NOTE BOOK

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CALCULUS 2

Date: 14 / 11 / 2022

RECAP

1. critical points $\Rightarrow$ $f_x = 0 = f_y = 0$
2. Find discriminant $D = f_{xx} f_{yy} - (f_{xy})^2$

3 cases

- $D > 0$
  - $f_{xx} > 0$ → local min
  - $f_{xx} < 0$ → local max
- $D < 0$ → saddle point
- $D = 0$ → Inconclusive

HOME EXERCISES :

1. $f(x,y) = x^2y + y^2 + xy$

Step 1 :

$f_x = 2xy + y = 0 \Rightarrow y(2x+1) = 0 \Rightarrow y = 0 \text{ or } x = -\frac{1}{2}$

$f_y = x^2 + 2y + x = 0$

for $y = 0$;

$$\left.\begin{array}{l} x^2 + x = 0 \\ x(x+1) = 0 \\ x = 0 \text{ or } x = -1 \end{array}\right\} \Rightarrow \text{crit. pts. } (0,0) \text{ and } (-1,0)$$

for $x = -\frac{1}{2}$;

$$\left.\begin{array}{l} \left(-\frac{1}{2}\right)^2 + 2y - \frac{1}{2} = 0 \\ 2y = \frac{1}{4} \Rightarrow y = \frac{1}{8} \end{array}\right\} \text{crit. pt. } \left(-\frac{1}{2}, \frac{1}{8}\right)$$

Step 2 :

$f_{xx} = 2y$

$f_{yy} = 2$

$f_{xy} = 2x + 1$

$D = (2y)(2) - (2x+1)^2$

$D(0,0) = -1 < 0 \Rightarrow (0,0)$ is a saddle point

$D(-1;0) = -1 < 0 \Rightarrow (-1,0)$ 〃 〃 〃 〃

$D(-\frac{1}{2}; \frac{1}{8}) = \frac{1}{2} > 0$ and $f_{yy} > c$

$\Rightarrow (-\frac{1}{2}; \frac{1}{8})$ local min pt.

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(2) $f(x,y) = e^{-x^2-y^2}$

Step 1

$f_x = -2x\,e^{-x^2-y^2} = 0$

$f_y = -2y\,e^{-x^2-y^2} = 0$

e is always positive
so it means
$-2x=0$ and $-2y=0$
so $x=0$ and $y=0$

$(0,0)$ is the critical point

Step 2

$\boxed{f_{xx} =} -2e^{-x^2-y^2} + (-2x)(-2xe^{-x^2-y^2}) = -2e^{-x^2-y^2}(1-2x^2)$

[apply product rule since x is involved in both quantities
$\underbrace{-2x}\,\underbrace{e^{-x^2-y^2}}$]

$\boxed{f_{yy} =} -2e^{-x^2-y^2}(1-2y^2)$

$\boxed{f_{xy} =} 4xy\,e^{-x^2-y^2}$

$$D = (-2e^{-x^2-y^2})^2(1-2x^2)(1-2y^2) - [4xye^{-x^2-y^2}]^2$$

$D(0,0) = 4 > 0$ and $f_{xx}(0,0) = -2 < 0 \Rightarrow (0,0)$ local max

THM: EXTREME VALUE THEOREM

If $f(x,y)$ is continuous fct., on a closed, bounded set S in $\mathbb{R}^2$.
Then f has an absolute max and an absolute min in S.

Def:

(1) A region D in $\mathbb{R}^2$ is called bounded if it lies in a disk of finite Radius.
In other words, a region will be bounded if its finite.

(2) A region in $\mathbb{R}^2$ is called closed if it includes all its boundary points. A region is called open if it consists only of interior points.

(3) A pt. (a,b) in a Region D in $\mathbb{R}^2$ is an interior point of D if it is the center of a disk that lies entirely in D.
A pt. (a,b) is a boundary pt. if every disk centered at (a,b) Contains pts.

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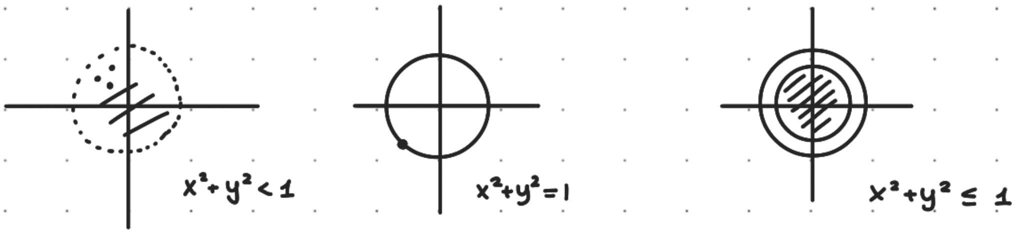
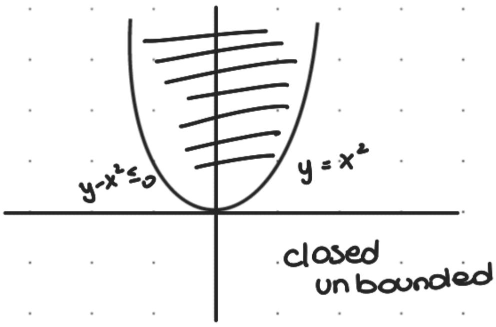


EX $f(x,y) = \sqrt{y-x^2}$

D: $y-x^2 \geq 0$

$y \geq x^2$



Steps :

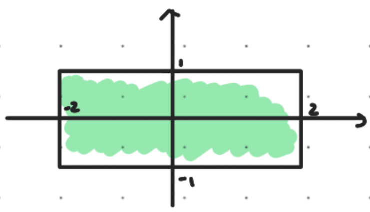
1) List the interior points of R where f may have local max, local min and evaluate f at these points.

2) List the boundary points of R where f has local max and min and evaluate f at these points. This usually involves calculus 1 approach for this work.

3) The largest and smallest values found in the 1st two steps are the absolute max and min of the fct.

EX1 Let $f(x,y) = x^2 + xy + y^2$

on $R = \left\{ (x,y) \,\middle|\, \begin{array}{l} -2 \le x \le 2 \\ -1 \le y \le 1 \end{array} \right\}$

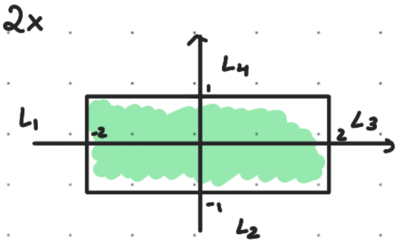


Step 1 : $f_x = 2x + y = 0 \implies y = -2x$

$f_y = 2y + x = 0$

$-4x + x = 0 \implies x = 0$

$f(0,0) = 0$



4 boundaries

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The boundary of the Rectangle is given by :

- L_1: $x=-2$ $\quad -1 \le y \le 1$
- L_2: $x=2$ $\quad -1 \le y \le 1$
- L_3: $y=-1$ $\quad -2 \le x \le 2$
- L_4: $y=1$ $\quad -2 \le x \le 2$

Along L_1

$x=-2 \quad = g(y) = f(-2,y) = 4-2y+y^2$

$g'(y) = -2+2y = 0 \implies \boxed{y=1} \longrightarrow$ Crit. Pt. $(-2,1)$

$\boxed{f(-2,1) = 3}$

Endpoints are $(-2;1)$ and $(-2,-1)$ because $-1 \le y \le 1$. so $(-2,1)$ and $(-2;-1)$

$(-2;1) \xrightarrow{f(-2;1)} 3 \qquad (-2,-1) \xrightarrow{f(-2;-1)} 7$

Along L_2

$x=2 \implies f(2,y) = 4+2y+y^2$

$f'(2,y) = 2+2y = 0 \implies y=-1$

C.P. $(2,-1) \implies f(2,-1) = 3$

E.P. $(2,-1) \longrightarrow 3$

$(2,1) \longrightarrow 7$

Along L_3

$y=-1 \implies f(x,-1) = x^2-x+1$

$f'(x,-1) = 2x-1 = 0 \implies x = \frac{1}{2}$

C.P. $(\frac{1}{2},-1) \implies f(\frac{1}{2},-1) = \frac{3}{4}$

E.P. $(-2;-1) \longrightarrow 7$

$(2;-1) \longrightarrow 7$

Along L_4

C.P. $(-\frac{1}{2};1) \longrightarrow \frac{3}{4}$

E.P. $(-2,1) \longrightarrow 3$

$(2,1) \longrightarrow 7$

So $f(0,0)=0$ is a global min value and $(-2,-1)$ and $(2,1)$ are global maxs

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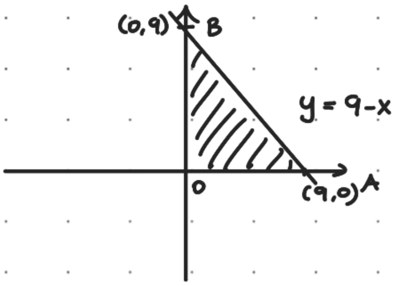
Ex2: Find the absolute max or min of $f(x,y) = 2+2x+2y-x^2-y^2$ on the Δ region in the 1^{st} quadrant bounded by $x=0$, $y=0$, and $y=9-x$



Sol.

$f_x = 2-2x = 0 \Rightarrow x=1$

$f_y = 2-2y = 0 \Rightarrow y=1$

$\boxed{f(1,1) = 4}$

Along OA

$y=0 \quad 0 \le x \le 9$

$f(x,0) = 2+2x-x^2 \rightarrow f'(x) = 2-2x = 0 \downarrow x=1$

C.P. : $(1,0) \longrightarrow 3$

E.P. : $(0,0) \longrightarrow 2$

$(9,0) \longrightarrow -61$

Along OB

$x=0, \quad 0 \le y \le 9$

$f(0,y) = 2y - y^2 + 2$

C.P : $(0,1) \rightarrow 3$

E.P. : $(0,0) \rightarrow 2$

$(0,9) \rightarrow -61$

Along BA

$y = 9-x$

$f(x,y) = f(x, 9-x)$

$\left(\frac{9}{2}, \frac{9}{2}\right) \rightarrow -\frac{41}{2}$

$\boxed{\begin{array}{l} g.\ \text{min}\ (0,9) \rightarrow -61 \\ g.\ \text{max}\ (1,1) \rightarrow 4 \end{array}}$