NOTE BOOK

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CALCULUS 2

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EX1: $f(x,y) = 3x - x^2y^2 + 2x^3y$

$$f_x = \frac{\partial f}{\partial x} = 3 - 2xy^2 + 6x^2y \qquad f_y = \frac{\partial f}{\partial y} = -2x^2y + 2x^3$$

EX2: $f(x,y) = y\sin(xy)$ *

$$f_x = y \cdot y\cos(xy) = y^2\cos(xy)$$

$$f_y = \frac{\partial y}{\partial y} \cdot \sin(xy) + y\,\frac{\partial \sin(xy)}{\partial y} = 1 \cdot \sin(xy) + yx\cos(xy)$$

Here we have 2 y so we can use product rule

EX3: $f(x,y) = \dfrac{2y}{y + \cos x}$

$$f_x = \frac{\frac{\partial}{\partial x}\cancel{2y}^{\,0}(y+\cos x) - 2y\frac{\partial}{\partial x}(y+\cos x)}{(y+\cos x)^2} = \frac{2y\sin x}{(y+\cos x)^2}$$

$$f_y = \frac{\frac{\partial(2y)}{y}\cdot(y+\cos x) - 2y\frac{\partial}{\partial y}(y+\cos x)}{(y+\cos x)^2} =$$

$$= \frac{2(y+\cos x) - 2y}{(y+\cos x)^2} = \frac{2\cos x}{(y+\cos x^2)}$$

EX4: $f(x,y,z) = z^3 - x^2y$

$$f_x = 2xy \qquad f_y = -x^2 \qquad f_z = 3z^2$$

EX.6: $f(x,y) = y^3\sin(x) + x^2\tan(y)$

$$f_x = y^3\cos(x) + 2x\tan(y)$$

$$f_y = 3y^2\sin(x) + x^2\sec^2(y)$$

so no need of product rule

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ex. 7:

$$f(x,y) = x\ln(xy) + y\sin(xy)$$

partial $\frac{y}{yx}$

$$f_x = \frac{\partial}{\partial x}(x)\ln(xy) + x\,\frac{\partial(\ln(xy))}{\partial x} + y^2\cos(xy) =$$

$$= \ln(xy) + 1 + y^2\cos(xy)$$

$$f_y = \cancel{x}\cdot\frac{x}{\cancel{x}y} + \sin(xy) + yx\cos(xy)$$

ex. 8:

$$z = y^x$$

$$\frac{\partial z}{\partial x} = y^x \cdot \ln(y) \qquad \frac{\partial z}{\partial x} = xy^{x-1}$$

ex. 9:

$$f(x,y,z,w) = \frac{xw^2}{y+\sin(zw)} *$$

$$f_x = \frac{w^2}{y+\sin(zw)} \qquad f_y = \frac{2xw\,(y+\sin(zw)) - xw^2(z\cos(zw))}{y+\sin(zw)^2}$$

$$\hookrightarrow f_x = \frac{\frac{\partial(xw^2)}{\partial y}(*) - xw^2(1)}{*^2}$$

(BEHIND THE SCENES)

Implicit Differentiation

ex: $x^2y + xz + yz^2 = 8$ where $z = f(x,y)$

Find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$

$$\frac{\partial}{\partial x}\left(x^2y + xz + yz^2\right) = 0$$

remember, in this case, z isn't constant cuz $z = f(x,y)$

$$2xy + z + x\frac{\partial z}{\partial x} + 2yz\frac{\partial z}{\partial x} = 0$$

$$2xy + z\ (x+2yz)\frac{\partial z}{\partial y} = 0$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{-2xy - z}{x+2yz}$$

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ex: $\frac{\partial}{\partial y}\left(x^2y + xz + yz^2\right)\left(\frac{\partial}{\partial y} 8\right)$ where $z = f(x,y)$

Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

$$x^2 + x\frac{\partial z}{\partial y} + z^2 + y2z\frac{\partial z}{\partial y} = 0 \rightarrow x^2 + z^2 + (x+2yz)\frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial y} = \frac{-x^2 - z^2}{x+2yz}$$

ex3: $2\cos(x+2y) + \sin(yz) - 1 = 0$

$-2\cdot 2\cos(x+2y)$ $\quad \cos(yz)\left(z + y\frac{\partial z}{\partial y}\right)$ [for $\frac{\partial z}{\partial y}$]

$$-2\sin(x+2y) + y\cos(yz)\cdot\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = \frac{2\sin(x+2y)}{y\cos(yz)}$$

$$-4\sin(x+2y) + z\cos(yz) + y(\cos(yz))\frac{\partial z}{\partial y} = 0$$

$$\Rightarrow \frac{\partial z}{\partial y} = \frac{4\sin(x+2y) - z\cos(yz)}{y\cos(yz)}$$

ex4: $\frac{\partial}{\partial x}(yz - \ln z) = \frac{\partial}{\partial x}(x+y)$ $\quad z = f(x,y)$!

$$y\frac{\partial z}{\partial x} - \frac{1}{z}\frac{\partial z}{\partial x} = 1 \Rightarrow \frac{\partial z}{\partial x} = \frac{z}{yz-1}$$

Higher Order partial Derivatives

$$z = f(x,y) < \begin{matrix} f_x \\ f_y \end{matrix}$$

$$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} \qquad f_{yy} = (f_y)_y = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x} \qquad (f_y)_x = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\,\partial y}$$

mixed partial der
(the order is important)

EX1: $f(x,y) = x\cos y + ye^x$

$$f_x = \cos(y) + ye^x \longrightarrow f_{xx} = ye^x$$
$$f_y = -x\sin y + e^x \longrightarrow f_{yy} = -x\cos y$$
$$f_{xy} = -\sin y + e^x \qquad f_{yx} = -\sin y + e^x$$

EX 2. $f(x,y,z) = 1 - 2xy^2z + x^2y$, Find $f_{yxyz} = \dfrac{\partial^4 f}{\partial z\,\partial y\,\partial x\,\partial y}$

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$$f_y = -4xyz + x^2$$
$$f_{yx} = -4yz + 2x$$
$$f_{yxy} = -4z$$
$$f_{yxyz} = -4$$

Schwartz's Theorem

It states that symmetry of 2^{nd} partial derivatives will always hold at a point if the second partial derivatives are continuous around that point.

EX :

$$\text{det } f(x,y) = \begin{cases} \overbrace{\dfrac{xy(x^2-y^2)}{x^2+y^2}}^{u} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

claim : $\dfrac{\partial^2 f}{\partial x\,\partial y}(0,0) \neq \dfrac{\partial^2 f}{\partial y\,\partial x}(0,0)$

For $(x,y) \neq (0,0)$

u

$$\frac{\partial f}{\partial x} = \frac{[y(x^2-y^2) + xy(2x)](x^2+y^2) - 2x(xy)(x^2-y^2)}{(x^2+y^2)^2}$$

$$= \frac{y(x^4-y^4) + 2x^2y(x^2+y^2) - 2x^2y(x^2-y^2)}{(x^2+y^2)}$$

so $\frac{\partial f}{\partial x} = \frac{y(x^4+4x^2y^2-y^4)}{(x^2+y^2)^2}$ $(x,y) \neq (0,0)$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h+0,0) - \cancel{f(0,0)}^{0}}{h} = 0$$

$$\boxed{\frac{\partial f}{\partial x} = \begin{cases} \frac{y(x^4+4x^2y^2-y^4)}{(x^2+y^2)^2} & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}}$$

1^{st} partial derivative in respect to x

For $(x,y) \neq 0,0$:

$$\frac{\partial f}{\partial y} = \frac{[x(x^2-y^2)+xy(-2y)](x^2+y^2) - 2y(xy)(x^2-y^2)}{(x^2+y^2)^2} =$$

$$= \frac{x(x^4-4x^2y^2-y^4)}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,0+h) - \cancel{f(0,0)}^{0}}{h} = 0$$

so

$$\boxed{\frac{\partial f}{\partial y} = \begin{cases} \frac{x(x^4-4x^2y^2-y^4)}{(x^2+y^2)^2} & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}}$$

1^{st} partial derivative in respect to y

$$\Rightarrow \frac{\partial^2 f}{\partial y \partial x}(0,0) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(0,0) = \lim_{h\to 0} \frac{\frac{\partial f}{\partial x}(0,h) - \cancel{\frac{\partial f}{\partial x}(0,0)}^{0}}{h} =$$

$$= \frac{-h}{h} = -1$$

$$\Rightarrow \frac{\partial^2 f}{\partial x \partial y}(0,0) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(0,0) = \lim_{h\to 0} \frac{\overset{h}{\frac{\partial f}{\partial y}(h,0)} - \cancel{\frac{\partial f}{\partial y}(0,0)}^{0}}{h} = 1$$

initial claim: $\frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y \partial x}(0,0)$

$\rightarrow$ $1 \neq -1$ because not continuous at $(0,0)$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) =$$

$$= \frac{[(x^4-4x^2y^2-y^4)+x(4x^3-8xy^2)][x^2+y^2]^2 - x(x^4-4x^2y^2-y^4)(2)(x^2+y^2)(2x)}{(x^2+y^2)^4}$$

$$= \frac{x^6+9x^4y^2-9x^2y^4-y^6}{(x^2+y^2)^3}$$

$$\boxed{\frac{\partial^2 f}{\partial x \partial y} = \begin{cases} \frac{x^6+9x^4y^2-9x^2y^4-y^6}{(x^2+y^2)^3} & (x,y)\neq(0,0) \\ 1 & (x,y)=(0,0) \end{cases}}$$

$*$

Second partial derivative

Now we have to see if 2nd partial derivative is cont.:

Along $y=mx$

$$* \quad \frac{x^6+9x^4m^2x^2-9x\,m^4x^4-m^6x^6}{(x^2+m^2x^2)^3} = \frac{\cancel{x^6}(1+9m^2-9m^4-m^6)}{\cancel{x^6}(1+m^6)} =$$

$$= \frac{1+9m^2-9m^4-m^6}{1+m^6}$$

so $\lim\limits_{(x,y)\to(0,0)} \frac{\partial^2 f}{\partial x \partial y} = \frac{1+9m^2-9m^4-m^6}{1+m^6}$

limit doesn't exist since it depends on m

NOTE: Since we know $\frac{\partial^2 f}{\partial x \partial y}$ isn't contin., we don't need to calculate the other one by definition of theorem