NOTE BOOK

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CALCULUS 2

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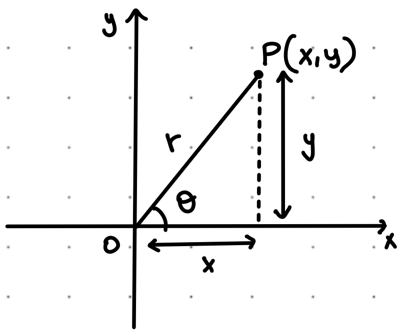
$$\iint_{-\sqrt{1-y^2}}^{\sqrt{1+y^2}} e^{-x^2-y^2}\,dx$$

when you're in a situation like this, we have to use polar coordinates.

DOUBLE INTEGRAL IN POLAR COORDINATES

In cartesian coordinates the pt (x,y).
In the polar system, the coordinates are (r,θ) where r is the distance from origin and θ is the angle that OP makes with



CONVERSION FORMULA

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \begin{array}{l} x^2+y^2=r^2 \\ \tan\theta = \dfrac{y}{x} \\ \theta \in [0;2\pi] \end{array}$$

$$\sin\theta = \frac{\text{opp.}}{\text{hyp}} = \frac{y}{r} \qquad \cos\theta = \frac{x}{r}$$

Now, convert the double integral from cartesian into polar coordinates;

$$\iint_R f(x,y)\,dA = \iint_R f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

$dA = dxdy = dydx \rightsquigarrow$ $\boxed{dA = r\,dr\,d\theta}$

FINDING LIMITS OF INTEGRATION

To evaluate $\iint_R f(r,\theta)\,dA$ over a region R in polar coordinates, integrate 1st w.r.t. r and then w.r.t. θ, and take the following steps.

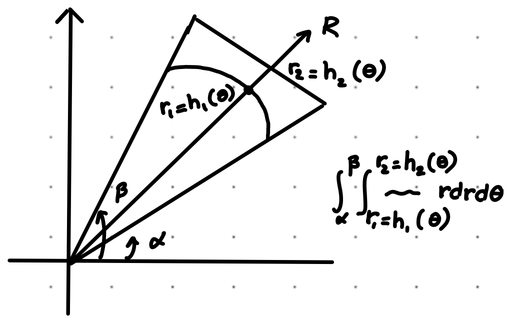
1. Sketch the Region
2. Find the limits of r and θ

* For r limits of integration:

Draw a ray L from the origin to the Region R and see where it enters the Region $(r = h_1(\theta))$ and where it leaves $(r = h_2(\theta))$

* θ-limits:

The smallest and largest values of θ that bound R. $(\theta = \alpha$ and $\theta = \beta)$

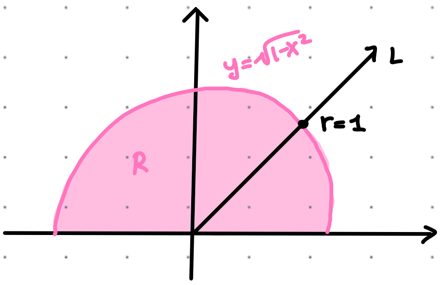


EX1: Evaluate $I = \iint_R e^{x^2+y^2}\,dx\,dy$ where R is the region bounded by the x-axis and the curve $y = \sqrt{1-x^2}$

$$I = \iint e^{r^2} r\,dr\,d\theta$$

1. Sketch R

$$y = \sqrt{1-x^2}$$
$$y^2 = 1 - x^2$$
$$x^2 + y^2 = 1$$



2. Solve

Let $u = 2r\,dr$

$r\,dr = \frac{1}{2}du$

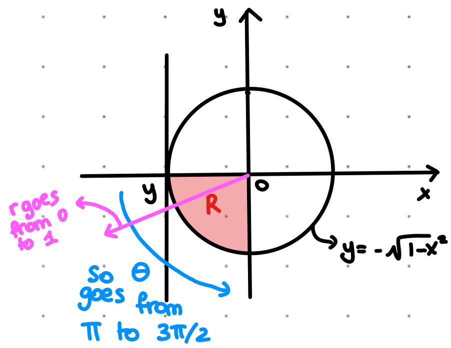
$$I = \int_0^\pi \int_0^1 \frac{1}{2} e^u\,du\,d\theta = \int_0^\pi \frac{1}{2} e^u \Big]_0^1 \,d\theta$$

$$= \int_0^\pi \frac{1}{2}\left[e - 1\right] d\theta = \frac{1}{2}(e-1)\theta \Big]_0^\pi = \frac{1}{2}\pi(e-1)$$

EX2: Evaluate $I = \int_{-1}^{0} \int_{-\sqrt{1-x^2}}^{0} \frac{1}{\sqrt{x^2+y^2}}\,dx\,dy$

- So we know, $I = \iint \frac{1}{r} r\,dr\,d\theta$ and that $-\sqrt{1-x^2} \leq y \leq 0$ and $-1 \leq x \leq 0$

$$I = \int_\pi^{\frac{3\pi}{2}} \int_0^1 \frac{1}{r} r\,dr\,d\theta = \frac{\pi}{2}$$



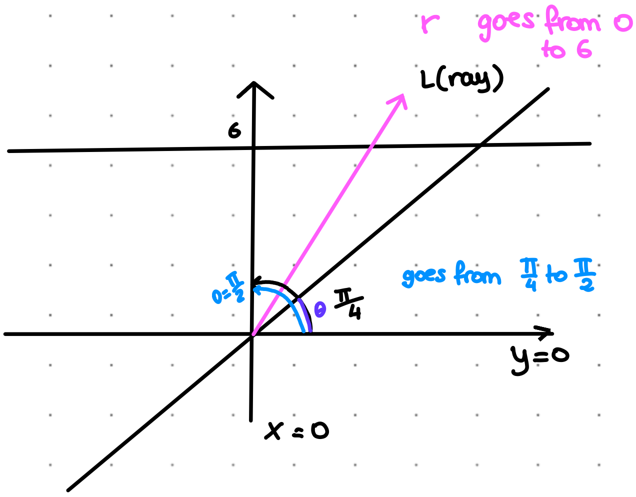
EX3:

$$I = \int_0^6 \int_0^y x\,dx\,dy$$

$$\Downarrow$$

$$I = \iint r\cos\theta\, r\,dr\,d\theta$$

Bounds: $0 \le x \le y$
$0 \le y \le 6$



Coordinate of r from 0 to 6
we can't put 6 so

$y = r\sin\theta = 6$
$r = \frac{6}{\sin\theta} = 6\csc\theta$

$$I = \int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} \int_{r=0}^{r=6\csc\theta} r\cos\theta\, r\,dr\,d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos\theta}{\sin^3\theta}\,d\theta =$$

$u = \sin\theta$
$du = \cos\theta\, d\theta$

$u = \sin\frac{\pi}{2} = 1$

$$= \int_{\frac{\sqrt{2}}{2}}^{1} \frac{du}{u^3} = \int_{\frac{\sqrt{2}}{2}}^{1} u^{-3}\,du = 36$$

$u = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

EX4

$\iint_R 2xy\,dA$ where R is the portion of the Region between the circles of radius 2 and radius 5 centered at the origin that lies in 1^{st} quadrant.

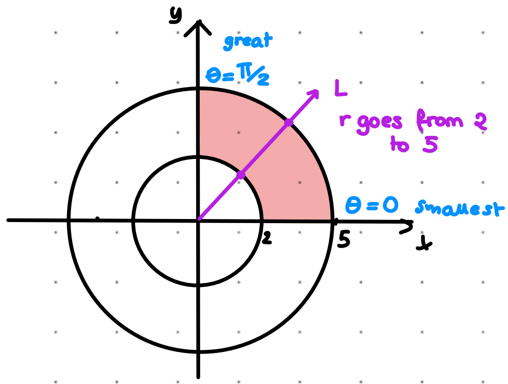
$$I = \iint 2(r\cos\theta)(r\sin\theta)\,r\,dr\,d\theta =$$

$$= \int_0^{\pi/2} \int_0^5 2(r\cos\theta)(r\sin\theta)\,r\,dr\,d\theta =$$

$$= \int_0^{\pi/2} \int_0^5 \sin 2\theta \left.\frac{r^4}{4}\right]_2^5 d\theta = \int_0^{\frac{\pi}{2}} \sin 2\theta \left(\frac{5^4}{4} - \frac{2^4}{4}\right) d\theta$$

do substitution

$$= -\frac{1}{2}\cos(2\theta)\Big]_0^{\pi/2} = \frac{609}{4}$$



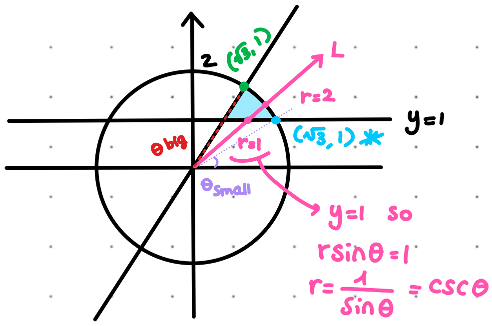
AREA IN POLAR COORDINATES

The area of a closed bounded region R is:

$$\iint_R dA = \iint_R dx\,dy = \iint_R dr\,d\theta$$

EX5: Find the area of the Region R in (xy) plane enclosed by the circle $x^2+y^2=4$ above the line $y=1$ and below the line $y=\sqrt{3}x$ (straight line)

$$I=\int\int_{\csc\theta}^{2} r\,dr\,d\theta$$



A

$y=1 \rightsquigarrow x^2+y^2=4$

$x^2=3$

$x=\pm\sqrt{3}$

$(\sqrt{3},1)$ * look in graph

$\tan\theta_1=1/\sqrt{3}$

$\theta=\frac{\pi}{6}$

→ Smallest angle θ

B

$y=\sqrt{3}x \rightarrow x^2+y^2=4$

$x^2+3x^2=4$

$x=1$

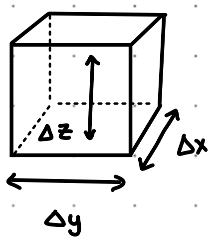
$B(1,\sqrt{3})$ * graph

$\tan\theta_2=\sqrt{3} \Rightarrow \theta=\frac{\pi}{3}$

→ Largest angle θ

TRIPLE INTEGRALS

Let $f(x,y,z)$ be a fct., be a fct. defined over a region D in space. Divide D into cubes.



The volume of each cube is $\Delta V=\Delta x\Delta y\Delta z$

$$S_n=\sum_{i=1}^{n} \underbrace{f(x_k,y_k,z_k)}_{\text{arbitrary pt.}}\Delta V_k \qquad \lim_{n\to\infty} S_n=\iiint f(x,y,z)\,\underbrace{dV}_{dxdydz \text{ or } dydxdz}$$

Def: The volume of a closed bounded region D in the space is:

$$\boxed{\iiint_D dV}$$

EX6:

$$\int_0^2 \int_0^{\pi/3} \int_0^3 xy^2 \cos(z)\, dy\, dz\, dx = \int_0^2 \int_0^{\pi/2} x\cos(z) \underbrace{\left[\frac{y^3}{3}\right]_0^3}_{9} dz\, dx$$

$$= \int_0^2 \underbrace{9x\sin(z)\Big]_0^{\pi/2}}_{=1} = 9\int_0^2 x\, dx = 9\left[\frac{x^2}{2}\right]_0^2 = 9\left(\frac{4}{2} - 0\right) = 18$$