NOTE BOOK

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$$\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases}$$

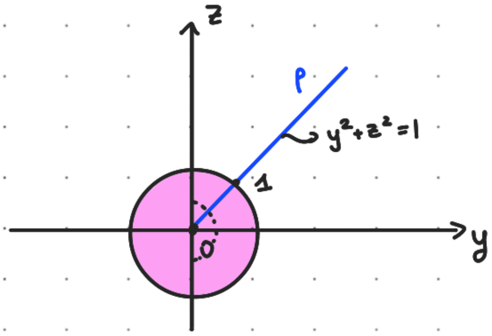
$0 \le \phi \le \pi$
$0 \le \theta \le 2\pi$

$x^2+y^2+z^2 = \rho^2$

$dxdydz = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

EX1: Evaluate $\iiint_D \sqrt{x^2+y^2+z^2}\,dV$ using spherical coordinates where D is the sphere $x^2+y^2+z^2 \le 1$.

$x=0$



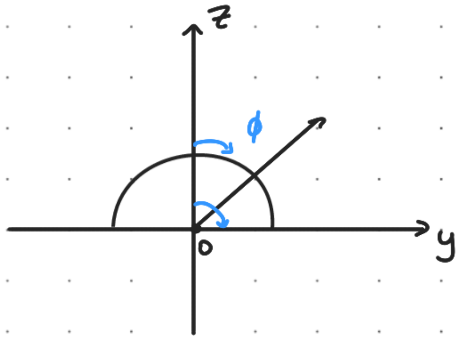
$0 \le \rho \le 1$
$0 \le \phi \le \pi$
$0 \le \theta \le 2\pi$

So $\int_0^{2\pi}\int_0^{\pi}\int_0^{1} \rho \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi}\int_0^{\pi} \frac{1}{4}\sin\phi \, d\phi \, d\theta =$

$= \int_0^{2\pi} \frac{1}{4}\left[-\cos\phi\right]_0^{\pi} d\theta$ $= \pi$

EX2: $\iiint_D y\,dV$ where D is the region bounded between $z=\sqrt{1-x^2-y^2}$ and (xy) plane.

Set $x=0$ $\leadsto$ $z=\sqrt{1-y^2}$



θ $z=0$

$\Rightarrow \sqrt{1-x^2-y^2} = 0 \rightarrow x^2+y^2=1$

Since it's circle, $0 \le \theta \le 2\pi$

so $\iiint_D y \, dv = \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{1} \rho\sin\phi\sin\phi\,\rho^2\sin\phi \, d\rho\, d\phi\, d\theta$ $= 0$

$\frac{1-\cos(2\phi)}{2}$

CALCULUS 2

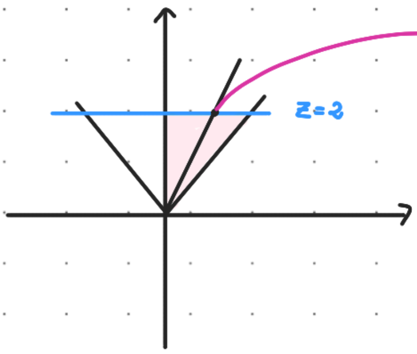
Date: 5 / 12 / 2022

EX3: Find the volume of the solid bounded by $z=2$ and $z=\sqrt{x^2+y^2}$

$$V=\iiint_D dv$$

$x=0$ $\quad \rho, \phi$

$$z=\sqrt{y^2}=|y|$$



it leaves at $z=2$ but we cant say $0\le\rho\le2$ so we have to express $z=2$ in terms of ρ

$z=2 \longrightarrow \rho\cos\phi=2$

$\rho=\frac{2}{\cos\phi}=2\sec\phi$

so $\left[0\le\rho\le 2\sec\phi \ ,\ 0\le\phi\le\frac{\pi}{4} \ ,\ \underbrace{0\le\theta\le2\pi}\right]$

we have $z=2$ and $z=\sqrt{x^2+y^2}$

so $\sqrt{x^2+y^2}=2$

$x^2+y^2=4$

(circle)

so $$V=\iiint_D dv=\int_0^{2\pi}\int_0^{\pi/4}\int_0^{2\sec\phi}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

EX4: Find the volume of the solid between $\rho=\cos\phi$ and the hemisphere $\rho=2$; $z\ge0$

$$\boxed{x^2+y^2+z^2=4}$$

$\rightarrow \rho=\cos\phi \ ,\quad \rho^2=\rho\cos\phi$

express in terms of x,y,z

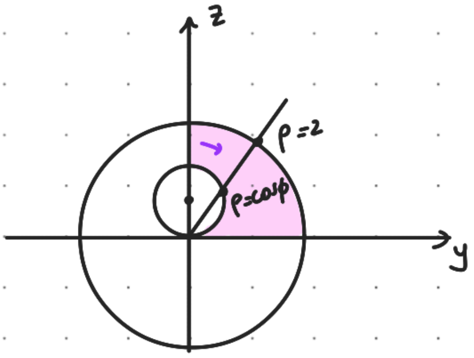
$$x^2+y^2+z^2=z \quad \Rightarrow \quad \boxed{x^2+y^2+\left(z-\tfrac{1}{2}\right)^2=\tfrac{1}{4}}$$

method of completing the square

$z^2-z=0$

Set $x=0$

$$y^2+z^2=4$$
$$y^2+\left(z-\tfrac{1}{2}\right)^2=\tfrac{1}{4}$$



Change of variable

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_c^d f(u)\,du$$

let $u = g(x)$
$du = g'(x)\,dx$

EX1: $R \rightarrow x^2 + \frac{y^2}{36} = 1$ (ellipse)

use the change $\begin{cases} x = \frac{u}{2} \\ y = 3v \end{cases}$

$$\left(\frac{u}{2}\right)^2 + \frac{(3v)^2}{36} = 1 \Rightarrow \boxed{u^2 + v^2 = 4}$$

DEF: $x = g(u,v)$, $y = h(u,v)$

$\Rightarrow$ The Jacobian $J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$ do determinant

we want to integrate $f(x,y)$ over a region R
under $x = g(u,v)$ $y = h(u,v)$

$R \xrightarrow{\text{transformed}} S$

$$\iint_R f(x,y)\,dx\,dy = \iint f(g(u,v), h(u,v))\,|J(u,v)|\,\underbrace{du\,dv}_{d\bar{A}}$$

$$dA = |J(u,v)|\,d\bar{A}$$

EX2: Show that $dA = r\,dr\,d\theta$ in polar coordinates

$\begin{cases} x = \cos\theta \\ y = r\sin\theta \end{cases}$

$dA = |J(r,\theta)|\,dr\,d\theta = r\,dr\,d\theta$

$$J(r,\theta) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} =$$

$$= r\cos^2\theta + r\sin^2\theta = r$$

$$R \downarrow S \qquad \begin{cases} x = g(u,v,w) \\ y = h(u,v,w) \\ z = k(u,v,w) \end{cases} \qquad \underset{dxdydz}{dV} = |J(u,v,w)|\, \underset{dudvdw}{dV}$$

$$\iiint_R f(x,y,z)\, dV \longrightarrow \iiint_S f(g,h,k)\, |J(u,v,w)|\, dV$$

where $J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$

EX3: Show that $dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$ in spherical coordinates

$$\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\theta \end{cases} \qquad J(\rho,\phi,\theta) = \begin{array}{c} \begin{matrix} \rho & \phi & \theta \end{matrix} \\ \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix} \end{array}$$

$$= \cos\phi\,\overbrace{\left(\rho^2\sin\phi\cos\phi\cos^2\theta + \rho^2\sin\phi\cos\phi\sin^2\theta\right)}^{\rho^2\sin\phi\cos\phi} +$$

$$\rho\sin\phi\,\underbrace{\left(\rho\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta\right)}_{\rho\sin^2\theta} =$$

$$= \rho^2\sin\phi\left(\cos^2\phi + \sin^2\phi\right) =$$