LINEAR ALGEBRA

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A linear equation of n variables can be expressed as:

$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n = b$$

ex:

$9x + 2y = 7$ linear

$\frac{x}{2} + 3y - \pi z = 1$ not lin.

$xy + z = 3$ not linear

$\frac{1}{x} + \frac{1}{y} = 4$ not linear

A solution always makes an equation true when placed

ex:

$x_1 + 2x_2 = 4$ $\quad x_1 = 0 \ ; \ x_2 = 2$ ✓

There is always a set of solutions, can be written with the **t** parameter, where one variable depends on others

ex:

$x_1 = 4 - 2x_2$

Let $x_2 = t$

$$\begin{cases} x_1 = 4 - 2x_2 \\ x_2 = t \end{cases} \qquad t \in \mathbb{R}$$

matrices

With $m, n > 0$, an $(m \times n)$ matrix is a rectangular array of values:

m rows

n columns

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ a_{m1} & \cdots & \cdots & a_{mn} \end{bmatrix}$$

common notation

$A[a_{ij}]$

row, column

ex:

$$A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -2 & 4 & 6 \\ 3 & 1 & 1 & 0 \end{bmatrix}$$

size of matrix = (3×4) matrix

$a_{11} = 1$, $a_{22} = -2$

$A\,[3;4]$

$$B = \begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix}$$

square matrix (2×2)

$\left.\begin{array}{l} b_{11} = 1 \\ b_{22} = -3 \end{array}\right\}$ main diagonal

$C = [1 \quad 2 \quad 3 \quad 4]$ row matrix (1×4)

$$D = \begin{bmatrix} 0 \\ -4 \\ 2 \end{bmatrix}$$

column matrix (3×1)

Operations with matrices

$A = [a_{ij}] \qquad B = [b_{ij}]$

$a_{ij} = b_{ij} \;, \; \forall ij \rightarrow$ everything is equal

ex:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad C = [1 \quad 3] \quad D = \begin{bmatrix} 1 & 2 \\ x & 4 \end{bmatrix}$$

$3 = x$

$A \neq B \qquad A \neq C \qquad A = D$

$A = D$ if $x = 3$

Matrix Addition

★ Only if the matrices are the same size

ex:

$$A = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} + B = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} \longrightarrow A + B = \begin{bmatrix} 0 & 5 \\ -1 & 3 \end{bmatrix}$$

(2×2) (2×2)

ex:

$$\begin{bmatrix} 0 & 1 & -2 \\ 1 & 2 & 3 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{zero matrix}} = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 2 & 3 \end{bmatrix}$$

ex:

$$\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 2 \end{bmatrix} = \text{not defined}$$

Scalar Multiplication

$A = [a_{ij}]$, $c \in \mathbb{R}$ $\hookrightarrow$ constant

$$\begin{bmatrix} x & y \\ z & t \end{bmatrix} \times c = \begin{bmatrix} x \cdot c & y \cdot c \\ z \cdot c & t \cdot c \end{bmatrix}$$

ex:

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} \times -1 = -A = \begin{bmatrix} -1 & -2 \\ 3 & -4 \end{bmatrix}$$

ex:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix} \quad \text{find } 3A - B$$

$$\begin{bmatrix} 3\cdot 1-2 & 3\cdot 2-0 & 3\cdot 4-0 \\ \vdots & \vdots & \vdots \end{bmatrix} = \begin{bmatrix} 1 & 6 & 12 \\ -6 & 4 & -6 \\ 7 & 0 & 4 \end{bmatrix}$$

Matrix Multiplication

No same size is required

$$(A)_{m\times \underline{\underline{n}}} \qquad (B)_{\underline{\underline{n}}\times r} \longrightarrow (A\times B)_{m\times r}$$

ex:

$$A_{(3\times \underline{\underline{4}})} \times B_{(\underline{\underline{4}}\times 2)} = (AB)_{(3\times 2)}$$ ✓ is defined

$$B_{(4\times \underline{\underline{2}})} \times A_{(\underline{\underline{3}}\times 4)} \neq (BA)$$ ✗ not defined

we don't make the assumption that $A\times B \neq B\times A$

ex:

$$A = \begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix}_{(3\times 2)} \quad \text{and} \quad B = \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix}_{(2\times 2)}$$

$$AB = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \\ C_{31} & C_{32} \end{bmatrix}_{(3\times 2)}$$

$$C_{11} = (-1)\cdot(-3) + 3\cdot(-4) = -9$$

C_{21} ← second row of A
← first column of B

$C_{21} = 4 \cdot (-3) + (-2)(-4) = -4$

$A \cdot B = \begin{bmatrix} -9 & 1 \\ -4 & 6 \\ -15 & 10 \end{bmatrix}$

ex:

$\begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -2 \end{bmatrix} \begin{bmatrix} -2 & 4 & 2 \\ 1 & 0 & 0 \\ -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -5 & 7 & -1 \\ -3 & 6 & 6 \end{bmatrix}$

ex:

$\begin{bmatrix} 1 & -2 & -3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}$

The Identity Matrix

$(n \times n)$ square matrix

$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ this is an identity matrix

when multiplied by identity matrix → $A \cdot I = A$ remains the same

ex:

$\begin{bmatrix} 3 & -2 \\ -6 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -6 & 3 \\ 4 & 1 \end{bmatrix}$

If $A_{m \times n}$:

$A \cdot I_n = A$

$I_m \cdot A = A$

A is a square matrix $(n \times n)$ → A^2, A^3, A^n is defined

Transpose of a Matrix

$A_{m \times n}$, $A^T = A_{n \times m}$

ex:

$B = \begin{bmatrix} 1 & 2 \end{bmatrix} \rightarrow B^T = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

ex:

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \rightarrow A^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$

Properties of Transposing

- $(A^T)^T = A$
- $(A+B)^T = A^T + B^T$
- $(A \cdot B)^T = B^T \cdot A^T$
- $(c \cdot A)^T = c \cdot A^T$

Remark!

A is called **symmetric** if $A = A^T$

ex:

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow A^T = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A is a symmetric matrix

$$A \cdot A^T = \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 9 \\ 9 & 14 \end{bmatrix}$$

Skew-symmetric matrix

$A^T = -A \rightarrow$ A is skew symmetric matrix

ex:

$$A = \begin{bmatrix} 0 & 2 & 45 \\ -2 & 0 & -4 \\ -45 & 4 & 0 \end{bmatrix}$$

ex:

$$A = \begin{bmatrix} 1 & 3 \\ 0 & -2 \\ -2 & 1 \end{bmatrix}$$

$B = AA^T$

$$\underset{A}{\begin{bmatrix} 1 & 3 \\ 0 & -2 \\ -2 & 1 \end{bmatrix}} \cdot \underset{A^T}{\begin{bmatrix} 1 & 0 & -2 \\ 3 & -2 & 1 \end{bmatrix}} = \begin{bmatrix} 10 & -6 & -5 \\ -6 & 4 & 2 \\ -5 & 2 & 5 \end{bmatrix} = B$$

(3×2) (2×3)

Since $B = B^T$, B is a symmetric

Trace of a matrix

Sum of entries on the main diagonal $\rightsquigarrow$ only for squares $(n \times n)$

$$tr(A) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} \dots + a_{nn}$$

ex:

$$B = \begin{bmatrix} 10 & -6 & -5 \\ -6 & 4 & 2 \\ -5 & 2 & 5 \end{bmatrix} \qquad tr(B) = 10 + 4 + 5 = 19$$

- $tr(A+B) = tr(A) + tr(B)$
- $tr(A^T) = tr(A)$
- $tr(AB) = tr(BA)$

Properties of matrix

→ Zero matrix $\qquad A_{(m \times n)} + 0_{(m \times n)} = A$

$0_{(m \times n)} \longrightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & \cdots & \end{bmatrix}$ all is zero

ex: $3X + A = B$

$$A = \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & 4 \\ 2 & 1 \end{bmatrix}$$

$$3X = B - A$$

$$X = \frac{B-A}{3}$$

$$B - A = \begin{bmatrix} -4 & 6 \\ 2 & -2 \end{bmatrix} \qquad \frac{B-A}{3} = \begin{bmatrix} -4/3 & 2 \\ 2/3 & -2/3 \end{bmatrix} = X$$

Properties of matrix multiplication

→ $ABC = (AB)C = A(BC) \neq (AC)B$

→ $A(B+C) = AB + AC$ → $(A+B)C = AC + BC$

be careful! these two are different!

→ if $AB = AC \not\Rightarrow B = C$

Row – echelon form of a matrix (r.e.f)

1. All zero rows are at the bottom of a matrix, if no zero rows not necessary.

2. For every non-zero row, the 1^{st} non-zero is 1 (leading 1)

3. For 2 successive rows, the leading 1 in the higher row is to to left of leading 1 in the lower row.

ex:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ \cancel{0} & \cancel{0} & 1 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

in r.e.f ✓

$$\begin{bmatrix} \cancel{0} & \cancel{0} & 1 & 2 \\ 1 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

leading 1 is to the left ✗

ex:

$$A^2 = A \cdot A = \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 10 \\ 5 & 18 \end{bmatrix}$$

we check the conditions in order

ex:

$$A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$ ✓ in r.e.f

$$B = \begin{bmatrix} 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ ✓ in r.e.f.

$$C = \begin{bmatrix} 1 & -5 & 2 & -1 & 3 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$ ✓ in r.e.f

$$D = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ ✓ in r.e.f.

$$E = \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & \boxed{2} & 1 & -1 \\ 0 & 0 & 1 & -3 \end{bmatrix} \rightarrow$$ not in r.e.f ✗

$$F = \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 4 \end{bmatrix} \rightarrow$$ not in r.e.f ✗

Reduced Row-echelon form of a matrix (r.r.e.f

There should be zeros above and below leading 1.

ex:

$$D = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ ✓ in r.r.e.f.

$$B = \begin{bmatrix} 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ ✓ in r.r.e.f.

$$A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$ not in r.e.f ✗

Augmented & Coefficient matrix

$$\begin{cases} x - 4y + 3z = 5 \\ -x + 3y - z = -3 \\ 2x \quad\quad - 4z = 6 \end{cases} \Rightarrow \begin{bmatrix} 1 & -4 & 3 \\ -1 & 3 & -1 \\ 2 & 0 & -4 \end{bmatrix}$$ coefficient matrix

$$\Rightarrow \begin{bmatrix} 1 & -4 & 3 & 5 \\ -1 & 3 & -1 & -3 \\ 2 & 0 & -4 & 6 \end{bmatrix}$$ augmented matrix

Elementary Row Operations

1. Interchange 2 rows
2. multiply a row by a non-zero
3. Add multiple of row to another

Gauss Elimination with Back Substitution

1. Write augmented matrix of system
2. Use elementary row operations to convert the matrix into row-echelon form (r.e.f.)
3. write the corresponding system

ex:

$$\begin{cases} x - 2y + 3z = 9 \\ -x + 3y = -4 \\ 2x - 5y + 5z = 17 \end{cases} \Rightarrow \begin{bmatrix} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{bmatrix}$$

make everything 0 below

$\times(-2)+$

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{bmatrix} \xrightarrow[-2R_1 + R_3 \to R_3]{R_1 + R_2 \to R_2} \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{bmatrix} \xrightarrow{R_2 + R_3 \to R_3} \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

create a leading 1

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{bmatrix} \xrightarrow{\frac{1}{2}R_3 \to R_3} \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$\div 2$

★ to create a leading 1 you either $\times$ or $\div$

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix} = \begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ z &= 2 \end{aligned}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

Gauss Jordan method (with r.r.e.f elimination)

ex:

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow[-3R_3+R_1 \to R_1]{-3R_3+R_2 \to R_2} \begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \checkmark$$

this system has one solution!

$$\begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{2R_2+R_1 \to R_1} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$x=1$

$y=-1$

$z=2$

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for a system of linear equations:

☆ compatible and consistent means that the system has at least one solution.

☆ may have exactly 1 solution, an infinite number of solutions or no solutions

ex:

Solve the system

$$\begin{cases} x_1 - x_2 + 2x_3 = 4 \\ x_1 + x_3 = 6 \\ 2x_1 - 3x_2 + 5x_3 = 4 \\ 3x_1 + 2x_2 - x_3 = 1 \end{cases} \longrightarrow \begin{bmatrix} 1 & -1 & 2 & 4 \\ 1 & 0 & 1 & 6 \\ 2 & -3 & 5 & 4 \\ 3 & 2 & -1 & 1 \end{bmatrix}$$

augmented matrix needs to be transfered to r.e.f.

make 0 below

$$\begin{bmatrix} 1 & -1 & 2 & 4 \\ 1 & 0 & 1 & 6 \\ 2 & -3 & 5 & 4 \\ 3 & 2 & -1 & 1 \end{bmatrix} \xrightarrow[\substack{R_3 - 2R_1 = R_3 \\ R_4 - 3R_1 = R_4}]{R_2 - R_1 = R_2} \begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & -1 & 1 & -4 \\ 0 & 5 & -7 & -11 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & -1 & 1 & -4 \\ 0 & 5 & -7 & -11 \end{bmatrix} \xrightarrow[R_4 - 5R_2 = R_4]{R_3 + R_2 = R_3} \begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & -2 & -21 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & -2 & -21 \end{bmatrix} \longrightarrow$$

the system has no solution!

$$0x_1 + 0x_2 + 0x_3 = -2$$
$$0 \neq -2$$

ex:

use gauss-jordan

$$\begin{cases} 2x_1 + 4x_2 - 2x_3 = 0 \\ 3x_1 + 5x_2 = 1 \end{cases} \longrightarrow \begin{bmatrix} 2 & 4 & -2 & 0 \\ 3 & 5 & 0 & 1 \end{bmatrix}$$

divide by 2 to get leading 1

augmented matrix needs to be transfered to r.e.f.

$$\begin{bmatrix} 2 & 4 & -2 & 0 \\ 3 & 5 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \cdot \frac{1}{2} \to R_1} \begin{bmatrix} 1 & 2 & -1 & 0 \\ 3 & 5 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 & 0 \\ 3 & 5 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 - 3R_1 = R_2} \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & -1 & 3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & -1 & 3 & 1 \end{bmatrix} \xrightarrow{-1 \cdot R_2 = R_2} \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -3 & -1 \end{bmatrix} \xrightarrow{R_1 - 2R_2 = R_1} \begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

zero above all leading 1's

$$\left.\begin{aligned} x_1 + 5x_3 &= 2 \\ x_2 - 3x_3 &= -1 \end{aligned}\right\} \longrightarrow \begin{aligned} x_1 &= 2 - 5t \\ x_2 &= 3t - 1 \\ x_3 &= t \end{aligned}$$

the system has infinite solutions

ex: given the following system, use gauss-elimination

$$\begin{cases} x - y + 3z = 1 \\ 2x + y = 5 \\ -x - 5y + 9z = -7 \end{cases} \longrightarrow \begin{bmatrix} 1 & -1 & 3 & 1 \\ 2 & 1 & 0 & 5 \\ -1 & -5 & 9 & -7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 3 & 1 \\ 2 & 1 & 0 & 5 \\ -1 & -5 & 9 & -7 \end{bmatrix} \xrightarrow[R_3 + R_1 = R_3]{R_2 - 2R_1 = R_2} \begin{bmatrix} 1 & -1 & 3 & 1 \\ 0 & 3 & -6 & 3 \\ 0 & -6 & 12 & -6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 3 & 1 \\ 0 & 3 & -6 & 3 \\ 0 & -6 & 12 & -6 \end{bmatrix} \xrightarrow{R_2 \cdot \frac{1}{3} = R_2} \begin{bmatrix} 1 & -1 & 3 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & -6 & 12 & -6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 3 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & -6 & 12 & -6 \end{bmatrix} \xrightarrow{R_3 + 6R_2 = R_3} \begin{bmatrix} 1 & -1 & 3 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\boxed{t \in \mathbb{R}}$

$$\left.\begin{aligned} x - y + 3z &= 1 \\ y - 2z &= 1 \end{aligned}\right\} \Longrightarrow$$

Let $z = t$

$y = 1 + 2z = 1 + 2t$

$x = 1 + y - 3z = 2t$

$$\begin{cases} x = 2 - t \\ y = 1 + 2t \\ z = t \end{cases}$$

$\boxed{\text{inf. soln.}}$

ex: Solve the system using gauss elimination

$$\begin{cases} x + 5y = 7 \\ -2x - 7y = -5 \end{cases}$$

$$\begin{bmatrix} 1 & 5 & 7 \\ -2 & -7 & -5 \end{bmatrix} \xrightarrow{R_2 + 2R_1 = R_2} \begin{bmatrix} 1 & 5 & 7 \\ 0 & 3 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & 7 \\ 0 & 3 & 9 \end{bmatrix} \xrightarrow{R_2 \cdot \frac{1}{3} = R_2} \begin{bmatrix} 1 & 5 & 7 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\underbrace{\begin{matrix} x + 5z = 7 \\ z = 3 \end{matrix}}_{x = -8}$$

ex:

$$\left.\begin{matrix} x + y + z = 150 \\ x + 2y + 3z = 100 \\ 2x + 3y + 4z = 200 \end{matrix}\right\} \longrightarrow \begin{bmatrix} 1 & 1 & 1 & 150 \\ 1 & 2 & 3 & 100 \\ 2 & 3 & 4 & 200 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 150 \\ 1 & 2 & 3 & 100 \\ 2 & 3 & 4 & 200 \end{bmatrix} \xrightarrow[R_3 - 2R_1 = R_3]{R_2 - R_1 = R_2} \begin{bmatrix} 1 & 1 & 1 & 150 \\ 0 & 1 & 2 & -50 \\ 0 & 1 & 2 & -100 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 150 \\ 0 & 1 & 2 & -50 \\ 0 & 1 & 2 & -100 \end{bmatrix} \xrightarrow{R_3 - R_2 = R_3} \begin{bmatrix} 1 & 1 & 1 & 150 \\ 0 & 1 & 2 & -50 \\ 0 & 0 & 0 & -50 \end{bmatrix}$$

$0 \neq -50$

no soln

ex: gauss jordan (rref)

$$\left.\begin{matrix} x_1 - 2x_2 + x_3 - 3x_4 = 0 \\ 3x_1 - 6x_2 + 2x_3 - 7x_4 = 0 \end{matrix}\right\} \quad \begin{bmatrix} 1 & -2 & 1 & -3 & 0 \\ 3 & -6 & 2 & -7 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 & -3 & 0 \\ 3 & -6 & 2 & -7 & 0 \end{bmatrix} \xrightarrow{R_2 - 3R_1 = R_2} \begin{bmatrix} 1 & -2 & 1 & -3 & 0 \\ 0 & 0 & -1 & 2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 & -3 & 0 \\ 0 & 0 & -1 & 2 & 0 \end{bmatrix} \xrightarrow{-R_2 = R_2} \begin{bmatrix} 1 & -2 & 1 & -3 & 0 \\ 0 & 0 & 1 & -2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 & -3 & 0 \\ 0 & 0 & 1 & -2 & 0 \end{bmatrix} \xrightarrow{R_1 - R_2 = R_1} \begin{bmatrix} 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & -2 & 0 \end{bmatrix}$$

needs to be 0

$$\begin{cases} x_1 - 2x_2 - x_4 = 0 \\ x_3 - 2x_4 = 0 \end{cases} \Longrightarrow x_3 = 2x_4$$

Let $x_2 = s$

Let $x_4 = t$

$x_1 = 2x_2 + x_4 = 2s + t$

$\boxed{\text{inf. soln.}}$

Homogenous systems of linear equations

↳ system in which each constant term is zero

↳ must have atleast one soln. which is all zeros. this solution is called trivial solution.

ex:

$$\begin{cases} x_1 - 2x_2 = 0 \\ 2x_1 + 3x_2 = 0 \end{cases}$$

this system is homogenous

Diagonal Matrix

all entries are on the diagonal

ex: $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$\begin{bmatrix} 1 & 4 & 1 \\ 0 & 6 & 4 \\ 0 & 0 & 1 \end{bmatrix}$ upper triangular matrix $\rightarrow$ all entries below the diagonal are 0

$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 6 & 0 \\ 1 & 4 & 1 \end{bmatrix}$ lower triangular matrix $\rightarrow$ all entries above the diagonal are 0

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ex: unrelated to diagonal matrices

let $A = \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 3 \\ 2 & 2 \end{bmatrix}$, $3(X+B) = A^T$

$$3(X+B) = A^T$$

$$X = \frac{1}{3}A^T - B$$

$$X = \frac{1}{3}\begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix} - \begin{bmatrix} -1 & 3 \\ 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4/3 & -8/3 \\ -4/3 & -2/3 \end{bmatrix}$$

ex:

$$X^T - A = 2B$$

$$X^T = 2B + A$$

$$(X^T)^T = (2B+A)^T$$

$$X = (2B+A)^T$$

$$X = \left(2\begin{bmatrix} -1 & 3 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix}\right)^T$$

$$X = \begin{bmatrix} -1 & 8 \\ 5 & 8 \end{bmatrix}^T = \begin{bmatrix} -1 & 5 \\ 8 & 8 \end{bmatrix}$$

ex:

Find $(A+B)^2$

method 1 $A+B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} -1 & 3 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ 3 & 6 \end{bmatrix}$

$$(A+B)^2 = \begin{bmatrix} 0 & 5 \\ 3 & 6 \end{bmatrix} \cdot \begin{bmatrix} 0 & 5 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 15 & 30 \\ 18 & 51 \end{bmatrix}$$

$0 \cdot 0 + 5 \cdot 3 = 15$

method 2 $(a+b)^2 = a^2 + 2ab + b^2$

for matrices $\rightarrow (A+B)^2 \neq A^2 + 2AB + B^2$

$$(A+B)^2 = (A+B)(A+B) = AA + AB + BA + BB$$

ex:

$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 2 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 0 \end{bmatrix}$ $C = \begin{bmatrix} 2 & -4 \\ 1 & 6 \end{bmatrix}$

1) $BA + C^T - 2I = \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 2 & 2 \end{bmatrix}} + \begin{bmatrix} 2 & -4 \\ 1 & 6 \end{bmatrix}^T - 2 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 3 & 6 \\ 1 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ -4 & 6 \end{bmatrix} + \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ -3 & 8 \end{bmatrix}$$

2) $(2X+B)^T = A$

$2X + B = A^T$

$2X = A^T - B$

$X = \frac{1}{2}(A^T - B)$ — — — solve onwards

Inverse of a Matrix

for rational numbers

inverse of $2 \longrightarrow \frac{1}{2}$

$$2 \cdot \frac{1}{2} = 1$$

for matrices

inverse of $A \longrightarrow B$

$$A \cdot B = \mathbf{I} \left(\begin{matrix} \text{identity} \\ \text{matrix} \end{matrix} \right)$$

Definition: Let A be $(n \times n)$ matrix

If there exists a matrix B such that $A \cdot B = B \cdot A = I_n$

then A is invertible and B is the inverse of A.

$$\boxed{A^{-1} = B}$$

ex: $A = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$

$$\underbrace{A \cdot B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad B \cdot A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\text{therefore } B = A^{-1}}$$

Inverse of a (2×2) matrix

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a 2×2.

Then A is invertible if $ad - bc \neq 0$

$$\boxed{A^{-1} = \frac{1}{ad-bc} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$$

★ not invertible
= singular matrix

ex:

Find the inverse of $A = \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$ if possible

$$\begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$$

$3 \cdot 2 - (-1) \cdot (-2) = 4 \neq 0$

A is invertible

$$A^{-1} = \frac{1}{4} \cdot \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/4 \\ 1/2 & 3/4 \end{bmatrix}$$

ex:

Find the inverse of $B = \begin{bmatrix} 3 & -1 \\ -6 & 2 \end{bmatrix}$ if possible

$$ad - bc = 3 \cdot 2 - (-6) \cdot (-1) = 6 - 6 = 0$$

B is not invertible $\rightarrow$ B is a singular matrix

Inverse of a square matrix

$$[A \,\vdots\, I] \xrightarrow[\text{row operations}]{\text{using elementary}} [I \,\vdots\, A^{-1}]$$

ex:

Find the inverse of $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix}$ if possible

make 0 below

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 + 6R_1 = R_3]{R_2 - R_1 = R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & -4 & 3 & 6 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & -4 & 3 & 6 & 0 & 1 \end{array}\right] \xrightarrow{R_3 + 4R_2 = R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 & 4 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 & 4 & 1 \end{array}\right] \xrightarrow{-1\cdot R_3 = R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right]$$

create leading 1

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right] \xrightarrow{R_2 + R_3 = R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right]$$

zeros above

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right] \xrightarrow{R_1 + R_2 = R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & -1 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right]$$

zeros above leading 1

I $\qquad$ A^{-1}

ex: Find the inverse of $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ -2 & 3 & -2 \end{bmatrix}$ if possible

$$[A : I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 3 & -1 & 2 & 0 & 1 & 0 \\ -2 & 3 & -2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 + 2R_1]{R_2 - 3R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -7 & 2 & -3 & 1 & 0 \\ 0 & 7 & -2 & 2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 + R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -7 & 2 & -3 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right]$$

The matrix A is singular

Properties of Inverse matrices

A invertible, K = positive integer, $c \neq 0$ scalar, then:

- $(A^{-1})^{-1} = A$
- $(A^{K})^{-1} = (A^{-1})^{K} = (A^{-1})(A^{-1}) \dots (A^{-1}) = A^{-K}$
- $(c \cdot A)^{-1} = \frac{1}{c} \cdot A^{-1}$
- $(A^{T})^{-1} = (A^{-1})^{T}$

ex: let $A = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}$, Find A^{-2}

$$A^{-2} = (A^{-1})^{2} = (A^{2})^{-1}$$

method 1

$$A^{2} = A \cdot A = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} = \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} 3 & 5 \\ 10 & 18 \end{bmatrix} \\ \begin{array}{cc} c & d \end{array} \end{array}$$

$$(A^{2})^{-1} \longrightarrow ad - bc = 3 \cdot 18 - 5 \cdot 10 = 4 \neq 0$$

$$A^{-1} = \frac{1}{ad-bc} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{3 \cdot 18 - 5 \cdot 10} \begin{bmatrix} 18 & -10 \\ -5 & 3 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 18 & -10 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 9/2 & -5/4 \\ -5/2 & 3/4 \end{bmatrix}$$

method 2

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 4 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1/2 \\ -1 & 1/2 \end{bmatrix}$$

$$(A^{-1})^2 = A^{-1}\cdot A^{-1} = \begin{bmatrix} 2 & -1/2 \\ -1 & 1/2 \end{bmatrix}\begin{bmatrix} 2 & -1/2 \\ -1 & 1/2 \end{bmatrix} = \begin{bmatrix} 9/2 & -5/4 \\ -5/2 & 3/4 \end{bmatrix}$$

Inverse of Product

Let A, B be invertible matrices of size n, then $A\cdot B$ is also invertible and $(A\cdot B)^{-1} = B^{-1}\cdot A^{-1}$

order is reversed !

ex: Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}$ Find $(AB)^{-1}$

$$A^{-1} = \frac{1}{3-2}\cdot\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

$$B^{-1} = \frac{1}{4-2}\cdot\begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1/2 \\ -1 & 1 \end{bmatrix}$$

$$B^{-1}\cdot A^{-1} = \begin{bmatrix} 7/2 & -5/2 \\ -4 & 3 \end{bmatrix}$$

theorem for systems with only one soln.

If A is invertible, then the system of linear equations $Ax = b$ has a unique solution given by $\boxed{x = A^{-1}b}$

ex:

$$\begin{cases} 2x + 3y - z = 1 \\ 3x + 3y + z = 1 \\ 2x + 4y + z = -2 \end{cases}$$

$$\underbrace{\begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}}_{b}$$

$$A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}$$

$\downarrow$

after

$$[A \vdots I] = [I \vdots A^{-1}]$$

$$\left[\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 & 1 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{1/2\, R_1} \left[\begin{array}{ccc|ccc} 1 & 3/2 & 1/2 & 1/2 & 0 & 0 \\ 3 & 3 & 1 & 0 & 1 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_2 - 3R_1 = R_2$

$R_3 - 2R_1 = R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 3/2 & 1/2 & 1/2 & 0 & 0 \\ 0 & -3/2 & -1/2 & -3/2 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right] \xrightarrow{\text{swap } R_2 \,\&\, R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 3/2 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & -3/2 & -1/2 & -3/2 & 1 & 0 \end{array}\right] \xrightarrow[R_1 - \frac{3}{2} R_2 = R_1]{R_3 + \frac{3}{2} R_2 = R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1/2 & 2 & 0 & -3/2 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1/2 & -3 & 1 & 3/2 \end{array}\right] \xrightarrow{R_3 \cdot -2 = R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1/2 & 2 & 0 & -3/2 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 6 & -2 & -3 \end{array}\right] \xrightarrow{R_1 - \frac{1}{2} R_3 = R_1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 6 & -2 & -3 \end{array}\right]$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}}_{A^{-1}} \cdot \underset{\substack{\downarrow \\ b}}{\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}}$$

$$= \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix}$$

$$\boxed{X = \frac{b}{A}}$$

ex:

$$\begin{cases} 2x+3y+z=4 \\ 3x+3y+z=8 \\ 2x+4y+z=5 \end{cases}$$

$$\underbrace{\begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 4 \\ 8 \\ 5 \end{bmatrix}}_{b}$$

$$A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix} \Longrightarrow x = \underbrace{\begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}}_{A^{-1}} \cdot \underbrace{\begin{bmatrix} 4 \\ 8 \\ 5 \end{bmatrix}}_{b} = \underbrace{\begin{bmatrix} 4 \\ 1 \\ -7 \end{bmatrix}}_{\text{soln.}}$$

Remark!

$AX=0 \rightarrow$ at least 1 soln. which is the trivial soln. (0)

since $X = A^{-1} \cdot b$ if A is invertible, the system has only one solution which is the trivial soln. (zero)

---

ex:

$$\begin{cases} x+2y-3z=0 \\ 2x+5y+2z=0 \\ 3x-y-4z=0 \end{cases}$$

$$\underbrace{\begin{bmatrix} 1 & 2 & -3 \\ 2 & 5 & 2 \\ 3 & -1 & -4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{b}$$

$$A^{-1} = \begin{bmatrix} -18/61 & 11/61 & 19/61 \\ 14/61 & 5/61 & -8/61 \\ -17/61 & 7/61 & 1/61 \end{bmatrix}$$

↓ done by prof

$X = A^{-1} \cdot b = 0$

$x=0, \ y=0, \ z=0$

ex: Given $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, Find X such that $(A^T)^{-1} + 2X = I$

$$2X = I - (A^T)^{-1}$$

$$X = \frac{1}{2}\left(I - (A^T)^{-1}\right)$$

$$A^T = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$(A^T)^{-1} = \frac{1}{1\cdot 1 - 2\cdot 0}\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

$$X = \frac{1}{2}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Find $(2A)^{-3}$

$$2\cdot A = \begin{bmatrix} 2 & 4 \\ 0 & 2 \end{bmatrix}$$

$$(2\cdot A)^{-1} = \frac{1}{2\cdot 2 - 4\cdot 0}\begin{bmatrix} 2 & -4 \\ 0 & 2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2 & -4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -1 \\ 0 & 1/4 \end{bmatrix}$$

$$((2\cdot A)^{-1})^3 = \begin{bmatrix} 1/2 & -1 \\ 0 & 1/4 \end{bmatrix}\begin{bmatrix} 1/2 & -1 \\ 0 & 1/4 \end{bmatrix}\begin{bmatrix} 1/2 & -1 \\ 0 & 1/4 \end{bmatrix} = \begin{bmatrix} 1/8 & -3/4 \\ 0 & 1/8 \end{bmatrix}$$

Determinant of a matrix

Every $A_{(n\times n)}$ can be associated with a real number called its **determinant**, denoted by $|A|$ or $\det(A)$

Determinant of a 2x2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \det(A) = ad - bc$$

ex: $A = \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}$

$$\det(A) = 4 - (-3) = 7$$

a determinant can be $+$, 0, $-$

Determinant of a Triangular Matrix

If A is a triangular matrix of order n, the determinant is the product of entries on main diagonal

$$|A| = a_{11} \cdot a_{22} \cdots \cdot a_{nn}$$

ex:

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ 5 & 6 & 1 & 0 \\ 1 & 5 & 3 & 3 \end{bmatrix} \longrightarrow$$

lower triangular matrix
(the lower part forms a triangle)

$$|A| = 2 \cdot (-2) \cdot 3 = -12$$

Minors and Cofactors of a matrix

- If $A_{(n \times n)}$ then the minor (M_{ij}) of the element a_{ij} is the determinant of the matrix obtained by deleting the i^{th} row and j^{th} column of A
- The cofactor c_{ij} is given by $c_{ij} = (-1)^{i+j} M_{ij}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & \boxed{a_{22}} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\underline{\underline{C_{ij}}} = (-1)^{i+j} + \underline{\underline{M_{ij}}}$$

equal except the sign

$a_{22} \rightarrow M_{22}$

ex.

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$$

$$M_{11} = \det \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix} = \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1$$

$$C_{11} = (-1)^{1+1} M_{11} \implies C_{11} = M_{11} = -1$$

$$M_{12} = \det \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = 3 \cdot 1 - 4 \cdot 2 = -5$$

$$C_{12} = (-1)^{1+2} M_{12} = -M_{12} \implies C_{12} = -M_{12} = 5$$

$$M_{13} = \det \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix} = \begin{vmatrix} 3 & -1 \\ 4 & 0 \end{vmatrix} = 3 \cdot 0 - 4 \cdot (-1) = 4$$

$$C_{13} = (-1)^{1+3} M_{13} = M_{13} \implies C_{13} = M_{13} = 4$$

Determinant of a matrix

The determinant of a square matrix is the sum of the entries in the 1^{st} row multiplied by their cofactors

ex: A is the previous example

$$a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13}$$

$$|A| = 0(-1) + 2(5) + 1(4)$$

$$\boxed{\det(A) = 14}$$

any row or column would do

$\star$ we use the row with more zeros bc. easier

ex:

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$$

expansion through row 1

$$\det(A) = 0 - 2\begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} + 1\begin{vmatrix} 3 & -1 \\ 4 & 0 \end{vmatrix} = 14$$

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$$

expansion through column 1

$$\det(A) = 0 - 3\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} + 4\begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = -3\cdot 2 + 4\cdot 5$$
$$= 14$$

ex:

$$A = \begin{bmatrix} 1 & -2 & 3 & 0 \\ -1 & 1 & 0 & 2 \\ 0 & 2 & 0 & 3 \\ 3 & 4 & 0 & -2 \end{bmatrix}$$

expansion through 3rd column

$$\det(A) = 3\begin{vmatrix} -1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 4 & -2 \end{vmatrix}$$

$$\begin{vmatrix} -1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 4 & -2 \end{vmatrix} = -1\begin{vmatrix} 2 & 3 \\ 4 & -2 \end{vmatrix} + 3\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}$$
$$= -1\left[(2\cdot(-2)) - (3\cdot 4)\right] + 3\left[(3\cdot 1) - (2\cdot 2)\right]$$
$$= +16 - 3 = 13$$

$$\det(A) = 3\cdot 13 = 39$$

ex:

Find the det of

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & -1 & 2 \\ 3 & 1 & 1 \end{bmatrix}$$

$$|A| = 1\begin{vmatrix} -1 & 2 \\ 1 & 1 \end{vmatrix} - 0 + 2\begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix}$$
$$= -3 + 8 = 5$$

Zero Determinant

$\det(A) = 0$ if:

- An entire row/column consists of zeros.
- A row/column is a multiple of another
- Two row/columns are equal

ex:

$$A = \begin{bmatrix} -1 & 2 & 2 \\ 3 & -6 & 4 \\ 5 & -10 & -3 \end{bmatrix}$$

$\text{column}_2 = -2 \cdot \text{column}_1$

$\therefore \det(A) = 0$

$$B = \begin{bmatrix} \overset{+}{0} & 4 & 4 \\ \overset{-}{2} & -1 & 0 \\ \overset{+}{0} & 4 & 4 \end{bmatrix}$$

$R_1 = R_3 \quad \therefore \quad \det(B) = 0$

lets test it $\longrightarrow -2 \cdot \begin{vmatrix} 4 & 4 \\ 4 & 4 \end{vmatrix} = -2 \cdot 0 = 0$

$$C = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 4 \\ 3 & 5 & 0 \end{bmatrix}$$

$\det(C) = 0$

Elementary Row Operations and Determinant

If A and B are square matrices,

1. If B is obtained from A by interchanging 2 rows of A, then $\det(A) = -\det(B)$

ex:

$$|A| = \begin{vmatrix} 2 & -3 \\ 1 & 4 \end{vmatrix} = 2 \cdot 4 + 3 = 11$$

$$|B| = \begin{vmatrix} 1 & 4 \\ 2 & -3 \end{vmatrix} = -3 - 8 = -11$$

$\det(A) = -\det(B)$

$11 = -(-11)$ ✓

② If B is obtained from A by adding a multiple of row A to another row of A, then $\det(B) = \det(A)$

ex:

$$|A| = \begin{vmatrix} 1 & -3 \\ 2 & -4 \end{vmatrix} = 2 \xrightarrow{-2R_1 + R_2 = R_2} |B| = \begin{vmatrix} 1 & -3 \\ 0 & 2 \end{vmatrix} = 2$$

③ If B is obtained from A by multiplying a row of A by a non-zero constant, $\det(B) = c \cdot \det(A)$

ex:

$$|A| = \begin{vmatrix} 2 & -8 \\ -2 & 9 \end{vmatrix} = 2 \cdot 9 - 16 = 2 \xrightarrow{\frac{1}{2}R_1 = R_1} |B| = \begin{vmatrix} 1 & -4 \\ -2 & 9 \end{vmatrix} = 9 - 8 = 1$$

ex:

$$A = \begin{bmatrix} 2 & -3 & 10 \\ 1 & 2 & -2 \\ 0 & 1 & -3 \end{bmatrix}$$

$$\begin{vmatrix} 2 & -3 & 10 \\ 1 & 2 & -2 \\ 0 & 1 & -3 \end{vmatrix} \xrightarrow{R_1 \leftrightarrow R_2} - \begin{vmatrix} 1 & 2 & -2 \\ 2 & -3 & 10 \\ 0 & 1 & -3 \end{vmatrix}$$

$$- \begin{vmatrix} 1 & 2 & -2 \\ 2 & -3 & 10 \\ 0 & 1 & -3 \end{vmatrix} \xrightarrow{R_2 - 2R_1 \to R_2} - \begin{vmatrix} 1 & 2 & -2 \\ 0 & -7 & 14 \\ 0 & 1 & -3 \end{vmatrix} \xrightarrow{R_2 \leftrightarrow R_3} + \begin{vmatrix} 1 & 2 & -2 \\ 0 & 1 & -3 \\ 0 & -7 & 14 \end{vmatrix}$$

$$+ \begin{vmatrix} 1 & 2 & -2 \\ 0 & 1 & -3 \\ 0 & -7 & 14 \end{vmatrix} \xrightarrow{R_3 + 7R_2 \to R_3} + \begin{vmatrix} 1 & 2 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & -7 \end{vmatrix}$$

lower triangular matrix
↳ entries on the main diag. are $= \det(A)$

$\det = -7$

ex:

$$A = \begin{bmatrix} 1 & 4 & 1 \\ 2 & -1 & 0 \\ 0 & 18 & 4 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 4 & 1 \\ 2 & -1 & 0 \\ 0 & 18 & 4 \end{vmatrix} \xrightarrow{R_2 - 2R_1 \to R_2} + \begin{vmatrix} 1 & 4 & 1 \\ 0 & -9 & -2 \\ 0 & 18 & 4 \end{vmatrix}$$

$\det(A) = 0$

but lets continue as if we didn't notice that $-2 \cdot R_2 = R_3$

$$+\begin{vmatrix} 1 & 4 & 1 \\ 0 & -9 & -2 \\ 0 & 18 & 4 \end{vmatrix} \xrightarrow{R_3 + 2R_2 \to R_3} + \begin{vmatrix} 1 & 4 & 1 \\ 0 & -9 & -2 \\ 0 & 0 & 0 \end{vmatrix}$$

$\det(A) = 1 \cdot (-9) \cdot 0 = 0$

ex:

$$A = \begin{bmatrix} -3 & 5 & 2 \\ 2 & -4 & -1 \\ -3 & 0 & 6 \end{bmatrix}$$

Using column operations to create more zeros in row 3, find $|A|$ after

$$\xrightarrow{C_3 + 2C_1 = C_3} \begin{vmatrix} -3 & 5 & -4 \\ 2 & -4 & 3 \\ -3 & 0 & 0 \end{vmatrix}$$

$$-3 \begin{vmatrix} 5 & -4 \\ -4 & 3 \end{vmatrix} - 0 + 0 = -3\left[5 \cdot 3 - (-4 \cdot -4)\right]$$

$$= -3 \cdot (15 - 16) = +3$$

ex:

$$A = \begin{bmatrix} 2 & 0 & 1 & 3 & -2 \\ -2 & 1 & 3 & 2 & -1 \\ 1 & 0 & -1 & 2 & 3 \\ 3 & -1 & 2 & 4 & -3 \\ 1 & 1 & 3 & 2 & 0 \end{bmatrix}$$

easier to work with since more zeros

Find det (A)

$$|A| = \begin{vmatrix} 2 & 0 & 1 & 3 & -2 \\ -2 & 1 & 3 & 2 & -1 \\ 1 & 0 & -1 & 2 & 3 \\ 3 & -1 & 2 & 4 & -3 \\ 1 & 1 & 3 & 2 & 0 \end{vmatrix} \xrightarrow[R_5 - R_2 \to R_5]{R_4 + R_2 \to R_4} \begin{vmatrix} 2 & 0 & 1 & 3 & -2 \\ -2 & 1 & 3 & 2 & -1 \\ 1 & 0 & -1 & 2 & 3 \\ 1 & 0 & 5 & 6 & -4 \\ 3 & 0 & 0 & 0 & 1 \end{vmatrix}$$

$$\begin{vmatrix} 2 & 0 & 1 & 3 & -2 \\ -2 & 1 & 3 & 2 & -1 \\ 1 & 0 & -1 & 2 & 3 \\ 1 & 0 & 5 & 6 & -4 \\ 3 & 0 & 0 & 0 & 1 \end{vmatrix} = 1 \begin{vmatrix} 2 & 1 & 3 & -2 \\ 1 & -1 & 2 & 3 \\ 1 & 5 & 6 & -4 \\ 3 & 0 & 0 & 1 \end{vmatrix} \xrightarrow{C_1 - 3C_4 = C_1} \begin{vmatrix} 8 & 1 & 3 & -2 \\ -8 & -1 & 2 & 3 \\ 13 & 5 & 6 & -4 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$\begin{vmatrix} 8 & 1 & 3 & -2 \\ -8 & -1 & 2 & 3 \\ 13 & 5 & 6 & -4 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1 \begin{vmatrix} 8 & 1 & 3 \\ -8 & -1 & 2 \\ 13 & 5 & 6 \end{vmatrix} \xrightarrow{R_2 + R_1 = R_2} \begin{vmatrix} 8 & 1 & 3 \\ 0 & 0 & 5 \\ 13 & 5 & 6 \end{vmatrix}$$

$$\begin{vmatrix} 8 & 1 & 3 \\ 0 & 0 & 5 \\ 13 & 5 & 6 \end{vmatrix} = -5 \begin{vmatrix} 8 & 1 \\ 13 & 5 \end{vmatrix} = -5 \cdot (8 \cdot 5 - 13 \cdot 1) = -5 \cdot 27 = -135$$

Properties of Determinants

#1 - If A and B are square matrices of size n, then $\det(AB) = \det(A) \cdot \det(B)$

ex:

$|A| = \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} = 2$ $\quad A \cdot B = \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} \cdot \begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 2 & 7 \end{vmatrix} = 14 - 2 = 12$

$|B| = \begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} = 6$ $\quad |AB| = 2 \cdot 6 = 12$

#2 - Determinant of a scalar multiple of a matrix

$$\det(c \cdot A_{(n\times n)}) = c^n \cdot \det(A_{(n\times n)})$$

ex:

$$A = \begin{bmatrix} 1 & 5 & -10 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix} \qquad |A| = 3\cdot 2\cdot 1 = 6$$

$$B = 2A = \begin{bmatrix} 2 & 10 & -20 \\ 0 & 4 & 4 \\ 0 & 0 & 6 \end{bmatrix} \qquad |B| = 2^3 \cdot 6 = 8\cdot 6 = 48$$
$$= 2\cdot 4 \cdot 6 = 48$$

#3 - A square matrix $A_{(n\times n)}$ is invertible if $\det(A) \neq 0$

$\boxed{\star \text{ not invertible } = \text{ singular matrix}}$

ex: Find if invertible.

$$A = \begin{bmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & -1 \end{bmatrix} \qquad \text{since } C_2 = -2\cdot C_1 \longrightarrow \det(A) = 0$$

A is singular

#4 - Determinant of an inverse matrix:

If A is invertible, then $\det(A^{-1}) = \dfrac{1}{\det(A)}$

ex:

$$A = \begin{bmatrix} 2 & 1 \\ -2 & 2 \end{bmatrix} \qquad |A| = 6 \neq 0 \qquad A \text{ is invertible}$$

$$A^{-1} = \frac{1}{ad-bc} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1/3 & -1/6 \\ 1/3 & 1/3 \end{bmatrix}$$

$$|A^{-1}| = \frac{1}{3}\cdot\frac{1}{3} - \left(\frac{-1}{6}\cdot\frac{1}{3}\right) = \frac{1}{9} + \frac{1}{18} = \frac{3}{18} = \frac{1}{6}$$

#5 - Determinant of a transpose

If A is a square matrix then $\det(A) = \det(A^T)$

ex:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 1 \\ 0 & 0 & 2 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -4 & 0 \\ 3 & 1 & 2 \end{bmatrix}$$

$\longrightarrow |A| = -8 \qquad \longrightarrow |A^T| = -8$

If A is a square matrix, the following are equivalent:

1) A is invertible
2) $\det(A) \neq 0$
3) $Ax = 0$ has a trivial solution
4) $Ax = b$ has one unique solution which is $x = A^{-1} \cdot b$

ex:

Which of the following systems have a unique soln?

$$\begin{cases} 2x_2 - x_3 = -1 \\ 3x_1 - 2x_2 + x_3 = 4 \\ 3x_1 + 2x_2 - x_3 = -4 \end{cases} \Rightarrow \begin{bmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & -1 \end{bmatrix}$$

since $C_2 = -2 \cdot C_3$

$\det(A) = 0$

the system doesn't have a unique soln.

$$\begin{cases} 2x_2 - x_3 = -1 \\ 3x_1 - 2x_2 + x_3 = 4 \\ 3x_1 + 2x_2 + x_3 = -4 \end{cases} \Rightarrow \begin{bmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

$\det(A) = 12 \neq 0$

the system has unique soln.

Applications of Determinants

If A is a square matrix, then the matrix of cofactors of A has the form:

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \\ C_{n1} & C_{n2} & & C_{nn} \end{bmatrix}$$

The **transpose** of this matrix is called the **adjoint** of A denoted by $adj(A)$

ex: Find the adjoint of matrix $A = \begin{bmatrix} -1^{+} & 3^{-} & 2^{+} \\ 0^{-} & -2^{+} & 1^{-} \\ 1^{+} & 0^{-} & -2^{+} \end{bmatrix}$

$$\begin{bmatrix} +\begin{vmatrix} -2 & 1 \\ 0 & -2 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 1 & -2 \end{vmatrix} & +\begin{vmatrix} 0 & -2 \\ 1 & 0 \end{vmatrix} \\ -\begin{vmatrix} 3 & 2 \\ 0 & -2 \end{vmatrix} & +\begin{vmatrix} -1 & 2 \\ 1 & -2 \end{vmatrix} & -\begin{vmatrix} -1 & 3 \\ 1 & 0 \end{vmatrix} \\ +\begin{vmatrix} 3 & 2 \\ -2 & 1 \end{vmatrix} & -\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} & +\begin{vmatrix} -1 & 3 \\ 0 & -2 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 4 & 1 & 2 \\ 6 & 0 & 3 \\ 7 & 1 & 2 \end{bmatrix}$$

$$C^T = \begin{bmatrix} 4 & 1 & 2 \\ 6 & 0 & 3 \\ 7 & 1 & 2 \end{bmatrix}^T = \begin{bmatrix} 4 & 6 & 7 \\ 1 & 0 & 1 \\ 2 & 3 & 2 \end{bmatrix} = adj(A)$$

Inverse of a matrix using adjoint

If A is $(n \times n)$ invertible, then $\boxed{A^{-1} = \frac{1}{\det(A)} \cdot \operatorname{adj}(A)}$

ex: A from prev. ex.

If $\det(A) = 3$, $A^{-1} = \frac{1}{3}\begin{bmatrix} 4 & 6 & 7 \\ 1 & 0 & 1 \\ 2 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 4/3 & 2 & 7/3 \\ 1/3 & 0 & 1/3 \\ 2/3 & 1 & 2/3 \end{bmatrix}$

Cramer's Rule

A = coefficient $\longrightarrow \det(A) \neq 0$

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_3)}{\det(A)} \quad \dots \quad x_n = \frac{\det(A_n)}{\det(A)}$$

ex:
$$\begin{cases} 4x_1 - 2x_2 = 10 \\ 3x_1 - 5x_2 = 11 \end{cases} \qquad A = \begin{bmatrix} 4 & -2 \\ 3 & -5 \end{bmatrix} \longrightarrow |A| = -14$$

$$A_1 = \begin{bmatrix} 10 & -2 \\ 11 & -5 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 4 & 10 \\ 3 & 11 \end{bmatrix} \longrightarrow \begin{matrix} |A_1| = -28 \\ |A_2| = 14 \end{matrix}$$

ex:

Use Cramer's rule to solve

$$\begin{cases} -x + 2y - 3z = 1 \\ 2x + z = 0 \\ 3x - 4y + 4z = 2 \end{cases} \qquad A = \begin{bmatrix} -1 & 2 & -3 \\ 2 & 0 & 1 \\ 3 & -4 & 4 \end{bmatrix} \qquad |A| = 10$$

$$A_1 = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 1 \\ 2 & -4 & 4 \end{bmatrix} \qquad A_2 = \begin{bmatrix} -1 & 1 & -3 \\ 2 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} \qquad A_3 = \begin{bmatrix} -1 & 2 & 1 \\ 2 & 0 & 0 \\ 3 & -4 & 2 \end{bmatrix}$$

$$|A_1| = -\begin{vmatrix} 1 & 2 \\ 2 & -4 \end{vmatrix} = 8 \qquad |A_2| = -15 \qquad |A_3| = -16$$

$$x = \frac{|A_1|}{|A|} = \frac{8}{10} = \frac{4}{5} \qquad y = \frac{|A_2|}{|A|} = \frac{-15}{10} = \frac{-3}{2} \qquad z = \frac{|A_3|}{|A|} = \frac{-16}{10} = \frac{-8}{5}$$

ex: $A = \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 3 \\ 2 & 2 \end{bmatrix}$, find X $\quad AX - 3B = 2I_2$

$$\boxed{A \cdot A^{-1} = I}$$

$$AX - 3B = 2I_2$$

$$AX = 2I_2 + 3B$$

$$X = (2I_2 + 3B) \cdot A^{-1}$$

$$X = \begin{bmatrix} 2 & -1 \\ -1/2 & 1/2 \end{bmatrix} \cdot \left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} -3 & 9 \\ 6 & 6 \end{bmatrix} \right)$$

$$X = \begin{bmatrix} 2 & -1 \\ -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} -1 & 9 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} -8 & 10 \\ 7/2 & -1/2 \end{bmatrix}$$

$$\boxed{A^{-1} = \frac{1}{ad - bc} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$$

$$A^{-1} = \frac{1}{1 \cdot 4 - 2 \cdot 1} \begin{bmatrix} 4 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1/2 & 1/2 \end{bmatrix}$$

ex:

Let $A = \begin{bmatrix} 2 & m \\ m & 2 \end{bmatrix}$ for what values of m is A invertible ?

invertible $\iff$ $\det(A) \neq 0$

$$|A| = \begin{vmatrix} 2 & m \\ m & 2 \end{vmatrix} \longrightarrow 4 - m^2 \neq 0$$

$$4 \neq m^2$$

$m \neq 2$ or $m \neq -2$

$m \in \mathbb{R} - \{-2, 2\}$

ex:

Solve for n $\begin{vmatrix} x-1 & x & x+2 \\ 1 & 2 & 1 \\ 1 & x & 2 \end{vmatrix} = 0$

$$\begin{array}{c} \begin{matrix} \;\;+ & \;\;\;\;- & \;\;\;\;\;+ \end{matrix} \\ \begin{vmatrix} x-1 & x & x+2 \\ 1 & 2 & 1 \\ 1 & x & 2 \end{vmatrix} \end{array} = (x-1)\begin{vmatrix} 2 & 1 \\ x & 2 \end{vmatrix} - (x)\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} + (x+2)\begin{vmatrix} 1 & 2 \\ 1 & x \end{vmatrix}$$

$$= (x-1)(4-x) - x + (x+2)(x-2)$$

$$= 4x \cancel{- x^2} - 4 \cancel{+ x} \cancel{- x} \cancel{+ x^2} \cancel{- 2x} \cancel{+ 2x} - 4 = 4x - 8 = 0 \Longrightarrow x = 2$$

Chapter 4 : Vector Space

Let V be a set on two operations :

1 – Scalar multiplication
2 - Vector addition

the 10 rules

a $u + v \in V$ (closure)

b. $u + v = v + u$ (commuttative)

c. $u + (v + w) = (u + v) + w$ (associative)

d. there is a zero vector such that

$$0 + u = u + 0 = u \quad \text{(additive identity)}$$

e. there is $u \in V$ such that $u + (-u) = -u + u = 0$

f. $c \cdot u \in V$

g. $c(u + w) = cu + cw$

h. $c(du) = (cd)u$

i. $1(u) = u$

Proofs

The set $V = \mathbb{R}^2$ is a vector space under usual addition and scalar multiplication. Let $u, v, w \in V$

$u = (u_1, u_2)$ $v = (v_1, v_2)$ $w = (w_1, w_2)$

a $u + v = (u_1 + v_1, u_2 + v_2) \in V$

b $u + v = (u_1 + v_1, u_2 + v_2) = (v_1 + u_1, v_2 + u_2) = v + u$ ✓

c $(u+v)+w = (u_1+v_1, u_2+v_2) + (w_1, w_2)$

$= (u_1 + (v_1+w_1), u_2 + (v_2+w_2))$

$= (u_1+u_2) + (v_1+w_1, v_2+w_2) = u + (v+w)$ ✓

d $0 = (0,0) \Rightarrow u+0 = (u_1+0, u_2+0) = (u_1, u_2) = u$ ✓

e $u+(-u) = (u_1 - u_1, u_2 - u_2) = (0,0) = 0$ ✓

f $c \cdot u = c\,(u_1, u_2) = (c \cdot u_1, c \cdot u_2) \in V$ ✓

g $c\,(u+v) = c \cdot (u_1+v_1, u_2+v_2) = (cu_1 + cv_1, cu_2 + cv_2)$

$= (cu_1, cu_2) + (cv_1, cv_2) = cu + cv$ ✓

h $c\,(du) = c\,(du_1, du_2) = (cd\,(u_1), cd\,(u_2)) =$

$= cd\,(u_1, u_2) = cd\,(u)$ ✓

i. $1u = 1 \cdot (u_1, u_2) = (1 \cdot u_1, 1 \cdot u_2) = (u_1, u_2) = u$ ✓

ex:

Prove that the set of all (2×2) matrices (M_{22}) under usual matrix addition and scalar multiplication is a vector space (V).

Let $u, v, w \in V$ $\quad u = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad v = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \quad w = \begin{bmatrix} a'' & b'' \\ c'' & d'' \end{bmatrix}$

a) $u+v = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = \begin{bmatrix} a+a' & b+b' \\ c+c' & d+d' \end{bmatrix} \in V$ ✓

b) $u + v = \begin{bmatrix} a+a' & b+b' \\ c+c' & d+d' \end{bmatrix} = \begin{bmatrix} a'+a & b'+b \\ c'+c & d'+d \end{bmatrix} = v + u$ ✓

c) $(u+v) + w = \begin{bmatrix} a+a' & b+b' \\ c+c' & d+d' \end{bmatrix} + \begin{bmatrix} a'' & b'' \\ c'' & d'' \end{bmatrix} = \begin{bmatrix} a+(a'+a'') & b+(b'+b'') \\ c+(c'+c'') & d+(d'+d'') \end{bmatrix}$

$= \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a'+a'' & b'+b'' \\ c'+c'' & d'+d'' \end{bmatrix} = u + (v+w)$ ✓

d) $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Longrightarrow u + 0 = \begin{bmatrix} a+0 & b+0 \\ c+0 & d+0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = u$ ✓

e) $u + (-u) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ ✓

f) $k \cdot u = k \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix} \in V$ ✓

g) $k(u+v) = k \cdot \begin{bmatrix} a+a' & b+b' \\ c+c' & d+d' \end{bmatrix} = \begin{bmatrix} k(a+a') & k(b+b') \\ k(c+c') & k(d+d') \end{bmatrix}$

$= \begin{bmatrix} ka+ka' & kb+kb' \\ kc+kc' & kd+kd' \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix} + \begin{bmatrix} ka' & kb' \\ kc' & kd' \end{bmatrix} = ku + kv$

$(k+L)u = ku + kL \longrightarrow$

h) $k(Lu) = k \begin{bmatrix} La & Lb \\ Lc & Ld \end{bmatrix} = kL \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = kL(u)$ ✓

i) $1u = 1 \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = u$ ✓

★ In general, the set of all $(m \times n)$ matrices denoted by $M_{(m \times n)}$ is a vector space ex: The set of all polynomials of degree 2 or less denoted by P_2 is a vector space under usual addition and scalar multiplication

$a_0 + a_1 x + a_2 x^2$

$p(x) = a_0 + a_1 x + a_2 x^2 \in P_2$
$f(x) = b_0 + b_1 x + b_2 x^2 \in P_2$ $\Big\} (a_0 + b_0) + (a_1 + b_1)x \ldots$

$cP(x) = ca_0 + ca_1 x + ca_2 x^2$

$(-\infty, +\infty)$ is the set of all real valued continuous function defined where $(f+g)x = f(x) + g(x)$ and $(cf)(x) = cf(x)$ is a VS

ex: show that the set of integers is not a VS

$\mathbb{Z}$ is not closed under scalar mult.

ex: when $c = \frac{1}{2}$

$c = 1/2 \qquad u = 1 \quad \Rightarrow \quad c \cdot u = \frac{1}{2} \notin \mathbb{Z}$

ex: Let $V = \mathbb{R}^2$ and $u = (u_1, u_2)$ $v = (v_1, v_2)$ such that $u + v = (u_1 + v_1, u_2 + v_2)$ and $ku = (ku_1, 0)$. Show that V is not a vector space.

$1 \cdot u = 1 \cdot (u_1, u_2) = (u_1, 0) \neq (u_1, u_2)$

ex: Show that the set V of all second degree polynomials is not a vector space.

$$\begin{array}{r} x^2 + x + 1 \\ + \quad -x^2 + 2x + 3 \\ \hline 3x + 4 \end{array}$$

not in second degree anymore
$\downarrow$
not closed under addition

ex: Let $V = \left\{ \begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}, \; a, b \in \mathbb{R} \right\}$ with the usual matrix addition and scalar multiplication. Show whether or not V is a vector space and show proof.

$$\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix} + \begin{bmatrix} a' & 1 \\ 1 & b' \end{bmatrix} = \begin{bmatrix} a+a' & 2 \\ 2 & b+b' \end{bmatrix} \notin V$$

Section 4.3 Subspaces of Vector Spaces

Def: A non-empty subset W of a vector space V is called a subspace of V if W itself is a v.s. under the same (+) and (x) operations defined in V.

Test for subspaces:

if $W \neq \emptyset$ is a subset of V, then W is a subspace under the following conditions

check for zero first

1- if u and v are in W then $u+v \in W$

2- if u in W and c any scalar then $cu \in W$

ex: Let W be the set of all (2×2) symmetric matrices. Show that W is a subspace of the vector space $M_{2,2}$ with usual matrix addition and scalar multiplication.

1) Let $A, B \in W$

$\Rightarrow A^T = A$ and $B^T = B$

$(A+B) \in W$?

$(A+B)^T = A^T + B^T = A + B$ is symmetric ✓

2) $c \in \mathbb{R}$

$A \in W \Rightarrow A^T = A$

$c \cdot A \in W$

$(cA)^T = cA^T = cA \in W$ ✓

ex: Let W be the set of all (2×2) singular matrices. ($\det(W)=0$)
Show that W isn't subspace of the vector space $M_{2,2}$ with usual matrix addition and scalar multiplication.

$$0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in W \qquad A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

1) Let $A, B \in W$

$$\det(A+B) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0 \implies A+B \notin W$$

* W is not a subspace since W is not closed under addition

ex: Show that $W = \{(x_1, x_2) \,/\, x_1 \geq 0 \text{ and } x_2 \geq 0\}$ with standard operations is not a subspace of $\mathbb{R}^2$

$0 = (0,0) \in W$

1) Let $u(x_1, x_2) \in W$ and $v(y_1, y_2) \in W$

Then $u + v = (x_1 + y_1, x_2 + y_2) \in W$

2) Let $\left.\begin{array}{l} (1,1) \in u \\ c = -2 \end{array}\right\}$ $cu = (-2, -2) \notin W$

$\therefore$ not a subspace

ex: Show if $W = \{(x_1, x_2, 1) \,/\, x_1, x_2 \in \mathbb{R}\}$ is/ is not a subspace of $\mathbb{R}^2$ with standard operations

$(0,0,1) \in W$

2) For any $c \neq 1 \implies c(x_1, x_2, 1) = (cx_1, cx_2, c1)$
$c_2 \neq 1$

$\therefore$ not a subspace

ex: Show that $W = \{(x_1, x_1 + x_3, x_3) ; x_1, x_2 \in \mathbb{R}\}$ with standard operations is / isn't subspace of $\mathbb{R}^2$

$0 = (0,0,0) \in W$ ✓

1) Let $u = (x_1, x_1 + x_3, x_3)$ and $v = (y_1, y_1 + y_3, y_3)$

$u + v = (x_1 + y_1, x_1 + y_1 + x_3 + y_3, x_3 + y_3) \in W$ ✓

2) Let $c \in \mathbb{R}$

$c \cdot u = c(x_1, x_1 + x_3, x_3) = (cx_1, cx_1 + cx_3, cx_3)$ ✓

$\therefore$ is a subspace

ex: Show that $W = \{(x, y, z) \in \mathbb{R}^3 / 2x + y + z = 3\}$ with standard operations is / isn't subspace of $\mathbb{R}^3$

$0 \rightarrow (0,0,0) \notin W$ since $2 \cdot 0 + 0 + 0 \neq 3$ $\therefore$ not a subspace

ex: Show that $W = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 / x_1 \geq 0\}$ is standard operations is / isn't subspace of $\mathbb{R}^4$

$0 \rightarrow (0,0,0,0) \in W$

2) Let $c = -1 \Rightarrow c \cdot u = (-x_1, -x_2, -x_3, -x_4)$

$\hookrightarrow$ since new $x_1 < 0$

$\notin W$

$\therefore$ not a subspace

ex: $W = \{ (x, y, z) \in \mathbb{R}^3 \ / \ x+z > 3 \}$

Let $x=0$ and $z=0$ $\Rightarrow$ $x+z = 0 < 3$

$\therefore$ not a subspace

ex: $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2,2} \ / \ b + 2a = 0 \right\}$

$0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Longrightarrow b+2a = 0+0=0$

1) Let $u = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ and $v = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$ and $a_1 + 2b_1 = 0$ and $a_2 + 2b_2 = 0$

$u + v = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix}$ ✓

2) $k \cdot w = k \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$ $\quad k(a+2b) = 0$ ✓

W is a subspace of $M_{2,2}$ under matrix addition and scalar multiplication

$\therefore$ is a subspace

ex: $W = \{ a_0 + a_1 x + a_2 x^2 \in P_2 \ / \ a_0 = 0 \}$

$0 = 0 + 0x + 0 \cdot x^2$ ✓

1) Let $u = (u_0 + u_1 x + u_2 x^2)$ and $v = (v_0 + v_1 x + v_2 x^2)$

$u_0 \to 0$, $v_0 \to 0$

$u+v = (u_0 + v_0 + (u_1 + u_2)x + (u_2 + v_2) x^2)$ ✓

2) $c \cdot u = cu_1 x + cu_2 x^2$ ✓

$\therefore$ is a subspace

ex: $W = \{ a_0 + a_1 x + a_2 x^2 \in P_2 \; ; \; a_1 \geq 0 \}$

$0 = 0 + 0 \cdot x + 0 \cdot x^2$ ✓

2) Let $c = -1$ $\implies$ $c \cdot u = cu_0 + cu_1 x + cu_2 x^2$

$= -u_0 - u_1 x - u_2 x^2 \notin W$

since $-u_1 < 0$ $\therefore$ not a subspace under scalar multiplication

ex: $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2,2} \; / \; a > 0 \right\}$

$0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \notin W$

W is not a subspace of $M_{2,2}$ since the zero matrix is not contained in W

Spanning Sets and Linear Independence

Linear Combination of vectors

A vector in a VS is called a linear comb. of vectors if v_n are vectors and c_n are scalors $\rightarrow$ $v = c_1 v_1 + c_2 v_2 + \dots$

ex: Is it possible to write $w = (1, 1, 1)$ as a LC of $S = \{\underbrace{(1,2,3)}_{v_1}, \underbrace{(0,1,2)}_{v_2}, \underbrace{(-1,0,1)}_{v_3}\}$

$$w = c_1 v_1 + c_2 v_2 + c_3 v_3 = (c_1, 2c_1, 3c_1) + (0, c_2, 2c_2) + (-c_3, 0, c_3)$$

$$(1, 1, 1) = (c_1 - c_3, \; 2c_1 + c_2, \; 3c_1 + 2c_2 + c_3)$$

$$\begin{cases} c_1 - c_3 = 1 \\ 2c_1 + c_2 = 1 \\ 3c_1 + 2c_2 + c_3 = 1 \end{cases} \Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} - \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 1 - 1 = 0$$

the system doesn't have a unique soln.

from this point you can solve however you want

ex: Is it possible to write $w = (1, -2, 2)$ as a LC of $S = \{(1,2,3), (0,1,2), (-1,0,1)\}$

$$w = c_1 v_1 + c_2 v_2 + c_3 v_3 = (c_1, 2c_1, 3c_1) + (0, c_2, 2c_2) + (-c_3, 0, c_3)$$

$$(1, -2, 2) = (c_1 - c_3, \; 2c_1 + c_2, \; 3c_1 + 2c_2 + c_3)$$

$$\begin{cases} c_1 - c_3 = 1 \\ 2c_1 + c_2 = -2 \\ 3c_1 + 2c_2 + c_3 = 2 \end{cases} \Rightarrow \begin{bmatrix} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & -2 \\ 3 & 2 & 1 & 2 \end{bmatrix}$$

use gauss to solve

$$\left[\begin{array}{cccc}1&0&-1&1\\2&1&0&-2\\3&2&1&2\end{array}\right]\xrightarrow[R_3-3R_1=R_3]{R_2-2R_1=R_2}\left[\begin{array}{cccc}1&0&-1&1\\0&1&2&-4\\0&2&4&-1\end{array}\right]\xrightarrow{R_3-2R_2=R_3}\left[\begin{array}{cccc}1&0&-1&1\\0&1&2&-4\\0&0&0&7\end{array}\right]$$

no soln since $0 \neq 7$

Spanning Set of a Vector Space

Let $S = \{v_1, v_2 \dots v_k\}$ be a subset of a VS V.

The set S is called a spanning set of V if every vector in V can be written as a linear combination of vectors in S. We can say that S spans V.

ex: The following is a spanning set $\{(1,0,0),(0,1,0),(0,0,1)\} \rightsquigarrow \mathbb{R}^3$

S spans $\mathbb{R}^3$ since every vector in $\mathbb{R}^3$, say $u=(u_1, u_2, u_3)$ is written as a LC of the vector of S.

ex: The set $S=\{1, x, x^2\}$ spans P_2 since for every polynomial $f(x)$ in P_2:

standard spanning for P_2

$$f(x) = a_0 + a_1 x + a_2 x^2$$

ex: Show that $S=\{\underbrace{(1,2,3)}_{v_1}, \underbrace{(0,1,2)}_{v_2}, \underbrace{(-2,0,1)}_{v_3}\}$ spans $\mathbb{R}^3$

Let $u = (u_1, u_2, u_3) \in \mathbb{R}^3$

$u = c_1 v_1 + c_2 v_2 + c_3 v_3$ *

$$(u_1, u_2, u_3) = c_1(1,2,3) + c_2(0,1,2) + c_3(-2,0,1)$$

$$= (c_1 - 2c_3,\ 2c_1 + c_2,\ 3c_1 + 2c_2 + c_3)$$

* you can't solve a spanning question without writing this

$$\begin{cases} c_1 - 2c_3 = u_1 \\ 2c_1 + c_2 = u_2 \\ 3c_1 + 2c_2 + c_3 = u_3 \end{cases}$$

The coefficient matrix is $A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$

if $\det(A) \neq 0 \implies A$ is solveable

$|A| = \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = -1 \neq 0 \implies$ system has soln

$\therefore$ S spans $\mathbb{R}^3$

ex:

Show that S doesn't span $\mathbb{R}^3$:

$S = \{ \underbrace{(1,2,3)}_{v_1}, \underbrace{(0,1,2)}_{v_2}, \underbrace{(-1,0,1)}_{v_3} \}$, $w = (1,-2,2) \in \mathbb{R}^3$

w can't be a LC of v_1, v_2, v_3 $\therefore$ S doesn't span $\mathbb{R}^3$

Steps to determine spanning ($S = \{v_1, v_2 \ldots v_k\}$ spans V ??)

1. take a general vector u in the vector space
2. write general vector $u = c_1 v_1 + c_2 v_2 \ldots$
3. system solveable? → spans , not solveable? → doesn't span

ex:

Show that $v = (2,4,6)$ is not in the span $\{v_1, v_2\}$ where $v_1 = (0,1,2)$ and $v_2 = (1,2,0)$ can't do det for this since not sqr matrix

Let $v = c_1 v_1 + c_2 v_2 \implies (2,4,6) = (0, c_1, 2c_1) + (c_2, 2c_2, 0)$

$$\begin{cases} c_2 = 2 \\ c_1 + 2c_2 = 4 \\ 2c_1 = 6 \end{cases} \implies A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 4 \\ 2 & 0 & 6 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ 2 & 0 & 6 \end{bmatrix} \xrightarrow{R_3 - 2R_1 = R_3} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ 0 & -4 & -2 \end{bmatrix}$$

$$\xrightarrow{R_3 + 4R_2 = R_3} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 6 \end{bmatrix} \qquad \cancel{0c_1 + 0c_2 = 6} \longrightarrow$$

system has no soln

v is not in the span

ex:

Determine whether $S = \{ \underbrace{(1,-1,4)}_{v_1}, \underbrace{(2,1,3)}_{v_2}, \underbrace{(4,-3,5)}_{v_3} \}$ spans $\mathbb{R}^3$

Let $u = (u_1, u_2, u_3) \in \mathbb{R}^3 \implies u = c_1 v_1 + c_2 v_2 + c_3 v_3$

$(u_1, u_2, u_3) = (c_1, -c_1, 4c_1) + (2c_2, c_2, 3c_2) + (4c_3, -3c_3, 5c_3)$

$\begin{bmatrix} 1 & 2 & 4 \\ -1 & 1 & -3 \\ 4 & 3 & 5 \end{bmatrix} \rightsquigarrow \det(A) = -28 \neq 0 \quad \therefore$ S has soln. S spans $\mathbb{R}^3$

ex:

Show that $\underbrace{(-1,7)}_{u}$ is in the span $\{ \underbrace{(1,2)}_{v_1}, \underbrace{(-1,1)}_{v_2} \}$

Let $u = (u_1, u_2) \implies u = c_1 v_1 + c_2 v_2$

$\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 7 \end{bmatrix} \rightsquigarrow \begin{array}{l} c_1 = 2 \\ c_2 = 3 \end{array} \qquad u = 2v_1 + 3v_2 \quad \therefore$ u has soln
u in the span

Linear Dependence and Linear Independence

A set of vectors $S = \{ v_1, v_2 \ldots \}$ in a vector space V is called linear independence (LI) if the equation

$$c_1 v_1 + c_2 v_2 + \ldots + c_k v_k = 0$$

has only the trivial solution i.e. $c_1 = c_2 = \ldots = c_k = 0$

If there are non-trivial solutions then linear dependence (LD)

ex:

The set $S = \{ (1,2), (2,4) \}$ in $\mathbb{R}^2$ is LD

$-2v_1 + v_2 = -2(1,2) + (2,4) = (0,0)$

$2v_1 = v_2$

ex:

The set $S = \{\underbrace{(0,0,0)}_{v_1}, \underbrace{(1,2,3)}_{v_2}\}$ in $\mathbb{R}^3$ is LD

$3v_1 + 0v_2 = 0$

★ Rule: Any set that contains the zero vector is LD

Steps to test for LI & LD

1. Write the eqn. $c_1v_1 + c_2v_2 + \dots + c_kv_k = 0$

2. Check for unique soln in system (solve by gauss, det etc.)

a) If the coeff. matrix A is $(n \times n)$: $AX = 0$

if $\det(A) \neq 0$ only unique soln. $\Rightarrow c_1 = c_2 = c_k = 0$
↳ LI

if $\det(A) = 0$ more than one soln. → LD

b) If $A_{(m \times n)}$ (not square): Use Gauss elimination or Jordan

If $c_1 = c_2 = c_k = 0$ LI   If not, LD

ex:

Determine whether the set $S = \{\underbrace{(1,2,3)}_{v_1}, \underbrace{(0,1,2)}_{v_2}, \underbrace{(-2,0,1)}_{v_3}\}$ is LI or LD in $\mathbb{R}^3$.

$c_1v_1 + c_2v_2 + c_3v_3 = 0$ $(0,0,0) = c_1(1,2,3) + c_2(0,1,2) + c_3(-2,0,1)$

$$\begin{cases} c_1 - 2c_3 = 0 \\ 2c_1 + c_2 = 0 \\ 3c_1 + 2c_2 + c_3 = 0 \end{cases}$$

$$\begin{array}{ccc} v_1 & v_2 & v_3 \end{array}$$
$$\begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

method 1:

$\det(A) = \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = -1 \neq 0$

resulting sentence

we only have the trivial soln.

$c_1 = c_2 = c_3 = 0$

S is LI

method 2:

Using Gauss – elimination

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$c_1 = c_2 = c_3 = 0$

S is LI

ex:

Check if $S = \{\underbrace{1+x-2x^2}_{v_1}, \underbrace{2+5x-x^2}_{v_2}, \underbrace{x+x^2}_{v_3}\}$ is LD or LI in P_2

$$c_1v_1 + c_2v_2 + c_3v_3 = 0$$

$$c_1(1+x-2x^2) + c_2(2+5x-x^2) + c_3(x+x^2) = 0$$

$$f(x) = a_0 + a_1x + a_2x^2 = 0 \implies a_0 = a_1 = a_2 = 0$$

$$(c_1 + 2c_2) + x(c_1 + 5c_2 + c_3) + x^2(-2c_1 - c_2 + c_3) = 0$$

$$\begin{cases} c_1 + 2c_2 = 0 \\ c_1 + 5c_2 + c_3 = 0 \\ -2c_1 - c_2 + c_3 = 0 \end{cases} \implies A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 5 & 1 \\ -2 & -1 & 1 \end{bmatrix}$$

$\det(A) = 0$

S is LD

method 1

method 2

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 5 & 1 & 0 \\ -2 & -1 & 1 & 0 \end{bmatrix} \xrightarrow[\text{elim.}]{\text{Gauss}} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \implies \begin{cases} c_1 + 2c_2 = 0 \\ c_2 + \frac{1}{3}c_3 = 0 \end{cases}$$

Let $\boxed{c_3 = t}$

$c_2 = 1/3\, c_3$

$\boxed{c_2 = 1/3\, t}$

$c_1 = -2c_2$

$c_1 = -2(-1/3\, t)$

$\boxed{c_1 = \frac{2}{3}t}$

ex:

Determine if $S = \left\{ \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \right\}$ is LD or LI in $M_{2\times2}$

$$c_1 \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad c_1 = c_2 = c_3 = 0$$

$$\begin{cases} 2c_1 + 3c_2 + c_3 = 0 \\ c_1 = 0 \\ c_2 + c_3 = 0 \\ c_1 + c_2 = 0 \end{cases} \Longrightarrow \begin{bmatrix} 2 & 3 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow[R_4 - 2R_1 = R_4]{R_3 - R_1 = R_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \end{bmatrix} \xrightarrow[R_4 - 3R_2 = R_4,\ -1 \cdot R_3]{R_3 - R_2 = R_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 0 \end{bmatrix} \xrightarrow[R_2 - R_3 = R_2]{R_4 \div -\frac{1}{2}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$c_1 = c_2 = c_3 = 0$

S is LI

ex:

Determine if $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \end{bmatrix} \right\}$ is LD or LI in $M_{4\times1}$

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 \\ -1 & 0 & 1 & -1 \\ 6 & 2 & 2 & 2 \end{bmatrix} \xrightarrow[R_4 - 6R_1 = R_4]{R_3 + R_1 = R_3} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & -4 & 2 & 2 \end{bmatrix} \xrightarrow[R_4 + 4R_2 = R_4]{R_3 - R_2 = R_3} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 14 & 6 \end{bmatrix} \xrightarrow{R_4 + 7R_3 = R_4} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & -8 \end{bmatrix}$$

$$\det(A) = -2 \ -8 = 16 \neq 0 \Longrightarrow S \text{ is LI in } M_{4\times1}$$

ex:

Determine if $S = \{(1,1,1), (2,2,2), (3,3,3)\}$ is LD or LI in $\mathbb{R}^3$

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

Since $R_1 = R_2 = R_3 \Rightarrow \det = 0 \Longrightarrow S$ is LD

ex:

Determine if $S = \{\underbrace{(1,2,-1)}_{v_1}, \underbrace{(2,4,-2)}_{v_2}\}$ is LD or LI in $\mathbb{R}^3$

since $v_1 = 2v_2$, S is LD

ex:

Determine if $S = \{2-x,\ 2x-x^2,\ 6-5x,\ x^2\}$ is LD or LI in P_2 (P up to 2nd deg.)

$$\begin{bmatrix} \cancel{2}^{1} & 0 & \cancel{6}^{3} & 0 \\ -1 & 2 & -5 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \xrightarrow[R_2 \cdot 1/2]{R_2+R_1=R_2} \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \xrightarrow[-R_3]{R_3+R_2=R_3} \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow[R_1-3R_3=R_1]{R_2+R_3=R_2}$$

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} (R.R.E.F) \Longrightarrow \quad c_1 + 3c_4 = 3 \quad \vdots$$

S is LD

$c_1 \neq 0$
$c_2 \neq 0$ ...

ex:

Determine if $S = \{1+3x+x^2,\ 3x-1,\ 4x\}$ is LD or LI in P_2

$$\begin{array}{c} \\ x \\ x^2 \end{array} \begin{bmatrix} 1 & -1 & 0 \\ 3 & 3 & 4 \\ 1 & 0 & 0 \end{bmatrix} \qquad \det(A) = \begin{vmatrix} -1 & 0 \\ 3 & 4 \end{vmatrix} = -4 \neq 0$$

S is LI

ex:

Determine if $S = \{(1,2), (-2,2)\}$ is LD or LI in $\mathbb{R}^2$

Since none are scalar multiple of another , S is LI

Basis & Dimension of vector spaces : Basis

A set of vectors $S = \{v_1, v_2 \dots v_k\}$ in a vs V is called a basis for V if :

1) S spans V 2) S is LI

ex:

$S = \{ \underbrace{(1,0,0)}_{v_1}, \underbrace{(0,1,0)}_{v_2}, \underbrace{(0,0,1)}_{v_3} \}$ is basis for $\mathbb{R}^3$

S spans $\mathbb{R}^3$ and $c_1v_1 + c_2v_2 + c_3v_3 = 0$

#1

$(c_1, c_2, c_3) = (0,0,0)$ #2

S is the standard basis for $\mathbb{R}^3$

$\overset{n}{\longleftrightarrow}$

$(1,0,0 \dots 0)$
$(0,1,0 \dots 0)$
$(0,0,1 \dots 0)$
$(0,0,0 \dots 1)$

st. basis for $\mathbb{R}^n$

ex:

Show that $S = \{ \overset{v_1}{(1,1)}, \overset{v_2}{(1,-1)} \}$ is a basis for $\mathbb{R}^2$

1) S spans $\mathbb{R}^2$?

Let $u = (u_1, u_2) \in \mathbb{R}^2$
such that $u = c_1v_1 + c_2v_2$

$(u_1, u_2) = (c_1 + c_2, c_1 - c_2)$

$$\begin{cases} c_1 + c_2 = u_1 \\ c_1 - c_2 = u_2 \end{cases}$$

$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \longrightarrow \det = -1 \neq 0 \quad \therefore$ S spans $\mathbb{R}$

2) $c_1v_1 + c_2v_2 = 0$

$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \longrightarrow \det = -1 \neq 0 \quad \therefore$ S is LI

S is a non - Standard basis for $\mathbb{R}^2$

ex:

A Basis for Polynomials

Show that the v.s. P_2 has the Basis $S = \{1, x, x^2\}$

1) S spans P_2 $\quad f(x) = a_0 + a_1 x + a_2 x^2$

2) S is LI $\quad (c_1(1) + c_2(x) + c_3(x^2)) = 0$

$$c_1 = c_2 = c_3 = 0$$

$\hookrightarrow$ $S = \{1, x, x^2\}$ is the standard basis for P_2

ex:

A Basis for $M_{2,2}$

Show that $S = \left\{ \underset{v_1}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}, \underset{v_3}{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}, \underset{v_4}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \right\}$ is a basis for $M_{2,2}$

1) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2,2}$

$$A = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$A = av_1 + bv_2 + cv_3 + dv_4 \quad \therefore S$ spans $M_{2,2}$

2) $c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = 0$

$$c_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c_4\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{cases} c_1 = 0 \\ c_2 = 0 \\ c_3 = 0 \\ c_4 = 0 \end{cases} \quad \therefore S \text{ is LI.}$$

$\hookrightarrow$ S is the standard basis for $M_{2,2}$

The standard basis for $M_{m,n}$

consists of $(m \times n)$ matrices having 1 and all other entries are 0 $\rightarrow$ all combinations like this

Uniqueness of Basis Representation

* If $S = \{v_1, v_2 \dots v_k\}$ is the basis for a vs V, then every vector in V can be written in one and only one way as L.C. of vectors in S.

$\hookrightarrow$ Proof S spans V then:

$u_1 = c_1 v_1 + c_2 v_2 + c_3 v_3 \dots + c_n v_n$

Assume that $u = b_1 v_1 + b_2 v_2 + b_3 v_3 + \dots b_n v_n$

$u - u = (c_1 - b_1) v_1 + (c_2 - b_2) v_2 + \dots + (c_n - b_n) v_n$

$c_1 - b_1 = 0$, $c_2 - b_2 = 0$, $\dots$, $c_n - b_n = 0$

* If $S = \{v_1, v_2 \dots v_k\}$ is the basis for a vs V, then every set containing more than n vectors in V is LD

ex:

Is $\{(1,2,-1), (1,1,0), (2,3,0), (5,9,-1)\}$ LD or LI in $\mathbb{R}^3$?

$\mathbb{R}^3$ has a basis of 3 vectors $\{(1,0,0), (0,1,0), (0,0,1)\}$

order of S $|S| = 4 > 3$ $\therefore$ LD

ex:

$S = \{1, 1+x, 1-x, 1+x+x^2, 1-x+x^2\}$ in P_3

$\therefore$ LD because $|P_3| = 4 < 5 = |S|$

* If a v.s. V has one basis with n-vectors, then every other basis for V should have n-vectors

ex:

$S_1 = \{(1,2,0), (0,1,3)\}$

$\mathbb{R}^3$ has $\{(1,0,0), (0,1,0), (0,0,1)\} \rightarrow$ standard basis
3 vectors

$\therefore$ S_1 not basis of $\mathbb{R}^3$

ex:

$S = \{x, 1, x^2, x^3, 1+x\}$

$|S| = 5$ however $|P_3| = 4 \longrightarrow \{1, x, x^2, x^3\}$ standard basis

Basis & Dimension of vector spaces : Dimension

If a vs V has a Basis consisting of n-vectors, then the number of "n" is the dimension of V, denoted by $\dim(V) = n$

ex:

$\dim(\mathbb{R}^2) = 2$ $\quad$ $\dim(P_2) = 2+1 = 3$ $\quad$ $\dim(M_{2,3}) = 6$

$\vdots$ $\quad$ $\vdots$ $\quad$ $\vdots$

$\dim(\mathbb{R}^n) = n$ $\quad$ $\dim(P_n) = n+1$ $\quad$ $\dim(M_{m \times n}) = m \cdot n$

Finding the dimension of a subspace

ex:

$W = \{ (d, c-d, c) \; ; \; c$ and d are real numbers$\}$ a subspace of $\mathbb{R}^3$

$(d, c-d, c) = c(0,1,1) + d(1,-1,0)$

$S = \{ (0,1,1), (1,-1,0) \}$ is a basis of W

1) S spans W 2) S is LI (none are scalar mults. of eachother)

$\therefore$ S is basis of W

$\dim(W) = 2$

ex:

$W = \{ (2b, b, 0) \,/\, b \in \mathbb{R} \}$

$(2b, b, 0) = b(2,1,0)$ $S = \{(2,1,0)\}$ spans W and is LI

$\therefore$ S is basis for W, $|S| = 1 \implies \dim(W) = 1$

ex:

Let $W = \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix}, \; a,b,c \in \mathbb{R} \right\}$ be the subspace of all symmetric matrices in $M_{2,2}$. What is $\dim(W) = ?$

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$S = \left\{ \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{v_2}, \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{v_3} \right\}$$

S spans W and S is LI

$\therefore$ $\dim(W) = 3$

$$\begin{cases} c_1 v_1 + c_2 v_2 + c_3 v_3 = 0 \\ c_1 = c_2 = c_3 = 0 \end{cases}$$

Let V be a v.s. of $\dim(V) = n$

1) If $S = \{v_1, v_2, \dots, v_n\}$ is LI, then S is basis for V

2) If $S = \{v_1, \dots, v_n\}$ spans V, then S is basis for V

ex:

Show that $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -2 \end{bmatrix} \right\}$ is a basis for $M_{5,1}$

$\dim(M_{5,1}) = 5$

$|S| = 5$

$c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 + c_5v_5 = 0$

$c_1 = c_2 \dots = 0$

Section 4.6 - Rank of a Matrix

Given $A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$

$$\left(\begin{bmatrix} (a_{11} & a_{12} & \cdots & a_{1n}) \\ (a_{21} & a_{22} & & a_{2n}) \\ & \vdots & & \\ (a_{m1} & a_{m2} & \cdots & a_{mn}) \end{bmatrix} \right) \text{ Row vectors } \in \mathbb{R}^n$$

$$\left(\left[\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \cdots, \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \right] \right) \text{ Column vectors } \in \mathbb{R}^m$$

Row space and Column space

Let A be $m\times n$ matrix

1) The Row space of A is a subspace of $\mathbb{R}^n$ spanned by row vectors of A;

2) The Column space of A is a subspace of $\mathbb{R}^n$ spanned by column vectors of A.

Finding Basis for a Row space (R.S.)

Elementary row operations are used to convert matrix A to r.e.f. The non-zero row vectors in r.e.f form a basis for the row space of A.

ex:

Find a basis for the row space

$$A = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{bmatrix}$$

step convert to r.e.f

$$B = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \to w_1 \\ \to w_2 \\ \to w_3 \\ \left.\begin{matrix} \\ \\ \end{matrix}\right\} \text{zero rows} \end{matrix}$$

$$\left.\begin{aligned} w_1 &= (1,3,1,3) \\ w_2 &= (0,1,1,0) \\ w_3 &= (0,0,0,1) \end{aligned}\right\} \text{form a basis for the row space } A$$

Note: This technique can also be used to find a Basis for the subspace spanned by the set.

$$S = \{v_1, v_2, \ldots, v_n\} \text{ in } \mathbb{R}^N$$

The idea is to use the vectors in S as the rows of a matrix A, then rewrite A in r.e.f. The non-zero rows of this matrix will then form a basis for the subspace spanned by S.

ex:

Find a basis for the subspace of $\mathbb{R}^3$ spanned by

$$S = \{\underbrace{(-1,2,5)}_{v_1}, \underbrace{(3,0,3)}_{v_2}, \underbrace{(5,1,8)}_{v_3}\}$$

$$A = \begin{bmatrix} -1 & 2 & 5 \\ 3 & 0 & 3 \\ 5 & 1 & 8 \end{bmatrix} \xrightarrow{\text{r.e.f}} B = \begin{bmatrix} -1 & 2 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \to w_1 \\ \to w_2 \\ \\ \end{matrix}$$

w_1 & w_2 form the basis for the R.S. of A

w_1 & w_2 form the basis for subspaces spanned $\{v_1, v_2, v_3\}$

Finding Basis for a Column Space (C.S.)

method 1 CS of A = RS of A^T

method 2 Let B be the r.e.f of A. Then the column of B having leading ones are LI, and these cols. in A form the basis for CS of A.

ex:

Find the basis for CS of A

$$A = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{bmatrix}$$

method #1

$$A^T = \begin{bmatrix} 1 & 0 & -3 & 3 & 2 \\ 3 & 1 & 0 & 4 & 0 \\ 1 & 1 & 6 & -2 & -4 \\ 3 & 0 & -1 & 1 & -2 \end{bmatrix}$$

r.e.f. A^T

$$\begin{array}{cc} \begin{bmatrix} 1 & 0 & -3 & 3 & 2 \\ 0 & 1 & 9 & -5 & -6 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{matrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{matrix} \end{array}$$

$(w_1, w_2, w_3) \rightarrow$ basis for RS of A^T

$\therefore\ C_1\, C_2\, C_3 \rightarrow$ basis for CS of A

method #2 The r.e.f of A is

$$B = \begin{bmatrix} \boxed{1} & 3 & 1 & 3 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{matrix} \downarrow & \downarrow & \downarrow & \downarrow \\ c_1 & c_2 & c_3 & c_4 \end{matrix}$$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \longrightarrow \text{form basis for CS of } A$$

Rank of a matrix

The dimension of CS = RS $\longrightarrow$ rank (A)

ex:

Find the rank of the matrix A.

$$A = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 2 & 1 & 5 & -3 \\ 0 & 1 & 3 & 5 \end{bmatrix} \xrightarrow{\text{r.e.f}} \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$\{w_1, w_2, w_3\} \longrightarrow$ Basis for RS

rank (A) = dim (RS) = 3

Null Space

A(m×n), the set of all solns of the homogenous system $\boxed{AX=0}$ is a subspace of $\mathbb{R}^n$ called the null space N(A)

$$N(A) = \{ x \in \mathbb{R}^n \mid Ax = 0 \}$$

Dimension of null space is called nullity of A

ex: Find the null space of A

$$A = \begin{bmatrix} 1 & 2 & -2 & 1 \\ 3 & 6 & -5 & 4 \\ 1 & 2 & 0 & 3 \end{bmatrix} \xrightarrow{\text{r.e.f.}} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{cases} x_1 + 2x_2 + 3x_4 = 0 \\ x_3 + x_4 = 0 \end{cases}$$

Let $\boxed{x_4 = t}$, $\boxed{x_3 = -t}$ Let $\boxed{x_2 = s}$, $\boxed{x_1 = -2s - 3t}$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2s - 3t \\ s \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

The basis for N(A) consists of

dim(N(A)) = 2 nullity!

Theorem: $A_{(m\times n)} \longrightarrow \text{rank}(A) + \text{nullity}(A) = n$

ex:

$$A = \begin{array}{c} \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 3 & 9 & 0 & -12 \end{bmatrix} \\ \begin{array}{ccccc} c_1 & c_2 & \quad & c_4 & \quad \end{array} \end{array} \xrightarrow{\text{r.e.f.}} \begin{array}{c} w_1 \\ w_2 \\ w_3 \\ \end{array} \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 3 & 0 & -4 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

a) Basis for RS $\{w_1, w_2, w_3\} \rightarrow$ Basis for RS

b) Basis for CS $\{c_1, c_2, c_3\} \rightarrow$ Basis for CS

c) Rank and nullity rank $= 3$

$5 - \text{rank} = \text{nullity}$ $\quad\therefore$ nullity $= 2$

★ The solution of the homogenous system $Ax=0$ is a subspace called "null space". But the set of soln. of non-homogenous system $Ax=b$ is not a subspace. However, there's a relation between the soln. of $Ax=0$ and $Ax=b$ is as follows

Theorem: If x_p is a particular soln of $Ax=b$, then every soln can be written as

$$x = x_p + x_h$$

x_h is the soln of $Ax=0$ (homogenous soln)

ex: Find the set of all soln.s of

$$\begin{cases} x_1 - 2x_3 + x_4 = 5 \\ 3x_1 + x_2 - 5x_3 = 8 \\ x_1 + 2x_2 - 5x_4 = -9 \end{cases}$$

$$\begin{bmatrix} 1 & 0 & -2 & 1 & 5 \\ 3 & 1 & -5 & 0 & 8 \\ 1 & 2 & 0 & -5 & -9 \end{bmatrix} \xrightarrow{\text{r.e.f.}} \begin{bmatrix} 1 & 0 & -2 & 1 & 5 \\ 0 & 1 & 1 & -3 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} x_1 - 2x_3 + x_4 = 5 \\ x_2 + x_3 - 3x_4 = -7 \end{cases}$$

we expect 2 free parameters since 2 eqn.

Let $x_3 = s$ and $x_4 = t$

$\Rightarrow x_1 = 5 + 2s - t$

$x_2 = 3t - s - 7$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5+2s-t \\ 3t-s-7 \\ s \\ t \end{bmatrix} = \underbrace{s \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 3 \\ 0 \\ 1 \end{bmatrix}}_{x_h} + \begin{bmatrix} 5 \\ 7 \\ 0 \\ 0 \end{bmatrix} \rightarrow x_p$$

Coordinate Representation Relative to Basis

Let $B = \{v_1, v_2, \dots, v_n\}$ be a basis for a vector space V and let x be a vector in V

$\Rightarrow x = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$

The scalars $c_1, c_2, \dots, c_n$ are called the coordinates of x relative to the basis B. The coordinate matrix of x relative to B is the column matrix in R^n whose components are the coordinates of x.

$$\Rightarrow [x]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

ex. below

standard basis

ex:

$$S = \{(1,0,0), (0,1,0), (0,0,1)\}, \quad x = (-2,1,3) \in \mathbb{R}^3$$

$$x = -2(1,0,0) + 1(0,1,0) + 3(0,0,1) \qquad [x]_S = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$$

$$= -2v_1 + 1v_2 + 3v_3$$

non-standard basis

ex:

$B = \{v_1, v_2\} = \{(1,0), (1,2)\}$ is $[x]_B = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

Find the coordinates relative to the standard

$$[x]_B = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \implies x = 3v_1 + 2v_2 = 3(1,0) + 2(1,2)$$

$$x = (3,0) + (2,4) = (5,4)$$

since $x = (5,4) \implies [x]_{B'} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$

ex:

Find the coordinate matrix of $x = (1,2,-1)$ in $\mathbb{R}^3$ relative to the non-standard Basis

$$B' = \{u_1, u_2, u_3\} = \{(1,0,1), (0,-1,2), (2,3,-5)\}$$

$$x = c_1u_1 + c_2u_2 + c_3u_3$$

$$(1,2,-1) = c_1(1,0,1) + c_2(0,-1,2) + c_3(2,3,-5) \qquad \begin{cases} c_1 + 2c_3 = 1 \\ -c_2 + 3c_3 = 2 \\ c_1 + 2c_2 - 5c_3 = -1 \end{cases}$$

$$c_1 = 5, \; c_2 = -8, \; c_3 = -2 \implies [x]_{B'} = \begin{bmatrix} 5 \\ -8 \\ -2 \end{bmatrix}$$

Eigen Values and Eigen Vectors

Let A be a $(n \times n)$ matrix. The scalar λ is called an eigen value of A if there is a non-zero vector x such that $Ax = \lambda x$

the vector x is called an eigen vector of A corresponding to λ

ex:

Let $A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$, show that $x_1 = (1,0)$ is an eigen vector of A and $x_2 = (0,1)$ is an eigen vector of A corresponding to $\lambda = -1$

Check if $A x_1 = \lambda x_1$

$$A x_1 = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \underbrace{2}_{\lambda} \cdot \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{x_1}$$

$\lambda = 2$ is the eigen value and $x_1 = (1,0)$ is the eigen vector corresponding to $\lambda = 2$

$$A x_2 = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \underbrace{-1}_{\lambda} \cdot \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{x_2}$$

$\lambda_2 = -1$ is the eigen val. x_2 corres. to $\lambda_2 = -1$

ex: Let $A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ Show that $x_1 = (-3,-1,1)$ and $x_2 = (1,0,0)$ are the eigen vectors of A and find their corresponding eigen values

$$Ax_1 = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} = \underset{(\lambda_1)}{0} \cdot x_1$$

x_1 is an eigen vector corr. to the eigen val $\lambda_1 = 0$

x_2 is an eigen vector corr. to the eigen val $\lambda_2 = 1$

$$Ax_2 = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \underset{(\lambda_2)}{1} \cdot x_2$$

Finding Eigen Values and Eigen Vectors

Let A be a $(n \times n)$ matrix.

1) An eigen value of A is scalar λ such that

$\boxed{\det(\lambda I - A) = 0}$ $\Rightarrow$ " the characteristic polynomial "

2) The eigen vector corresponding to λ are the non-zero solutions of $(\lambda I - A)x = 0$

$$\lambda x = Ax$$

order matters! $AB \neq BA$

$$\lambda x - Ax = 0$$

$$(\lambda - A)x = 0 \longrightarrow (\lambda I - A)x = 0 \checkmark$$

~~scalar — vector ??~~

ex:

Find the eigen values and corresponding eigen vectors of

$$A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$$

Step #1 find the characteristic polynomial

$$\det(\lambda I - A) = 0$$

$$\lambda I - A = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix} = \begin{bmatrix} \lambda-2 & 12 \\ -1 & \lambda+5 \end{bmatrix}$$

$$\det(\lambda I - A) = \begin{vmatrix} \lambda-2 & 12 \\ -1 & \lambda+5 \end{vmatrix} = (\lambda-2)(\lambda+5) + 12 = 0$$

$$\lambda^2 + 5\lambda - 2\lambda - 10 + 12 = 0$$

$$\lambda^2 + 3\lambda + 2 = 0$$

$$\begin{matrix} \lambda & 2 \\ \lambda & 1 \end{matrix}$$

$$(\lambda+1)(\lambda+2) = 0$$

$$\lambda_1 = -1 \ \& \ \lambda_2 = -2$$

Step # 2 for $\lambda_1 = -1$ $\Rightarrow (\lambda_1 I - A)x = 0$

$$\begin{bmatrix} -3 & 12 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 12 \\ -1 & 4 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} -1 & 4 \\ -3 & 12 \end{bmatrix} \xrightarrow{-R_1} \begin{bmatrix} 1 & -4 \\ -3 & 12 \end{bmatrix} \xrightarrow{R_2 + 3R_1 = R_2} \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix}$$

$x_1 - 4x_2 = 0$

Let $x_2 = t$

$x_2 = t$
$x_1 = 4t$

$$\Rightarrow x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4t \\ t \end{bmatrix} = t \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

is an eigen vector corr. to $\lambda_1 = -1$

$B = \{(4,1)\}$ the basis for the eigen space for $\lambda_2 = -2 \Rightarrow (\lambda_1 I - A)x = \begin{bmatrix} -4 & 12 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

converted to r.e.f $\Longrightarrow \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$ $\{ x_1 - 3x_2 = 0$

Let $x_2 = t$

$x_1 = 3t$
$x_2 = t$ $\Rightarrow x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3t \\ t \end{bmatrix} = t \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is the eigen vector corr. to $\lambda_2 = -2$

$B = \{(3,1)\}$ the basis for the eigen space

Remark! The homogenous system that is formed when finding the eigen vectors will always reduce to a matrix having atleast one row of zeros

Theorem: If A $(n \times n)$ is a triangular matrix, then its eigen values are the entries on the main diagonal

ex: Find the eigen values of $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 5 & 3 & -3 \end{bmatrix}$ upper triangular matrix

$\det(\lambda I - A) = 0$

$$\begin{vmatrix} \lambda-2 & 0 & 0 \\ 1 & \lambda-1 & 0 \\ -5 & -3 & \lambda+3 \end{vmatrix} = (\lambda-2)(\lambda-1)(\lambda+3) = 0 \Rightarrow \begin{cases} \lambda_1 = 2 \\ \lambda_2 = 1 \\ \lambda_3 = -3 \end{cases}$$

ex: Find the eigen values of

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

diagonal matrix

$\lambda_1 = -1, \quad \lambda_2 = 2 \quad \ldots \quad \lambda_5 = 3$

$$\lambda(\lambda+1)(\lambda-2)(\lambda+4)(\lambda-3) = 0$$

ex: Find the eigen values of $A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

$$|\lambda I - A| = \begin{vmatrix} \lambda-1 & -3 & 0 \\ -3 & \lambda-1 & 0 \\ 0 & 0 & \lambda+2 \end{vmatrix} = (\lambda+2)\begin{vmatrix} \lambda-1 & -3 \\ -3 & \lambda-1 \end{vmatrix}$$

$$= (\lambda+2)\left[(\lambda-1)^2 - 9\right] = (\lambda+2)(\lambda^2 - 2\lambda - 8)$$

$$= (\lambda+2)(\lambda-4)(\lambda+2) = 0 \quad \begin{cases} \lambda_1 = -2 \\ \lambda_2 = 4 \\ \cancel{\lambda_3 = -2} \end{cases}$$

★ $\lambda_1 = 2$ is an eigen value with multiplicity 2 → will produce 2 eigen vectors

for $\lambda_1 = -2$

$$(-2I - A)x = 0 \implies \begin{bmatrix} -3 & -3 & 0 \\ -3 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

ref

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \implies \{x_1 + x_2 = 0$$

Let $x_2 = s$ and $x_3 = t$

$$\implies \begin{matrix} x_1 = -s \\ x_2 = s \\ x_3 = t \end{matrix} \implies x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -s \\ s \\ t \end{bmatrix}$$

$$= t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \implies \therefore B = \{(0,0,1), (-1,1,0)\}$$

basis for $\lambda = -2$

for $\lambda_2 = 4$

$$(4I - A)x = 0 \Rightarrow \begin{bmatrix} 3 & -3 & 0 \\ -3 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} \xrightarrow{\text{ref.}} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{cases} x_1 - x_2 = 0 \\ x_3 = 0 \end{cases}$$

Let $x_2 = t \Rightarrow \begin{matrix} x_1 = t \\ x_2 = t \end{matrix} \Rightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\therefore B = \{(1,1,0)\}$ basis for $\lambda = 4$

Diagonalization of A matrix

An $(n\times n)$ matrix A is diagonalizable if there exists an invertible matrix P such that $P^{-1}AP = D$ where D is a diagonal matrix

ex:

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \quad P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad P^{-1} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \rightsquigarrow \text{diagonal matrix} \quad \therefore A \text{ is diagonalizable}$$

Condition :

An $(n\times n)$ matrix is diagonalizable if it has n LI eigen vectors

ex:

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

(result found previously, 2 pg before)

$\lambda = 4 \Rightarrow P_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\lambda = -2 \Rightarrow P_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\Longrightarrow P = [P_1 \mid P_2 \mid P_3] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

if $\det(P) \neq 0$
P is LI

$\det(P) = -2 \neq 0$
$\therefore P_1, P_2, P_2$ are LI
$\therefore$ A is diagonalizable

Steps to check for diagonalizability

Let A be $(n \times n)$

1) Find the eigen vectors $P_1, P_2 \dots P_n$ and their corresponding eigen values $\lambda_1, \lambda_2 \dots \lambda_n$

2) If A has n LI eigen vectors, then A is diagonalizable
Define $P = [P_1 \mid P_2 \mid P_3 \dots]$

3) $D = P^{-1} \cdot A \cdot P$ will have the eigen values $\lambda_1, \lambda_2 \dots \lambda_n$ on its main diagonal and zeros in the rest

$$\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}\dots$$

ex:

Show that $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ is NOT diagonalizable

$\downarrow$

$\lambda_1 = \lambda_2 = 1$

The eigen vector

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow$ 1 eigen vector

$\therefore$ A is not dia.

ex:

Check for the diagonalizability of $A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix}$

$$\det(\lambda I - A) = (\lambda - 2)(\lambda + 2)(\lambda - 3) = 0 \quad \begin{cases} \lambda_1 = 2 \\ \lambda_2 = -2 \\ \lambda_3 = 3 \end{cases}$$

for $\lambda_1 = 2$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \rightarrow P_1$$

for $\lambda_2 = -2$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \rightarrow P_2$$

for $\lambda_3 = 3$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \rightarrow P_3$$

$$\Rightarrow \quad P = [P_1 | P_2 | P_3] = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 4 & 1 & 1 \end{bmatrix} \rightarrow \det(P) \neq 0$$

$\begin{array}{ccc} P_2 & P_3 & P_1 \end{array}$

$\therefore$ A is diagonalizable

$$D = P^{-1} \cdot A \cdot P = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Remark:

If all λ are distinct (multiplicity of each is 1) $\Rightarrow$ diagonalizable

Dot Product in $\mathbb{R}^n$

Length of a vector in $\mathbb{R}^n$:

The length / magnitude / norm of a vector $v = (v_1, v_2 \dots v_n)$ is

$$\| v \| = \sqrt{v_1^2 + v_2^2 \dots + v_n^2}$$

→ If $\| v \| = 1$, v is a unit vector

→ The length of a scalar vector cV is $\| cV \| = |c| \cdot \| v \|$

ex:

Let $v = (0, -2, 1, 4, -2) \in \mathbb{R}^5$, find $\| v \|$

$$\| v \| = \sqrt{0^2 + 2^2 + 1^2 + 4^2 + 2^2} = 5$$

→ To find a unit vector we have to divide every element of the vector v for its length $\| v \|$

→ If v is a non-zero vector in $\mathbb{R}^n$, then the vector $u = \dfrac{v}{\| v \|}$ has length $= 1$ and has the same direction as v. This vector u is called a unit vector in the direction of v.

Distance between 2 vectors

Let $\vec{u}$ and $\vec{v} \in \mathbb{R}^n$, then the distance between $\vec{u}$ and $\vec{v}$ is denoted by $d(\vec{u}, \vec{v})$ and it's equal to:

$$d(\vec{u}, \vec{v}) = \| \vec{u} - \vec{v} \|$$

ex:

Let $u = (0, 2, 2)$ and $v = (2, 0, 1)$

$$u - v = (0-2, 2-0, 2-1) = (-2, 2, 1)$$

$$\| u - v \| = \sqrt{(-2)^2 + (2)^2 + (1)^2} = 3$$

Dot Product in $\mathbb{R}^n$

Given two (or more) vectors $\vec{u}$ and $\vec{v}$, then the dot product is:

$$u \cdot v = u_1 v_1 + u_2 v_2 \dots + u_n v_n$$

ex:

$u = (1, 2, 0, -3)$, $v = (3, -2, 4, 2)$

$$u \cdot v = (1 \cdot 3) + (2 \cdot (-2)) + (0 \cdot 4) + (-3 \cdot 2) = -7$$

Properties of Dot Product

Let u, v and $w \in \mathbb{R}^n$ and c be any scalar

1) $u \cdot v = v \cdot u$
2) $u \cdot (v+w) = (u \cdot v) + (u \cdot w)$
3) $c \cdot (u \cdot v) = cu \cdot v = cv \cdot u$
4) $v \cdot v = \|v\|^2$
5) $v \cdot v \geq 0$ $\longrightarrow$ $(=0 \text{ iff } v=0)$

ex:

Assume that $u \cdot u = 39$, $u \cdot v = -3$, $v \cdot v = 79$

Evaluate $(u+2v) \cdot (3u+v)$

$$(u+2v) \cdot (3u+v) = u \cdot (3u) + u \cdot v + (2v)(3u) + (2v)v$$

$$= 3 \cdot (u \cdot u) + (u \cdot v) + 6 \cdot (v \cdot u) + 2 \cdot (v \cdot v)$$

$$= 3 \quad 39 + (-3) + 6 \cdot (-3) + 2 \cdot 79 = 254$$

Angle Between 2 Vectors

The angle between two non-zero vectors is given by :

$$\boxed{\theta = \arccos\left(\frac{u \cdot v}{\|u\| \cdot \|v\|}\right)}$$

★ $\vec{u} \cdot \vec{v} = \|u\| \cdot \|v\| \cdot \cos\theta$

ex:

$u = (-4, 0, 2, -2)$, $v = (2, 0, -1, 1)$

$$u \cdot v = (-4 \cdot 2) + (0 \cdot 0) + (2 \cdot -1) + (-2 \cdot 1) = -12$$

$$\|u\| = \sqrt{(-4)^2 + 0^2 + 2^2 + (-2)^2} = \sqrt{24}$$

$$\|v\| = \sqrt{2^2 + 0^2 + 1^2 + (-1)^2} = \sqrt{6}$$

$$\theta = \arccos\left(\frac{-12}{\sqrt{24} \cdot \sqrt{6}}\right) = \arccos(-1) = \pi$$

Theorem: 2 vectors $\vec{u}, \vec{v}$ are orthogonal iff $\vec{u} \cdot \vec{v} = 0$
when $\vec{u} \cdot \vec{v} = 0$, $\theta = \pi/2$

If $u, v \in \mathbb{R}^n$, then $|u \cdot v| \leq \|u\| \cdot \|v\|$
This is the Cauchy - Schwarz theorem

Proof : $|u \cdot v| = \|u\| \cdot \|v\| \cdot \underbrace{|\cos\theta|}_{-1 \leq \cos(\theta) \leq 1} \leq \|u\| \, \|v\|$

Triangular Inequality

$\|u + v\| \leq \|u\| + \|v\|$, not necessarily they are equal

Proof : $\|u + v\|^2 = (u+v)(u+v)$

$$= (v \cdot u) + (u \cdot v) + (v \cdot u) + (u \cdot v)$$

$$= \|v\| + 2(u \cdot v) + \|v\| \leq \|u\|^2 + 2|u \cdot v| + \|v\|^2$$

$$\leq \underbrace{\|u\|^2 + 2\|u\| \, \|v\| + \|v\|^2}_{= (\|u\| + \|v\|)^2}$$

→ If u and v are orthogonal then

$$\|u\|^2 + \|v\|^2 = \|u+v\|^2$$

→ we can write the vectors as matrices

$u, v \in \mathbb{R}^n$

$$\Rightarrow \quad u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \quad v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$ then $u\,v =$ matrix mult.

Inner Product

Also known as "eucledian inner product" and denoted by $\langle u, v \rangle$

→ Let $u, v, w \in V$. An inner product in V is a function that associates a real number $\langle u, v \rangle$ with each pair of vectors u, v with the conditions:

1) $\langle u, v \rangle = \langle v, u \rangle$

2) $\langle u, v+w \rangle = \langle u, v \rangle + \langle u, w \rangle$

3) $c \cdot \langle u, v \rangle = \langle cu, v \rangle$

4) $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0$ iff $v = 0$

ex:

Let $u = (u_1, u_2)$, $v = (v_1, v_2) \in \mathbb{R}^n$

Define $\langle u, v \rangle = u_1 v_1 + 2 u_2 v_2$

1) $\langle u, v \rangle = u_1 v_1 + u_2 v_2 = v_1 u_1 + 2 v_2 u_2$

2) $\langle u, v+w \rangle = u_1 (v_1 + w_1) + 2u_2 (v_2 , w_2)$

$= u_1 v_1 + u_1 w_1 + 2u_2 v_2 + 2u_2 w_2$

$= \langle u, v \rangle + \langle u, w \rangle$

3) $c \langle u, v \rangle = c(u_1 v_1 + 2u_2 v_2) = cu_1 v_1 + c2u_2 v_2$

$= \langle cu, v \rangle$

4) $\langle v, v \rangle = v_1 v_1 + 2v_2 v_2 = v_1^2 + 2v_2^2 \geq 0$

and $\langle v, v \rangle = 0$ iff $v_1^2 + 2v_2^2 = 0$

$v_1 = v_2 = 0 \quad \therefore \quad v = 0$

ex:

Let $V = M_{2,2}$, $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

Define $\langle A, B \rangle = a_{11} \cdot b_{11} + a_{12} \cdots + a_{22} \cdot b_{22}$

ex:

$u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3) \in \mathbb{R}^n$

check that $\langle u, v \rangle = u_1 v_1 - 2 u_2 v_2 + u_3 v_3$ is not an inner product

lets check the 4^{th} property / condition

for example $\Rightarrow$ $v = (1, 2, 1)$

$\langle u, v \rangle = 1 - 2(2)^2 + 1 = -6 < 0$

Norm, Distance and Angles

1) The Norm (length) of u is $\|u\| = \sqrt{\langle u, u \rangle}$

2) The distance between u and v,

$$d(u,v) = \|u-v\| = \sqrt{\langle u-v, v-u \rangle}$$

3) The

ex:

Let $f = a_0 + a_1 x + \dots + a_n x^n$

$g = b_0 + b_1 x + \dots + b_n x^n$

Let $\langle f, g \rangle = a_0 b_0 + a_1 b_1 + \dots + a_n b_n$

Let $f(x) = 1 - 2x^2$, $g(x) = 4 - 2x + x^2$, $r(x) = x + 2x^2 \in P_2$

$$\langle f, g \rangle = (1)(4) + (-2)(1) + (0)(-2) = 2$$

$$\langle g, r \rangle = (4)(0) + (-2)(1) + (1)(2) = 0 \Rightarrow$$ g and r are or thagonal

$$\|g\| = \sqrt{\langle g, g \rangle} = \sqrt{(4)(4) + (-2)(-2) + (1)(1)}$$

$$d(f,g) = \|f - g\| = \|-3 - 3x^2 + 2x\|$$

$$= \sqrt{(-3)(-3) + (-3)(-3) + (2)(2)} = \sqrt{22}$$

Theorem: Let u, v be vectors in inner product V

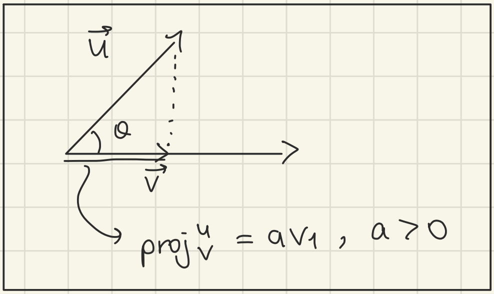
1) Cauchy Schwarz $\quad |\langle u,v\rangle| \leq \|u\| \cdot \|v\|$

2) Triangular Inequality $\quad \|u+v\| \leq \|u\| + \|v\|$

3) If u and v are orthagonal : $\|u+v\|^2 = \|u\|^2 + \|v\|^2$

Orthogonal Projection

→ Let u, v be 2 non-zero vectors in plane.



→ The projection of u over v is $proj_v^u$

→ The length of $proj_v^u$ is

$$\|av\| = |a| \cdot \|v\| = a\|v\| = \cos\theta \, \|u\|$$

$$= \frac{\|u\| \cdot \|v\| \cdot \cos\theta}{\|v\|} = \frac{u, v}{\|v\|}$$

$$\Rightarrow \boxed{a = \frac{u, v}{\|v\|^2} = \frac{u, v}{v, v}} \qquad \boxed{proj_v^u = \frac{u, v}{v, v} \cdot v}$$

Orthogonal and Orthonormal sets

A set S of vectors in an inner product space V is orthogonal if every paired vectors in S is orthogonal

If each vector is also an unit vector, S is orthonormal

$S = \{v_1, v_2 \dots, v_n\}$

1) S orthogonal if $\langle v_i, v_j \rangle = 0 \quad (\forall i \neq j)$

2) S is orthonormal if $\langle v_i, v_j \rangle = 0 \quad (\forall i \neq j)$ and
$\|v_i\| = 1 \quad (\forall i)$

ex:

the basis of R^n is orthonormal

$$S = \Big\{ \underbrace{\left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0\right)}_{v_1}, \underbrace{\left(\tfrac{-\sqrt{2}}{6}, \tfrac{\sqrt{2}}{6}, \tfrac{2\sqrt{2}}{3}\right)}_{v_2}, \underbrace{\left(\tfrac{2}{3}, \tfrac{-2}{3}, \tfrac{1}{3}\right)}_{v_3} \Big\}$$

$$\left.\begin{aligned} v_1, v_2 &= \left(\tfrac{1}{\sqrt{2}}\right)\left(\tfrac{-\sqrt{2}}{6}\right) + \left(\tfrac{1}{\sqrt{2}}\right)\left(\tfrac{\sqrt{2}}{6}\right) + 0 = 0 \ \checkmark \\ v_1, v_3 &= \left(\tfrac{1}{\sqrt{2}}\right)\left(\tfrac{2}{3}\right) + \left(\tfrac{1}{\sqrt{2}}\right)\left(\tfrac{-2}{3}\right) + 0 = 0 \ \checkmark \\ v_2, v_3 &= 0 \end{aligned}\right\} \text{S is orthogonal}$$

$$\left.\begin{aligned} \|v_1\| &= \sqrt{\left(\tfrac{1}{\sqrt{2}}\right)^2 + \left(\tfrac{1}{\sqrt{2}}\right)^2 + 0^2} = 1 \ \checkmark \\ \|v_2\| &= 1 \ \checkmark, \quad \|v_3\| = 1 \ \checkmark \end{aligned}\right\} \text{S is also orthonormal}$$

Theorem: If V is an inner product space of $\dim = n$, then any orthogonal set of n-nonzero vectors is a basis for V.

ex: Let $S = \{ \underbrace{(2,3,2,-2)}_{v_1}, \underbrace{(1,0,0,1)}_{v_2}, \underbrace{(-1,0,2,1)}_{v_3}, \underbrace{(-1,2,-1,1)}_{v_4} \}$

$|S| = 4$, $\dim(\mathbb{R}^4) = 4$

v_1, v_2 , v_2, v_3 , v_3, v_4 , v_1, v_3 , v_2, v_4 , v_1, v_4 $= 0$

Theorem: If $B = \{ v_1, v_2, \dots, v_n \}$ is orthonormal basis for V, then the coordinate representation of a vector w relative to B is :

$$\boxed{w = \langle w_1 v_1 \rangle v_1 + \langle w_1 v_2 \rangle v_2 \dots + \langle w_1 v_n \rangle v_n}$$

proof

$$w = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$$

$$\langle w_1, v_i \rangle = \langle c_1 v_1 + c_2 v_2 + \dots + c_n v_n , v_i \rangle$$

$$= \underbrace{\langle c_1 \cancel{v_1, v_i} \rangle}_{0} + c_2 \underbrace{\langle \cancel{v_2, v_i} \rangle}_{0} + \dots + c_i \underbrace{\langle v_i, v_i \rangle}_{\|v_i\|^2}$$

$$\therefore \langle w_1, v_i \rangle = c_i$$

ex:

$B = \{ (\frac{3}{5}, \frac{4}{5}, 0), (\frac{-4}{5}, \frac{3}{5}, 0), (0,0,1) \}$ is an orthonormal basis for $\mathbb{R}^3$. Let $w = (5,-5,2)$

$$[w]_B = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \qquad \begin{aligned} c_1 &= w, v_1 = -1 \\ c_2 &= w, v_2 = -7 \\ c_3 &= w, v_3 = 2 \end{aligned} \implies [w]_B = \begin{bmatrix} -1 \\ -7 \\ 2 \end{bmatrix}$$

Gramm – Schmidt Orthonormalization Process

Let $B = \{ v_1, v_2, \dots, v_n \}$ be a basis for inner product space and let $B' = \{ w_1, w_2, \dots, w_n \}$, w_i is given by:

$$w_1 = v_1 \qquad w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1$$

$$w_3 = v_3 - \frac{\langle v_3, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_3, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2$$

$$w_n = v_n - \frac{\langle v_n, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_n, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2 \cdots - \frac{\langle v_n, w_{n-1} \rangle}{\langle w_{n-1}, w_{n-1} \rangle} w_{n-1}$$

B' is orthogonal basis for V

Let $B'' = \{ u_1, u_2, \dots, u_n \}$ where $u_i = \dfrac{w_i}{\| w_i \|}$

B'' is orthonormal basis for V

ex:

Apply Gramm- Schmidt process to $B = \{ \underbrace{(1,1)}_{v_1}, \underbrace{(0,1)}_{v_2} \}$ for $\mathbb{R}^3$

$B' = \{ w_1, w_2 \}$

$w_1 = v_1 = (1,1)$ and

$$w_2 = v_2 - \frac{\overbrace{\langle v_2, w_1 \rangle}^{=1}}{\underbrace{\langle w_1, w_1 \rangle}_{= \|w_1\|^2 = 2}} w_1 = (0,1) - \frac{1}{2}(1,1) = \left(\frac{-1}{2}, \frac{1}{2}\right)$$

$\therefore$ $B' = \{ (1,1), \left(\frac{-1}{2}, \frac{1}{2}\right) \}$ is orthogonal basis for $\mathbb{R}^3$

$B'' = \{ u_1, u_2 \}$

$u_1 = \frac{w_1}{\|w_1\|} = \frac{1}{\sqrt{2}} \cdot (1,1)$ and

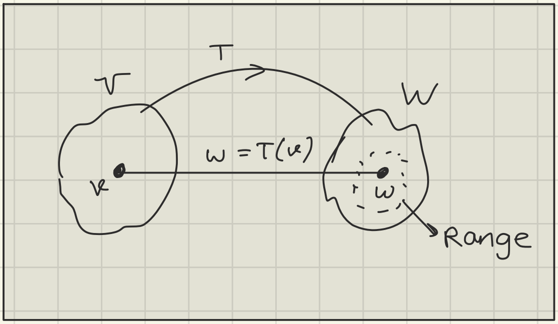
$$u_2 = \frac{w_2}{\|w_2\|} = \frac{1}{\frac{1}{\sqrt{2}}} \left(\frac{-1}{2}, \frac{1}{2}\right) = \left(\frac{-\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

$\therefore$ $B'' = \{ (1,1), \left(\frac{-\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \}$ is orthonormal basis for $\mathbb{R}^3$

Introduction to Linear Transformation

$T : V \longrightarrow W$, V : domain W : codomain

$v \in V_1$ $\exists$ $w \in W$ such that $T(v) = w$ then w is called the max of v under T



ex: $T : \mathbb{R}^2 \to \mathbb{R}^2$, $(v_1, v_2) \to T(v_1, v_2) = (v_1 - v_2 , v_1 + 2v_2)$

$T(-1, 2) = (-1-2 , -1 + 2 \cdot 2) = (-3, 3)$

Definition: The map $T : V \longrightarrow W$ is called Linear Transformation if $\forall\, u, v \in V$ and scalar c:

1) $T(u + v) = T(u) + T(v)$ 2) $T(cu) = c \cdot T(u)$

ex:

Let $T : M_{n,m} \to M_{n,m}$ be defined by $T(A) = A^T$

Let $A, B \in M_{n,m}$ and c is a scalar

$T(A+B) = (A+B)^T = A^T + B^T = T(A) + T(B)$ ✓

$T(cA) = (cA)^T = c \cdot A^T = c \cdot T(A)$ ✓

$\therefore$ The map $T : M_{n,m} \to M_{n,m}$ is a linear trans.

ex:

$f: \mathbb{R} \to \mathbb{R}$ is defined as $x \to \sin x$ / $f(x) = \sin x$

$f(x+y) = \sin(x+y) \neq \sin(x) + \sin(y)$ ∴ not linear

ex:

$f: \mathbb{R} \to \mathbb{R}^2$ is defined as $f(x) = x^2$

$f(x+y) = (x+y)^2 \neq x^2 + y^2$ ∴ not linear

$f(cx) = (cx)^2 = c^2 x^2 = c^2 \cdot f(x)$ ∴ not linear

ex:

$f: \mathbb{R} \to \mathbb{R}$ is defined as $f(x) = x+1$

$f(x+y) = (x+y) + 1 \neq f(x) + f(y)$

$f(x) + f(y) = x+1+y+1$ ∴ not linear

Properties of Linear Transformation

1) $T(0) = 0$ $\longrightarrow$ $T(0) = T(0 \cdot u) = 0 \cdot T(u) = 0$

2) $T(-v) = -T(v)$ $\longrightarrow$ $T(-v) = T(-1 \cdot v) = -1 \cdot T(v)$

3) $T(u-v) = T(u) - T(v) \to T(u-v) = T(u+(-v))$

$= T(u) + T(-v) = T(u) - T(v)$

from #2

ex:

Show that $T(v) = Av$ is a linear from $R^n \to R^m$ when A is $(m \times n)$ matrix

v in $R^n \to v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ $Av = A \cdot \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$

$$T(u+v) = A(u+v) = Au + Av = T(u) + T(v)$$

$$T(cu) = A(cu) = c\,Au = c \cdot T(u)$$

Standard matrix for Linear Transformation

Let $T: R^n \to R^m$ be a linear transformation

$$T(e_1) = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} \quad T(e_2) = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} \quad \cdots \quad T(e_n) = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Then the $(m \times n)$ matrix

$$A_{m \times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is such that $T(v) = Av$ for every v in R^n.

A is called the standard matrix for linear transformation T.

ex:

Let $T: R^3 \longrightarrow R^2$ is a lin. trans. defined by

$$T(x, y, z) = (x - 2y,\ 2x + y)$$

$$T(e_1) = T(1,0,0) = (1,2)$$

$$T(e_2) = T(0,1,0) = (-2,1)$$

$$T(e_3) = T(0,0,1) = (0,0)$$

$$B = \{(1,0,0), (0,1,0), (0,0,1)\}$$

$$A = \left[\, T(e_1) \mid T(e_2) \mid T(e_3) \,\right] = \begin{array}{c} \begin{bmatrix} \boxed{1 \quad -2 \quad 0} \\ 2 \quad\ 1 \quad\ 0 \end{bmatrix} \\ \ \ x \quad\ y \quad\ z \end{array} \rightarrow 1 \cdot x - 2 \cdot y + 0 \cdot z$$

$$\boxed{T(v) = Av}$$

ex:

$$T(1,0,1) = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = (1,2)$$

ex:

$T: R^3 \longrightarrow R^3$ is a linear transformation defined by

$$T(x_1, x_2, x_3) = (x_1 - 2x_2 + 3x_3,\ 2x_1 + x_3,\ 4x_1 + x_2 - 2x_3)$$

$$A = \begin{bmatrix} 1 & -2 & 5 \\ 2 & 0 & 1 \\ 4 & 1 & -2 \end{bmatrix}$$

Composition of Linear Transformation

Let $\overset{\curvearrowright A_1}{T_1}: R^n \to R^m$, $\overset{\curvearrowright A_2}{T_2}: R^m \longrightarrow R^p$

Then the composition T denoted by $T = T_2 \circ T_1$ is $T(v) = (T_2 \circ T_1)(v) = T_2(T_1(v))$

$\rightarrow$ Standard matrix $\Rightarrow A = A_2 \cdot A_1$ (order is important!)

Inverse of Linear Transformation

Let $T_1: R^n \to R^n$, $T_2: R^n \longrightarrow R^n$ are linear transformations such that for every $v \in R^n$:

$$T_2(T_1(v)) = v \quad \text{and} \quad T_1(T_2(v)) = v$$

Then T_2 is the inverse of T_1 and T_1 is said to be invertible

Theorem: $T: R^n \to R^n$ is invertible $\Rightarrow A$ is invertible

ex:

$T: R^3 \to R^3$ is a linear transformation defined by

$T(x_1, x_2, x_3) = (2x_1 + 3x_2 + x_3,\ 3x_1 + 3x_2 + x_3,\ 2x_1 + 4x_2 + x_3)$

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}$$

A is invertible bc $\det(A) \neq 0$

T is invertible bc A is invertible

$$T^{-1}(v) = A^{-1} v = \overbrace{\begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}}^{A^{-1}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_1 + x_2 \\ -x_1 + x_3 \\ 6x_1 - 2x_2 - 3x_3 \end{bmatrix}$$

Kernel of a Linear Transformation

Let $T: V \to W$ be a linear transformation

$$\ker(T) = \{ v \in V \mid T(v) = 0 \}$$

all vectors that go to 0

★ Kernel of T is a subspace of V

ex:

$T: M_{3,2} \to M_{2,3}$ be a linear trans. defined by $T(A) = A^T$

$$\ker(T) = \{ A \in M_{3,2} \mid T(A) = 0 \}$$

$$= \{ A \in M_{3,2} \mid A^T = 0 \} = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

Theorem: Let $T: R^n \to R^m$ be a lin. tr. given by $T(x) = Ax$, then $\underline{Ker(T)}$ is the soln of nullspace $Ax=0$

↳ since kernel is a subspace we can do basis operations

ex:

Find Ker (T) where $T: R^3 \to R^2$ defined by $T(x) = Ax$ where $A = \begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{bmatrix}$

$$Ker(T) = \{ x \,/\, Ax = 0 \}$$

$$A = \begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{bmatrix} \xrightarrow{\text{row ech form}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \quad \begin{cases} x_1 - x_3 = 0 \\ x_2 + x_3 = 0 \end{cases}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$B = \{ (1, -1, 1) \}$ basis for kernel

$dim(Ker(T)) = 1$

Range of a Linear Transformation

$T: V \to W$

$Range(T) = \{ T(v) \,/\, v \in V \}$ is a subspace of W

$T(x) = Ax$

The column space of A = Range of T

ex:

Let $T: \mathbb{R}^5 \longrightarrow \mathbb{R}^4$ is a lin. tr. defined by $T(x) = Ax$

$$A = \begin{array}{c} \begin{bmatrix} 1 & 2 & 0 & 1 & -1 \\ 2 & 1 & 3 & 1 & 0 \\ -1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 8 \end{bmatrix} \\ \begin{array}{ccccc} c_1 & c_2 & & c_3 & \end{array} \end{array} \xrightarrow{\text{r.e.f}} \begin{array}{c} A^T \\ \begin{bmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & -1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccccc} w_1 & w_2 & & w_4 & \end{array} \end{array}$$

basis for the column space

dim (range) = 3