NOTE BOOK

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CALCULUS 2

Date: 28 / 10 / 2022

CHAIN RULE FOR MULTI-VARIABLE FUNCTION

Aim: How to find derivatives of multivariable fct involving parametrics or composition

The chain rule for functions of a single variable says that when $y = f(x)$ is a differentiable fct. of x and $x = g(t)$ is a differentiable fct. of t, then y is a differentiable fct. of t and

$$\boxed{\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}}$$

Another notation which is probably familiar to most people is:

$$F(x) = f(g(x)) \implies F'(x) = f'(g(x)) \cdot g'(x)$$

For functions of 2 or more variables, the Chain rule has several forms. The form depends on how many are involved, but once this is taken into account, it works like Chain Rule in single variable.

FUNCTION OF TWO VARIABLES: (case of 1 ind. var and 2 intermediate var.)

THM: If $z = f(x,y)$ is differentiable and if $x = x(t)$, $y \neq y(t)$ are differentiable functions of f, then the composite $z = f(x(t), y(x))$ is a differentiable fct. of t and

$$\frac{dz}{dt} = \overset{=\frac{\partial z}{\partial x}}{\frac{\partial f}{\partial x}} \cdot \frac{dx}{dt} + \overset{=\frac{\partial z}{\partial y}}{\frac{\partial f}{\partial y}} \cdot \frac{dy}{dt} \quad \text{or}$$

$$\frac{dz}{dt} = f_x(x(t), y(t)) \cdot x'(t) + f_y(x(t), y(t)) \cdot y'(t)$$

The branch diagram provides a convenient way to remember the Chain Rule. To find $\frac{dz}{dt}$, start at z, and read down each route to t, multiplying derivatives along the way, then add the products

$$\begin{tikzcd} & z=f(x,y) \arrow[dl, "\frac{\partial z}{\partial x}"'] \arrow[dr, "\frac{\partial z}{\partial y}"] & \\ x \arrow[dr, dash, "\frac{dx}{dt}"'] & & y \arrow[dl, dash, "\frac{dy}{dt}"] \\ & t & \end{tikzcd}$$

Dependent variable

Intermediate vars

Independent var

NOTE: The meaning of the dependant variable "z" is different on each side of the preceding eqn.
On the left-hand side, it refers to the composite function $z = f(x(t), y(t))$ as a fct. of a single variable t.
On the right-hand side, it refers to the function $z = f(x, y)$ as a fct. of 2 variables x, y.

Moreover, the single derivatives, $\frac{dw}{dt}$, $\frac{dx}{dt}$, and $\frac{dy}{dt}$ are being evaluated at a pt. "t", whereas the partial derivatives $\frac{\partial w}{\partial x}$, $\frac{\partial w}{\partial y}$ are being evaluated at the pt (x_0, y_0), with $x_0 = x(t)$, $y_0 = y(t_0)$.

EX1 Find the derivative of $z = xy$ w.r.t. "t" along the path $x = \cos t$, $y = \sin t$.

Sol:
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} =$$
(using Chain Rule)
$$= y(-\sin t) + x(\cos t) =$$
$$= (\sin t)(-\sin t) + (\cos t)(\cos t) =$$
$$= \cos 2t$$

→ We can check the results with direct calculation and in this case, it might be easier to just substitute in for x and y in the original fct and just compute the derivative as we normally do.

$$z = xy = \cos t \sin t = \frac{1}{2}\sin(2t)$$
$$\frac{dz}{dt} = \frac{d}{dt}\left(\frac{1}{2}\sin 2t\right) = \frac{1}{2}\cdot 2\cos(2t) = \cos(2t)$$
$$\rightarrow \left.\frac{dz}{dt}\right|_{t=\frac{\pi}{2}} = \cos\left(2\cdot\frac{\pi}{2}\right) = -1$$

EX2: $z = x^2y^3 + y\cos x$; $x = \ln(t^2)$, $y = \sin(4t)$

In this case, it would almost definetely be more work to do the substitution first, so we'll use the chain Rule and then substitute

$$\frac{dz}{dt} = (2xy^3 - y\sin x)\left(\frac{2t}{t^2}\right) + (3x^2y^2 + \cos x)(4\cos 4t) =$$
$$= \frac{[(\ln(t^2)\sin^3 4t) - (\sin 4t \sin \ln t^2)]}{t} + 4\cos 4t(3\ln^2 t^2 \sin^2 4t + \cos(\ln t^2))$$

CALCULUS 2

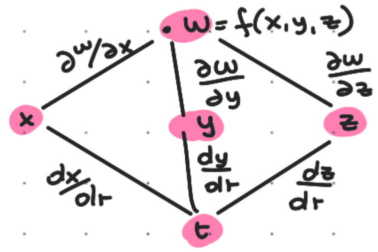
Date: 28 / 10 / 2022

FUNCTIONS OF 3 VARIABLES

THM: If w is diff. and x, y, z are diff. functions of t, then w is diff. function of t and:

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

EX: $w = \tan^{-1}(xz) + \frac{z}{y}$



$x = t, \quad y = t^2, \quad z = \sin t$

$$\boxed{\frac{dw}{dt}} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} =$$

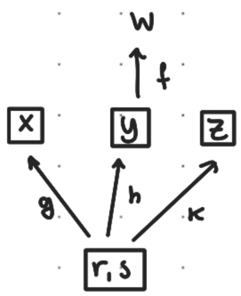
$$= \boxed{\left(\frac{z}{1+x^2z^2}\right)(1) + \left(\frac{-z}{y^2}\right)(2t) + \left[\left(\frac{x}{1+x^2z^2}\right) + \left(\frac{1}{y}\right)\right]\cos t}$$

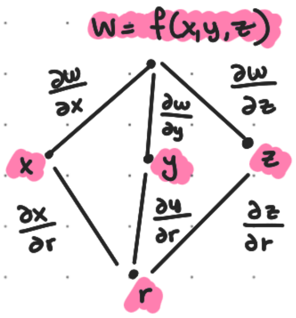
THM: CHAIN RULE FOR TWO INDEPENDENT VARIABLES AND THREE INTERMEDIATE VARIABLES

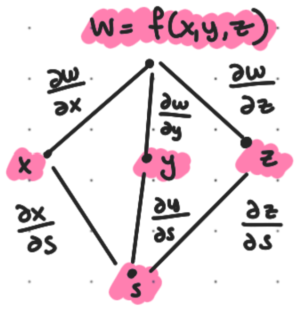
Suppose that $w = f(x, y, z)$, $x = g(r,s)$, $y = h(r,s)$ and $z = k(r,s)$.

If all 4 functions are diff., then w has partial derivatives w.r.t. "r" and "s" given by the formulas:

$$\boxed{\begin{aligned} \frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} \\ \frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} \end{aligned}}$$







EX3 Let $w = x + 2y + z^2$, $x = \frac{r}{s}$, $y = r^2 + \ln s$, $z = 2r$ *

$\rightarrow \frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} =$

$= 1\left(\frac{1}{s}\right) + 2(2r) + (2\underbrace{z}_{2r\,*})(2) =$

$= \frac{1}{s} + 4r + 8r =$

$= 1/s + 12r$

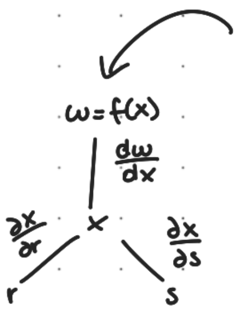
$\rightarrow \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} =$

$= (1)\left(-\frac{r}{s^2}\right) + 2\left(\frac{1}{s}\right) + 2z(0) =$

$= -\frac{r}{s^2} + \frac{2}{s}$

NOTE: If f was a fct. of 2 independent var. (i.e. $w = f(x,y)$) instead of 3, then

$$\boxed{\begin{aligned} \frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} \\ \frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} \end{aligned}}$$



and if f was a fct. of single var. x, then (i.e $w = f(x)$, $x = g(r,s)$)

$$\boxed{\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r}} \quad \text{and} \quad \boxed{\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s}}$$

where $\frac{dw}{dx} = f'(x)$ is the ordinary derivative.

Now that we've seen couple of cases for the chain rule, let's see the general version.

CHAIN RULE

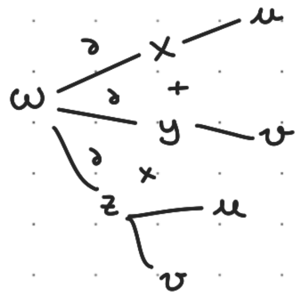
Suppose that z is a fct. of n variables $x_1, x_2 \cdots x_n$ and each one of these variables in turn are fct. of m variables $t_1, t_2, \cdots t_m$. Then for any variable $t_1, t_2, \cdots t_m$ we have the following:

$$\frac{\partial z}{\partial t_i} = \frac{\partial z}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial z}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial z}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

CALCULUS 2

Date: 28 / 10 / 2022

EX1: $\omega = x\tan^{-1}(yz)$; $x = \sqrt{u}$, $y = e^{-2v}$, $z = v\cos u$



$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{dx}{du} + \frac{\partial w}{\partial z}\frac{dz}{du}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial y}\frac{dy}{dv} + \frac{\partial w}{\partial z}\frac{dz}{dv}$$

$$\frac{\partial w}{\partial u} = \tan^{-1}(yz) \cdot \frac{1}{2\sqrt{u}} + \frac{xy}{1+y^2z^2}(-v\sin u)$$

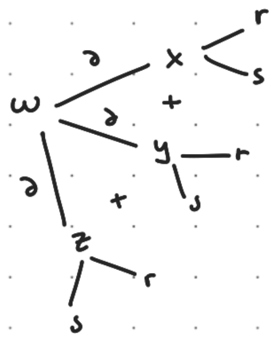
$$\frac{\partial w}{\partial v} = \frac{xz}{1+y^2z^2}(-2e^{-2v}) + \frac{xy}{1+y^2z^2}(\cos u) =$$

$$= \frac{-2xze^{-2v} + xy\cos u}{1+y^2z^2}$$

EX2: $\omega = x^2y + y^2z^3$, $x = r\cos(s)$, $y = r\sin(s)$, $z = re^s$

Find $\left.\frac{\partial w}{\partial s}\right|_{r=1,\ s=0}$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$



$$\Rightarrow \frac{\partial w}{\partial s} = 2xy(-r\sin(s)) + (x^2 + 2yz^3) \cdot (r\cos s) + 3y^2z^2(re^s)$$

$r=1,\ s=0 \Rightarrow x=1$, $y=0$, $z=1$

$$\Rightarrow \left.\frac{\partial w}{\partial s}\right|_{r=1,\ s=0} = 1$$

IMPLICIT DIFFERENTIATION REVISITED

With these forms of the chain rule, implicit differentiation becomes a fairly simple process.

Given the fct. in the form $w = f(x,y) = 0$ where y is implicitly defined function of x, say $y = g(x)$

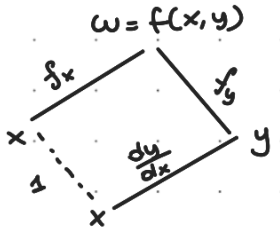
$$0 = \frac{\partial w}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} \implies f_y \frac{dy}{dx} = -f_x$$

$$\implies \boxed{\frac{dy}{dx} = \frac{-f_x}{f_y}}$$

Branch Diagram

$$\frac{dw}{dx} = f_y \frac{dy}{dx} + f_x(1)$$

$$\implies \frac{dy}{dx} = \frac{-f_x}{f_y}$$



EX1 $2x^2 + 3\sqrt{xy} - 2y - 4 = 0$ $(\iff f(x,y)=0)$

$y = g(x)$. Find $\frac{dy}{dx}$?

$$\implies \frac{dy}{dx} = -\frac{4x + \frac{3y}{2\sqrt{xy}}}{\frac{3x}{2\sqrt{xy}} - 2} \cdot \frac{2\sqrt{xy}}{2\sqrt{xy}} =$$

$$= -\frac{8x\sqrt{xy} + 3y}{3x - 4\sqrt{xy}}$$

EX2 Find $\frac{dy}{dx}$ for $x\cos(3y) + x^3y^5 = 3x - e^{xy}$ and $y = g(x)$

So $\underbrace{x\cos(3y) + x^3y^5 - 3x + e^{xy}}_{f(x,y)} = 0$

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{\cos(3y) + 3x^2y^5 - 3 + ye^{xy}}{-3x\sin(3y) + 5x^3y^4 + xe^{xy}}$$

[This calculation is significantly shorter than a single-variable calculation using Implicit diff.]

CALCULUS 2 Date: 28 / 10 / 2022

IMPLICIT DIFFERENTIATION FOR FCT. OF 3 VARIABLES

Assume $f(x,y,z) = 0$ and z is implicitly defined as $z = g(x,y)$ and we want to find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$

$$f(x,y,z) = 0$$

- Differentiate both sides w.r.t. x and we'll need to remember to treat y as a const.

$$\frac{\partial f}{\partial x} \cdot 1 + \frac{\partial f}{\partial y} \cdot \underbrace{\cancel{\frac{\partial y}{\partial x}}}_{0} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial x} = 0$$

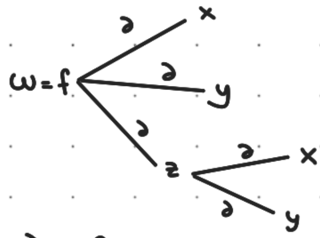
$$f_x + f_z \frac{\partial z}{\partial x} = 0 \quad \Rightarrow \quad \boxed{\frac{\partial z}{\partial x} = -\frac{f_x}{f_z}}$$

Similarly, $\boxed{\frac{\partial z}{\partial y} = -\frac{f_y}{f_z}}$

EX1: $x^2 e^{xy} + xy - x^2 z + yz^2 = 0$ and $z = g(x,y)$

$$\Rightarrow \boxed{\frac{\partial z}{\partial x} = -\frac{2xe^{xy} + x^2 y e^{xy} + y - 2xz}{-x^2 + 2yz}}$$

and $\boxed{\frac{\partial z}{\partial y} = -\frac{x^3 e^{xy} + x + z^2}{-x^2 + 2yz}}$



$\frac{\partial w}{\partial x} = 0$

$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 0$

$f_x + f_y \cdot \frac{\partial z}{\partial x} = 0$

$\frac{\partial z}{\partial x} = -\frac{f_x}{f_z}$

EX2: Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for $x^2 \sin(2y - 5z) = 1 + y\cos(6zx)$

$$\rightarrow x^2 \sin(2y - 5z) - y\cos(6zx) - 1 = 0$$

$$\boxed{\begin{aligned} \frac{\partial z}{\partial x} &= -\frac{2x\sin(2y-5z) + 6zy\sin(6zx)}{-5x^2\cos(2y-5z) + 6yx\sin(6zx)} \\ \frac{\partial z}{\partial y} &= \frac{2x^2\cos(2y-5z) - \cos(6zx)}{-5x^2\cos(2y-5z) + 6yx\sin(6zx)} \end{aligned}}$$