t), y(t))$ as a fct. of a single variable t.
On the right-hand side, it refers to the function $z = f(x,y)$ as a fct. of 2 variables x, y.
Moreover, the single derivatives, $\frac{dw}{dt}$, $\frac{dx}{dt}$, and $\frac{dy}{dt}$ are being evaluated at a pt. "t", whereas the partial derivatives $\frac{\partial w}{\partial x}$, $\frac{\partial w}{\partial y}$ are being evaluated at the pt (x_0, y_0), with $x_0 = x(t)$, $y_0 = y(t_0)$.

EX1 Find the derivative of $z = xy$ w.r.t. "t" along the path $x = \cos t$, $y = \sin t$.

Sol:
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} =$$
(using Chain Rule)
$$= y(-\sin t) + x(\cos t) =$$
$$= (\sin t)(-\sin t) + (\cos t)(\cos t) =$$
$$= \cos 2t$$

→ We can check the results with direct calculation and in this case, it might be easier to just substitute in for x and y in the original fct and just compute the derivative as we normally do.

$$z = xy = \cos t \sin t = \frac{1}{2}\sin(2t)$$
$$\frac{dz}{dt} = \frac{d}{dt}\left(\frac{1}{2}\sin 2t\right) = \frac{1}{2}\cdot 2\cos(2t) = \cos(2t)$$
$$\rightarrow \left.\frac{dz}{dt}\right|_{t=\frac{\pi}{2}} = \cos\left(2\cdot\frac{\pi}{2}\right) = -1$$

EX2: $z = x^2y^3 + y\cos x$; $x = \ln(t^2)$, $y = \sin(4t)$

In this case, it would almost definetely be more work to do the substitution first, so we'll use the chain Rule and then substitute

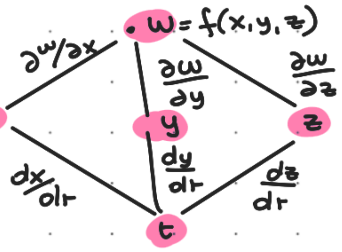
$$\frac{dz}{dt} = (2xy^3 - y\sin x)\left(\frac{2t}{t^2}\right) + (3x^2y^2 + \cos x)(4\cos 4t) =$$
$$= \frac{\left[(\ln(t^2)\sin^3 4t) - (\sin 4t \sin \ln t^2)\right]}{t} + 4\cos 4t(3\ln^2 t^2 \sin^2 4t + \cos(\ln t^2))$$

FUNCTIONS OF 3 VARIABLES

THM: If w is diff. and x, y, z are diff. functions of t, then w is diff. function of t and:

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

EX: $w = \tan^{-1}(xz) + \frac{z}{y}$



$x = t, \quad y = t^2, \quad z = \sin t$

$$\boxed{\frac{dw}{dt}} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} =$$

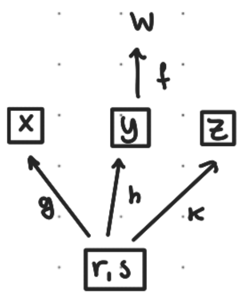
$$= \boxed{\left(\frac{z}{1+x^2z^2}\right)(1) + \left(\frac{-z}{y^2}\right)(2t) + \left[\left(\frac{x}{1+x^2z^2}\right) + \left(\frac{1}{y}\right)\right]\cos t}$$

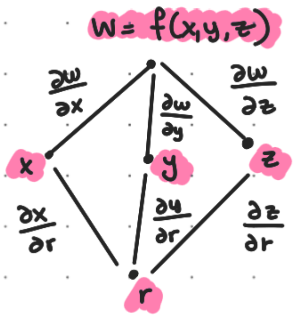
THM: CHAIN RULE FOR TWO INDEPENDENT VARIABLES AND THREE INTERMEDIATE VARIABLES

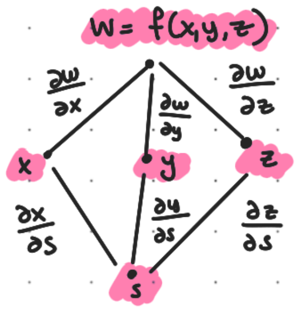
Suppose that $w = f(x, y, z)$, $x = g(r, s)$, $y = h(r, s)$ and $z = k(r, s)$.

If all 4 functions are diff., then w has partial derivatives w.r.t. "r" and "s" given by the formulas:

$$\boxed{\begin{aligned} \frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} \\ \frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} \end{aligned}}$$







EX3 Let $w = x + 2y + z^2$, $x = \frac{r}{s}$, $y = r^2 + \ln s$, $z = 2r$ *

$$\to \frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} =$$

$$= 1\left(\frac{1}{s}\right) + 2(2r) + (2\underbrace{z}_{2r^*})(2) =$$

$$= \frac{1}{s} + 4r + 8r =$$

$$= 1/s + 12r$$

$$\to \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} =$$

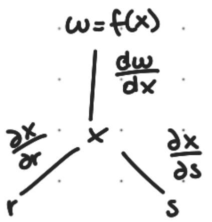
$$= (1)\left(-\frac{r}{s^2}\right) + 2\left(\frac{1}{s}\right) + 2z(0) =$$

$$= -\frac{r}{s^2} + \frac{2}{s}$$

NOTE: If f was a fct. of 2 independent var. (i.e. $w = f(x,y)$) instead of 3, then

$$\boxed{\begin{aligned} \frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} \\ \frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} \end{aligned}}$$

and if f was a fct. of single var. x, then (i.e $w = f(x)$, $x = g(r,s)$)



$$\boxed{\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r}} \text{ and } \boxed{\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s}}$$

where $\frac{dw}{dx} = f'(x)$ is the ordinary derivative.

Now that we've seen couple of cases for the chain rule, let's see the general version.

CHAIN RULE

Suppose that z is a fct. of n variables $x_1, x_2 \cdots x_n$ and each one of these variables in turn are fct. of m variables $t_1, t_2, \cdots t_m$. Then for any variable $t_1, t_2, \cdots t_m$ we have the following:

$$\frac{\partial z}{\partial t_i} = \frac{\partial z}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial z}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial z}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

CALCULUS 2

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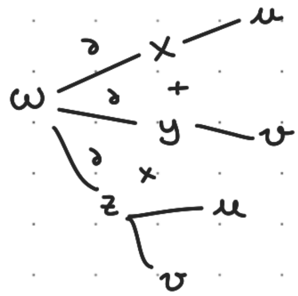
EX1: $w = x\tan^{-1}(yz)$; $x = \sqrt{u}$, $y = e^{-2v}$, $z = v\cos u$



$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{dx}{du} + \frac{\partial w}{\partial z}\frac{dz}{du}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial y}\frac{dy}{dv} + \frac{\partial w}{\partial z}\frac{dz}{dv}$$

$$\frac{\partial w}{\partial u} = \tan^{-1}(yz)\cdot\frac{1}{2\sqrt{u}} + \frac{xy}{1+y^2z^2}(-v\sin u)$$

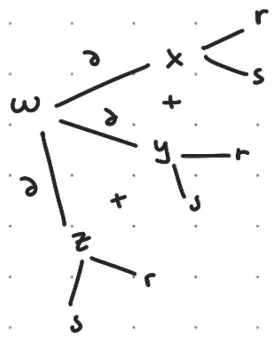
$$\frac{\partial w}{\partial v} = \frac{xz}{1+y^2z^2}(-2e^{-2v}) + \frac{xy}{1+y^2z^2}(\cos u) =$$

$$= \frac{-2xze^{-2v} + xy\cos u}{1+y^2z^2}$$

EX2: $w = x^2y + y^2z^3$, $x = r\cos(s)$, $y = r\sin(s)$, $z = re^s$

Find $\left.\frac{\partial w}{\partial s}\right|_{r=1,\ s=0}$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$



$$\Rightarrow \frac{\partial w}{\partial s} = 2xy(-r\sin(s)) + (x^2+2yz^3)\cdot(r\cos s) + 3y^2z^2(re^s)$$

$r=1,\ s=0 \Rightarrow x=1$, $y=0$, $z=1$

$$\Rightarrow \left.\frac{\partial w}{\partial s}\right|_{r=1,\ s=0} = 1$$

IMPLICIT DIFFERENTIATION REVISITED

With these forms of the chain rule, implicit differentiation becomes a fairly simple process.

Given the fct. in the form $w = f(x,y) = 0$ where y is implicitly defined function of x, say $y = g(x)$

CALCULUS 2

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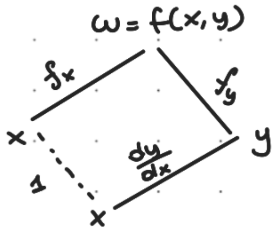
$$0 = \frac{\partial w}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} \Rightarrow f_y \frac{dy}{dx} = -f_x$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{-f_x}{f_y}}$$

Branch Diagram

$$\frac{dw}{dx} = f_y \frac{dy}{dx} + f_x(1)$$

$$\Rightarrow \frac{dy}{dx} = \frac{-f_x}{f_y}$$



EX1 $2x^2 + 3\sqrt{xy} - 2y - 4 = 0$ $\quad (\Leftrightarrow f(x,y)=0)$

$y = g(x)$. Find $\frac{dy}{dx}$?

$$\Rightarrow \frac{dy}{dx} = -\frac{4x + \frac{3y}{2\sqrt{xy}}}{\frac{3x}{2\sqrt{xy}} - 2} \cdot \frac{2\sqrt{xy}}{2\sqrt{xy}} = -\frac{8x\sqrt{xy} + 3y}{3x - 4\sqrt{xy}}$$

EX2 Find $\frac{dy}{dx}$ for $x\cos(3y) + x^3y^5 = 3x - e^{xy}$ and $y = g(x)$

So $\underbrace{x\cos(3y) + x^3y^5 - 3x + e^{xy}}_{f(x,y)} = 0$

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{\cos(3y) + 3x^2y^5 - 3 + ye^{xy}}{-3x\sin(3y) + 5x^3y^4 + xe^{xy}}$$

[This calculation is significantly shorter than a single-variable calculation using Implicit diff.]

CALCULUS 2

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IMPLICIT DIFFERENTIATION FOR FCT. OF 3 VARIABLES

Assume $f(x,y,z) = 0$ and z is implicitly defined as $z = g(x,y)$ and we want to find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$

$$f(x,y,z) = 0$$

- Differentiate both sides w.r.t. x and we'll need to remember to treat y as a const.

$$\frac{\partial f}{\partial x} \cdot 1 + \frac{\partial f}{\partial y} \cdot \cancel{\frac{\partial y}{\partial x}}_{0} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial x} = 0$$

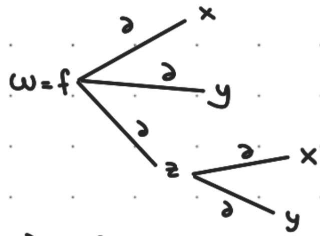
$$f_x + f_z \frac{\partial z}{\partial x} = 0 \quad \Rightarrow \quad \boxed{\frac{\partial z}{\partial x} = -\frac{f_x}{f_z}}$$

Similarly, $\boxed{\frac{\partial z}{\partial y} = -\frac{f_y}{f_z}}$

EX1: $x^2 e^{xy} + xy - x^2 z + yz^2 = 0$ and $z = g(x,y)$

$$\Rightarrow \boxed{\frac{\partial z}{\partial x} = -\frac{2xe^{xy} + x^2 y e^{xy} + y - 2xz}{-x^2 + 2yz}}$$

and $\boxed{\frac{\partial z}{\partial y} = -\frac{x^3 e^{xy} + x + z^2}{-x^2 + 2yz}}$



$$\frac{\partial w}{\partial x} = 0$$

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 0$$

$$f_x + f_y \cdot \frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = -\frac{f_x}{f_z}$$

EX2: Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for $x^2 \sin(2y - 5z) = 1 + y\cos(6zx)$

$$\rightarrow x^2 \sin(2y - 5z) - y\cos(6zx) - 1 = 0$$

$$\boxed{\begin{aligned} \frac{\partial z}{\partial x} &= -\frac{2x\sin(2y-5z) + 6zy\sin(6zx)}{-5x^2\cos(2y-5z) + 6yx\sin(6zx)} \\ \frac{\partial z}{\partial y} &= \frac{2x^2\cos(2y-5z) - \cos(6zx)}{-5x^2\cos(2y-5z) + 6yx\sin(6zx)} \end{aligned}}$$

NOTE BOOK

Rokshana Ahmed

DIRECTIONAL DERIVATIVE AND GRADIENT

- The partial derivatives of f w.r.t. x give the slope of the tangent line to the intersection of the graph of f with the plane $y = y_0$ at (x_0, y_0) in the direction of x.

  Also $\frac{\partial f}{\partial y}$ gives the slope to the tan. line in the y-direction

- We can generalize the partial derivatives to calculate the slope in any direction. The result is called the Directional Derivative. It also allows us to find the rate of change of f if we allow both x and y to change simultaneously

- The first step in taking a directional derivative is to specify the direction. One way to specify a direction is with a vector $\vec{u} = (u_1, u_2)$ that points in the direction in which we want to compute the slope. For simplicity, we'll assume that $\vec{u}$ is a unit vector.

- Sometimes the direction of changing x and y is given as an angle θ. The unit vector that points in this direction is given by $\vec{u} = (\cos\theta \sin\theta)$

DEF: The derivative of f in the direction of the unit vector $\vec{u} = u_1\vec{i} + u_2\vec{j}$ is called the directional derivative and denoted by $D_{\vec{u}} f(x,y)$ and given by:

$$\boxed{D_{\vec{u}} f(x,y) = \lim_{h \to 0} \frac{f(x + u_1 h, y + u_2 h) - f(x,y)}{h}}$$

→ if $\vec{u} = \vec{j} = (1,0)$ $\quad D_i = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h} = f_x$

→ if $\vec{u} = j = (0,1)$ $\quad D_j = \lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{h} = f_y$

The partial derivative $f_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$ and $f_y(x_0, y_0)$ are the directional derivative of f at $P_0 = (x_0, y_0)$ in the $\vec{i}$ and $\vec{j}$ directions.

EX1: Using the definition, find the derivative of $f(x,y) = x^2 + xy$ at $P_0(1,2)$ in the direction of the unit vector $\vec{u} = \frac{1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} \vec{j}$

Sol: $D_{\vec{u}} f(x,y) = \lim_{h \to 0} \frac{f(x+u_1 h,\ y+u_2 h) - f(x,y)}{h}$

$$= \lim_{h\to 0} \frac{f(x+\frac{1}{\sqrt{2}}h,\ y+\frac{1}{\sqrt{2}}h) - f(x,y)}{h} =$$

$$= \lim_{h\to 0} \frac{(x+\frac{h}{\sqrt{2}})^2 + (x+\frac{h}{\sqrt{2}})(y+\frac{h}{\sqrt{2}}) - x^2 - xy}{h} =$$

$$= \lim_{h\to 0} \frac{\cancel{x^2} + \cancel{\frac{h^2}{2}} + \frac{2xh}{\sqrt{2}} + \cancel{xy} + \frac{xh}{\sqrt{2}} + \frac{yh}{\sqrt{2}} + \cancel{\frac{h^2}{2}} - \cancel{x^2} - \cancel{xy}}{h}$$

$$= \lim_{h\to 0} \frac{h^2 + \frac{3xh}{\sqrt{2}} + \frac{yh}{\sqrt{2}}}{h} = \lim_{h\to 0} h + \frac{3x}{\sqrt{2}} + \frac{y}{\sqrt{2}} =$$

$$= \frac{3x+y}{\sqrt{2}}$$

So $D_{\vec{u}} f(1,2) = \frac{3}{\sqrt{2}} + \frac{2}{\sqrt{2}} = \frac{5}{\sqrt{2}}$. So, the rate of change of $f(x,y) = x^2+xy$ at $(1,2)$ in the direction of $\vec{u}$ is $\frac{5}{\sqrt{2}}$

- In practice it could be difficult to compute the limit, so we need an easier way to evaluate the directional derivative
- To see how, let's define a new fct of a single var:

$$g(z) = f(x_0 + u_1 z,\ y_0 + u_2 z)$$

where x_0, y_0, u_1, u_2 are some fixed numbers.

Then $g'(z) = \lim_{h\to 0} \frac{g(z+h) - g(z)}{h}$ and

$$g'(0) = \lim_{h\to 0} \frac{g(h) - g(0)}{h} =$$

$$= \lim_{h\to 0} \frac{f(x_0+u_1 h,\ y_0+u_2 h) - f(x_0,y_0)}{h} = D_{\vec{u}} f(x_0,y_0)$$

So $g'(0) = D_{\vec{u}} f(x_0, y_0)$

Now, let's write $g(z) = f(x,y)$ where $\begin{cases} x = x_0 + u_1 z \\ y = y_0 + u_2 z \end{cases}$ *

$\Rightarrow$ $g'(z) = \frac{dg}{dz} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dz} + \frac{\partial f}{\partial y} \frac{dy}{dz} = f_x^{(x,y)} u_1 + f_y^{(x,y)} u_2$

Let $z=0$ i.e. $x=x_0$ and $y=y_0$ *

Then $\boxed{g'(0) = f_x(x_0,y_0)u_1 + f_y(x_0,y_0)u_2}$

$\boxed{D_{\vec{u}} f(x_0,y_0) = f_x^{(x_0,y_0)} u_1 + f_y^{(x_0,y_0)} u_2}$

So, $D_{\vec{u}} f(x,y) = f_x(x,y)u_1 + f_y(x,y)u_2$

There are similar formulas that can be derived by the same type of argument for functions with more than two var. The directional derivative of $f(x,y,z)$ in the direction of the unit vector $\vec{u} = (u_1, u_2, u_3)$ is given by:

$$D_{\vec{u}} f(x,y,z) = f_x u_1 + f_y u_2 + f_z u_3$$

EX1 $f(x,y) = x^2 + xy$

we are picking up from the EX1 on page 3

$f_x = 2x + y$ $\qquad u_1 = \frac{1}{\sqrt{2}} = u_2$

$f_y = x$

Then $D_{\vec{u}} f(x,y) = f_x u_1 + f_y u_2 =$

$= \frac{1}{\sqrt{2}} x + \frac{1}{\sqrt{2}}(2x+y) = \frac{3x+y}{\sqrt{2}}$

and $D_{\vec{u}} f(1,2) = \frac{5}{\sqrt{2}}$

EX2: Let $f(x,y,z) = x^2 z + y^3 z^2 - xyz$ $\qquad \vec{u} = (-1,0,3)$

Notice that $\|\vec{u}\| = \sqrt{1+3^2} = \sqrt{10} \neq 1$ ($\vec{u}$ is not a unit vector)

Let $\vec{u} = \left(-\frac{1}{\sqrt{10}}, 0, \frac{3}{\sqrt{10}}\right)$

$\Rightarrow$ $D_{\vec{u}} f(x,y,z) = f_x u_1 + f_y u_2 + f_z u_3$ where

$$f_x = 2x - yz$$
$$f_y = 3y^2z^2 - xz$$
$$f_z = x^2 + y^3z - xy$$

$$D_{\vec{u}} f = \left(-\frac{1}{\sqrt{10}}\right)(2x - yz) + 0 + \frac{3}{\sqrt{10}}(x^2 + 2y^3z - xy) =$$
$$= \frac{1}{\sqrt{10}}(3x^2 + 6y^3z - 3xy - 2xz + yz)$$

EX3 Find $D_{\vec{u}} f(2,0)$ where $f(x,y) = xe^{xy} + y$ where $\vec{u}$ is the unit vector in the direction of $\theta = 2\pi/3$

$\longrightarrow$ $\vec{u} = \left(\cos\frac{2\pi}{3}; \sin\frac{2\pi}{3}\right) = \left(-\frac{1}{2}; \frac{\sqrt{3}}{2}\right)$

$f_x = e^{xy} + xye^{xy}$ $\quad$ $f_y = x^2e^{xy} + 1$

$$D_{\vec{u}} f = (e^{xy} + xye^{xy})\left(-\tfrac{1}{2}\right) + \frac{\sqrt{3}}{2}(x^2e^{xy} + 1)$$

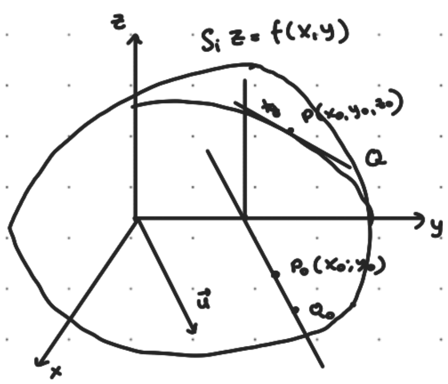
$\Rightarrow$ $D_{\vec{u}} f(2,0) = (1+0)\left(-\frac{1}{2}\right) + \frac{\sqrt{3}}{2}(4+1) = \frac{5\sqrt{3} - 1}{2}$

GRAPHICAL INTERPRETATION OF DIRECTIONAL DERIVATIVE

The equation $z = f(x,y)$ represents a surface S in space. If $z_0 = f(x_0, y_0)$, then the pt $(x_0, y_0, z_0) = P$ lies on S.

The vertical plane that passes thru P and $P_0(x_0, y_0)$ // to the (uz) plane intersects S in a curve C.

The rate of change of f in the direction of u is the slope of the tangent to C at P.



When $\vec{u} = \vec{i} = (1,0)$, the directional derivative at P_0 is $\frac{\partial f}{\partial x}$ evaluated at (x_0, y_0). When $\vec{u} = \vec{j} = (0,1)$ " " $\frac{\partial f}{\partial y}$ " " at $P_0 = (x_0, y_0)$

$\rightarrow$ The Directional Derivative generalizes the two partial derivatives. We can now ask for the rate of change of f in any direction of $\vec{u}$, not just $\vec{i}$ or $\vec{j}$

We now develop an efficient formula to calculate the directional derivative for a differentiable fct, of f.

$$D_{\vec{u}} f = f_x u_1 + f_y u_2 = \underbrace{(f_x, f_y)}_{\substack{\text{Gradient of } f \\ \nabla f}} \cdot \underbrace{(u_1, u_2)}_{\substack{\text{direction} \\ \vec{u}}}$$

So the derivative f in the direction of $\vec{u}$ is the dot product of $\vec{u}$ with the special vector called the gradient of f

DEF: The gradient vector of $f(x,y)$ at a pt $(x_0, y_0) = P_0$ is the vector $\boxed{\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}}$ obtained by evaluating the partial derivatives of f at P_0

⇒ With the definition of the gradient we can now say that the directional derivative is given by:

$$\boxed{D_{\vec{u}} f = \nabla f \cdot \vec{u}}$$

EX Find the derivative of $f(x,y) = xe^y + \cos(xy)$ at the pt $(2,0)$ in the direction $\vec{v} = 3\vec{i} - 4\vec{j}$

$\|v\| = \sqrt{3^2+4^2} = 5$

let $\vec{u} = \frac{3}{5}\vec{i} - \frac{4}{5}\vec{j}$ (unit vector)

$f_x = e^y - y\sin(xy)$; $f_x(2,0) = 1$

$f_y = xe^y - x\sin(xy)$; $f_y(2,0) = 2$

The gradient of f at (2,0) is

$$\nabla f|_{(2,0)} = f_x(2,0)\vec{i} + f_y(2,0)\vec{j} = \vec{i} + 2\vec{j}$$

The Directional derivative of f at (2,0) is :

$$D_u f|_{(2,0)} = \nabla f|_{(2,0)} \cdot \vec{u} = (\vec{i} + 2\vec{j}) \cdot \left(\frac{3}{5}\vec{i} - \frac{4}{5}\vec{j}\right) =$$

$$= \frac{3}{5} - \frac{8}{5} = -1$$

PROPERTIES OF THE DIRECTIONAL DERIVATIVE

$$D_{\vec{u}} f = \nabla f \cdot \vec{u} = \|\nabla f\| \cdot \|\vec{u}\| \cos\theta = \|\nabla f\| \cos\theta$$

θ is the angle between the vector $\vec{u}$ and ∇f

CALCULUS 2

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① The maximum value of $D_{\vec{u}}f$ (hence the max. rate of change of the function $f(x,y)$) is given by $\|\nabla f\|$ and will occur in the direction of ∇f. So, f (at each pt. P in the domain) increases most rapidly in the direction of the gradient vector ∇f at P. The derivative in this direction is $D_{\vec{u}}f = |\nabla f|$

PROOF $D_{\vec{u}}f = \|\nabla f\| \cos\theta$

The largest possible value of $\cos\theta$ is 1 which occurs at $\theta = 0$. The max value of $D_{\vec{u}}f$ is $\|\nabla f\|$. Also the max value occurs when the angle between the gradient and $\vec{u}$ is zero, or in otherwords when $\vec{u}$ is pointing in the same direction as the gradiant ∇f.

② Similarly, f decreases most rapidly in the direction $-\nabla f$. The derivative in this direction is $D_u f = |\nabla f| \cos\pi = -|\nabla f|$

③ Any direction $\vec{u}$ orthogonal to a gradient vector $\nabla f \neq 0$ is a direction of zero change in f because θ then equals $\frac{\pi}{2}$ and $D_u f = |\nabla f| \cos(\frac{\pi}{2}) = |\nabla f| \cdot 0 = 0$

EX Find the direction in which $f(x,y) = \frac{x^2}{2} + \frac{y^2}{2}$

a) ↗ most rapidly at the pt. (1,1)

b) ↙ " " at (1,1)

c) what are the directions of zero change in f at (1,1)?

Sol: a) The gradient $\nabla f = f_x \vec{i} + f_y j = x\vec{i} + y\vec{i}$

$\nabla f|_{(1,1)} = i + j$ and the rate of change in this case is $\|\nabla f_{(1,1)}\| = \sqrt{2} = D_{\vec{u}}f$

It's direction is $\vec{u} = \frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}j$

b) $-\vec{u} = -\frac{1}{\sqrt{2}}\vec{i} - \frac{1}{\sqrt{2}}\vec{j}$, rate of change is $D_{\vec{u}}f = -\|\nabla f\| = -\sqrt{2}$

c) $\vec{n} = -\frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}$ $\quad -\vec{n} = \frac{1}{\sqrt{2}}\vec{i} - \frac{1}{\sqrt{2}}\vec{j}$

CALCULUS 2

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ALGEBRAIC RULES FOR GRADIENT

- $\nabla(f+g) = \nabla f + \nabla g$ (Sum/Difference)
- $\nabla(kf) = k\nabla f$ (Constant multiple)
- $\nabla(fg) = f\nabla g + g\nabla f$ (product)
- $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$ (Quotient)

FUNCTIONS OF 3 VARIABLES

For a differentiable fct. $f(x,y,z)$ and a unit vector $u = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$:

$$\nabla f = f_x\vec{i} + f_y\vec{j} + f_z\vec{k}$$

$$D_{\vec{u}}f = \nabla f \cdot \vec{u} = f_x u_1 + f_y u_2 + f_z u_3$$

EX: Find the derivative of $f(x,y,z) = x^3 - xy^2 - z$ at $P_0(1,1,0)$

a) in the direction of $\vec{v} = 2\vec{i} - 3\vec{j} + 6\vec{k}$

b) In what direction does f change most rapidly at P_0 and what are the rates of change in these directions.

a) $\|\vec{v}\| = \sqrt{2^2+3^2+6^2} = \sqrt{49} = 7$ $\quad \vec{u} = \left(\frac{2}{7}, -\frac{3}{7}, \frac{6}{7}\right)$

$f_x = 3x^2 - y^2$; $\quad f_y = -2xy$; $\quad f_z = -1$

$f_x|_{(1,1,0)} = 3-1 = 2$ $\quad f_y|_{(1,1,0)} = -2$ $\quad f_z = -1$

and $\nabla f|_{(1,1,0)} = (2,-2,-1)$

$\Rightarrow D_{\vec{u}}f|_{(1,1,0)} = (2,-2,-1)\cdot\left(\frac{2}{7}, -\frac{3}{7}, \frac{6}{7}\right) =$

$= (2)\left(\frac{2}{7}\right) + (-2)\left(-\frac{3}{7}\right) + (-1)\left(\frac{6}{7}\right) =$

$= \frac{4}{7} + \frac{6}{7} - \frac{6}{7} = \frac{4}{7}$

b) The fct ↗ most rapidly in the direction of $\vec{\nabla f} = 2\vec{i} - 2\vec{j} - \vec{k}$ and the rate of change in this direction is $\|\nabla f\| = \sqrt{2^2+2^2+1} = 3$ and ↙ most rapidly in direction $-\nabla f$ and the rate of change in this direction is $-|\nabla f| = -3$

$-2\vec{i} + 2\vec{j} + \vec{k}$

NOTE BOOK

Rokshana Ahmed

$f(x,y,z)$ is a fct. of 3 variables and $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$ is a unit vector. Then:

$$\nabla f = f_x\vec{i} + f_y\vec{j} + f_z\vec{k} \quad \text{and}$$

$$D_{\vec{u}}f = \nabla f \cdot \vec{u} = f_x u_1 + f_y u_2 + f_z u_3$$

EX: Find the derivative of $f(x,y,z) = x^3 - xy^2 - z$ at $P_0(1,1,0)$

a) in the direction of $\vec{v} = 2\vec{i} - 3\vec{j} + 6\vec{k}$

b) In what direction does f change most rapidly at P and what are the rates of change in this direction.

1. $\vec{v} = (2, -3, 6)$ $\quad \|\vec{v}\| = \sqrt{2^2 + (-3)^2 + 6^2} = \sqrt{49} = 7$

$\Rightarrow \vec{u} = \frac{1}{\|\vec{v}\|} v = \left(\frac{2}{7}; -\frac{3}{7}; \frac{6}{7}\right)$

$f_x = 3x^2 - y^2$ $\quad f_y = -2xy$ $\quad f_z = -1$

$f_x|_{(1,1,0)} = 2$ $\quad f_y|_{(1,1,0)} = -2$

$\nabla f(1,1,0) = (2, 2, -1)$

$Df(1,1,0) = \nabla f \cdot \vec{u} = (2,2,-1)\cdot\left(\frac{2}{7}, \frac{3}{7}, \frac{6}{7}\right) = \frac{4}{7} + \frac{6}{7} - \frac{6}{7} = \frac{4}{7}$

2. The fct. f ↗ most rapidly in the direction of $\vec{\nabla f} = 2\vec{i} - 2\vec{j} - \vec{k}$ and the rate of change in this direction is

$$\|\nabla f\| = \sqrt{2^2 + (-2)^2 + (-1)^2} = 3$$

↙ most rapidly in the direction of

$-\vec{\nabla f} = -2\vec{i} + 2\vec{j} + \vec{k}$ $\qquad -\|\nabla f\| = -3$

TANGENT PLANES AND NORMAL LINE

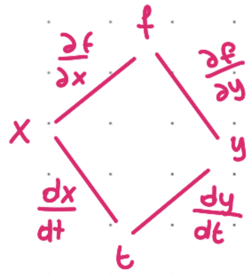
$z = f(x,y)$ $\qquad \underline{z = c}$ plane $\parallel$ to (xy) plane

↳ intersect a surface

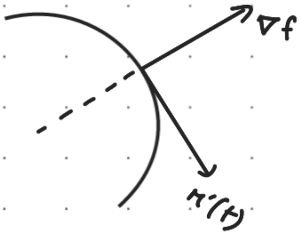
$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$ $\qquad f(x(t), y(t)) = c$

$$\frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} = 0$$

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \cdot (x'(t), y'(t)) = 0$$



$$\nabla f \cdot r'(t) = 0$$



If $\vec{r}(t) = g(t)\vec{i} + h(t)\vec{j} + k(t)\vec{k}$ is a curve on the level surface $f(x,y,z) = c$ of a differentiable fct. f then $f(g(t), h(t), k(t)) = c$

diff. both sides

$$\frac{\partial f}{\partial x}\frac{dg}{dt} + \frac{\partial f}{\partial y}\frac{dh}{dt} + \frac{\partial f}{\partial z}\frac{dk}{dt} = 0$$

$$\underbrace{\left(\frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}\right)}_{\nabla f} \cdot \underbrace{\left(\frac{dg}{dt}\vec{i} + \frac{dh}{dt}\vec{j} + \frac{dk}{dt}\vec{k}\right)}_{\frac{dr}{dt}} = 0$$

At any point along the curve ∇f is orthogonal to the curves velocity vector.

DEF: The tangent plane at the point (x_0, y_0, z_0) on the level surface $f(x,y,z) = c$ of a differentiable fct. f is the plane through P_0 and normal to $\nabla f|_{P_0}$

The tangent plane to $f(x,y,z) = c$ at $P_0(x_0, y_0, z_0)$ is:

$$f_x|_{P_0}(x - x_0) + f_y|_{P_0}(y - y_0) + f_z|_{P_0}(z - z_0) = 0$$

where $(f_x, f_y, f_z)_{P_0} = \nabla f|_{P_0}$

CALCULUS 2

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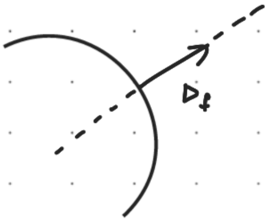

(x_0, y_0, z_0) $\vec{v} = (v_1, v_2, v_3)$

$$\begin{cases} x - x_0 = t v_1 \\ y - y_0 = t v_2 \\ z - z_0 = t v_3 \end{cases}$$

Normal line to $f(x,y,z) = c$ at (x_0, y_0, z_0) is :

$$\begin{cases} x = x_0 + f_x^{(P_0)} t \\ y = y_0 + f_y^{P_0} t \\ z = z_0 + f_z^{P_0} t \end{cases}$$

$$\frac{x - x_0}{f_x|_{P_0}} = \frac{y - y_0}{f_y|_{P_0}} = \frac{z - z_0}{f_z|_{P_0}}$$

Cartesian equation

EX: Find the equation of the tangent plane and normal line to the surface $f(x,y,z) = x^2 + y^2 + z - 9 = 0$ at the point $P_0(1,2,4)$

$$\left.\begin{array}{l} f_x = 2x \\ f_y = 2y \\ f_z = 1 \end{array}\right\} \nabla f = (2x, 2y, 1)$$

$\underline{\nabla f(1,2,4)} = (2,4,1) \Rightarrow 2(x-1) + 4(y-2) + (z-4) = 0$

Normal vector

$\boxed{2x + 4y + z = 14}$ eq. of tangent plane

$$\frac{x-1}{2} = \frac{y-2}{4} = \frac{z-4}{1} \rightarrow \begin{cases} x = 2t + 1 \\ y = 2 + 4t \\ z = 4 + t \end{cases}$$

cartesian eq. of normal line

EX2

$$\underbrace{xz^2 + yx^2 + y^2 - 2x + 3y = -6}_{f(x,y,z) = 6} \quad \text{in} \quad P(-2,1,3)$$

$$\left.\begin{array}{l} f_x = z^2 + 2yx - 2 \\ f_y = x^2 + 2y + 3 \\ f_z = 2zx \end{array}\right\} \quad \nabla f = \left.\begin{array}{l} (z^2 + 2yx - 2)\vec{i} + \\ (x^2 + 2y + 3)\vec{j} + \\ 2zx\ \vec{k} \end{array}\right\}$$

Normal vector to family of level surfaces

$\nabla f|_{(-2,1,3)} = 3\vec{i} + 9\vec{j} - 12\vec{k}$

$3(x+2) + 9(y-1) - 12(z-3) = 0 \rightarrow \boxed{x + 3y - 4z = -11}$ tangent plane eq.

Normal line $\longrightarrow$ $\dfrac{x-2}{3} = \dfrac{y-1}{9} = \dfrac{z+3}{-12}$

ex3 $f(x,y,z) = \tan^{-1}\left(\frac{y}{x}\right) - z$ a $P(1,1,\frac{\pi}{4})$

① $f_x = \dfrac{-\frac{y}{x^2}}{1+\left(\frac{y}{x}\right)^2} = \dfrac{-y}{x^2+y^2}$

$f_y = \dfrac{1/x}{1+\left(\frac{y}{x}\right)^2} = \dfrac{x}{x^2+y^2}$

$f_z = -1$

② $\nabla f = \dfrac{-y}{x^2+y^2}\vec{i} + \dfrac{x}{x^2+y^2}\vec{j} - \vec{K}$

③ $\nabla f|(1,1,\frac{\pi}{4}) = -\frac{1}{2}\vec{i} + \frac{1}{2}\vec{j} - \vec{K}$

$-\frac{1}{2}(x-1) + \frac{1}{2}(y-1) - 1\left(z-\frac{\pi}{4}\right) = 0$

tangent plane eq.

ex4 $xyz = -4$ $P(2,-1,2)$

$\rightarrow$ let $f(x,y,z) = xyz$

$\nabla f(x,y,z) = yz\vec{i} + xz\vec{j} + xy\vec{k}$ $\longrightarrow$ this is normal vector to family of level surfaces for $f(x,y,z)$

$\rightarrow$ $\nabla f(2,-1,2) = -2\vec{i} + 4\vec{j} - 2\vec{k}$

Tangent plane

$-2(x-2) + 4(y+1) - 2(z-2) = 0$

$\iff \boxed{x - 2y + z = 6}$

$\longrightarrow$ The pt. $P(2,-1,2)$ gives a specific normal $(-2,4,-2)$ to a specific level surface $xyz = -4$

Normal line

$\dfrac{x-2}{-2} = \dfrac{y+1}{4} = \dfrac{z-2}{-2}$

ex5: $z = x\cos y - ye^x$ at $(0,0,0)$

Let $f(x,y,z) = x\cos y - ye^x - z = 0$

$\nabla f = (\cos y - ye^x)\vec{i} + (-x\sin y - e^x)\vec{j} - \vec{k}$

$\nabla f(0,0,0) = \vec{i} - \vec{j} - \vec{k}$

Tg plane: $1(x-0) + (-1)(y-0) - 1(z-0) = 0$

$x - y - z = 0$

CALCULUS 2

Date: 7 / 11 / 2022

SECTION 14.7. EXTREME VALUES AND SADDLE POINT

Def: Let $f(x,y)$ be defined on a Region R containing the pt (a,b) then :

1. $f(x,y)$ has relative min. at pt (a,b) if $f(a,b) \leq f(x,y)$ $\forall$ (x,y) in an open disk centered at (a,b)
2. $f(a,b)$ is a local max. if $(a,b) \geq f(x,y)$ $\forall$ points in an open disk centered at (a,b)
3. f has an absolute max at (a,b) if $f(a,b) \geq f(x,y)$ for all (x,y) in R
4. f has absolute min at (a,b) if $f(a,b) \leq f(x,y)$ $\forall (x,y)$ in R

NOTE BOOK

Rokshana Ahmed

CALCULUS 2

Date: 11 / 11 / 22

- Partial derivative
  - extremes of a fct.
    - local max and min
    - absolute max and min
  - Chain rule
  - Directional Derivative
  - Tangent plane and Normal line

SECTION 14.7. EXTREME VALUES AND SADDLE POINT

Def: Let $f(x,y)$ be defined on a Region R containing the pt (a,b) then :

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3. f has an absolute max at (a,b) if $f(a,b) \geq f(x,y)$ for all (x,y) in R
4. f has absolute min at (a,b) if $f(a,b) \leq f(x,y)$ $\forall (x,y)$ in R

DEF : Critical Points

A point (a,b) of the domain of a function $f(x,y)$ is called a critical point (or stationary point) if

1. $f_x(a,b) = f_y(a,b) = 0$
2. Either $f_x(a,b)$ or $f_y(a,b)$ doesn't exist

NOTE THAT both partial derivatives must be zero at (a,b).
If only one of the first order partial derivatives is zero at the point then the point will not be critical.
The value of the function of the critical point is called critical value.

CALCULUS 2

Date: 11 / 11 / 2022

EX: Find the critical point for $f(x,y) = x^2 + 2xy - 4y^2 + 4x - 6y + 4$

$f_x = 2x + 2y + 4 \qquad f_y = 2x - 8y - 6$

Now we set them to zero

$2x + 2y + 4 = 0 \qquad 2x - 8y - 6 = 0$

$2x = 8y + 6$

$8y + 6 + 2y + 4 = 0$

$10y = -10$

$\boxed{y = -1}$ and $\boxed{x = -1}$

$(-1,-1)$ is only critical point for $f(x,y)$

$\Rightarrow$ The main goal of determining critical points is to locate local max and local min.

THM: First derivative test for Local Extreme Values

If the point (a,b) is a (local) relative of the function $f(x,y)$ and the first order derivatives of $f(x,y)$ exist at (a,b), then $f_x(a,b) = f_y(a,b) = 0$ and hence $(a,b]$ is a critical point.

Proof $\qquad g(x) = f(x,b) \qquad g(a) = f(a,b) \geq \underbrace{f(x,b)}_{g(x)} \quad \forall (x,y)$

Assume that f has a relative extremum at (a,b) then $g(x)$ also has a " " at $x = a$ (of the same kind as $f(x,y)$ at $x = 0$)

$\Longrightarrow \quad g'(a) = 0 \quad \Rightarrow \quad \boxed{f_x(a,b) = 0}$

Similarly, we can define $h(y) = f(a,y)$ to show that $\boxed{f_y(a,b) = 0}$

The Thm says that critical points are the only possible points where $f(x,y)$ can assume extreme values.

→ If we substitute the values $f_x(a,b)=0$ and $f_y(a,b)=0$ into the equation

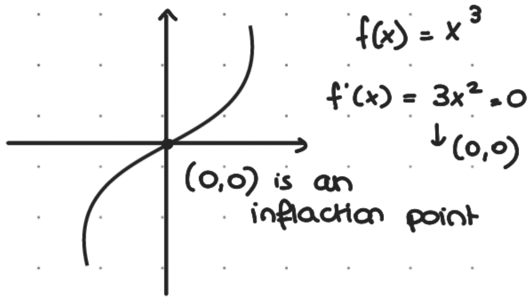
$$f_x(a,b)(x-a) + f_y(a,b)(y-b) - (z - f(a,b)) = 0$$

for the tangent plane to the surface $z = f(x,y)$ at (a,b) the eqn reduces to :

$$0(x-a) + 0(y-b) - z + f(a,b) = 0 \Rightarrow z = f(a,b)$$

Thus, the surface have a horizontal tangent plane at local extremum.

* For a fct. of single variable, NOT every critical point gives rise to local extremum ! *



$f(x) = x^3$

$f'(x) = 3x^2 = 0$

$\downarrow (0,0)$

DEF: Saddle Point

A function $f(x,y)$ has a saddle point at a critical point (a,b) if in every open disk centered at (a,b), there are domain points (x,y) wher $f(x,y) > f(a,b)$ and domain points (x,y) where $f(x,y) < f(a,b)$.

The corresponding point on the surface $z = f(x,y)$ is called saddle point on the surface.

EX: Find the local extreme value (if any) for $f(x,y) = y^2 - x^2$

$f_x = -2x = 0$
$f_y = 2y = 0$ > $(0,0)$ is the critical point and the only possible point where local extremas exist.

Along $y=0$, $f(0,x) = -x^2 \leq 0 = f(0,0)$

Along $x=0$, $f(y,0) = y^2 \geq 0 = f(0,0)$

$(0,0)$ is a saddle point as well

THM: Second Derivative Test

Suppose that $f(x,y)$ and its first and second part derivatives are continuous throughout a disk centered at (a,b) and that $f_x(a,b) = f_y(a,b) = 0$

Define the quantity:

$$D = f_{x,x}(a,b)\, f_{y,y}(a,b) - \left[f_{x,y}(a,b)\right]^2$$

Then:

1. If $D>0$ and $f_{x,x}(a,b) > 0$, f has a local minimum at (a,b)
2. If $D>0$ and $f_{x,x}(a,b) < 0$, f has a local max at (a,b)
3. If $D<0$, then f has a saddle point at (a,b)
4. If $D=0$, then the test is inconclusive.

The expression D is called the descriminant or the Hessian of f

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$

NOTE THAT if $D>0$, then both $f_{xx}(a,b)$ and $f_{yy}(a,b)$ will have the same sign, so in the first two cases, we could easily replace $f_{xx}(a,b)$ by $f_{yy}(a,b)$

STEPS

1. Determine the critical points (a,b) of the function where $f_{xx}(a,b) = f_{yy}(a,b) = 0$
2. Calculate the Descriminant for each critical point of f
3. Apply the 4 cases of the test to determine whether the point is a local max, local min or saddle point.

EX1: $f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$

Step 1 $\quad f_x = y - 2x - 2 = 0 \qquad f_y = x - 2y - 2 = 0$

$$\downarrow$$

$y = 2x + 2 \longrightarrow x - 2(2x+2) - 2 = 0 \Rightarrow x = -2 \quad y = -2$

$-3x - 6 = 0$

so $(-2,2)$ is a critical point

Step 2 $\quad f_{xx} = -2 \quad f_{yy} = -2 \quad f_{xy} = 1$

$D = (-2)(-2) - 1^2 = 3 > 0$

and $f_{xx}(-2,-2) = -2 < 0$

$\Rightarrow (-2,-2)$ is a local max pt. and the local max value is $f(-2,-2) = -2$

ex2 Let $f(x,y) = 3y^2 - 2y^3 - 3x^2 + 6xy$

Step 1. $\begin{cases} f_x = -6x + 6y = 0 \Rightarrow x = y \\ f_y = 6y - 6y^2 + 6x = 0 \end{cases}$

$6x - 6x^2 + 6x = 0$
$-6x^2 + 12x = 0$
$6x(-x+2) = 0$

$\Rightarrow x = 0$ or $x = 2$
$\Rightarrow y = 0$ or $y = 2$
$\Rightarrow (0,0)$ or $(2,2)$

Step 2 $\quad f_{xx} = -6 \quad f_{yy} = 6 - 12y \quad f_{xy} = 6$

$D = (-6)(6-12y) - (-6)^2 =$
$= -36 + 72y - 36 =$
$= 72y - 72 = 72(y-1)$

$D(0,0) = -72 < 0$ so $(0,0)$ is a saddle point

$D(2,2) = 72(2-1) = 72 > 0$ and

$f_{xx}(2,2) = -6 < 0$

so $(2,2)$ is a local max for f

EX3 $f(x,y) = \frac{1}{3}x^3 + y^2 + 2xy - 6x - 3y + 4$

1 $\quad f_x = x^2 + 2y - 6 = 0 \quad -$
$\quad f_y = 2y + 2x - 3 = 0$
$\quad x^2 - 2x - 3 = 0$

$(x-3)(x+1) = 0$

$\Rightarrow \boxed{x = 3 \text{ or } x = -1}$
$\Rightarrow \boxed{y = -\frac{3}{2} \text{ or } y = \frac{5}{2}}$

② $(3, -3/2)$ and $(-1, 5/2)$ critical points

$f_{xx} = 2x \quad f_{yy} = 2 \quad f_{xy} = 2$

$D = (f_{xx})(f_{yy}) - (f_{xy})^2 =$
$(2x)(2) - 2^2 = 4x - 4 = 4(x-1)$

$D(3, -3/2) = 4(3) - 4 = 8 > 0$ and $f_{yy} > 0$

then $(3, -3/2)$ is a local min pt.

$D(-1, 5/2) = 4(-1-1) = -8 < 0$

then $(-1, 5/2)$ is a saddle point

EX4 $\quad f_x = e^x \sin y \quad f_y = e^x \cos y$

① $f_x = e^x \sin y = 0 \implies \sin y = 0 \implies y = 0 + k\pi$
$f_y = e^x \cos y = 0 \implies \cos y = 0 \implies y = \frac{\pi}{2}$

we know e^x is always > 0

It has no critical point

EX ① $f(x,y) = x^2y + y^2 + xy$
② $f(x,y) = e^{-x^2-y^2}$
} TO DO AT HOME

REMEMBER TO REVISE INTEGRATION METHODS!

NOTE BOOK

Rokshana Ahmed

CALCULUS 2

Date: 14 / 11 / 2022

RECAP

① critical points $\Rightarrow$ $f_x = 0 = f_y = 0$

② Find discriminant $D = f_{xx} f_{yy} - (f_{xy})^2$

↓

3 cases

- $D > 0$
  - $f_{xx} > 0$ → local min
  - $f_{xx} < 0$ → local max
- $D < 0$ → saddle point
- $D = 0$ → Inconclusive

HOME EXERCISES :

① $f(x,y) = x^2 y + y^2 + xy$

Step 1 : $f_x = 2xy + y = 0 \Rightarrow y(2x+1) = 0 \Rightarrow y = 0$ or $x = -\frac{1}{2}$

$f_y = x^2 + 2y + x = 0$

for $y = 0$; $x^2 + x = 0$, $x(x+1) = 0$, $x = 0$ or $x = -1$ $\Rightarrow$ crit. pts. $(0,0)$ and $(-1,0)$

for $x = -\frac{1}{2}$; $\left(-\frac{1}{2}\right)^2 + 2y - \frac{1}{2} = 0$, $2y = \frac{1}{4} \Rightarrow y = \frac{1}{8}$ crit. pt. $\left(-\frac{1}{2}, \frac{1}{8}\right)$

Step 2 :

$f_{xx} = 2y$

$f_{yy} = 2$

$f_{xy} = 2x + 1$

$D = (2y)(2) - (2x+1)^2$

$D(0,0) = -1 < 0 \Rightarrow$ $(0,0)$ is a saddle point

$D(-1;0) = -1 < 0 \Rightarrow$ $(-1,0)$ 〃 〃 〃 〃

$D\left(-\frac{1}{2}; \frac{1}{8}\right) = \frac{1}{2} > 0$ and $f_{yy} > c$

$\Rightarrow \left(-\frac{1}{2}; \frac{1}{8}\right)$ local min pt.

CALCULUS 2

Date: 14 / 11 / 2022

② $f(x,y) = e^{-x^2-y^2}$

Step 1

$f_x = -2x e^{-x^2-y^2} = 0$
$f_y = -2y e^{-x^2-y^2} = 0$

e is always positive
so it means
$-2x=0$ and $-2y=0$
so $x=0$ and $y=0$

(0,0) is the critical point

Step 2

$\boxed{f_{xx} =} -2e^{-x^2-y^2} + (-2x)(-2xe^{-x^2-y^2}) = -2e^{-x^2-y^2}(1-2x^2)$

[apply product rule since x is involved in both quantities
$\underbrace{-2x}\,\underbrace{e^{-x^2-y^2}}$]

$\boxed{f_{yy} =} -2e^{-x^2-y^2}(1-2y^2)$

$\boxed{f_{xy} =} 4xy\, e^{-x^2-y^2}$

$D = (-2e^{-x^2-y^2})^2(1-2x^2)(1-2y^2) - [4xye^{-x^2-y^2}]^2$

$D(0,0) = 4 > 0$ and $f_{xx}(0,0) = -2 < 0 \Rightarrow (0,0)$ local max

THM: EXTREME VALUE THEOREM

If $f(x,y)$ is continuous fct., on a closed, bounded set S in $\mathbb{R}^2$.
Then f has an absolute max and an absolute min in S.

Def :

① A region D in $\mathbb{R}^2$ is called bounded if it lies in a disk of finite Radius.
In other words, a region will be bounded if its finite.

② A region in $\mathbb{R}^2$ is called closed if it includes all its boundary points. A region is called open if it consists only of interior points.

③ A pt. (a,b) in a Region D in $\mathbb{R}^2$ is an interior point of D if it is the center of a disk that lies entirely in D.
A pt. (a,b) is a boundary pt. if every disk centered at (a,b) contains pts. that lie outside D and inside D

CALCULUS 2

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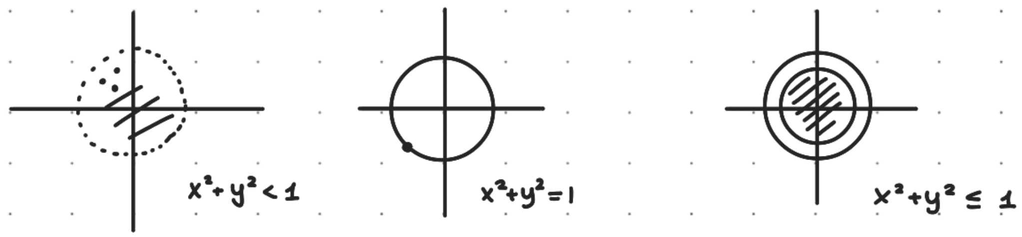
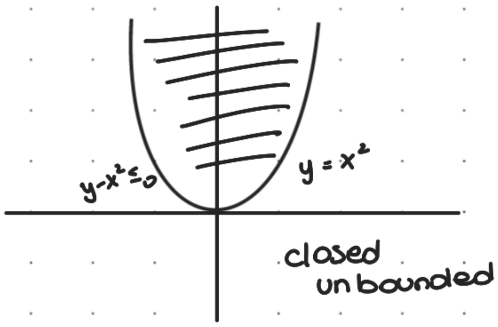


interior pts — bounded — finite, closed

EX $f(x,y) = \sqrt{y-x^2}$

D: $y - x^2 \geq 0$

$y \geq x^2$



Steps :

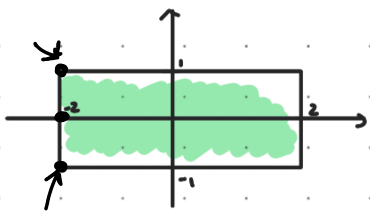
1) List the interior points of R where f may have local max, local min and evaluate f at these points.

2) List the boundary points of R where f has local max and min and evaluate f at these points. This usually involves calculus 1 approach for this work.

3) The largest and smallest values found in the 1st two steps are the absolute max and min of the fct.

EX1 Let $f(x,y) = x^2 + xy + y^2$

on $R = \left\{ (x,y) \;\middle|\; \begin{matrix} -2 \le x \le 2 \\ -1 \le y \le 1 \end{matrix} \right\}$

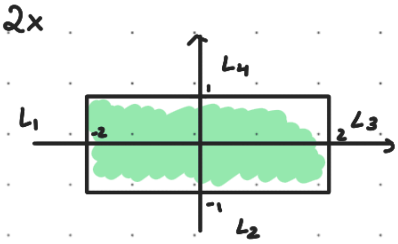


Step 1 : $f_x = 2x + y = 0 \Rightarrow y = -2x$

$f_y = 2y + x = 0$

$-4x + x = 0 \Rightarrow x = 0$

$f(0,0) = 0$



4 boundaries

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The boundary of the Rectangle is given by :

- L_1: $x=-2$ $\quad -1 \le y \le 1$
- L_2: $x=2$ $\quad -1 \le y \le 1$
- L_3: $y=-1$ $\quad -2 \le x \le 2$
- L_4: $y=1$ $\quad -2 \le x \le 2$

Along L_1

$x=-2 \quad = g(y) = f(-2,y) = 4-2y+y^2$

$g'(y) = -2+2y = 0 \implies \boxed{y=1} \longrightarrow$ Crit. Pt. $(-2,1)$

$\boxed{f(-2,1) = 3}$

Endpoints are $(-2;1)$ and $(-2,-1)$

because $-1 \le y \le 1$ so $(-2,1)$ and $(-2;-1)$

$(-2;1) \xrightarrow{f(-2;1)} 3 \qquad (-2,-1) \xrightarrow{f(-2;-1)} 7$

Along L_2

$x=2 \implies f(2,y) = 4+2y+y^2$

$f'(2,y) = 2+2y = 0 \implies y=-1$

C.P. $(2,-1) \implies f(2,-1) = 3$

E.P. $(2,-1) \longrightarrow 3$

$(2,1) \longrightarrow 7$

Along L_3

$y=-1 \implies f(x,-1) = x^2-x+1$

$f'(x,-1) = 2x-1 = 0 \implies x = \frac{1}{2}$

C.P. $(\frac{1}{2},-1) \implies f(\frac{1}{2},-1) = \frac{3}{4}$

E.P. $(-2;-1) \longrightarrow 7$

$(2;-1) \longrightarrow 7$

Along L_4

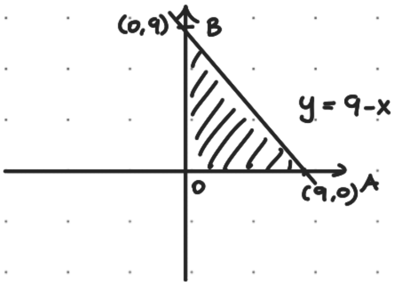
C.P. $(-\frac{1}{2};1) \longrightarrow \frac{3}{4}$

E.P. $(-2,1) \longrightarrow 3$

$(2,1) \longrightarrow 7$

So $f(0,0)=0$ is a global min value and $(-2,-1)$ and $(2,1)$ are global maxs

Ex2: Find the absolute max or min of $f(x,y) = 2+2x+2y-x^2-y^2$ on the Δ region in the 1^{st} quadrant bounded by $x=0$, $y=0$, and $y=9-x$



Sol.

$f_x = 2-2x = 0 \Rightarrow x=1$

$f_y = 2-2y = 0 \Rightarrow y=1$

$\boxed{f(1,1)=4}$

Along OA $\quad y=0 \quad 0 \le x \le 9$

$f(x,0) = 2+2x-x^2 \to f'(x) = 2-2x = 0 \downarrow x=1$

C.P. : $(1,0) \longrightarrow 3$

E.P. : $(0,0) \longrightarrow 2$

$(9,0) \longrightarrow -61$

Along OB $\quad x=0, \quad 0 \le y \le 9$

$f(0,y) = 2y-y^2+2$

C.P : $(0,1) \to 3$

E.P. : $(0,0) \to 2$

$(0,9) \to -61$

Along BA $\quad y = 9-x$

$f(x,y) = f(x, 9-x)$

$\left(\frac{9}{2}, \frac{9}{2}\right) \longrightarrow -\frac{41}{2}$

g. min $(0,9) \to -61$
g. max $(1,1) \to 4$

$x^2 = 4-y^2$

$x^2+y^2=4 \qquad x = \pm\sqrt{4-y^2}$

$y^2 = 4-x^2$

$y = \pm\sqrt{4-x^2}$

NOTE BOOK

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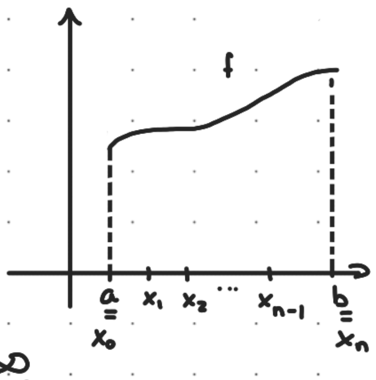
Multiple Integrals (Chap. 15)

RECAP:

Let f be a continuous fct. on $[a,b]$.
Divide the interval $[a,b]$ into equal n sub-intervals each of length $\Delta x = \frac{b-a}{n}$

x_k to be an arbitrary pt. $A \approx \sum_{k=1}^{n} f(x_k)\Delta x$
(called the approximation area)



To get the exact area, we have to send $n \to \infty$ $(\Delta x \to 0)$

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x = \int_a^b f(x)\,dx$$

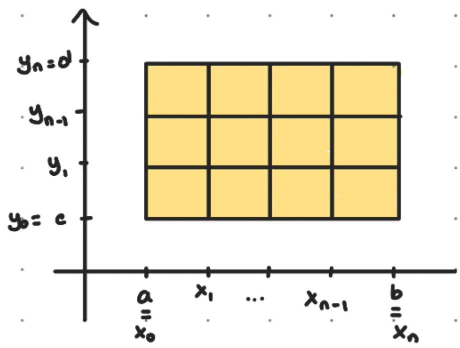
DOUBLE AND ITERATED INTEGRALS OVER RECTANGLES

DOUBLE INTEGRALS

Consider $R: \{(x,y) \mid a \le x \le y,\ c \le y \le d\}$

Divide $[a,b]$ into m subdivisions
Divide $[c,d]$ " n "

→ This will divide up our region into small rectangles



$R = [a,b] \times [c,d]$

These rectangles form a position of R $(\Delta A = \Delta x \Delta y)$
Δx = width of each rectangle along x-direction
Δy = height " " " y-direction

(x_k, y_k) be arbitrary pt. in each one of the rect.

→ Each of these rectangles have a base area $\Delta A = \Delta x \Delta y$

DOUBLE INTEGRALS AS VOLUMES

The volume under the surface is approximately

$$V \simeq \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i, y_j)\Delta A$$

$$V = \lim_{m,n\to+\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i, y_j)\Delta A = \iint_R f(x,y)\,dA$$

FUBINI'S THEOREM FOR CALCULATING DOUBLE INTEGRALS

$$\iint_R f(x,y)\,dA \to \int_{y=c}^{y=d}\left(\int_{x=a}^{x=b} f(x,y)\,dx\right)dy$$

$$\iint_R f(x,y)\,dA \to \int_{x=a}^{x=b}\int_{y=c}^{y=d} f(x,y)\,dy\,dx$$

THM: (Fubini's Thm)

If $f(x,y)$ is continuous fct. over a rectangular region
$R: a \le x \le b, \quad c \le y \le d$

$$\iint_R f(x,y)\,dA = \int_c^d\int_a^b f(x,y)\,dx\,dy = \int_a^b\int_c^d f(x,y)\,dy\,dx$$

these are called iterated integrals

EX1 $f(x,y) = 2xy$ (single integral w. constant) *

$$f(x,y) = \int f_x\,dx = \int 2x\underbrace{y}_{\text{constant}}\,dx = 2y\frac{x^2}{2} + c = yx^2 + c(y)$$

EX2 $\int_1^{2y} 2xy\,dx = \left[x^2y\right]_1^{2y} = (2y)^2y - y = 4y^3y - y$ *

EX3 Compute $\iint_R 6xy^2\,dx\,dy$ where $R = [2,4]\times[1,2]$

↓

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1st method (dx)

$$\int_1^2\left(\int_2^4 6xy^2\,dx\right)dy = \int_1^2 \left[3x^2y^2\right]_2^4 dy =$$

$$\int_1^2 3y^2(4^2-2^2)\,dy = 36\int_1^2 y^2\,dy =$$

$$= \frac{36}{3}\left[y^3\right]_1^2 = 84$$

2nd method (dy)

$$\int_2^4\left(\int_1^2 6xy^2\,dy\right)dx = \int_2^4 \left[2xy^3\right]_1^2 dy$$

$$= 14\int_2^4 x\,dx = \frac{14}{2}\left[x^2\right]_2^4 =$$

$$= 7(16-4) = 84$$

EX4 Compute $\iint_R xe^{xy}\,dA$ where $R=[-1,2]\times[0,1]$

1st method (dy)

$$\int_{-1}^2\left(\int_0^1 xe^{xy}\,dy\right)dx = \int_{-1}^2\left[e^{xy}\right]_0^1 dx =$$

$\int xe^{xy}\,dy$

$u=xy$ $\quad \frac{du}{dy}=x \to dy=\frac{dv}{x}$

$= e^{xy}+c$

$$= \int_{-1}^2 (e^x-1)\,dx = \left[e^x-x\right]_{-1}^2 = e^2-2-e^{-1}-1 =$$

$$= e^2-e^{-1}-3$$

2nd method (dx)

integr. by parts

$u=x \quad dv=e^{xy}dx$

$du=dx \quad v=\frac{1}{y}e^{xy}$

$$\int_0^1\int_{-1}^2 xe^{xy}\,dx\,dy = \int_0^1\left[\frac{x}{y}e^{xy} - \int\frac{1}{y}e^{xy}dx\right]_{-1}^2 dy =$$

$$= \int_0^1\left[\frac{x}{y}e^{xy}-\frac{1}{y^2}e^{xy}\right]_{-1}^2 dy = \int_0^1 \frac{2e^{2y}}{y} - \frac{1}{y^2}e^{2y} \ldots$$

too complicated

EX5: $\iint_R (x+y^2)\,dA$ where $R: \left\{(x,y) \,/\, \begin{matrix}0\le x\le 1\\ -1\le y\le 2\end{matrix}\right\}$

$[0,1]\times[-1,2]$

$$\int_{-1}^2\int_0^1 (x+y^2)\,dx\,dy = \int_{-1}^2\left[\frac{x^2}{2}+y^2x\right]_0^1 dy =$$

$$= \int_{-1}^2\left(\frac{1}{2}+y^2\right)dy = \left.\frac{1}{2}y+\frac{y^3}{3}\right]_{-1}^2 = 1+\frac{8}{3}+\frac{1}{2}+\frac{1}{3} = \frac{9}{2}$$

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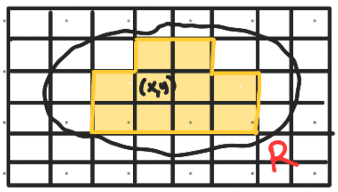
DOUBLE INTEGRAL OVER A GENERAL REGION

$$\lim_{m,n\to 0}\sum\sum f(x_i,y_i)\,\Delta x\Delta y$$

$$\Delta A=\Delta x\Delta y$$



We'll consider 2 types of Non-Rectangular Regions:

CASE 1: $R=\{(x,y)/\ a\le x\le b,\ g_1(x)\le y\le g_2(x)\}$

$$\iint_R f(x,y)\,dA = \int_{x=a}^{x=b}\int_{y=g_1(x)}^{y=g_2(x)} f(x,y)\,dy\,dx$$

CASE 2: $R: \{(x,y)/\ h_1(y)\le x\le h_2(y);\ c\le y\le d\}$

Then: $\iint_R f(x,y)\,dA = \int_c^d\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$

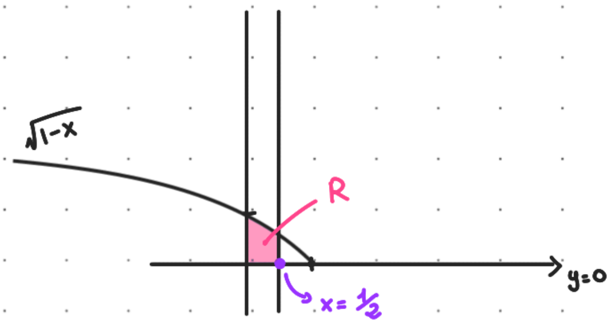
EX: $\iint_R 2xy\,dA$ $R: (x,y)/\ 0\le x\le \frac{1}{2};\ 0\le y\le\sqrt{1-x}\}$

We are in the 1° case. Set up the integral

$$\iint_R 2xy\,dA = \int_{x=0}^{1/2}\int_{y=0}^{y=\sqrt{1-x}} 2xy\,dy\,dx = \int_0^{1/2} xy^2\Big]_0^{\sqrt{1-x}}dx$$

$$= \int_0^{1/2} x(1-x)\,dx = \frac{x^2}{2}-\frac{x^3}{3}\Big]_0^{1/2} = \frac{1}{12}$$

Now we sketch the region ↓



FIBINUS THM FOR GENERAL REGION

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

y should be between fct. of x and x between constants

Bound x between fct. of y and y between constants

Ex Let R be the Region bounded by $y=2x$ and $y=x^2$

Sketch R

$y=y$

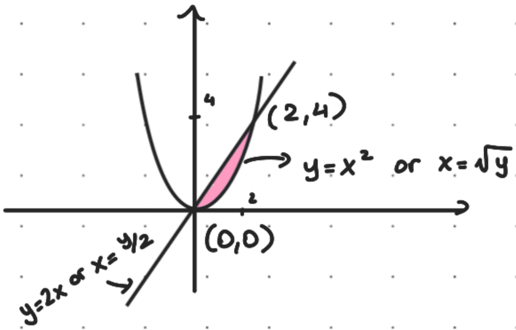
$x^2=2x$

$x^2-2x=0$

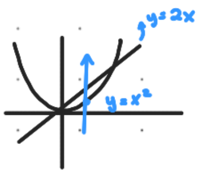
$x(x-2)=0$

$x=0$ or $x=2$

$(0,0)$ $(2,4)$



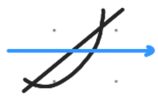
$$\iint_R f(x,y)\,dA = \int_{x=0}^{x=2} \int_{y=x^2}^{y=2-x} f(x,y)\,dy\,dx = \int_0^4 \int_{x=y/2}^{x=\sqrt{y}} f(x)\,dx\,dy$$



you go from bottom to up along y vertical line ↑

(from bottom to up because that will be the positive one)

we enter the R horizontally now, from left to right



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CALCULUS 2

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RECAP

$x \in [a,b]$
$y \in [c,d]$

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

Remember:

$$\int_a^b \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy \qquad \int_c^d \int_{h_1(x)}^{h_2(x)} f(x,y)\,dy\,dx$$

$h_1(y) \le x \le h_2(y)$, $a \le y \le b$ $\qquad$ $h_1(x) \le y \le h_2(x)$, $a \le x \le b$

EX1: Evaluate the following integral:

$$\iint_R \frac{x}{1+xy}\,dA \qquad R: \left\{(x,y) \,\middle|\, \begin{array}{l} 0 \le x \le 1 \\ 0 \le y \le 1 \end{array}\right\}$$

$$\int_0^1\int_0^1 \frac{x}{1+xy}\,dy\,dx = \int_0^1 \frac{x}{x}\ln|1+xy|\Big]_0^1$$

$$= \int_0^1 \ln|1+x|\,dx = (1+x)\ln(1+x) - (1+x)\Big]_0^1$$

REMEMBER
$\int \ln x = x\ln x - x$

$$= 2\ln 2 - 2 + 1 =$$
$$= 2\ln 2 - 1$$

EX2:

$$\iint_R \frac{\ln y}{y^2}\,dA, \qquad R: \left\{(x,y) : \begin{array}{l} 0 \le x \le \pi \\ e^{-2x} \le y \le e^{\cos x} \end{array}\right\}$$

$$\int_0^\pi \int_{e^{-2x}}^{e^{\cos x}} \frac{\ln(y)}{y}\,dy\,dx =$$

Let $u = \ln(y)$, $du = \frac{1}{y}dy$

$$= \int_0^\pi \int_{-2x}^{\cos x} u\,du\,dx =$$

For $y = e^{-2x}$, $u = \ln(e^{-2x}) = -2x$
For $y = e^{\cos x}$, $u = \ln(e^{\cos x}) = \cos x$

$$= \int_0^\pi \left[\frac{u^2}{2}\right]_{-2x}^{\cos x} dx = \frac{1}{2}\int_0^\pi (\cos^2 x - 4x^2)\,dx$$

$$= \frac{1}{2}\int_0^\pi \frac{4\cos 2x}{2} - 4x^2\,dx = \frac{1}{2}\left[\frac{x}{2} + \frac{\sin 2x}{4} - \frac{4}{3}x^3\right]_0^\pi = \boxed{\frac{\pi}{2} - \frac{2\pi^3}{3}}$$

$\cos(2x) = 2\cos^{2x} - 1$
$= 1 - 2\sin^2 x$

EX3: $\iint_R \frac{1}{xy}\,dA$; $R\left\{(x,y)\mid 1\le y\le e,\ y\le x\le y^2\right\}$

$$\int_1^e\int_y^{y^2}\frac{1}{xy}\,dx\,dy=\int_1^e \frac{1}{y}\ln|x|\Big]_y^{y^2}=\int_1^e \frac{1}{y}\left[\ln(y^2)-\ln(y)\right]dy$$

$$=\int_1^e \frac{\ln y}{y}\,dy=\frac{\ln y^2}{2}\Big]_1^e=\frac{1}{2}$$

EX4: $\iint_R (\sin(x)-y)\,dA$ R is the Region bounded by $y=\cos x$ and $y=0$, $x=\frac{\pi}{2}$ and $x=0$

$$\int_0^{\pi/2}\int_0^{\cos x}(\sin(x)-y)\,dy\,dx=\int_0^{\pi/2} y\sin x-\frac{y^2}{2}\Big]_0^{\cos x}dx=$$

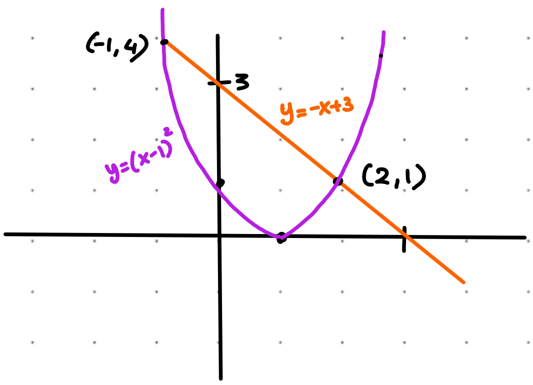
$$\int_0^{\pi/2}\sin x\cos x-\frac{\cos^2 x}{2}=\int_0^{\pi/2}\left[\frac{\sin 2x}{2}-\frac{1+\cos 2x}{4}\right]dx=$$

$\frac{\sin 2x}{2}$

REMEMBER
$\sin^2 x=2\sin x\cos x$

$$=\frac{-\cos 2x}{4}-\frac{1}{4}x-\frac{\sin 2x}{4}\Big]_0^{\pi/2}=\frac{1}{2}-\frac{\pi}{4}$$

EX5: $\iint_R 4x^3\,dA$ / R is the region bounded by $y=(x-1)^2$ and $y=-x+3$



Steps

1. sketch the R
2. Find the intersection

$(x-1)^2=-x+3$

$x^2-2x+1=-x+3$

$x=2$, $x=1$

$y=1$, $y=4$

$$=\int_{x=1}^{x=2}\int_{y=(x-1)^2}^{y=-x+3}4x^3\,dy\,dx=\int_{x=1}^{x=2}4x^3y\Big]_{(x-1)^2}^{-x+3}dx=\frac{72}{5}$$

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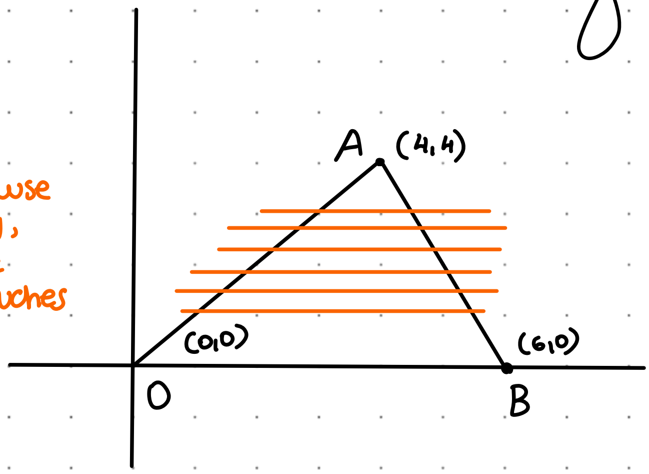
EX6: R is given by the points (0,0), (4,4), (6,0)

$$\iint_R f(x,y)\, dA$$



dx then dy because horizontally, every line always touches 2 fcts.

$-\frac{1}{2}x = y - 6$

$AB \rightarrow y = -\frac{1}{2}x + 6$

$x = -2y + 12$

EQ OF STRAIGHT LINE

$y - y_0 = m(x - x_0)$

$m = \frac{x_2 - x_1}{y_2 - y_1} = \frac{4-0}{4-6} = -\frac{4}{2} = -\frac{1}{2}$

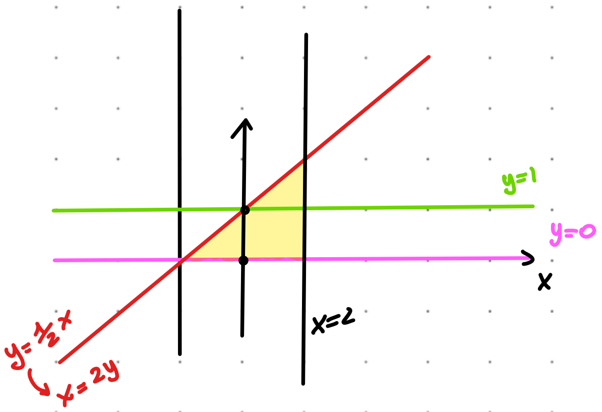
$y = -\frac{1}{2}x + 6$

EX7: $I = \int_0^1 \int_{2y}^2 e^{-x^2} dx\, dy$

we should switch from dxdy to dydx to make life easier.

$$= \int_0^2 \int_0^{\frac{1}{2}x} e^{-x^2} dy\, dy$$

we have to find the new bounds since we're switching so sketch R.



$$= \int_0^2 e^{-x^2} [y]_0^{\frac{1}{2}x} dx$$

$$= \int_0^2 \frac{1}{2}x e^{-x^2} = -\frac{1}{4}[e^{-4} - 1]$$

substitution

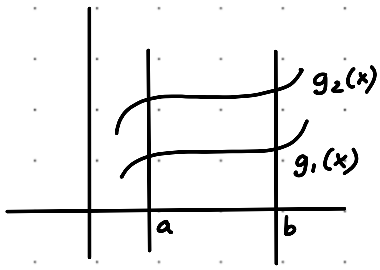
$u = -x^2$

$du = -2x\,du$

AREA BY DOUBLE INTEGRATION

So, we know Area of $R = \iint_R dA_{\substack{dxdy \\ dydx}}$

$$A = \int_a^b g_2(x) - g_1(x)\, dx$$



PROOF: $\iint_R dA = \int_a^b \int_{g_1(x)}^{g_2(x)} dy\, dx = \int_a^b g_2(x) - g_1(x)$

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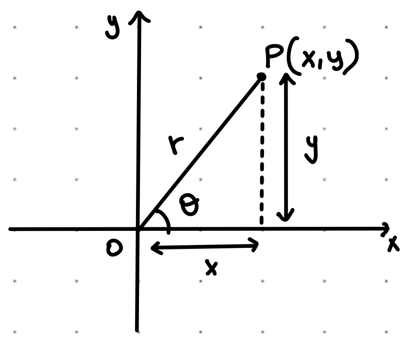
$$\int\int_{-\sqrt{1-y^2}}^{\sqrt{1+y^2}} e^{-x^2-y^2}\,dx$$

when you're in a situation like this, we have to use polar coordinates.

DOUBLE INTEGRAL IN POLAR COORDINATES

In cartesian coordinates the pt (x, y).
In the polar system, the coordinates are (r, θ) where r is the distance from origin and θ is the angle that OP makes with



$$\sin\theta = \frac{\text{opp.}}{\text{hyp}} = \frac{y}{r} \quad \cos\theta = \frac{x}{r}$$

CONVERSION FORMULA

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad \Bigg| \quad \begin{array}{l} x^2+y^2=r^2 \\ \tan\theta = \frac{y}{x} \\ \theta \in [0;2\pi] \end{array}$$

Now, convert the double integral from cartesian into polar coordinates;

$$\iint_R f(x,y)\,dA = \iint_R f(r\cos\theta, r\sin\theta)\, r\, dr\,d\theta$$

$dA = dxdy = dydx \rightsquigarrow$ $\boxed{dA = r\,dr\,d\theta}$

FINDING LIMITS OF INTEGRATION

To evaluate $\iint_R f(r,\theta)\,dA$ over a region R in polar coordinates, integrate 1st w.r.t. r and then w.r.t. θ, and take the following steps.

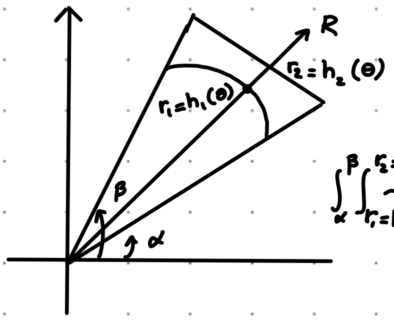
1. Sketch the Region
2. Find the limits of r and θ

* For r limits of integration:

Draw a ray L from the origin to the Region R and see where it enters the Region ($r = h_1(\theta)$) and where it leaves ($r = h_2(\theta)$)

* θ - limits:

The smallest and largest values of θ that bound R. ($\theta = \alpha$ and $\theta = \beta$)



$$\int_{\alpha}^{\beta}\int_{r_1=h_1(\theta)}^{r_2=h_2(\theta)} \sim r\,dr\,d\theta$$

EX1 : Evaluate $I = \iint_R e^{x^2+y^2}\,dx\,dy$ where R is the region bounded by the x-axis and the curve $y = \sqrt{1-x^2}$

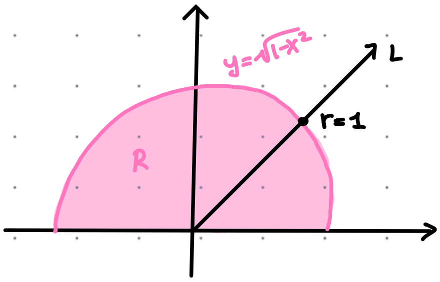
$$I = \iint e^{r^2} r\,dr\,d\theta$$

1. Sketch R

$y = \sqrt{1-x^2}$

$y^2 = 1 - x^2$

$x^2 + y^2 = 1$



2. Solve

Let $u = 2r\,dr$

$r\,dr = \frac{1}{2}du$

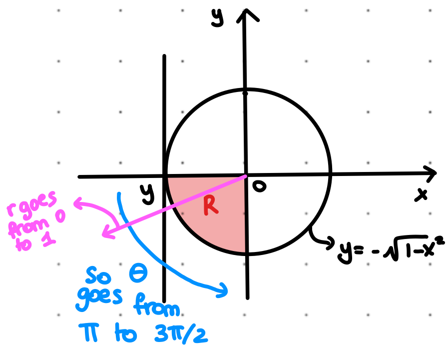
$$\boxed{I =} \int_0^{\pi}\int_0^{1} \frac{1}{2}e^{u}\,du\,d\theta = \int_0^{\pi} \frac{1}{2}e^{u}\Big]_0^{1}\,d\theta$$

$$= \int_0^{\pi} \frac{1}{2}\left[e-1\right]d\theta = \frac{1}{2}(e-1)\theta\Big]_0^{\pi} \boxed{= \frac{1}{2}\pi(e-1)}$$

EX2: Evaluate $I = \int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{0} \frac{1}{\sqrt{x^2+y^2}}\,dx\,dy$

- So we know, $I = \iint \frac{1}{r} r\,dr\,d\theta$ and that $-\sqrt{1-x^2} \leq y \leq 0$ and $-1 \leq x \leq 0$

$$I = \int_{\pi}^{\frac{3\pi}{2}}\int_0^{1} \frac{1}{r} r\,dr\,d\theta = \frac{\pi}{2}$$



EX3:

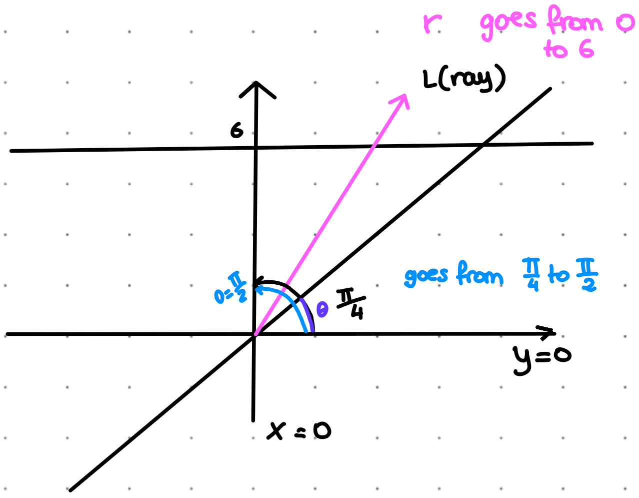
$$I = \int_0^6 \int_0^y x \, dx dy$$

$$\Downarrow$$

$$I = \iint r\cos\theta \, r dr d\theta$$

Bounds: $0 \le x \le y$

$0 \le y \le 6$



Coordinate of r from 0 to 6 we can't put 6 so

$y = r\sin\theta = 6$

$r = \frac{6}{\sin\theta} = 6\csc\theta$

$$I = \int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} \int_{r=0}^{r=6\cos\theta} r\cos\theta \, r dr d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos\theta}{\sin^3\theta} d\theta =$$

$u = \sin\theta$

$du = \cos\theta \, d\theta$

$u = \sin\frac{\pi}{2} = 1$

$$= \int_{\frac{\sqrt{2}}{2}}^{1} \frac{du}{u^3} = \int_{\frac{\sqrt{2}}{2}}^{1} u^{-3} du = 36$$

$u = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

EX4

$\iint_R 2xy \, dA$ where R is the portion of the Region between the circles of radius 2 and radius 5 centered at the origin that lies in 1^{st} quadrant.

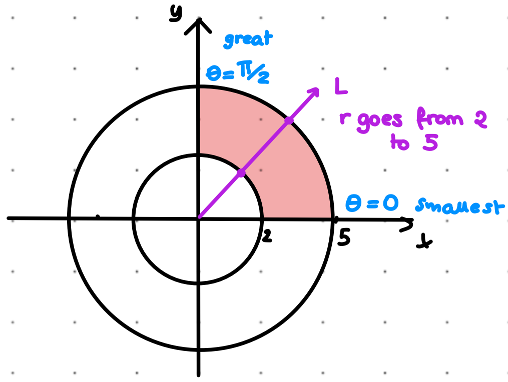
$$I = \iint 2(r\cos\theta)(r\sin\theta) \, r dr d\theta =$$

$$= \int_0^{\pi/2} \int_0^5 2(r\cos\theta)(r\sin\theta) \, r dr d\theta =$$

$$= \int_0^{\pi/2} \int_0^5 \sin 2\theta \left. \frac{r^4}{4} \right]_2^5 d\theta = \int_0^{\frac{\pi}{2}} \sin 2\theta \left(\frac{5^4}{4} - \frac{2^4}{4}\right) d\theta$$

u/o substitution

$$= -\frac{1}{2}\cos(2\theta) \Big]_0^{\pi/2} = \frac{609}{4}$$



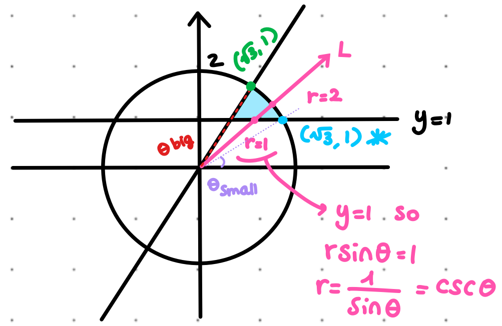
AREA IN POLAR COORDINATES

The area of a closed bounded region R is:

$$\iint_R dA = \iint_R dx dy = \iint_R dr d\theta$$

EX5: Find the area of the Region R in (xy) plane enclosed by the circle $x^2+y^2=4$ above the line $y=1$ and below the line $y=\sqrt{3}x$ (straight line)

$$I=\iint_{\csc\theta}^{2} r\,dr\,d\theta$$



A $y=1 \rightsquigarrow x^2+y^2=4$

$x^2=3$

$x=\pm\sqrt{3}$

$(\sqrt{3},1)$ * look in graph

$\tan\theta_1=1/\sqrt{3}$

$\theta=\frac{\pi}{6}$

Smallest angle θ

B $y=\sqrt{3}x \to x^2+y^2=4$

$x^2+3x^2=4$

$x=1$

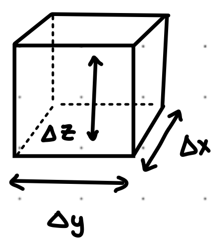
$B(1,\sqrt{3})$ * graph

$\tan\theta_2=\sqrt{3} \Rightarrow \theta=\frac{\pi}{3}$

Largest angle θ

TRIPLE INTEGRALS

Let $f(x,y,z)$ be a fct., be a fct. defined over a region D in space. Divide D into cubes.



The volume of each cube is $\Delta v=\Delta x\Delta y\Delta z$

$$S_n=\sum_{i=1}^{n}\underbrace{f(x_k,y_k,z_k)}_{\text{arbitrary pt.}}\Delta v_k \qquad \lim_{n\to\infty}S_n=\iiint f(x,y,z)\,dv$$

dv: dxdydz or dydxdz

Def: The volume of a closed bounded region D in the space is:

$$\boxed{\iiint_D dv}$$

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EX6: $\int_0^2\int_0^{\pi/3}\int_0^3 xy^2\cos(z)\,dy\,dz\,dx = \int_0^2\int_0^{\pi/2} x\cos(z)\underbrace{\left[\frac{y^3}{3}\right]_0^3}_{9} dz\,dx$

$= \int_0^2 \underbrace{9x\sin(z)\Big]_0^{\pi/2}}_{=1} = 9\int_0^2 x\,dx = 9\left[\frac{x^2}{2}\right]_0^2 = 9\left(\frac{4}{2}-0\right) = 18$

NOTE BOOK

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CALCULUS 2

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EX1: $\iiint_D 8\,xyz\; dv$ $D = [2,3] \times [1,2] \times [0,1]$

$2 \le x \le 3 \quad 1 \le y \le 2 \quad 0 \le z \le 1$

You can solve it in 6 ways ($dzdxdy$, $dzdydz$ etc)

- $\int_1^2 \int_2^3 \int_0^1 8\,xyz\; dzdxdy = \int_1^2 \int_2^3 \cancel{8}^{4} xy \left[\frac{z^2}{\cancel{2}_{1}}\right]_0^1 dxdy =$

$= \int_1^2 2x^2y \Big]_2^3 dy = \int_1^2 5 \times 2y\; dy = 5\left[y^2\right]_1^2 = 15$

[So when the boundaries all constants, you can choose 6 orders of "dv"]

CASE 1 : $h_1(x,y) \le z \le h_2(x,y)$

$$\underbrace{\iint_R}_{\text{projection on xy plane}} \left[\int_{z=h(x,y)}^{z=h_2(x,y)} f\; dz \right] \underbrace{dxdy}_{dA}$$

CASE 2: $h_1(y,z) \le x \le h_2(y,z)$

$$\underbrace{\iint_R}_{\text{projection on yz plane}} \left[\int_{x=h_1(y,z)}^{x=h_2(y,z)} f(x,y,z)\; dx \right] \underbrace{dydz}_{dA}$$

CASE 3 : $h_1(x,z) \le y \le h_2(x,z)$

$$\underbrace{\iint_R}_{\text{projection on xz plane}} \left[\int_{y=h_1(x,z)}^{y=h_2(x,z)} f(x,y,z)\; dy \right] \underbrace{dxdz}_{dA}$$

NOTE: when solving exercises, try to solve for dz first so that the projection is on the xy plane which is the most familiar plane for us.

EX2: Find the volume of the Region D enclosed by the surface $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$

$$V = \iiint dv = \iint_R \left[\int dz\right] dA$$

Step 1 Find z-limits of integration

$x^2+3y^2 \le z \le 8-x^2-y^2$ → to be sure if these are the bounds, pick a point in R and substitute. EX: P(0,0)

$0^2+3\cdot 0^2 \le z \le 8-0^2-0^2$ ✓

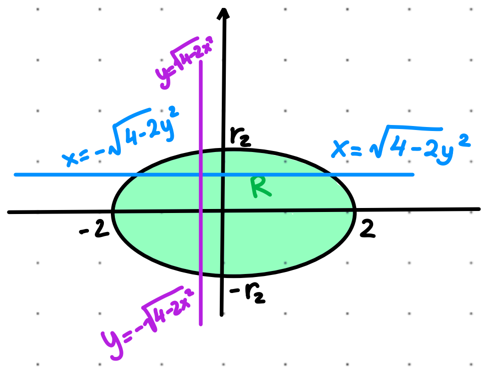
Step 2. Find "R" the projection of (xy) plane

- solve z = z

$x^2+3y^2 = 8-x^2-y^2$

$2x^2+4y^2 = 8$

$\boxed{\frac{x^2}{4}+\frac{y^2}{2}=1}$ ellipse



Find x → $\frac{x^2}{4}+\frac{y^2}{2}=1$

$x^2+2y^2=4$

$x^2=4-2y^2$

$x=\pm\sqrt{4-2y^2}$

so $V=\int_{-\sqrt{2}}^{\sqrt{2}}\int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}}\int_{x^2+3y^2}^{8-x^2-y^2} dz\,dx\,dy$

$V=\int_{-2}^{2}\int_{-\sqrt{4-2x^2}}^{\sqrt{4-2x^2}}\int_{x^2+3y^2}^{8-x^2-y^2} dz\,dy\,dx$

EX3: Evaluate $\iiint_D \sqrt{x^2+z^2}\,dV$ where D is the region bounded $y_1=x^2+z^2$ and $y_2=8-x^2-z^2$

Step 1 : Find y limits of integration

$x^2+z^2 \le y \le 8-x^2-z^2$

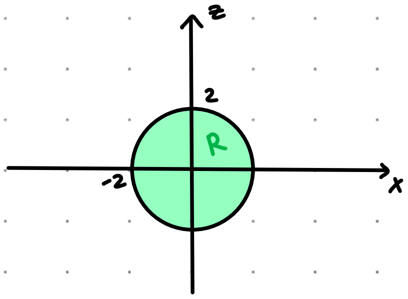
Step 2 : "R" the projection over x,z plane. Solve y=y

$x^2+z^2 = 8-x^2-z^2$

$2x^2+2z^2=8$

$\boxed{x^2+z^2=4}$

circle



$I=\iint_R\left[\left[\int_{y_1}^{y_2}\sqrt{x^2+z^2}\,dy\right]\right]dA$, $dA = dz\,dx$ or $dx\,dz$

↳ switch to polar coordinates as you're dealing with a circular R and it will be easier

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$$I = \iint_R \left[\left[\int_{y_1}^{y_2} \sqrt{x^2+z^2}\, dy\right]\right] dA \quad dA \to dxdz \quad dzdx \xrightarrow{\text{switch}}$$

$0 \le \theta \le 2\pi$
$0 \le r \le 2$
$x^2+z^2 = r^2$
$dxdz = r dr d\theta$

$y_1 = x^2+z^2 = r$
$y_2 = 8 - x^2 - z^2 = 8 - r^2$
$f(x,y,z) = \sqrt{x^2+z^2} = \sqrt{r^2} = r$

So

$$\boxed{I = \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=2}\int_{y_1=r}^{y_2=8-r^2} dy\, r dr d\theta}$$

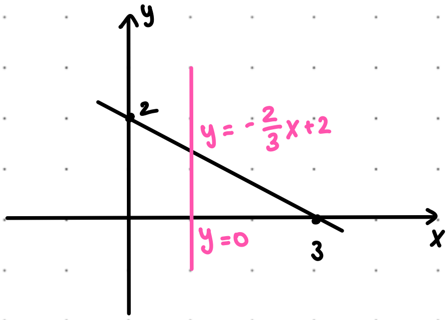
$$I = \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=2}\int_{y_1=r}^{y_2=8-r^2} dy\, r dr d\theta = [y]_{r^2}^{8-r^2} = (8-2r^2)r^2$$

EX4: Evaluate $\iiint_D 2x\, dv$ where D is the Region under the plane $2x+3y+z=6$ that lies in the 1^{st} octant $\to$ $x \ge 0$, $y \ge 0$, $z \ge 0$

$z = 6-2x-3y$ so z limits of integration $\to$ $0 \le z \le 6-2x-3y$

"P" projection over (xy) plane

$z=0 \Rightarrow 6-2x-3y=0 \Rightarrow \boxed{y = -\frac{2}{3}x+2}$



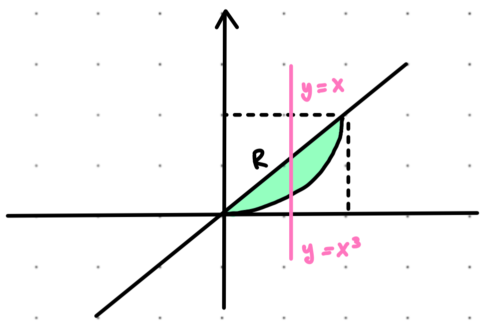
$$\Longrightarrow \iiint 2x\, dv = \int_{x=0}^{x=3}\int_{y=0}^{y=-\frac{2}{3}x+2}\int_{z=0}^{z=6-2x-3y} 2x\, dzdydx$$

EX4: D is given by $y=x^3$, $y=x$, $z=2x$, $z=0$, $x\ge 0$

Step 1 z - limits $0 \le z \le 2x$

Step 2 sketch R in (xy) plane

$x^3 \le y \le x$
$0 \le x \le 1$
dydx $\mapsto \int_0^1\int_{x^3}^{x}\int_0^{2x} f\, dzdydx$



2nd setup : y- limits → $x^3 \le y \le x$

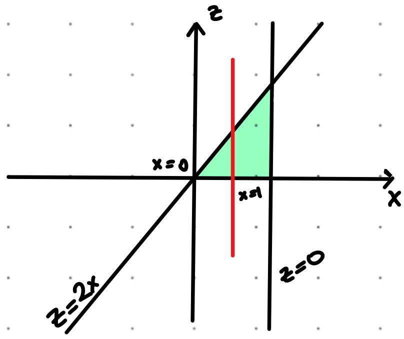
R on (xz) plane

$y = y \Rightarrow x^3 = x$

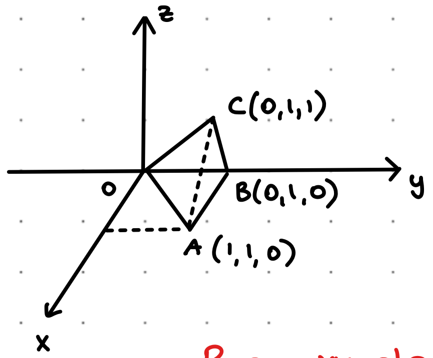
$x(x^2-1) = 0$

$x = 0 \quad x = \pm 1$

so $\int_0^1 \int_0^{2x} \int_{x^3}^{x} f \, dy\, dz\, dx \quad f = 2z$



EX5: Setup the limits of integration for evaluating the triple Integral of a fct. $f(x,y,z)$ over the tetrahedron D with vertices $O(0,0,0)$ $A(1,1,0)$ $B(0,1,0)$ $C(0,1,1)$



(OAC) ; $\vec{n} = \vec{OA} \times \vec{OC}$

$\vec{OM} \cdot (\vec{OA} \times \vec{OC}) = 0$

$x - y + z = 0 \rightarrow z = y - x$

so $0 \le z \le y - x$

R on xy plane

$z = 0 \Rightarrow y = x$

$y = 1$

$x = 0$

NOTE BOOK

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CALCULUS 2

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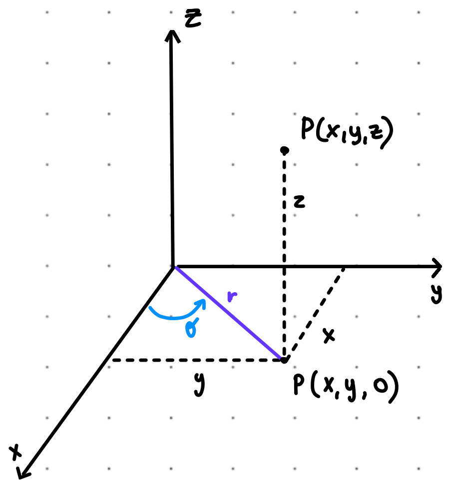
TRIPLE INTEGRALS IN CYLINDRICAL AND SPHERICAL COORDINATES

Integration in cylindrical Coordinates



Def: Cylindrical coordinates represent a pt. P by triples (r, θ, z)

1. r and θ are polar coordinates for the vertical projection of P on (xy) plane.
2. z is the rectangular vertical coordinate.

$$\Rightarrow \begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \qquad x^2+y^2=r^2 \qquad \tan\theta = \frac{y}{x}$$

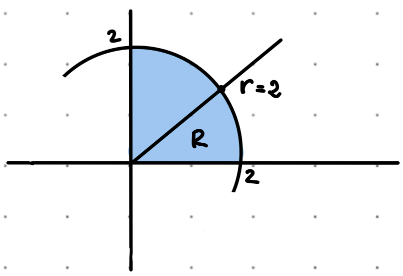
$$\boxed{\iiint_D f(x,y,z)\, dz\, dA = \int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2}\int_{z_1}^{z_2} r f(r\cos\theta, r\sin\theta, z)\, dz\, dr\, d\theta}$$

Ex: $\iiint_D y\, dV$ where D is the solid bounded by $z = 4-x^2-y^2$ in the first octant. $[dV = r\, dz\, dr\, d\theta]$

1. z-limits of integration $\rightarrow$ $0 \le z \le 4-x^2-y^2$

2. R projection on (xy) plane

$$4-x^2-y^2 = 0$$
$$x^2+y^2 = 4$$



$0 \le r \le 2$
$0 \le \theta \le \pi/2$

3. Solve

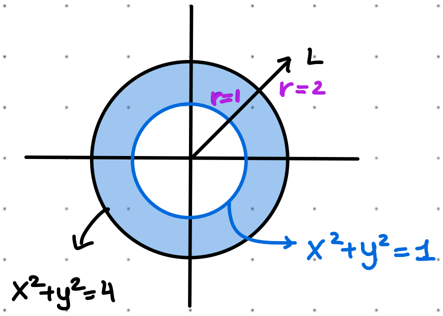
$$\int_0^{\pi/2}\int_0^2\int_0^{4-r^2} r(r\sin\theta)\, dz\, dr\, d\theta = \int_0^{\pi/2}\int_0^2 r^2\sin\theta(4-r^2)\, dr\, d\theta =$$

$$= \int_0^{\pi/2}\int_0^2 4r^2\sin\theta - r^4\sin\theta\,dr\,d\theta = \int_0^{\pi/2}\left[\frac{4}{3}r^3 - \frac{r^5}{5}\right]_0^2 \sin\theta\,d\theta =$$

$$= \int_0^{\pi/2} \frac{64}{15}\sin\theta\,d\theta = \frac{64}{15}$$

EX2: Evaluate $\iiint_D y\,dv$ where D is the region that lies below the plane $z = x+2$ and above (xy) plane and between the cylinders $x^2+y^2=1$ and $x^2+y^2=4$.

1. z-limits $\rightarrow$ $0 \le z \le x+2 \Rightarrow 0 \le z \le r\cos\theta + 2$
2. R projection on (xy) plane



$0 \le \theta \le 2\pi$
$1 \le r \le 2$

3. Solve

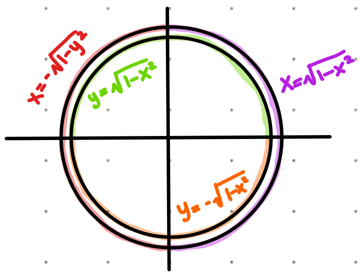
$$\iiint_D y\,dV = \int_0^{2\pi}\int_1^2\int_0^{r\cos\theta+2} r(r\sin\theta)\,dz\,dr\,d\theta = \int_0^{2\pi}\int_1^2 r^2\sin(r\cos+2)\,dr\,d\theta$$

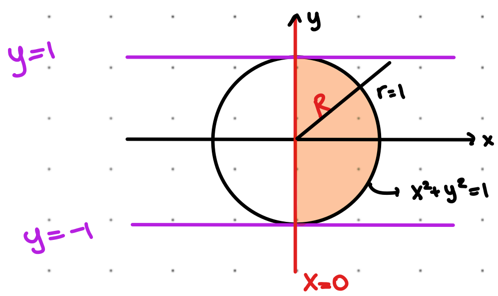
$$= \int_0^{2\pi}\int_1^2 \frac{1}{2}r^3\sin(2\theta) + 2r^2\sin\theta\,dr\,d\theta = \int_0^{2\pi}\left[\frac{1}{8}r^4\sin(2\theta) + \frac{2}{3}r^3\sin\theta\right]_1^2 d\theta$$

$$= \left[-\frac{15}{6}\cos 2\theta - \frac{14}{3}\cos\theta\right]_0^{2\pi}$$

EX3: Convert $I = \int_{-1}^{1}\int_0^{\sqrt{1-y^2}}\int_{x^2+y^2}^{\sqrt{x^2+y^2}} xyz\,dz\,dx\,dy$ into cylindrical coordinates

$x^2+y^2 \le z \le \sqrt{x^2+y^2}$
$0 \le x \le \sqrt{1-y^2}$
$-1 \le y \le 1$





$0 \le r \le 1$
$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
$r^2 \le z \le r$

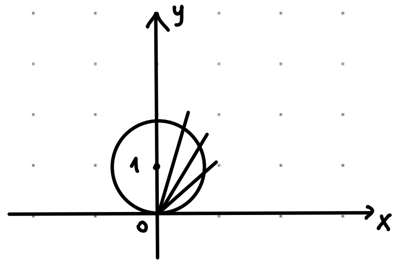
$$I = \int_{-\pi/2}^{\pi/2}\int_0^1\int_{r^2}^{r} r(r\cos\theta)(r\sin\theta)z\,dz\,dr\,d\theta$$

EX4: Find the limit of integration in cylindrical coordinates for integrating a fct. f over the region bounded below by $z=0$, $x^2+(y-1)=1$ and above $z=x^2+y^2$

$$\int_\theta \int_r \int_z f \, dz\,dr\,d\theta$$

1. z - limits $\rightarrow$ $0 \le z \le x^2+y^2 \Rightarrow 0 \le z \le r^2$
2. R projection on (xy) plane

$$x^2+(y-1)^2=1$$

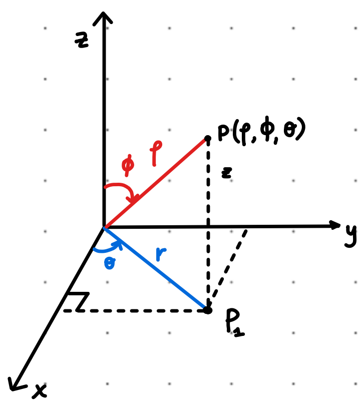


$$x^2+(y-1)^2=1$$
$$x^2+y^2-2y+1=1$$
$$\underbrace{x^2+y^2}_{r^2}-2y=0$$
$$r^2-2r\sin\theta=0$$
$$r(r-2\sin\theta)=0$$
$$r=0 \text{ or } r=2\sin\theta \Rightarrow 0 \le r \le 2\sin\theta$$
$$0 \le \theta \le \pi$$

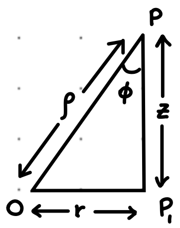
$$I=\int_0^\pi \int_0^{2\sin\theta} \int_0^{r^2} r f(r,\theta,z)\, dz\,dr\,d\theta$$

SPHERICAL COORDINATES AND INTEGRATION

Def: Spherical coordinates are determined by (ρ, ϕ, θ) with:
(ro)



1) ρ is the distance from origin to P ($\rho \ge 0$)
2) ϕ is the angle that $\overrightarrow{OP}$ makes with the positive z-axis $(0 \le \phi \le \pi)$
3) θ is the angle in polar coordinates $0 \le \theta \le 2\pi$ with $x=r\cos\theta$ and $y=r\sin\theta$



$\sin\phi = \frac{r}{\rho}$

$r=\rho\sin\theta$

$\Rightarrow$

$$\boxed{\begin{aligned} x&=\rho\sin\phi\cos\theta \\ y&=\rho\sin\phi\sin\theta \\ z&=\rho\cos\phi \\ r&=\rho\sin\theta \\ \rho^2&=x^2+y^2+z^2 \end{aligned}}$$

$$\iiint f(x,y,z)\,dV = \iiint f(r,\phi,\theta)\,\rho^2 \sin\theta \, d\rho\, d\phi\, d\theta$$

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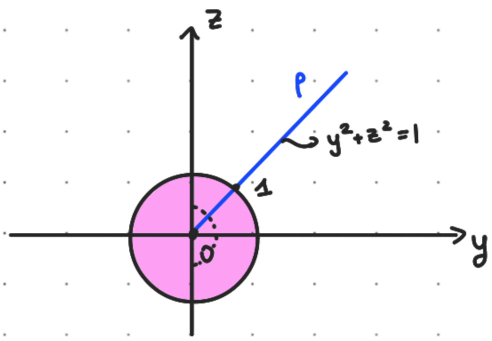
$$\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases} \qquad \begin{array}{l} 0 \le \phi \le \pi \\ 0 \le \theta \le 2\pi \end{array}$$

$$x^2 + y^2 + z^2 = \rho^2 \qquad dxdydz = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

EX1: Evaluate $\iiint_D \sqrt{x^2+y^2+z^2}\,dV$ using spherical coordinates where D is the sphere $x^2+y^2+z^2 \le 1$.

$x=0$



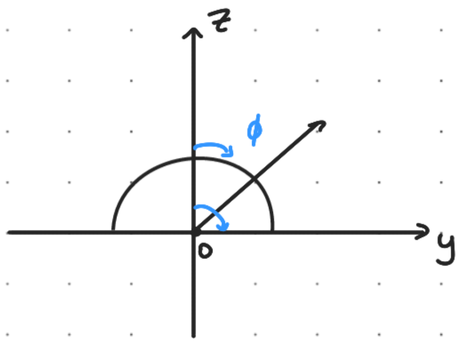
$$0 \le \rho \le 1$$
$$0 \le \phi \le \pi$$
$$0 \le \theta \le 2\pi$$

So $\int_0^{2\pi}\int_0^{\pi}\int_0^{1} \rho \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi}\int_0^{\pi} \frac{1}{4}\sin\phi \, d\phi \, d\theta =$

$= \int_0^{2\pi} \frac{1}{4}\left[-\cos\phi\right]_0^{\pi} d\theta = \boxed{\pi}$

EX2: $\iiint_D y\,dV$ where D is the region bounded between $z = \sqrt{1-x^2-y^2}$ and (xy) plane.

Set $x=0$ $\leadsto$ $z = \sqrt{1-y^2}$



θ $z=0$

$\Rightarrow \sqrt{1-x^2-y^2} = 0 \rightarrow x^2+y^2 = 1$

Since it's circle, $0 \le \theta \le 2\pi$

so $\iiint_D y \, dv = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{1} \rho\sin\phi \sin\phi \, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \boxed{0}$

$\frac{1-\cos(2\phi)}{2}$

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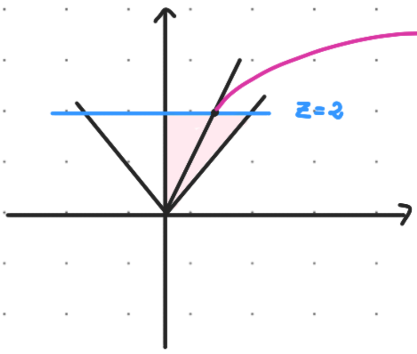
EX3: Find the volume of the solid bounded by $z=2$ and $z=\sqrt{x^2+y^2}$

$$V=\iiint_D dv$$

$x=0$ $\quad \rho, \phi$

$$z=\sqrt{y^2}=|y|$$



it leaves at $z=2$ but we can't say $0\le\rho\le 2$ so we have to express $z=2$ in terms of ρ

$z=2 \longrightarrow \rho\cos\phi=2$

$\rho=\frac{2}{\cos\phi}=2\sec\phi$

so $\left[0\le\rho\le 2\sec\phi,\ 0\le\phi\le\frac{\pi}{4},\ \underbrace{0\le\theta\le 2\pi}\right]$

we have $z=2$ and $z=\sqrt{x^2+y^2}$

so $\sqrt{x^2+y^2}=2$

$x^2+y^2=4$

(circle)

so $\quad V=\iiint_D dv=\int_0^{2\pi}\int_0^{\pi/4}\int_0^{2\sec\phi}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$

EX4: Find the volume of the solid between $\rho=\cos\phi$ and the hemisphere $\rho=2$; $z\ge 0$

$\rho=2 \to \boxed{x^2+y^2+z^2=4}$

$\to \rho=\cos\phi$, $\quad \rho^2=\rho\cos\phi$

express in terms of x,y,z

$x^2+y^2+z^2=z \quad \Rightarrow \quad \boxed{x^2+y^2+\left(z-\frac{1}{2}\right)^2=\frac{1}{4}}$

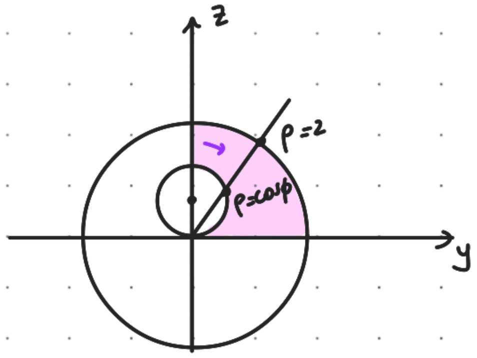
method of completing the square

$z^2-z=0$

Set $x=0$

$y^2+z^2=4$

$y^2+\left(z-\frac{1}{2}\right)^2=\frac{1}{4}$



Change of variable

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_c^d f(u)\,du$$

let $u = g(x)$
$du = g'(x)\,dx$

EX1: $R \to x^2 + \frac{y^2}{36} = 1$ (ellipse)

use the change $\begin{cases} x = \frac{u}{2} \\ y = 3v \end{cases}$

$$\left(\frac{u}{2}\right)^2 + \frac{(3v)^2}{36} = 1 \implies \boxed{u^2 + v^2 = 4}$$

DEF : $x = g(u,v)$, $y = h(u,v)$

$\implies$ The Jacobian $J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$ do determinant

we want to integrate $f(x,y)$ over a region R
under $x = g(u,v)$ $y = h(u,v)$ $\downarrow$ transformed
S

$$\iint_R f(x,y)\,dx\,dy = \iint f(g(u,v), h(u,v))\,|J(u,v)|\,\underbrace{du\,dv}_{d\bar{A}}$$

$$dA = |J(u,v)|\,d\bar{A}$$

EX2: Show that $dA = r\,dr\,d\theta$ in polar coordinates

$\begin{cases} x = \cos\theta \\ y = r\sin\theta \end{cases}$ $dA = |J(r,\theta)|\,dr\,d\theta = r\,dr\,d\theta$

$$J(r,\theta) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} =$$

$$= r\cos^2\theta + r\sin^2\theta = r$$

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$$R \downarrow S \qquad \begin{cases} x = g(u,v,w) \\ y = h(u,v,w) \\ z = k(u,v,w) \end{cases} \qquad \underbrace{dV}_{dxdydz} = |J(u,v,w)|\, \underbrace{dv}_{dudvdw}$$

$$\iiint_R f(x,y,z)\, dv \longrightarrow \iiint_S f(g,h,k)\, |J(u,v,w)|\, dv$$

where
$$J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

EX3: Show that $dv = \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$ in spherical coordinates

$$\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\theta \end{cases}$$

$$J(\rho,\phi,\theta) = \begin{array}{c} \begin{array}{ccc} \rho & \phi & \theta \end{array} \\ \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix} \end{array}$$

$$= \cos\phi\Big(\overbrace{\rho^2\sin\phi\cos\phi\cos^2\theta + \rho^2\sin\phi\cos\phi\sin^2\theta}^{\rho^2\sin\phi\cos\phi}\Big) + \rho\sin\phi\Big(\underbrace{\rho\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta}_{\rho\sin^2\theta}\Big) =$$

$$= \rho^2\sin\phi(\cos^2\phi + \sin^2\phi) =$$

NOTE BOOK

Rokshana Ahmed

INTEGRATION IN VECTOR FIELDS

VECTOR VALUED FUNCTION

The coordinates of a point A moving in space are given by $A(x(t), y(t), z(t))$. The position of A is determined by a vector called the position vector denoted by :

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$$

The point A traces a certain curve as it moves during time t. This curve is called the path of A and represented by $\vec{r}(t)$, where the parametric equations of this curves are :

$$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases} \qquad a \le t \le b$$

EX1 : Describe the vector function given by $r(t) = \cos t\,\vec{i} + \sin t\,\vec{j}$ for $0 \le t \le 2\pi$.

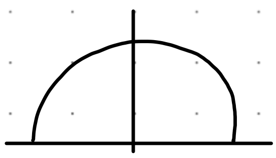
$$\begin{cases} x = \cos t \\ y = \sin t \end{cases} \qquad \text{But } \cos^2 t + \sin^2 t = 1$$

$$x^2 + y^2 = 1 \quad \text{and} \quad 0 \le t \le 2\pi$$

So circle - centred at $(0,0)$ of $r=1$

EX2 : $r(t) = \cos t\,\vec{i} + \sin t\,\vec{j}$ with $0 \le t \le \pi$

$$x^2 + y^2 = 1$$
$$0 \le t \le \pi$$



semi-circle

EX3 : $r(t) = 2\cos t\,\vec{i} + \sin t\,\vec{j}$ with $0 \le t \le 2\pi$

$$\begin{cases} x(t) = 2\cos t \implies \cos t = \frac{x}{2} \\ y(t) = \sin t \end{cases}$$

CALCULUS 2

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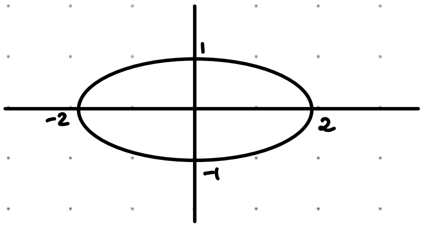
$$\cos^2 t + \sin^2 t = 1$$

$$\Downarrow$$

$$\left(\frac{x}{2}\right)^2 + y = 1$$

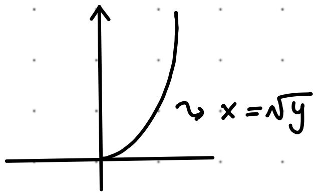
$$0 \le t \le 2\pi$$

$$\Longrightarrow$$



EX4 : $\vec{r}(t) = t\vec{i} + t^2\vec{j}$ $\quad t \geq 0$

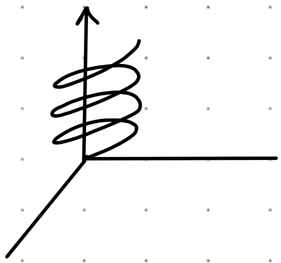
$$\begin{cases} x = t \\ y = t^2 \end{cases} \Longrightarrow \text{since } x = t \Rightarrow y = x^2 \text{ and } x \geq 0$$



EX5 $\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + t\,\vec{k}$

$$\begin{cases} x = \cos t \\ y = \sin t \end{cases} \Longrightarrow x^2 + y^2 = 1$$

$$z = t$$



DEF (VELOCITY, ACCELERATION)

If $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$ defines the position vector of a point M at any time t, then the velocity of the particle M at time t is :

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}$$

The velocity is the direction vector of the tangent line at any time t. The speed is the magnitude of the velocity given by :

$$|\vec{v}| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

The acceleration is the derivative of the velocity

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} = \frac{d^2x}{dt}\vec{i} + \frac{d^2y}{dt}\vec{j} + \frac{d^2z}{dt}\vec{k}$$

CALCULUS 2

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EX1: Find the equation of the tangent line $t_o(C)$ at $t = \pi/2$, where C is determined by
$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$

Recall $$(L) = \begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$$ where $(x_0, y_0, z_0) = M$
(a, b, c) = direction vector

$$\left.\begin{aligned} \vec{r}(\pi/2) &= \cos\frac{\pi}{2}\vec{i} + \sin\frac{\pi}{2}\vec{j} + \frac{\pi}{2}\vec{k} \\ &= 0\vec{i} + \vec{j} + \frac{\pi}{2}\vec{k} \end{aligned}\right\} \quad M(\overset{x_0}{0}, \overset{y_0}{1}, \overset{z_0}{\pi/2})$$

$$\vec{v}(\pi/2) = -\vec{i} + \vec{k} \quad \Rightarrow \quad \vec{v}(\overset{a}{-1}, \overset{b}{0}, \overset{c}{1})$$

$$\begin{cases} x = -t \\ y = 1 \\ z = \pi/2 + t \end{cases}$$

LENGTH OF A CURVE

The length of a curve $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$ with $a \leq t \leq b$ is:

$$L = \int_a^b |\vec{v}|\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

EX1: Find the length of the curve (C) given by $\vec{r}(t) = \cos(t)\vec{i} + \sin(t)\vec{j}$ with $0 \leq t \leq 2\pi$

$$\vec{v}(t) = \frac{dr}{dt} = -\sin t\,\vec{i} + \cos t\,\vec{j}$$

$$\|\vec{v}\| = \sqrt{(-\sin t)^2 + (\cos t)^2} \quad \Rightarrow \quad L = \int_0^{2\pi} 1\, dt = 2\pi$$

UNIT TANGENT VECTOR

Since the velocity is the tangent to the curve $\vec{r}(t)$, the unit vector tangent is:

$$\vec{T} = \frac{\vec{v}}{\|\vec{v}\|}$$

VECTOR FIELDS

Def: A vector field is a function $\vec{F}$ that assigns to each point (x,y,z) in its domain a vector given by:

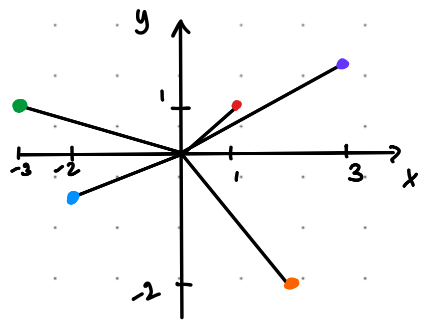
$$\vec{F}(x,y,z) = M(x,y,z)\vec{i} + N(x,y,z)\vec{j} + R(x,y,z)\vec{k}$$

* the field is continuous if the component functions M, N, R are continuous.

* It is differentiable if each of the components M, N and R are differentiable.

EX1 $F(x,y) = -x\vec{i} - y\vec{j}$

- $(1,1) \rightarrow F(1,1) = -\vec{i} - \vec{j}$
- $(3,2) \rightarrow F(3,2) = -3\vec{i} - 2\vec{j}$
- $(-3,1) \rightarrow F(-3,1) = 3\vec{i} - \vec{j}$
- $(2,-3) \rightarrow F(2,-3) = -2\vec{i} + 3\vec{j}$
- $(-2,-1) \rightarrow F(-2,-1) = 2\vec{i} + \vec{j}$



Recall that for a given fct. $f(x,y,z)$, the gradient vector is defined by

$$\nabla f = (f_x, f_y, f_z)$$

This is a vector field.

Def: A vector field is called a conservative vector field if there exists a fct. f such that:

$$\vec{F} = \nabla f$$

If F is conservative vector field, then the fct. f is called potential fct. of F.

EX1 : $f(x,y) = xy$

$\nabla f = y\vec{i} + x\vec{j}$

$[F(x,y) = y\vec{i} + x\vec{j}]$ is a conservative vector field with the potential fct $f(x,y) = xy$

LINE INTEGRALS

We integrate the function $f(x,y)$ by taking the points (x,y) that lie on a curve C. This integral is called Line or Curve integral, and is given by:

$$\int_C f(x,y)\,ds \qquad \int_C f(x,y,z)\,ds$$

Assume (C) is given by:

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$$

$\Rightarrow ds = \|\vec{v}\|\,dt$

$\Rightarrow \int_C f\,ds = \int f(x(t), y(t), z(t)) \cdot |\vec{v}|\,dt$

HOW TO EVALUATE A LINE INTEGRAL

1) parametrize (C) $\rightsquigarrow$ $\vec{r}(t) = ?$
2) Find $\vec{v}$ $\rightsquigarrow$ $\|\vec{v}\|$

$$\int_C f\,ds = \int f(x(t), y(t), z(t)) \cdot |\vec{v}|\,dt$$

WELL-KNOWN PARAMETRIC EQUATIONS

Ellipse $\longrightarrow$ $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $\quad \begin{cases} x = a\cos t \\ y = b\sin t \end{cases}$ $\quad 0 \le t \le 2\pi$

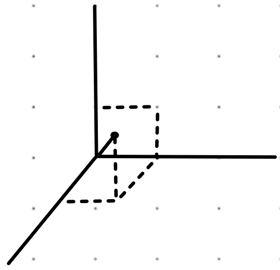
Circle $\rightarrow$ $x^2 + y^2 = r^2$ $\quad \begin{cases} x = r\cos t \\ y = r\sin t \end{cases}$ $\quad 0 \le t \le 2\pi$

function $\rightarrow$ $y = f(x)$ $\quad \begin{cases} x = t \\ y = f(t) \end{cases}$

NOTE BOOK

Rokshana Ahmed

EX1 Integrate $f(x,y,z) = x - 3y^2 + z$ over the line segment joining the point $O(0,0,0)$ to $A(1,1,1)$
origin



$$\vec{OA} = (\overset{a}{1}, \overset{b}{1}, \overset{c}{1})$$

$$\begin{cases} x - x_0 = at \\ y - y_0 = bt \\ z - z_0 = ct \end{cases} \quad a \le t \le b \implies \begin{cases} x = t \\ y = t \\ z = t \end{cases} \quad 0 \le t \le 1$$

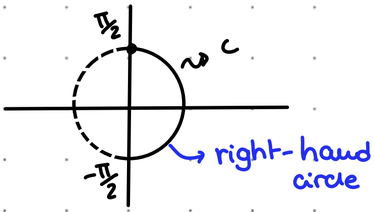
$$\vec{r}(t) = t\vec{i} + t\vec{j} + t\vec{k} \qquad \vec{v} = \frac{d\vec{r}}{dt} = \vec{i} + \vec{j} + \vec{k}$$

$$\|\vec{v}\| = \sqrt{1+1+1} = \sqrt{3}$$

$$f(x(t), y(t), z(t)) = f(t,t,t) = t - 3t^2 + t = 2t - 3t^2$$

$$\int_C f\, ds = \int_0^1 (2t - 3t^2)\underbrace{\sqrt{3}\,dt}_{ds} = \sqrt{3}\left[t^2 - t^3\right]_0^1 = 0$$

EX2 : Evaluate $\int_C \underbrace{xy^4}_{f(x(t),\,y(t))} ds$ where C is the right hand circle $x^2 + y^2 = 16$



$$\begin{cases} x = 4\cos t \\ y = 4\sin t \end{cases} \qquad -\frac{\pi}{2} \le t \le \frac{\pi}{2}$$

$$\vec{r}(t) = 4\cos t\,\vec{i} + 4\sin t\,\vec{j}$$

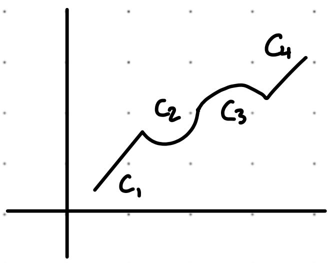
$$\vec{v} = \frac{d\vec{r}}{dt} = -4\sin t\,\vec{i} + 4\cos t\,\vec{j}$$

$$\|\vec{v}\| = 4$$

$$\int_C f\, ds = \int_{-\pi/2}^{\pi/2} 4^5 \cos t \sin^4 t \cdot \underbrace{4\, ds}_{ds} = \frac{8192}{5}$$

use $u = \sin t$

INTEGRAL OVER A PIECEWISE CURVE



$$\int_C f\, ds = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}$$

because $C = C_1 \cup C_2 \cup C_3 \cup C_4$

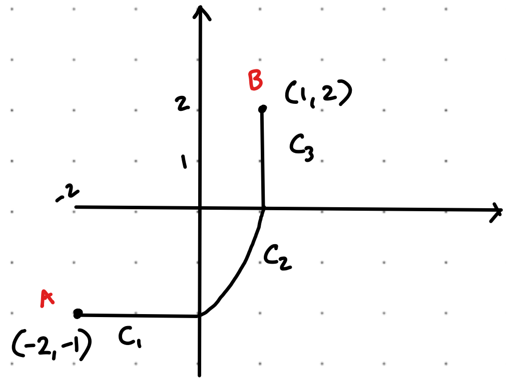
CALCULUS 2

Date: 19 / 12 / 2022

EX1 : $\int_C 4x^3\,ds$ where C is the curve shown below

$C_1 \to y = -1$
$C_2 \to y = x^3 - 1$
$C_3 \to x = 1$

$C = C_1 \cup C_2 \cup C_3$



① C_1 $\quad y = -1$

$$\begin{cases} x = t \\ y = -1 \end{cases} \quad -2 \le t \le 0$$

$\vec{r_1}(t) = t\vec{i} - \vec{j}$
$\vec{V_1}(t) = \vec{i}$
$\|\vec{V_1}\| = \sqrt{1} = 1$

$$\int_{C_1} 4x^3\,ds = \int_{-2}^{0} 4t^3 \cdot 1\,dt$$

$$= \left[\frac{4t^4}{4}\right]_{-2}^{0} = -2^4$$

② C_2 $\quad y = x^3 - 1$

$$\begin{cases} x = t \\ y = t^3 - 1 \end{cases} \quad 0 \le t \le 1$$

$\vec{r_2}(t) = t\vec{i} + (t^3 - 1)\vec{j}$
$\vec{V_2}(t) = \vec{i} + 3t^2\vec{j}$
$\|\vec{V_2}\| = \sqrt{1^2 + 9t^4}$

$$\int_{C_2} 4x^3\,ds = \int 4t^3 \underbrace{\sqrt{1+9t}}_{\text{use substitution}}\,dt$$

$$= \frac{2}{27}\left(\sqrt{10} - 1\right)$$

③ C_3 $\quad y = -1$

$$\begin{cases} x = 1 \\ y = t \end{cases} \quad 0 \le t \le 2$$

$\vec{r_3}(t) = \vec{i} + t\vec{j}$
$\vec{V_3}(t) = \vec{j}$
$\|\vec{V_3}\| = 1$

$$\int_{C_3} 4x^3\,ds = \int_0^2 4\,dt$$

$$= 4[t]_0^2 = 8$$

④ NOTE: Check by repeating same integral but (C) is line segment from $\underbrace{(-2,-1)}_{B}$ to $\underbrace{(1,2)}_{A}$

$\to \vec{AB} = (\overset{a}{3}, \overset{b}{3})$

$$(C) : \begin{cases} x = 3t - 2 \\ y = 3t - 1 \end{cases} \quad 0 \le t \le 1$$

$\vec{r}(t) = (3t-2)\vec{i} + (3t-1)\vec{j}$

$\vec{V} = \frac{d\vec{r}}{dt} = 3\vec{i} + 3\vec{j}$

$\|\vec{V}\| = \sqrt{3^2 + 3^2} = 3\sqrt{2}$

$$\int_C 4x^3\,dx = \int_0^1 4(3t-2)^3\, 3\sqrt{2}\,dt$$

$$= -15\sqrt{2}$$

LINE INTEGRALS OF VECTOR FIELD

Consider the vector field

$$\vec{F}(x,y,z) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$$

and the curve given by:

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k} \qquad a \leq t \leq b$$

Then, the line integral of $\vec{F}$ along C is:

$$\int_c \vec{F}\, d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \overbrace{\frac{d\vec{r}}{dt}}^{r'(t)} dt$$

STEPS:

1. Find $\vec{r}(t)$ of C and evaluate $\vec{F}$ in terms of the parametrized curve $\vec{F}(\vec{r}(t))$
2. Find $\frac{d\vec{r}}{dt}$
3. Evaluate $\int_c F \cdot dr$

EX1: Evaluate $\int_c \vec{F} d\vec{r}$ where $F(x,y,z) = 8x^2yz\vec{i} + 5z\vec{j} - 4xy\vec{k}$ and C is given by: $\vec{r}(t) = t\vec{i} + t^2\vec{j} + t^3\vec{k}$ with $0 \leq t \leq 1$

$F(t, t^2, t^3) = 8t^7\vec{i} + 5t^3\vec{j} - 4t^2\vec{k}$ $\qquad \vec{r}'(t) = \frac{dr}{dt} = \vec{i} + 2t\vec{j} + 3t^2\vec{k}$

$$\vec{F}\, \vec{r}'(t) = (8t^7)(1) + (5t^3)(2t) + (-4t^2)(3t^2) = \boxed{8t^7 + 10t^4 - 12t^5}$$

$$\int_c \vec{F}\, d\vec{r} = \int_0^1 (8t^7 + 10t^4 - 12t^5)\, dt = \left[t^8 + 2t^5 - 2t^6\right]_0^1 = 1$$

EX2: Evaluate $\int_c \vec{F} d\vec{r}$ where: $F(x,y,z) = xz\vec{i} - yz\vec{k}$ and C is the line segment from $\underbrace{(-1,2,0)}_{A}$ to $\underbrace{(3,0,1)}_{B}$

$\vec{AB}(4,-2,1)$

↓

(c) $\begin{cases} x = 4t - 1 \\ y = -2t + 2 \\ z = t \end{cases} \quad 0 \leq t \leq 1$

$$\vec{r}(t) = (4t-1)\vec{i} + (-2t+2)\vec{j} + t\vec{k}$$
$$\vec{r}'(t) = 4\vec{i} - 2\vec{j} + \vec{k}$$
$$\vec{F}(r(t)) = (4t-1)t\vec{i} - (-2t+2)t\vec{k}$$
$$\vec{F}(r(t)) \cdot \vec{r}'(t) = 18t^2 - 6t$$

$$\int_C \vec{F}\,d\vec{r} = \int_0^1 (18t^2 - 6t)\,dt = \left[6t^3 - 3t^2\right]_0^1 = 3$$

THM: Suppose that C is a smooth curve given by $\vec{r}(t)$, $a \leq t \leq b$. Assume f is a fct. whose gradient ∇f is continuous on C. Then:

$$\int_C \nabla f\,d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

Where $\vec{r}(a)$ is the initial point on C, while $\vec{r}(b)$ is the final point.

PROOF: $\int_C \nabla f\,d\vec{r} = \int_a^b \nabla f(\vec{r}(t)) \cdot r'(t)\,dt =$

$$= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial x}\frac{dz}{dt}\right) dt$$

chain rule → $\int_a^b \frac{d}{dt}\left(f(\vec{r}(t)\right) dt = \vec{f}(r(b)) - \vec{f}(r(a))$

EX1 Evaluate $\int_C \nabla f\,dr$ where $f(x,y,z) = \cos(\pi x) + \sin(\pi y) - xyz$ and C is any path that starts at $\underbrace{(1, \tfrac{1}{2}, 2)}_{\vec{r}(a)}$ and ends at $\underbrace{(2,1,-1)}_{\vec{r}(b)}$

$$\int_C \nabla f\,d\vec{r} = f(2,1,-1) - f(1, \tfrac{1}{2}, 2) =$$
$$= \cos(2\pi) + \sin(\pi) - 2(1)(-1) - [\cos(\pi) + \sin(\tfrac{\pi}{2}) - 1(\tfrac{1}{2})(2)] =$$
$$= 1 + 0 + 2 - [-1 + 1 - 1] = 4$$

PATH INDEPENDENCE, CONSERVATIVE FIELDS

Chap 16.3

Def:
Assume $\vec{F}$ is continuous vector field in D.
If $\int_{C_1} \vec{F} d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$ for any two paths C_1 and C_2 in D with some initial and final points then $\int_C \vec{F} d\vec{r}$ is independent of path

REMARKS:

1. $\int \vec{\nabla f}\, d\vec{r}$ is independent of path

   This is a direct consequence of the Fundamental Theorem of Calculus for line integrals because it tells us that we only need the initial and final point of the path

2. If $\vec{F}$ is a conservative vector field then $\int_C \vec{F} d\vec{r}$ is independent of path

   → F is conservative $\Rightarrow$ $\vec{F} = \nabla \vec{f}$ (f potential fct.)
   → $\int \vec{F} d\vec{r} = \int_C \nabla f\, dr$ which is path independent

3. If $\int_C \vec{F} d\vec{r} = 0$ for every closed path C, then $\int F d\vec{r}$ is independent of path.

TEST FOR CONSERVATIVE VECTOR FIELDS

Let $\vec{F} = M(x,y,z)\vec{i} + N(x,y,z)\vec{j} + P(x,y,z)\vec{k}$ be a field on a connected domain whose component fcts have continuous 1^{st} partial derivatives.

Then $\vec{F}$ is conservative if:

1. $\dfrac{\partial P}{\partial y} = \dfrac{\partial N}{\partial z}$
2. $\dfrac{\partial N}{\partial x} = \dfrac{\partial M}{\partial y}$
3. $\dfrac{\partial M}{\partial z} = \dfrac{\partial P}{\partial x}$

PROVE 1

$$\vec{F} = M\vec{i} + N\vec{j} + P\vec{k} = \Delta f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

$\Rightarrow$ $P = \dfrac{\partial f}{\partial z}$

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) = \frac{\partial^2 f}{\partial y \partial z} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial N}{\partial z}$$

NOTE BOOK

Rokshana Ahmed

CALCULUS 2

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RECAP : $\vec{F} = \nabla f$ $\quad \vec{F} = M(x,y,z)\vec{i} + N(x,y,z)\vec{j} + P(x,y,z)\vec{k} =$

$$= \vec{\nabla} f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z} \qquad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} \qquad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial y}$$

$$\left\{ \frac{\partial f}{\partial x} = M, \quad \frac{\partial f}{\partial y} = N, \quad \frac{\partial f}{\partial z} = P \right.$$

EX1 : $\vec{F} = \underbrace{(e^x \cos y + yz)}_{M}\vec{i} + \underbrace{(xz - e^x \sin y)}_{N}\vec{j} + \underbrace{(xy + z)}_{P}\vec{k}$

$\frac{\partial P}{\partial y} = x$ $\qquad \frac{\partial N}{\partial x} = z - e^x \sin y$ $\qquad \frac{\partial M}{\partial z} = y$

$\downarrow$ check $\qquad \downarrow$ check $\qquad \downarrow$ check

$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}$ ✓ $\qquad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$ ✓ $\qquad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}$ ✓

So $\vec{F}$ is conservative

$$\vec{F} = (e^x \cos y + yz)\vec{i} + (xz - e^x \sin y)\vec{j} + (xy + z)\vec{k} =$$

$$= \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

$$\left\{ \begin{array}{ll} (1) & \frac{\partial f}{\partial x} = e^x \cos y + yz \\ (2) & \frac{\partial f}{\partial y} = xz - e^x \sin y \\ (3) & \frac{\partial f}{\partial z} = xy + z \end{array} \right.$$

$f(x,y,z) = \int (e^x \cos y + yz)\, dx =$

$= e^x \cos y + xyz + g(y,z)$ (*)

(*) $\frac{\partial f}{\partial y} = -e^x \sin y + xz + \frac{\partial g}{\partial y} = xz - e^x \sin y$

$\Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow$ g is a fct, of z only

(*) $\Rightarrow \boxed{f(x,y,z) = e^x \cos y + xyz + h(z)}$ (**)

⊛⊛ $\frac{\partial f}{\partial z} = xy + h'(z) \overset{③}{=} xy + z$

$h'(z) = z \quad \Rightarrow \quad \boxed{h(z) = \frac{z^2}{2} + c}$

$\Rightarrow f(x,y,z) = e^x \cos y + xyz + \frac{z^2}{2} + c$

potential function

EX2 : $\vec{F} = \underbrace{(2x^3y^4 + x)}_{M}\vec{i} + \underbrace{(2x^4y^3 + y)}_{N}\vec{j}$

$\frac{\partial N}{\partial x} = 8x^3y^3 \Rightarrow \frac{\partial M}{\partial y} = 8x^3y^3$

$\vec{F}$ is conservative

$\vec{F} = (2x^3y^4 + x)\vec{i} + (2x^4y^3 + y)\vec{j} \overset{\nabla f}{\Rightarrow} \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$ *

$* \Rightarrow \begin{cases} ① \ \frac{\partial f}{\partial x} = 2x^3y^4 + x \\ ② \ \frac{\partial f}{\partial y} = 2x^4y^3 + y \end{cases}$

$\Rightarrow$ using ① $f(x,y) = \int (2x^3y^4 + x)\, dx^3 = \frac{2}{4}x^4y^4 + \frac{x^2}{2} + g(y)$

$\frac{\partial f}{\partial y} = 2x^4y^3 + g'(y) \overset{②}{=\!=} 2x^4y^3 + y$