NOTE BOOK

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EX 1

$$\lim_{(x,y)\to(0,0)} \frac{3xy}{3x^2+y^2}$$

1. Along $x=0$; $\lim_{y\to 0} \frac{3xy}{3x^2+y^2} = 0$

   Along $y=0$; $\lim_{x\to 0} \frac{0}{3x^2} = 0$ → L_1

   this doesn't imply that the limit exist

   Along $y=x$; $\lim_{(x,y)\to(0,0)} \frac{3x^2}{3x^2+x^2} = \frac{3}{4} \to L_2$

   $L_1 \neq L_2 \Rightarrow$ the limit doesn't exist

2. $y = mx$

$$\lim_{(x,y)\to(0,0)} \frac{3xy}{3x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{3mx^2}{3x^2+m^2x^2} = \frac{3m}{m^2+3}$$

for $m=0$ i.e. $y=0 \Longrightarrow L_1 = 0$

for $m=1$ i.e. $y=x \Rightarrow L_2 = 3/4$

$L_1 \neq L_2$ so limit doesn't exist

EX. 2

$$\lim_{(x,y)\to(1,0)} \frac{2xy-2y}{x^2+y^2-2x+1}$$

1. Along $y=0$, $\lim_{x\to 1} \frac{0}{x^2-2x+1} = 0$ → L_1

   Along $x=1$, $\lim_{y\to 0} \frac{2y-2y}{1^2+y^2-2+1} = \lim \frac{0}{y^2} = 0$

   if these were different, we could say lim doesn't exist right away

So, $\frac{2xy-2y}{x^2+y^2-2x+1} = \frac{2y(x-1)}{(x-1)^2+y^2}$

Along $y = x-1$:

$$\lim \frac{2(x-1)^2}{(x-1)^2+(x-1)^2} = \boxed{1} \curvearrowright L_2$$

So $L_1 \neq L_2 \implies$ Then limit doesn't exist

ex3 :

$$\lim_{(x,y)\to(0,0)} \frac{xy^3\cos x}{2x^2+y^6}$$

Along $x=0$, $\lim_{y\to 0} \frac{0}{y^6} = 0$

Along $y=0$, $\lim_{x\to 0} \frac{0}{2x^2} = 0$

L_1

Along $x=y^3$, $\lim_{(x,y)\to(0,0)} \frac{y^6\cos y^3}{3y^6} =$

$$\lim \frac{\cos y^3}{3} = \frac{1}{3} \leftarrow L_2$$

Continuity

REMEMBER: $\lim_{x\to x_0} f(x) = f(x_0)$

$\rightarrow$ $f(x_0)$ is defined

$=$

$\rightarrow$ $\lim_{x\to x_0} f(x)$ exists

$\rightarrow$ should be equal

A function $f(x,y)$ is continous at the point (x_0,y_0) if :

1. f is defined at (x_0,y_0)
2. $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists $\rightarrow$ so if you pick any path, along it, the limit should always be same.
3. $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$

So, a function is continuos if its continous at every point of it's domain.

EX1: Is the function $f(x,y) = \dfrac{x^2-y^2}{x^2-2xy-3y^2}$ contin. at $(1,-1)$?

Nope, it's not

EX2: Let $f(x,y) = \begin{cases} \dfrac{x^2-y^2}{x^2-2xy-3y^2} & x \neq -y \\ 5 & \text{if } x = -y \end{cases}$

① $f(1,-1) = 5$

② $\lim\limits_{(x,y)\to(1,-1)} f(x,y) = \lim\limits_{(x,y)\to(1,-1)} \dfrac{x^2-y^2}{x^2-2xy-3y^2}$ factorizing trial and error

$= \lim\limits_{(x,y)\to(1,-1)} \dfrac{(x-y)\cancel{(x+y)}}{\cancel{(x+y)}(x-3y)} = \dfrac{1+1}{1+3} = \dfrac{1}{2}$

$x^2 - 2xy - 3y^2$ -3: $-3,1$ / $3,-1$; 1, -1

$(x-y)(x+3y) \to x^2+2xy-3y^2$ NOT RIGHT

so $(x+y)(x-3y)$

③ $f(1,-1) = 5 \neq \lim\limits_{(x,y)\to(1,-1)} = \dfrac{1}{2}$

f is not continous at $(1,-1)$

Partial Derivatives

$f: \mathbb{R} \longrightarrow \mathbb{R}$

$$f'(x_0) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$$

CALCULUS 2

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Def: Let $z = f(x,y)$

The partial derivative of $f(x,y)$ w.r. to x at the point (x_0,y_0), denoted by

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} \quad \text{or} \quad f_x(x_0,y_0) \quad \text{or} \quad \frac{\partial z}{\partial x}(x_0,y_0)$$

is given by:

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \lim_{h \to 0} \frac{f(x_0+h,y_0) + f(x_0,y_0)}{h}$$

The partial derivative of $f(x,y)$ w.r. to y at (x_0,y_0) is:

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)}, \quad f_y(x_0,y_0), \quad \left.\frac{\partial z}{\partial y}\right|_{(x_0,y_0)}$$

Ex: Let $f(x,y) = x^2y + 2x + y^2$

Find $f_x(x,y)$ using the limit definition

$$f_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h} =$$

$$= \lim_{h \to 0} \frac{(x+h)^2y + 2(x+h) + y^3 - (x^2y + 2x + y^3)}{h} =$$

$$= \lim_{h \to 0} \frac{2xhy + h^2y + 2h}{h} = \lim_{h \to 0} 2xy + hy + 2 =$$

$$= 2xy + 2 \longrightarrow \quad \text{So } f_x(x,y) = 2xy + 2$$

$$\lim_{h \to 0} \frac{\cancel{x^2y} + 2xhy + h^2y \cancel{+2x} + 2h \cancel{+y^3} \cancel{-x^2y} \cancel{-2x} \cancel{-y^3}}{h}$$

In general, if $f: U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is a function. Then the partial derivative of f at the point $a = (a_1, a_2 \ldots a_n)$ w.r.t the i^{th} variable x is defined as:

$$\frac{\partial f(a)}{\partial x_i} = \lim_{h \to 0} \frac{f(a_1, a_2, \ldots, a_{i-1}, a_i+h, \ldots, a_n) - f(a_1, a_2, \ldots, a_n)}{h}$$