NOTE BOOK

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$f(x,y,z)$ is a fct. of 3 variables and $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$ is a unit vector. Then:

$$\nabla f = f_x\vec{i} + f_y\vec{j} + f_z\vec{k} \quad \text{and}$$
$$D_{\vec{u}}f = \nabla f \cdot \vec{u} = f_x u_1 + f_y u_2 + f_z u_3$$

Ex: Find the derivative of $f(x,y,z) = x^3 - xy^2 - z$ at $P_0(1,1,0)$

(a) in the direction of $\vec{v} = 2\vec{i} - 3\vec{j} + 6\vec{k}$

(b) In what direction does f change most rapidly at P and what are the rates of change in this direction.

1. $\vec{v} = (2,-3,6)$ $\quad \|\vec{v}\| = \sqrt{2^2+(-3)^2+6^2} = \sqrt{49} = 7$

$$\Rightarrow \vec{u} = \frac{1}{\|\vec{v}\|} v = \left(\frac{2}{7}; -\frac{3}{7}; \frac{6}{7}\right)$$

$f_x = 3x^2 - y^2$ $\quad f_y = -2xy$ $\quad f_z = -1$

$f_x|_{(1,1,0)} = 2$ $\quad f_y|_{(1,1,0)} = -2$

$$\nabla f(1,1,0) = (2,2,-1)$$
$$Df(1,1,0) = \nabla f \cdot \vec{u} = (2,2,-1)\cdot\left(\frac{2}{7}, \frac{3}{7}, \frac{6}{7}\right) = \frac{4}{7} + \frac{6}{7} - \frac{6}{7} = \frac{4}{7}$$

2. The fct. f $\nearrow$ most rapidly in the direction of $\vec{\nabla f} = 2\vec{i} - 2\vec{j} - \vec{k}$ and the rate of change in this direction is

$$\|\nabla f\| = \sqrt{2^2 + (-2)^2 + (-1)^2} = 3$$

$\swarrow$ most rapidly in the direction of

$$-\vec{\nabla f} = -2\vec{i} + 2\vec{j} + \vec{k} \qquad -\|\nabla f\| = -3$$

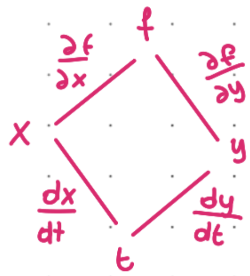
TANGENT PLANES AND NORMAL LINE

$z = f(x,y)$ $\qquad$ $Z = C$ plane $\parallel$ to (xy) plane

intersect a surface

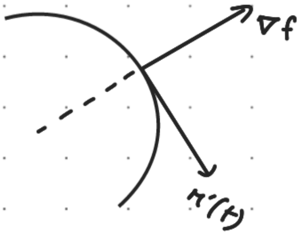
$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$ $\quad f(x(t), y(t)) = c$

$$\frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} = 0$$



$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \cdot (x'(t), y'(t)) = 0$$

$$\nabla f \cdot r'(t) = 0$$



If $\vec{r}(t) = g(t)\vec{i} + h(t)\vec{j} + k(t)\vec{k}$ is a curve on the level surface $f(x,y,z) = c$ of a differentiable fct. f then $f(g(t), h(t), k(t)) = c$

$$\frac{\partial f}{\partial x}\frac{dg}{dt} + \frac{\partial f}{\partial y}\frac{dh}{dt} + \frac{\partial f}{\partial z}\frac{dk}{dt} = 0$$

diff. both sides

$$\underbrace{\left(\frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}\right)}_{\nabla f} \cdot \underbrace{\left(\frac{dg}{dt}\vec{i} + \frac{dh}{dt}\vec{j} + \frac{dk}{dt}\vec{k}\right)}_{\frac{dr}{dt}} = 0$$

At any point along the curve ∇f is orthogonal to the curves velocity vector.

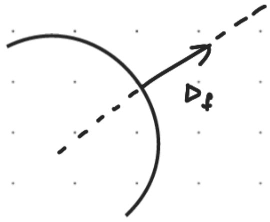
> **DEF:** The tangent plane at the point (x_0, y_0, z_0) on the level surface $f(x,y,z) = c$ of a differentiable fct. f is the plane through P_0 and normal to $\nabla f|_{P_0}$

↓

The tangent plane to $f(x,y,z) = c$ at $P_0(x_0, y_0, z_0)$ is:

$$f_x|_{P_0}(x - x_0) + f_y|_{P_0}(y - y_0) + f_z|_{P_0}(z - z_0) = 0$$

where $(f_x, f_y, f_z)_{P_0} = \nabla f|_{P_0}$



(x_0, y_0, z_0) $\vec{v} = (v_1, v_2, v_3)$

$$\begin{cases} x - x_0 = t v_1 \\ y - y_0 = t v_2 \\ z - z_0 = t v_3 \end{cases}$$

Normal line to $f(x,y,z) = c$ at (x_0, y_0, z_0) is :

$$\begin{cases} x = x_0 + f_x^{(P_0)} t \\ y = y_0 + f_y^{P_0} t \\ z = z_0 + f_z^{P_0} t \end{cases}$$

$$\frac{x - x_0}{f_x|_{P_0}} = \frac{y - y_0}{f_y|_{P_0}} = \frac{z - z_0}{f_z|_{P_0}}$$

Cartesian equation

EX:

Find the equation of the tangent plane and normal line to the surface $f(x,y,z) = x^2 + y^2 + z - 9 = 0$ at the point $P_0(1,2,4)$

$$\left.\begin{aligned} f_x &= 2x \\ f_y &= 2y \\ f_z &= 1 \end{aligned}\right\} \nabla f = (2x, 2y, 1)$$

$\underbrace{\nabla f(1,2,4)}_{\text{Normal vector}} = (2, 4, 1) \Rightarrow 2(x-1) + 4(y-2) + (z-4) = 0$

$\boxed{2x + 4y + z = 14}$ eq. of tangent plane

$$\frac{x-1}{2} = \frac{y-2}{4} = \frac{z-4}{1} \rightarrow \begin{cases} x = 2t + 1 \\ y = 2 + 4t \\ z = 4 + t \end{cases}$$

cartesian eq. of normal line

EX2

$\underbrace{xz^2 + yx^2 + y^2 - 2x + 3y}_{f(x,y,z) = 6} = -6$ in $P(-2, 1, 3)$

$$\left.\begin{aligned} f_x &= z^2 + 2yx - 2 \\ f_y &= x^2 + 2y + 3 \\ f_z &= 2zx \end{aligned}\right\} \nabla f = (z^2 + 2yx - 2)\vec{i} + (x^2 + 2y + 3)\vec{j} + 2zx\ \vec{k}$$

Normal vector to family of level surfaces

$\nabla f|_{(-2,1,3)} = 3\vec{i} + 9\vec{j} - 12\vec{k}$

$3(x+2) + 9(y-1) - 12(z-3) = 0 \rightarrow \boxed{x + 3y - 4z = -11}$ tangent plane eq.

CALCULUS 2

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Normal line $\longrightarrow$ $\dfrac{x-2}{3} = \dfrac{y-1}{9} = \dfrac{z+3}{-12}$

ex3 $f(x,y,z) = \tan^{-1}\left(\dfrac{y}{x}\right) - z$ a $P\left(1,1,\dfrac{\pi}{4}\right)$

① $f_x = \dfrac{-\frac{y}{x^2}}{1+\left(\frac{y}{x}\right)^2} = \dfrac{-y}{x^2+y^2}$

$f_y = \dfrac{\frac{1}{x}}{1+\left(\frac{y}{x}\right)^2} = \dfrac{x}{x^2+y^2}$

$f_z = -1$

② $\nabla f = \dfrac{-y}{x^2+y^2}\vec{i} + \dfrac{x}{x^2+y^2}\vec{j} - \vec{K}$

③ $\nabla f|\left(1,1,\frac{\pi}{4}\right) = -\frac{1}{2}\vec{i} + \frac{1}{2}\vec{j} - \vec{K}$

$-\frac{1}{2}(x-1) + \frac{1}{2}(y-1) - 1\left(z-\frac{\pi}{4}\right) = 0$

tangent plane eq.

ex4 $xyz = -4$ $P(2,-1,2)$

$\rightarrow$ let $f(x,y,z) = xyz$

$\nabla f(x,y,z) = yz\vec{i} + xz\vec{j} + xy\vec{K}$ $\longrightarrow$ this is normal vector to family of level surfaces for $f(x,y,z)$

$\rightarrow$ $\nabla f(2,-1,2) = -2\vec{i} + 4\vec{j} - 2\vec{K}$

Tangent plane

$-2(x-2) + 4(y+1) - 2(z-2) = 0$

$\Leftrightarrow$ $\boxed{x - 2y + z = 6}$

$\longrightarrow$ The pt. $P(2,-1,2)$ gives a specific normal $(-2,4,-2)$ to a specific level surface $xyz = -4$

Normal line

$\dfrac{x-2}{-2} = \dfrac{y+1}{4} = \dfrac{z-2}{-2}$

ex5: $z = x\cos y - ye^x$ at $(0,0,0)$

Let $f(x,y,z) = x\cos y - ye^x - z = 0$

$\nabla f = (\cos y - ye^x)\vec{i} + (-x\sin y - e^x)\vec{j} - \vec{K}$

$\nabla f(0,0,0) = \vec{i} - \vec{j} - \vec{K}$

Tg plane: $1(x-0) + (-1)(y-0) - 1(z-0) = 0$

$x - y - z = 0$

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SECTION 14.7. EXTREME VALUES AND SADDLE POINT

Def: Let $f(x,y)$ be defined on a Region R containing the pt (a,b) then :

1. $f(x,y)$ has relative min. at pt (a,b) if $f(a,b) \leq f(x,y)$ $\forall$ (x,y) in an open disk centered at (a,b)
2. $f(a,b)$ is a local max. if $(a,b) \geq f(x,y)$ $\forall$ points in an open disk centered at (a,b)
3. f has an absolute max at (a,b) if $f(a,b) \geq f(x,y)$ for all (x,y) in R
4. f has absolute min at (a,b) if $f(a,b) \leq f(x,y)$ $\forall (x,y)$ in R