NOTE BOOK

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DIRECTIONAL DERIVATIVE AND GRADIENT

- The partial derivatives of f w.r.t. x give the slope of the tangent line to the intersection of the graph of f with the plane $y = y_0$ at (x_0, y_0) in the direction of x.

  Also $\frac{\partial f}{\partial y}$ gives the slope to the tan. line in the y-direction

- We can generalize the partial derivatives to calculate the slope in any direction. The result is called <u>the Directional Derivative</u>. It also allows us to find the rate of change of f if we allow both x and y to change simultaneously

- The first step in taking a directional derivative is to specify the direction. One way to specify a direction is with a vector $\vec{u} = (u_1, u_2)$ that points in the direction in which we want to compute the slope. For simplicity, we'll assume that $\vec{u}$ is a unit vector.

- Sometimes the direction of changing x and y is given as an angle θ. The unit vector that points in this direction is given by $\vec{u} = (\cos\theta \sin\theta)$

DEF: The derivative of f in the direction of the unit vector $\vec{u} = u_1\vec{i} + u_2\vec{j}$ is called the directional derivative and denoted by $D_{\vec{u}} f(x,y)$ and given by:

$$\boxed{D_{\vec{u}} f(x,y) = \lim_{h \to 0} \frac{f(x+u_1h,\ y+u_2h) - f(x,y)}{h}}$$

→ if $\vec{u} = \vec{i} = (1,0)$ $\quad D_i = \lim_{h \to 0} \frac{f(x+h,\ y) - f(x,y)}{h} = f_x$

→ if $\vec{u} = j = (0,1)$ $\quad D_j = \lim_{h \to 0} \frac{f(x,\ y+h) - f(x,y)}{h} = f_y$

The partial derivative $f_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$ and $f_y(x_0, y_0)$ are the directional derivative of f at $P_0 = (x_0, y_0)$ in the $\vec{i}$ and $\vec{j}$ directions.

EX1: Using the definition, find the derivative of $f(x,y) = x^2 + xy$ at $P_0(1,2)$ in the direction of the unit vector $\vec{u} = \frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}$

Sol: $D_{\vec{u}} f(x,y) = \lim_{h \to 0} \frac{f(x+u_1 h,\, y+u_2 h) - f(x,y)}{h}$

$$= \lim_{h\to 0} \frac{f(x+\frac{1}{\sqrt{2}}h,\, y+\frac{1}{\sqrt{2}}h) - f(x,y)}{h} =$$

$$= \lim_{h\to 0} \frac{(x+\frac{h}{\sqrt{2}})^2 + (x+\frac{h}{\sqrt{2}})(y+\frac{h}{\sqrt{2}}) - x^2 - xy}{h} =$$

$$= \lim_{h\to 0} \frac{\cancel{x^2} + \cancel{\frac{h^2}{2}} + \frac{2xh}{\sqrt{2}} + \cancel{xy} + \frac{xh}{\sqrt{2}} + \frac{yh}{\sqrt{2}} + \cancel{\frac{h^2}{2}} - \cancel{x^2} - \cancel{xy}}{h}$$

$$= \lim_{h\to 0} \frac{h^2 + \frac{3xh}{\sqrt{2}} + \frac{yh}{\sqrt{2}}}{h} = \lim_{h\to 0} h + \frac{3x}{\sqrt{2}} + \frac{y}{\sqrt{2}} =$$

$$= \frac{3x+y}{\sqrt{2}}$$

So $D_{\vec{u}} f(1,2) = \frac{3}{\sqrt{2}} + \frac{2}{\sqrt{2}} = \frac{5}{\sqrt{2}}$. So, the rate of change of $f(x,y) = x^2+xy$ at $(1,2)$ in the direction of $\vec{u}$ is $\frac{5}{\sqrt{2}}$

- In practice it could be difficult to compute the limit, so we need an easier way to evaluate the directional derivative

- To see how, let's define a new fct of a single var:

$$g(z) = f(x_0 + u_1 z,\, y_0 + u_2 z)$$

where x_0, y_0, u_1, u_2 are some fixed numbers.

Then $g'(z) = \lim_{h\to 0} \frac{g(z+h) - g(z)}{h}$ and

$$g'(0) = \lim_{h\to 0} \frac{g(h) - g(0)}{h} =$$

$$= \lim_{h\to 0} \frac{f(x_0 + u_1 h,\, y_0 + u_2 h) - f(x_0, y_0)}{h} = D_{\vec{u}} f(x_0, y_0)$$

So $g'(0) = D_{\vec{u}} f(x_0, y_0)$

Now, let's write $g(z) = f(x,y)$ where $\begin{cases} x = x_0 + u_1 z \\ y = y_0 + u_2 z \end{cases}$ *

$\Rightarrow$ $g'(z) = \dfrac{dg}{dz} = \dfrac{\partial f}{\partial x} \cdot \dfrac{dx}{dz} + \dfrac{\partial f}{\partial y} \dfrac{dy}{dz} = f_x^{(x,y)} u_1 + f_y^{(x,y)} u_2$

Let $z=0$ i.e. $x = x_0$ and $y = y_0$ *

Then $\boxed{g'(0) = f_x(x_0, y_0) u_1 + f_y(x_0, y_0) u_2}$

$\boxed{D_{\vec{u}} f(x_0, y_0) = f_x^{(x_0,y_0)} u_1 + f_y^{(x_0,y_0)} u_2}$

So, $D_{\vec{u}} f(x,y) = f_x(x,y) u_1 + f_y(x,y) u_2$

There are similar formulas that can be derived by the same type of argument for functions with more than two var. The directional derivative of $f(x,y,z)$ in the direction of the unit vector $\vec{u} = (u_1, u_2, u_3)$ is given by:

$$D_{\vec{u}} f(x,y,z) = f_x u_1 + f_y u_2 + f_z u_3$$

EX1 $f(x,y) = x^2 + xy$

we are picking up from the EX1 on page 3

$f_x = 2x + y$ $\qquad u_1 = \dfrac{1}{\sqrt{2}} = u_2$

$f_y = x$

Then $D_{\vec{u}} f(x,y) = f_x u_1 + f_y u_2 =$

$= \dfrac{1}{\sqrt{2}} x + \dfrac{1}{\sqrt{2}}(2x+y) = \dfrac{3x+y}{\sqrt{2}}$

and $D_{\vec{u}} f(1,2) = \dfrac{5}{\sqrt{2}}$

EX2: Let $f(x,y,z) = x^2 z + y^3 z^2 - xyz$ $\qquad \vec{u} = (-1, 0, 3)$

Notice that $\|\vec{u}\| = \sqrt{1+3^2} = \sqrt{10} \neq 1$ ($\vec{u}$ is not a unit vector)

Let $\vec{u} = \left(-\dfrac{1}{\sqrt{10}}, 0, \dfrac{3}{\sqrt{10}}\right)$

$\Rightarrow$ $D_{\vec{u}} f(x,y,z) = f_x u_1 + f_y u_2 + f_z u_3$ where

$f_x = 2x - yz$
$f_y = 3y^2z^2 - xz$
$f_z = x^2 + y^3z - xy$

$$D_{\vec{u}} f = \left(-\frac{1}{\sqrt{10}}\right)(2x - yz) + 0 + \frac{3}{\sqrt{10}}(x^2 + 2y^3z - xy) =$$

$$= \frac{1}{\sqrt{10}}(3x^2 + 6y^3z - 3xy - 2xz + yz)$$

EX3 Find $D_{\vec{u}} f(2,0)$ where $f(x,y) = xe^{xy} + y$ where $\vec{u}$ is the unit vector in the direction of $\theta = 2\pi/3$

$\longrightarrow$ $\vec{u} = \left(\cos\frac{2\pi}{3}; \sin\frac{2\pi}{3}\right) = \left(-\frac{1}{2}; \frac{\sqrt{3}}{2}\right)$

$f_x = e^{xy} + xye^{xy}$ $\quad f_y = x^2e^{xy} + 1$

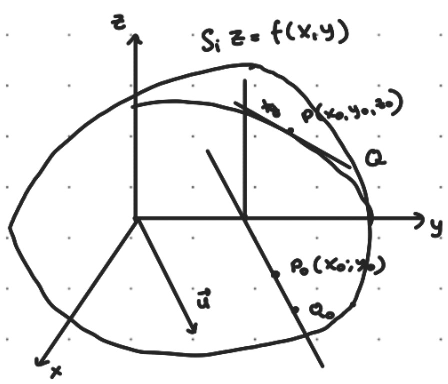
$D_{\vec{u}} f = (e^{xy} + xye^{xy})(-\frac{1}{2}) + \frac{\sqrt{3}}{2}(x^2e^{xy} + 1)$

$\Rightarrow$ $D_{\vec{u}} f(2,0) = (1+0)(-\frac{1}{2}) + \frac{\sqrt{3}}{2}(4+1) = \frac{5\sqrt{3} - 1}{2}$

GRAPHICAL INTERPRETATION OF DIRECTIONAL DERIVATIVE

The equation $z = f(x,y)$ represents a surface S in space. If $z_0 = f(x_0, y_0)$, then the pt $(x_0, y_0, z_0) = P$ lies on S.
The vertical plane that passes thru P and $P_0(x_0, y_0)$ // to the (uz) plane intersects S in a curve C.

The rate of change of f in the direction of u is the slope of the tangent to C at P.



When $\vec{u} = \vec{i} = (1,0)$, the directional derivative at P_0 is $\frac{\partial f}{\partial x}$ evaluated at (x_0, y_0). When $\vec{u} = \vec{j} = (0,1)$ " " $\frac{\partial f}{\partial y}$ " " at $P_0 = (x_0, y_0)$

$\rightarrow$ The Directional Derivative generalizes the two partial derivatives. We can now ask for the rate of change of f in any direction of $\vec{u}$, not just $\vec{i}$ or $\vec{j}$

We now develop an efficient formula to calculate the directional derivative for a differentiable fct, of f.

$$D_{\vec{u}} f = f_x u_1 + f_y u_2 = \underbrace{(f_x, f_y)}_{\substack{\text{Gradient of } f \\ \nabla f}} \cdot \underbrace{(u_1, u_2)}_{\substack{\text{direction} \\ \vec{u}}}$$

So the derivative f in the direction of $\vec{u}$ is the dot product of $\vec{u}$ with the special vector called the gradient of f

DEF: The gradient vector of $f(x,y)$ at a pt $(x_0, y_0) = P_0$ is the vector $\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$ obtained by evaluating the partial derivatives of f at P_0

⇒ With the definition of the gradient we can now say that the directional derivative is given by:

$$\boxed{D_{\vec{u}} f = \nabla f \cdot \vec{u}}$$

EX Find the derivative of $f(x,y) = xe^y + \cos(xy)$ at the pt $(2,0)$ in the direction $\vec{v} = 3\vec{i} - 4\vec{j}$

$\|v\| = \sqrt{3^2+4^2} = 5$

let $\vec{u} = \frac{3}{5}\vec{i} - \frac{4}{5}\vec{j}$ (unit vector)

$f_x = e^y - y\sin(xy)$; $f_x(2,0) = 1$

$f_y = xe^y - x\sin(xy)$; $f_y(2,0) = 2$

The gradient of f at $(2,0)$ is

$$\nabla f\big|_{(2,0)} = f_x(2,0)\vec{i} + f_y(2,0)\vec{j} = \vec{i} + 2\vec{j}$$

The Directional derivative of f at $(2,0)$ is:

$$D_u^f\big|_{(2,0)} = \nabla f\big|_{(2,0)} \cdot \vec{u} = (\vec{i} + 2\vec{j}) \cdot \left(\frac{3}{5}\vec{i} - \frac{4}{5}\vec{j}\right) =$$

$$= \frac{3}{5} - \frac{8}{5} = -1$$

PROPERTIES OF THE DIRECTIONAL DERIVATIVE

$$D_{\vec{u}} f = \nabla f \cdot \vec{u} = \|\nabla f\| \cdot \|\vec{u}\| \cos\theta = \|\nabla f\| \cos\theta$$

θ is the angle between the vector $\vec{u}$ and ∇f

CALCULUS 2

Date: 4 / 11 / 2022

① The maximum value of $D_{\vec{u}}f$ (hence the max. rate of change of the function $f(x,y)$) is given by $\|\nabla f\|$ and will occur in the direction of ∇f. So, f (at each pt. P in the domain) increases most rapidly in the direction of the gradient vector ∇f at P. The derivative in this direction is $D_{\vec{u}}f = |\nabla f|$

PROOF $D_{\vec{u}}f = \|\nabla f\| \cos\theta$

The largest possible value of $\cos\theta$ is 1 which occurs at $\theta = 0$. The max value of $D_{\vec{u}}f$ is $\|\nabla f\|$. Also the max value occurs when the angle between the gradient and $\vec{u}$ is zero, or in otherwords when $\vec{u}$ is pointing in the same direction as the gradiant ∇f.

② Similarly, f decreases most rapidly in the direction $-\nabla f$. The derivative in this direction is $D_u f = |\nabla f| \cos\pi = -|\nabla f|$

③ Any direction $\vec{u}$ orthogonal to a gradient vector $\nabla f \neq 0$ is a direction of zero change in f because θ then equals $\frac{\pi}{2}$ and $D_u f = |\nabla f| \cos(\frac{\pi}{2}) = |\nabla f| \cdot 0 = 0$

EX Find the direction in which $f(x,y) = \frac{x^2}{2} + \frac{y^2}{2}$

a) ↗ most rapidly at the pt. $(1,1)$
b) ↙ " " at $(1,1)$
c) what are the directions of zero change in f at $(1,1)$?

Sol: a) The gradient $\nabla f = f_x \vec{i} + f_y j = x\vec{i} + y\vec{i}$
$\nabla f|_{(1,1)} = i + j$ and the rate of change in this case is $\|\nabla f_{(1,1)}\| = \sqrt{2} = D_{\vec{u}}f$

It's direction is $\vec{u} = \frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}j$

b) $-\vec{u} = -\frac{1}{\sqrt{2}}\vec{i} - \frac{1}{\sqrt{2}}\vec{j}$, rate of change is $D_{\vec{u}}f = -\|\nabla f\| = -\sqrt{2}$

c) $\vec{n} = -\frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}$ $\quad -\vec{n} = \frac{1}{\sqrt{2}}\vec{i} - \frac{1}{\sqrt{2}}\vec{j}$

CALCULUS 2

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ALGEBRAIC RULES FOR GRADIENT

- $\nabla(f+g) = \nabla f + \nabla g$ (Sum / Difference)
- $\nabla(kf) = k\nabla f$ (Constant multiple)
- $\nabla(fg) = f\nabla g + g\nabla f$ (product)
- $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$ (Quotient)

FUNCTIONS OF 3 VARIABLES

For a differentiable fct. $f(x,y,z)$ and a unit vector $u = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$:

$$\boxed{\nabla f = f_x\vec{i} + f_y\vec{j} + f_z\vec{k}}$$

$$\boxed{D_{\vec{u}}f = \nabla f \cdot \vec{u} = f_x u_1 + f_y u_2 + f_z u_3}$$

EX: Find the derivative of $f(x,y,z) = x^3 - xy^2 - z$ at $P_0(1,1,0)$

ⓐ in the direction of $\vec{v} = 2\vec{i} - 3\vec{j} + 6\vec{k}$

ⓑ In what direction does f change most rapidly at P_0 and what are the rates of change in these directions.

ⓐ $\|\vec{v}\| = \sqrt{2^2+3^2+6^2} = \sqrt{49} = 7$ $\quad \vec{u} = \left(\frac{2}{7}, -\frac{3}{7}, \frac{6}{7}\right)$

$f_x = 3x^2 - y^2$; $\quad f_y = -2xy$; $\quad f_z = -1$

$f_x|_{(1,1,0)} = 3-1 = 2 \quad f_y|_{(1,1,0)} = -2 \quad f_z = -1$

and $\nabla f|_{(1,1,0)} = (2,-2,-1)$

$$\Rightarrow D_{\vec{u}}f|_{(1,1,0)} = (2,-2,-1)\cdot\left(\frac{2}{7}, -\frac{3}{7}, \frac{6}{7}\right) =$$
$$= (2)\left(\frac{2}{7}\right) + (-2)\left(-\frac{3}{7}\right) + (-1)\left(\frac{6}{7}\right) =$$
$$= \frac{4}{7} + \frac{6}{7} - \frac{6}{7} = \boxed{\frac{4}{7}}$$

ⓑ The fct $\nearrow$ most rapidly in the direction of $\vec{\nabla f} = 2\vec{i} - 2\vec{j} - \vec{k}$ and the rate of change in this direction is $\|\nabla f\| = \sqrt{2^2+2^2+1} = 3$ and $\swarrow$ most rapidly in direction $-\nabla f$ and the rate of change in this direction is $-|\nabla f| = -3$

$-2\vec{i} + 2\vec{j} + \vec{k}$